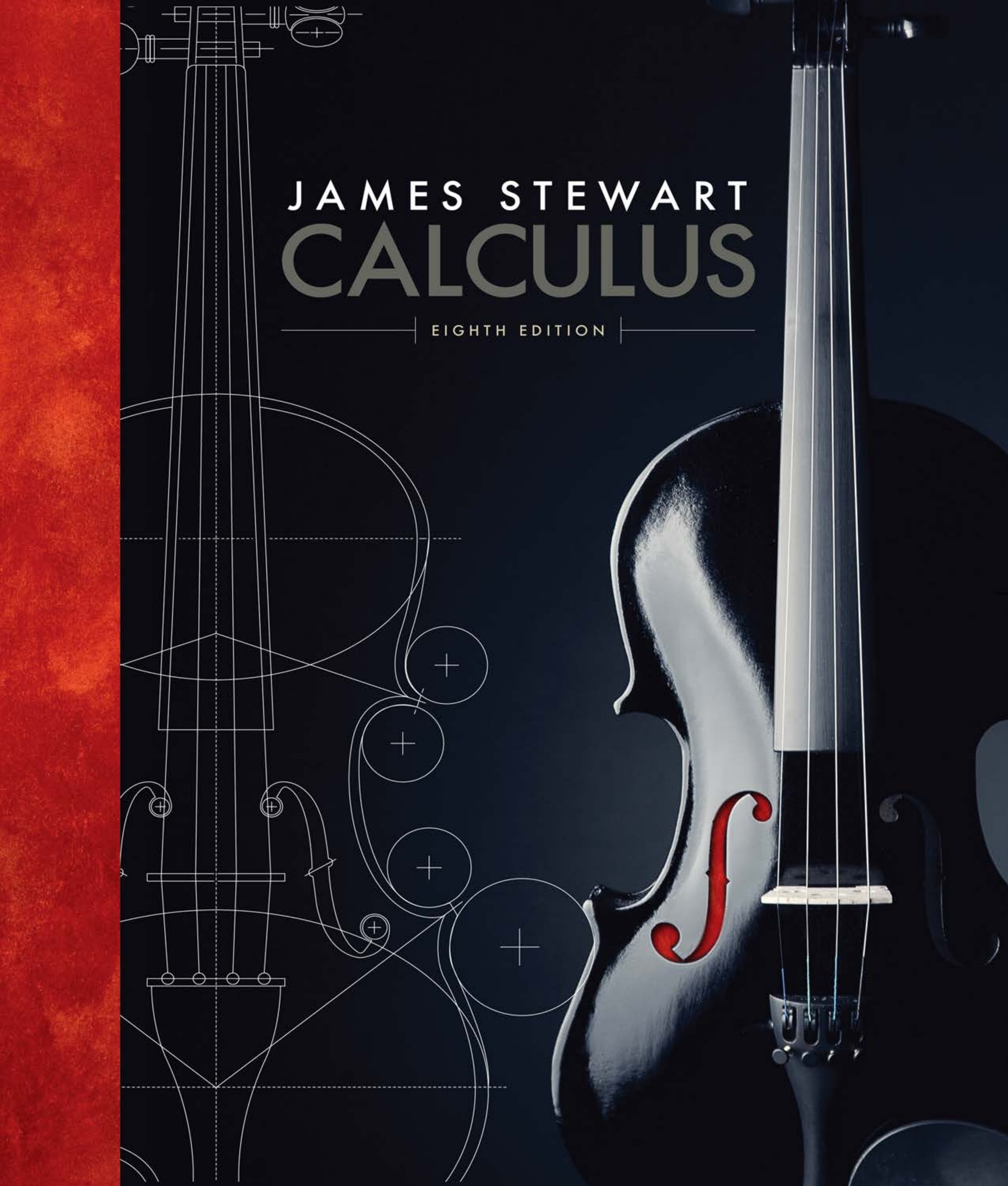
JAMES STEWART

CALCULUS

EIGHTH EDITION



CALCULUS

EIGHTH EDITION

JAMES STEWART

McMASTER UNIVERSITY

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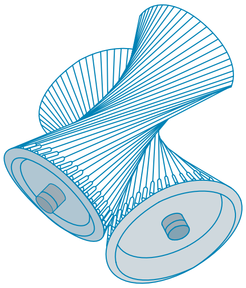
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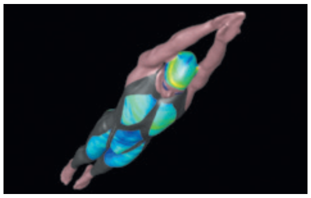
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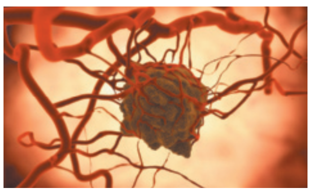
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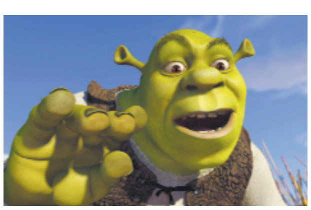
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Preface

> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
>
> GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. I have tried to write a book that assists students in discovering calculus—both for its practical power and its surprising beauty. In this edition, as in the first seven editions, I aim to convey to the student a sense of the utility of calculus and develop technical competence, but I also strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. I want students to share some of that excitement.

The emphasis is on understanding concepts. I think that nearly everybody agrees that this should be the primary goal of calculus instruction. In fact, the impetus for the current calculus reform movement came from the Tulane Conference in 1986, which formulated as their first recommendation:

Focus on conceptual understanding.

I have tried to implement this goal through the *Rule of Three:* "Topics should be presented geometrically, numerically, and algebraically." Visualization, numerical and graphical experimentation, and other approaches have changed how we teach conceptual reasoning in fundamental ways. More recently, the Rule of Three has been expanded to become the *Rule of Four* by emphasizing the verbal, or descriptive, point of view as well.

In writing the eighth edition my premise has been that it is possible to achieve conceptual understanding and still retain the best traditions of traditional calculus. The book contains elements of reform, but within the context of a traditional curriculum.

Alternate Versions

I have written several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus: Early Transcendentals,* Eighth Edition, is similar to the present textbook except that the exponential, logarithmic, and inverse trigonometric functions are covered in the first semester.
- *Essential Calculus*, Second Edition, is a much briefer book (840 pages), though it contains almost all of the topics in *Calculus,* Eighth Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.

- *Essential Calculus: Early Transcendentals,* Second Edition, resembles *Essential Calculus,* but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Calculus: Early Vectors* introduces vectors and vector functions in the first semester and integrates them throughout the book. It is suitable for students taking engineering and physics courses concurrently with calculus.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.
- *Biocalculus: Calculus for the Life Sciences* is intended to show students in the life sciences how calculus relates to biology.
- *Biocalculus: Calculus, Probability, and Statistics for the Life Sciences* contains all the content of *Biocalculus: Calculus for the Life Sciences* as well as three additional chapters covering probability and statistics.

What's New in the Eighth Edition?

The changes have resulted from talking with my colleagues and students at the University of Toronto and from reading journals, as well as suggestions from users and reviewers. Here are some of the many improvements that I've incorporated into this edition:

- The data in examples and exercises have been updated to be more timely.
- New examples have been added (see Examples 5.1.5, 11.2.5, and 14.3.3, for instance). And the solutions to some of the existing examples have been amplified.
- Three new projects have been added: The project *Planes and Birds: Minimizing Energy* (page 271) asks how birds can minimize power and energy by flapping their wings versus gliding. The project *Controlling Red Blood Cell Loss During Surgery* (page 473) describes the ANH procedure, in which blood is extracted from the patient before an operation and is replaced by saline solution. This dilutes the patient's blood so that fewer red blood cells are lost during bleeding and the extracted blood is returned to the patient after surgery. In the project *The Speedo LZR Racer* (page 976) it is explained that this suit reduces drag in the water and, as a result, many swimming records were broken. Students are asked why a small decrease in drag can have a big effect on performance.
- I have streamlined Chapter 15 (Multiple Integrals) by combining the first two sections so that iterated integrals are treated earlier.
- More than 20% of the exercises in each chapter are new. Here are some of my favorites: 2.1.61, 2.2.34–36, 3.3.30, 3.3.54, 3.7.39, 3.7.67, 4.1.19–20, 4.2.67–68, 4.4.63, 5.1.51, 6.2.79, 6.7.54, 6.8.90, 8.1.39, 12.5.81, 12.6.29–30, 14.6.65–66. In addition, there are some good new Problems Plus. (See Problems 10–12 on page 201, Problem 10 on page 290, Problems 14–15 on pages 353–54, and Problem 8 on page 1026.)

Features

Conceptual Exercises

The most important way to foster conceptual understanding is through the problems that we assign. To that end I have devised various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section. (See, for instance, the first few exercises in Sections 1.5, 1.8, 11.2, 14.2, and 14.3.) Similarly, all the review sections begin with a Concept Check and a True-False Quiz. Other exercises test conceptual understanding through graphs or tables (see Exercises 2.1.17, 2.2.33–36, 2.2.45–50, 9.1.11–13, 10.1.24–27, 11.10.2, 13.2.1–2, 13.3.33–39, 14.1.1–2, 14.1.32–38, 14.1.41–44, 14.3.3–10, 14.6.1–2, 14.7.3–4, 15.1.6–8, 16.1.11–18, 16.2.17–18, and 16.3.1–2).

Another type of exercise uses verbal description to test conceptual understanding (see Exercises 1.8.10, 2.2.64, 3.3.57–58, and 7.8.67). I particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercises 2.7.25, 3.4.33–34, and 9.4.4).

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises and skill-development problems to more challenging problems involving applications and proofs.

Real-World Data

My assistants and I spent a great deal of time looking in libraries, contacting companies and government agencies, and searching the Internet for interesting real-world data to introduce, motivate, and illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. See, for instance, Figure 1 in Section 1.1 (seismograms from the Northridge earthquake), Exercise 2.2.33 (unemployment rates), Exercise 4.1.16 (velocity of the space shuttle *Endeavour*), and Figure 4 in Section 4.4 (San Francisco power consumption). Functions of two variables are illustrated by a table of values of the wind-chill index as a function of air temperature and wind speed (Example 14.1.2). Partial derivatives are introduced in Section 14.3 by examining a column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity. This example is pursued further in connection with linear approximations (Example 14.4.3). Directional derivatives are introduced in Section 14.6 by using a temperature contour map to estimate the rate of change of temperature at Reno in the direction of Las Vegas. Double integrals are used to estimate the average snowfall in Colorado on December 20–21, 2006 (Example 15.1.9). Vector fields are introduced in Section 16.1 by depictions of actual velocity vector fields showing San Francisco Bay wind patterns.

Projects

One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. I have included four kinds of projects: *Applied Projects* involve applications that are designed to appeal to the imagination of students. The project after Section 9.3 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height. (The answer might surprise you.) The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired

velocity. *Laboratory Projects* involve technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer. *Writing Projects* ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance. Suggested references are supplied. *Discovery Projects* anticipate results to be discussed later or encourage discovery through pattern recognition (see the one following Section 7.6). Others explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.6), and intersections of three cylinders (after Section 15.7). Additional projects can be found in the *Instructor's Guide* (see, for instance, Group Exercise 4.1: Position from Samples).

Problem Solving

Students usually have difficulties with problems for which there is no single well-defined procedure for obtaining the answer. I think nobody has improved very much on George Polya's four-stage problem-solving strategy and, accordingly, I have included a version of his problem-solving principles following Chapter 1. They are applied, both explicitly and implicitly, throughout the book. After the other chapters I have placed sections called *Problems Plus*, which feature examples of how to tackle challenging calculus problems. In selecting the varied problems for these sections I kept in mind the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." When I put these challenging problems on assignments and tests I grade them in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant.

Dual Treatment of Exponential and Logarithmic Functions

There are two possible ways of treating the exponential and logarithmic functions and each method has its passionate advocates. Because one often finds advocates of both approaches teaching the same course, I include full treatments of both methods. In Sections 6.2, 6.3, and 6.4 the exponential function is defined first, followed by the logarithmic function as its inverse. (Students have seen these functions introduced this way since high school.) In the alternative approach, presented in Sections 6.2*, 6.3*, and 6.4*, the logarithm is defined as an integral and the exponential function is its inverse. This latter method is, of course, less intuitive but more elegant. You can use whichever treatment you prefer.

If the first approach is taken, then much of Chapter 6 can be covered before Chapters 4 and 5, if desired. To accommodate this choice of presentation there are specially identified problems involving integrals of exponential and logarithmic functions at the end of the appropriate sections of Chapters 4 and 5. This order of presentation allows a faster-paced course to teach the transcendental functions and the definite integral in the first semester of the course.

For instructors who would like to go even further in this direction I have prepared an alternate edition of this book, called *Calculus: Early Transcendentals,* Eighth Edition, in which the exponential and logarithmic functions are introduced in the first chapter. Their limits and derivatives are found in the second and third chapters at the same time as polynomials and the other elementary functions.

Tools for Enriching Calculus

TEC is a companion to the text and is intended to enrich and complement its contents. (It is now accessible in the eBook via CourseMate and Enhanced WebAssign. Selected Visuals and Modules are available at www.stewartcalculus.com.) Developed by Harvey Keynes, Dan Clegg, Hubert Hohn, and myself, TEC uses a discovery and exploratory

approach. In sections of the book where technology is particularly appropriate, marginal icons direct students to TEC Modules that provide a laboratory environment in which they can explore the topic in different ways and at different levels. **Visuals are animations of figures in text; Modules are more elaborate activities and include exercises**. Instructors can choose to become involved at several different levels, ranging from simply encouraging students to use the Visuals and Modules for independent exploration, to assigning specific exercises from those included with each Module, or to creating additional exercises, labs, and projects that make use of the Visuals and Modules.

TEC also includes Homework Hints for representative exercises (usually odd-numbered) in every section of the text, indicated by printing the exercise number in red. These hints are usually presented in the form of questions and try to imitate an effective teaching assistant by functioning as a silent tutor. They are constructed so as not to reveal any more of the actual solution than is minimally necessary to make further progress.

Enhanced WebAssign

Technology is having an impact on the way homework is assigned to students, particularly in large classes. The use of online homework is growing and its appeal depends on ease of use, grading precision, and reliability. With the Eighth Edition we have been working with the calculus community and WebAssign to develop an online homework system. Up to 70% of the exercises in each section are assignable as online homework, including free response, multiple choice, and multi-part formats.

The system also includes Active Examples, in which students are guided in step-by-step tutorials through text examples, with links to the textbook and to video solutions.

Website

Visit CengageBrain.com or stewartcalculus.com for these additional materials:

- Homework Hints
- Algebra Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to the better historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Formulas for the Remainder Term in Taylor Series, Rotation of Axes
- Archived Problems (drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources
- Selected Visuals and Modules from Tools for Enriching Calculus (TEC)

Content

Diagnostic Tests The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

A Preview of Calculus This is an overview of the subject and includes a list of questions to motivate the study of calculus.

1 Functions and Limits

From the beginning, multiple representations of functions are stressed: verbal, numerical, visual, and algebraic. A discussion of mathematical models leads to a review of the standard functions from these four points of view. The material on limits is motivated by a prior discussion of the tangent and velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Section 1.7, on the precise epsilon-delta defintion of a limit, is an optional section.

2 Derivatives

The material on derivatives is covered in two sections in order to give students more time to get used to the idea of a derivative as a function. The examples and exercises explore the meanings of derivatives in various contexts. Higher derivatives are introduced in Section 2.2.

3 Applications of Differentiation

The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. Graphing with technology emphasizes the interaction between calculus and calculators and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head 42° to see the top of a rainbow.

4 Integrals

The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix E.) Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.

5 Applications of Integration

Here I present the applications of integration—area, volume, work, average value—that can reasonably be done without specialized techniques of integration. General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral.

6 Inverse Functions: Exponential, Logarithmic, and Inverse Trigonometric Functions

As discussed more fully on page xiv, only one of the two treatments of these functions need be covered. Exponential growth and decay are covered in this chapter.

7 Techniques of Integration

All the standard methods are covered but, of course, the real challenge is to be able to recognize which technique is best used in a given situation. Accordingly, in Section 7.5, I present a strategy for integration. The use of computer algebra systems is discussed in Section 7.6.

8 Further Applications of Integration

Here are the applications of integration—arc length and surface area—for which it is useful to have available all the techniques of integration, as well as applications to biology, economics, and physics (hydrostatic force and centers of mass). I have also included a section on probability. There are more applications here than can realistically be covered in a given course. Instructors should select applications suitable for their students and for which they themselves have enthusiasm.

9 Differential Equations

Modeling is the theme that unifies this introductory treatment of differential equations. Direction fields and Euler's method are studied before separable and linear equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal

consideration. These methods are applied to the exponential, logistic, and other models for population growth. The first four or five sections of this chapter serve as a good introduction to first-order differential equations. An optional final section uses predator-prey models to illustrate systems of differential equations.

10 Parametric Equations and Polar Coordinates

This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to laboratory projects; the two presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 13.

11 Infinite Sequences and Series

The convergence tests have intuitive justifications (see page 759) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics. Error estimates include those from graphing devices.

12 Vectors and the Geometry of Space

The material on three-dimensional analytic geometry and vectors is divided into two chapters. Chapter 12 deals with vectors, the dot and cross products, lines, planes, and surfaces.

13 Vector Functions

This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

14 Partial Derivatives

Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, I introduce partial derivatives by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.

15 Multiple Integrals

Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute probabilities, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals.

16 Vector Calculus

Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

17 Second-Order Differential Equations

Since first-order differential equations are covered in Chapter 9, this final chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions.

Ancillaries

Calculus, Eighth Edition, is supported by a complete set of ancillaries developed under my direction. Each piece has been designed to enhance student understanding and to facilitate creative instruction. The tables on pages xxi–xxii describe each of these ancillaries.

Acknowledgments

The preparation of this and previous editions has involved much time spent reading the reasoned (but sometimes contradictory) advice from a large number of astute reviewers. I greatly appreciate the time they spent to understand my motivation for the approach taken. I have learned something from each of them.

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To the Student

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, I suggest that you cover up the solution and try solving the problem yourself. You'll get a lot more from looking at the solution if you do so.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix H. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from mine, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're right and rationalizing the denominator will show that the answers are equivalent.

The icon ⊞ indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software. But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol CAS is reserved for problems in which the full resources of a computer algebra system (like Maple, Mathematica, or the TI-89) are required.

You will also encounter the symbol ⊘, which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

Tools for Enriching Calculus, which is a companion to this text, is referred to by means of the symbol TEC and can be accessed in the eBook via Enhanced WebAssign and CourseMate (selected Visuals and Modules are available at www.stewartcalculus.com). It directs you to modules in which you can explore aspects of calculus for which the computer is particularly useful.

You will notice that some exercise numbers are printed in red: 5. This indicates that *Homework Hints* are available for the exercise. These hints can be found on stewartcalculus.com as well as Enhanced WebAssign and CourseMate. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. You need to pursue each hint in an active manner with pencil and paper to work out the details. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

I recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. I hope you will discover that it is not only useful but also intrinsically beautiful.

JAMES STEWART

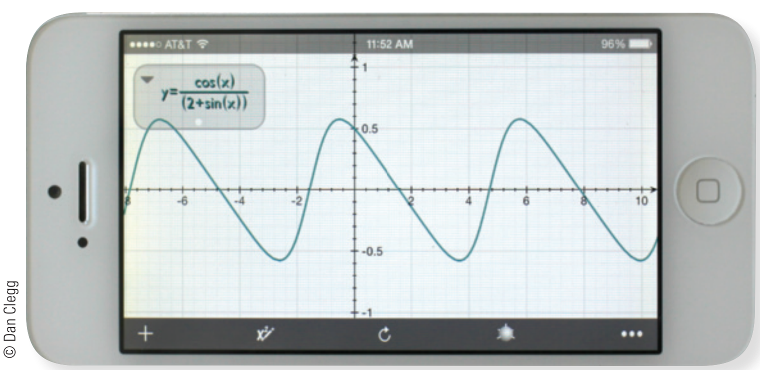
Calculators, Computers, and Other Graphing Devices



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Advances in technology continue to bring a wider variety of tools for doing mathematics. Handheld calculators are becoming more powerful, as are software programs and Internet resources. In addition, many mathematical applications have been released for smartphones and tablets such as the iPad.

Some exercises in this text are marked with a graphing icon, which indicates that the use of some technology is required. Often this means that we intend for a graphing device to be used in drawing the graph of a function or equation. You might also need technology to find the zeros of a graph or the points of intersection of two graphs. In some cases we will use a calculating device to solve an equation or evaluate a definite integral numerically. Many scientific and graphing calculators have these features built in, such as the Texas Instruments TI-84 or TI-Nspire CX. Similar calculators are made by Hewlett Packard, Casio, and Sharp.

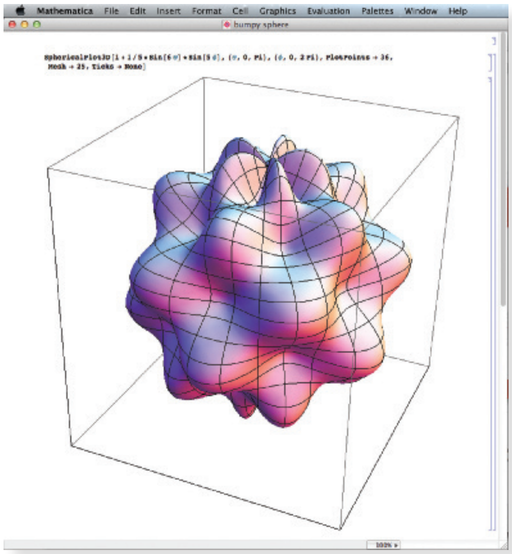


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You can also use computer software such as *Graphing Calculator* by Pacific Tech (www.pacifict.com) to perform many of these functions, as well as apps for phones and tablets, like Quick Graph (Colombiamug) or Math-Studio (Pomegranate Apps). Similar functionality is available using a web interface at WolframAlpha.com.



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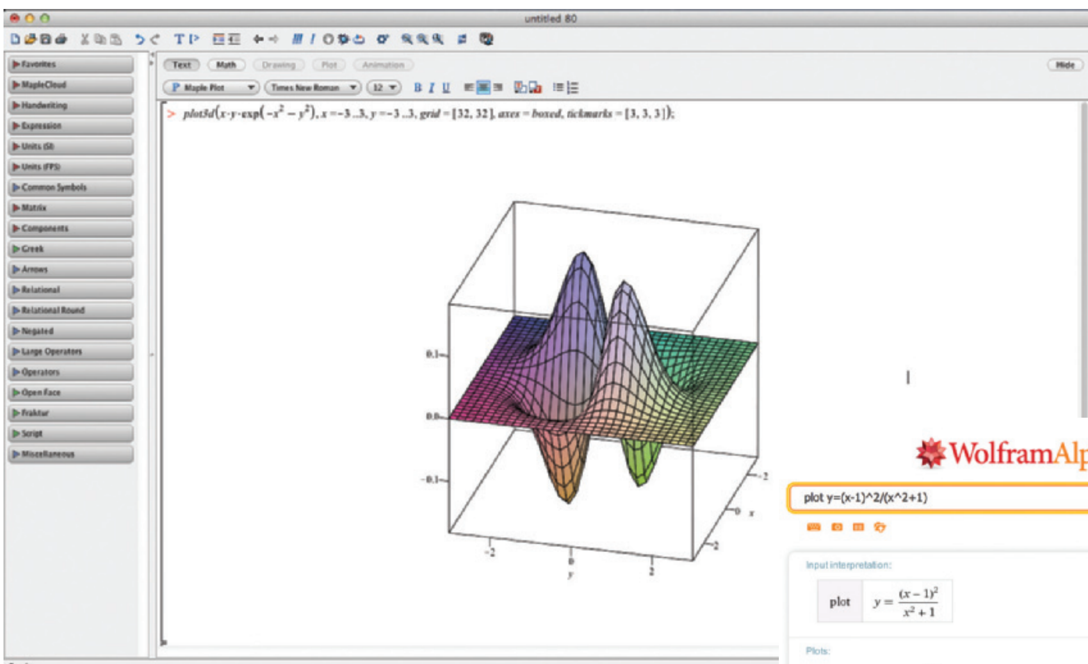
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In general, when we use the term "calculator" in this book, we mean the use of any of the resources we have mentioned.

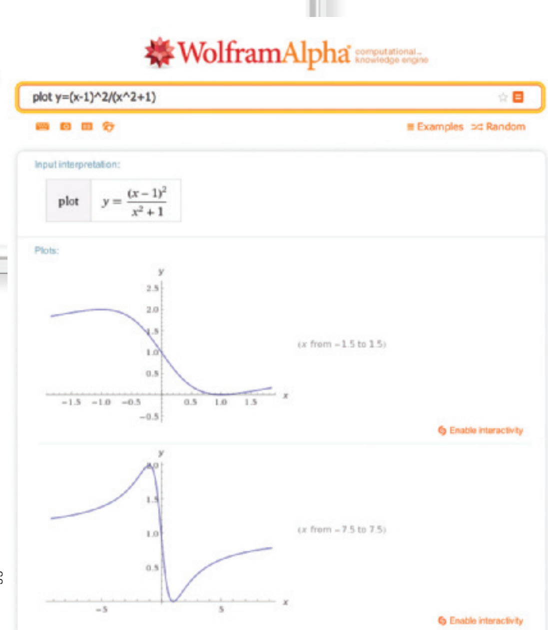
The CAS icon is reserved for problems in which the full resources of a *computer algebra system* (CAS) are required. A CAS is capable of doing mathematics (like solving equations, computing derivatives or integrals) *symbolically* rather than just numerically.

Examples of well-established computer algebra systems are the computer software packages Maple and Mathematica. The WolframAlpha website uses the Mathematica engine to provide CAS functionality via the Web.

Many handheld graphing calculators have CAS capabilities, such as the TI-89 and TI-Nspire CX CAS from Texas Instruments. Some tablet and smartphone apps also provide these capabilities, such as the previously mentioned MathStudio.



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Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$

(b) $(3a^3b^3)(4ab^2)^2$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right)$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5}-2}$ (b) $\dfrac{\sqrt{4+h}-2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leqslant 17$ (b) $x^2 < 2x + 8$

(c) $x(x - 1)(x + 2) > 0$ (d) $|x - 4| < 3$

(e) $\dfrac{2x-3}{x+1} \leqslant 1$

10. State whether each equation is true or false.

(a) $(p + q)^2 = p^2 + q^2$ (b) $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$ (d) $\dfrac{1 + TC}{C} = 1 + T$

(e) $\dfrac{1}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$ (f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a-b}$

ANSWERS TO DIAGNOSTIC TEST A: ALGEBRA

1. (a) 81 (b) -81 (c) $\frac{1}{81}$
(d) 25 (e) $\frac{9}{4}$ (f) $\frac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x - 5)(2x + 5)$ (b) $(2x - 3)(x + 4)$
(c) $(x - 3)(x - 2)(x + 2)$ (d) $x(x + 3)(x^2 - 3x + 9)$
(e) $3x^{-1/2}(x - 1)(x - 2)$ (f) $xy(x - 2)(x + 2)$

5. (a) $\dfrac{x+2}{x-2}$ (b) $\dfrac{x-1}{x-3}$
(c) $\dfrac{1}{x-2}$ (d) $-(x + y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4+h}+2}$

7. (a) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ (b) $2(x - 3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\frac{2}{3}, \frac{22}{3}$
(g) $\frac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4]$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you had difficulty with these problems, you may wish to consult the Review of Algebra on the website **www.stewartcalculus.com**.

B Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point $(2, -5)$ and
(a) has slope -3
(b) is parallel to the x-axis
(c) is parallel to the y-axis
(d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
(a) Find the slope of the line that contains A and B.
(b) Find an equation of the line that passes through A and B. What are the intercepts?
(c) Find the midpoint of the segment AB.
(d) Find the length of the segment AB.
(e) Find an equation of the perpendicular bisector of AB.
(f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.
(a) $-1 \leqslant y \leqslant 3$
(b) $|x| < 4$ and $|y| < 2$
(c) $y < 1 - \frac{1}{2}x$
(d) $y \geqslant x^2 - 1$
(e) $x^2 + y^2 < 4$
(f) $9x^2 + 16y^2 = 144$

ANSWERS TO DIAGNOSTIC TEST B: ANALYTIC GEOMETRY

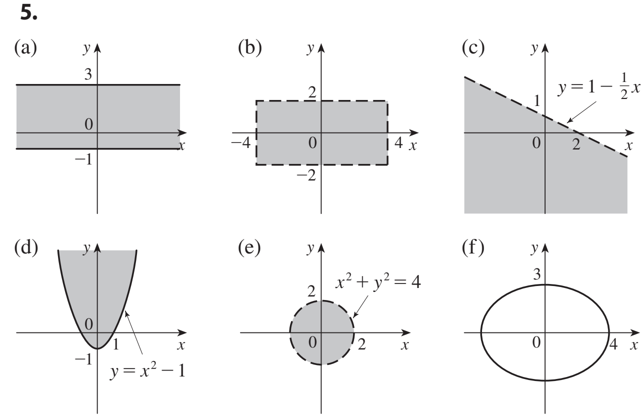
1. (a) $y = -3x + 1$ (b) $y = -5$
(c) $x = 2$ (d) $y = \frac{1}{2}x - 6$

2. $(x + 1)^2 + (y - 4)^2 = 52$

3. Center $(3, -5)$, radius 5

4. (a) $-\frac{4}{3}$
(b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
(c) $(-1, -4)$
(d) 20
(e) $3x - 4y = 13$
(f) $(x + 1)^2 + (y + 4)^2 = 100$

5.



If you had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendixes B and C.

C Diagnostic Test: Functions

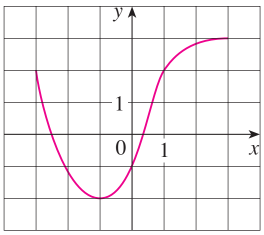


FIGURE FOR PROBLEM 1

1. The graph of a function f is given at the left.
  (a) State the value of $f(-1)$.
  (b) Estimate the value of $f(2)$.
  (c) For what values of x is $f(x) = 2$?
  (d) Estimate the values of x such that $f(x) = 0$.
  (e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2 + h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.
  (a) $f(x) = \dfrac{2x + 1}{x^2 + x - 2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2 + 1}$ (c) $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

4. How are graphs of the functions obtained from the graph of f?
  (a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x - 3) + 2$

5. Without using a calculator, make a rough sketch of the graph.
  (a) $y = x^3$ (b) $y = (x + 1)^3$ (c) $y = (x - 2)^3 + 3$
  (d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
  (g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$
  (a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
  (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

ANSWERS TO DIAGNOSTIC TEST C: FUNCTIONS

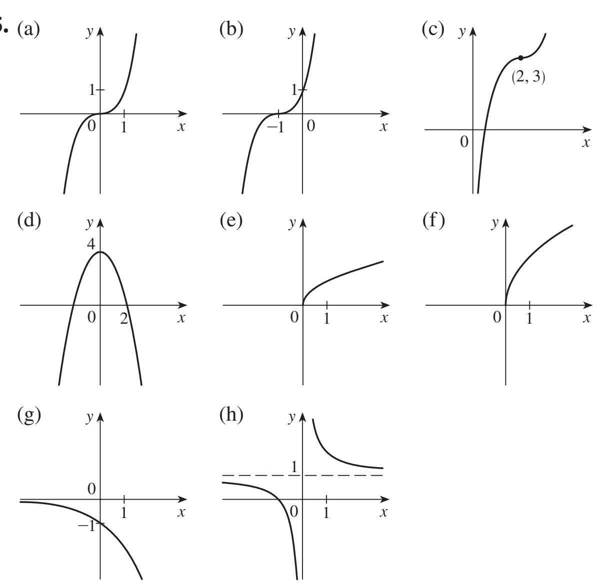
1. (a) -2 (b) 2.8
  (c) $-3, 1$ (d) $-2.5, 0.3$
  (e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

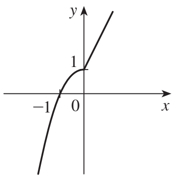
3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
  (b) $(-\infty, \infty)$
  (c) $(-\infty, -1] \cup [1, 4]$

4. (a) Reflect about the x-axis
  (b) Stretch vertically by a factor of 2, then shift 1 unit downward
  (c) Shift 3 units to the right and 2 units upward

5.



6. (a) $-3, 3$ (b)



7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$
(b) $(g \circ f)(x) = 2x^2 + 4x - 5$
(c) $(g \circ g \circ g)(x) = 8x - 21$

If you had difficulty with these problems, you should look at sections 1.1–1.3 of this book.

D Diagnostic Test: Trigonometry

1. Convert from degrees to radians.
(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.
(a) $5\pi/6$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of 30°.

4. Find the exact values.
(a) $\tan(\pi/3)$ (b) $\sin(7\pi/6)$ (c) $\sec(5\pi/3)$

5. Express the lengths a and b in the figure in terms of θ.

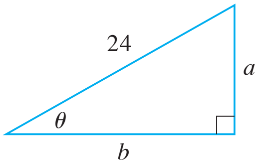


FIGURE FOR PROBLEM 5

6. If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.

7. Prove the identities.
(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$ (b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \leqslant x \leqslant 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

ANSWERS TO DIAGNOSTIC TEST D: TRIGONOMETRY

1. (a) $5\pi/3$ (b) $-\pi/10$

2. (a) $150°$ (b) $360°/\pi \approx 114.6°$

3. 2π cm

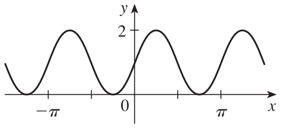
4. (a) $\sqrt{3}$ (b) $-\frac{1}{2}$ (c) 2

5. (a) $24\sin\theta$ (b) $24\cos\theta$

6. $\frac{1}{15}\left(4 + 6\sqrt{2}\right)$

8. $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



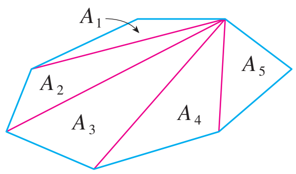
If you had difficulty with these problems, you should look at Appendix D of this book.

A Preview of Calculus

By the time you finish this course, you will be able to calculate the length of the curve used to design the Gateway Arch in St. Louis, determine where a pilot should start descent for a smooth landing, compute the force on a baseball bat when it strikes the ball, and measure the amount of light sensed by the human eye as the pupil changes size.



CALCULUS IS FUNDAMENTALLY DIFFERENT FROM the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. Here we give a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.



$A = A_1 + A_2 + A_3 + A_4 + A_5$

FIGURE 1

The Area Problem

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

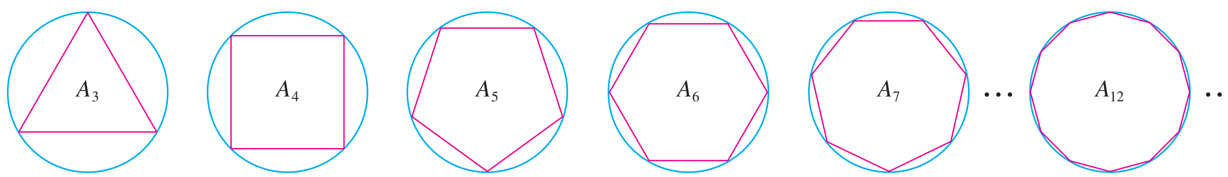


FIGURE 2

Let A_n be the area of the inscribed polygon with n sides. As n increases, it appears that A_n becomes closer and closer to the area of the circle. We say that the area of the circle is the limit of the areas of the inscribed polygons, and we write

TEC In the Preview Visual, you can see how areas of inscribed and circumscribed polygons approximate the area of a circle.

$$A = \lim_{n \to \infty} A_n$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 4 to find areas of regions of the type shown in Figure 3. We will approximate the desired area A by areas of rectangles (as in Figure 4), let the width of the rectangles decrease, and then calculate A as the limit of these sums of areas of rectangles.

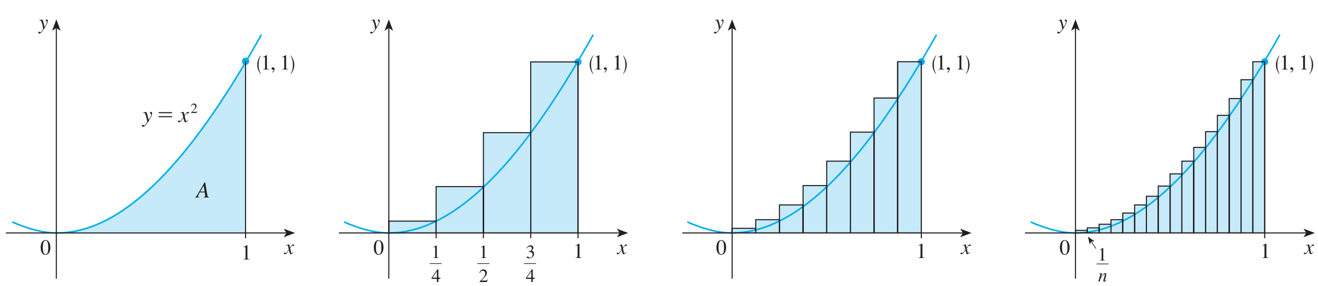


FIGURE 3

The area problem is the central problem in the branch of calculus called *integral calculus*. The techniques that we will develop in Chapter 4 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

The Tangent Problem

Consider the problem of trying to find an equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P. (We will give a precise definition of a tangent line in

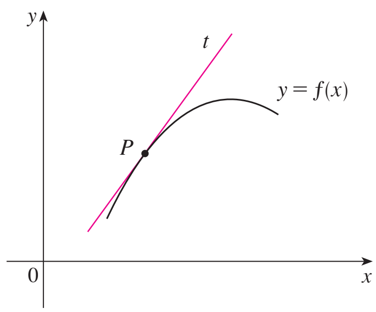


FIGURE 5
The tangent line at P

Chapter 1. For now you can think of it as a line that touches the curve at P as in Figure 5.) Since we know that the point P lies on the tangent line, we can find the equation of t if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on t. To get around the problem we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ. From Figure 6 we see that

$$\boxed{1} \qquad m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

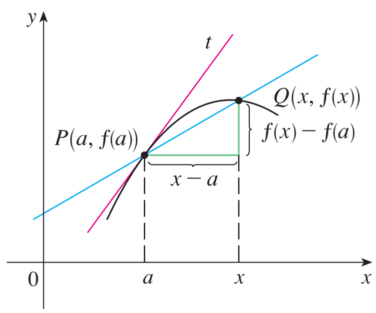


FIGURE 6
The secant line at PQ

Now imagine that Q moves along the curve toward P as in Figure 7. You can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve. Because x approaches a as Q approaches P, we could also use Equation 1 to write

$$\boxed{2} \qquad m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

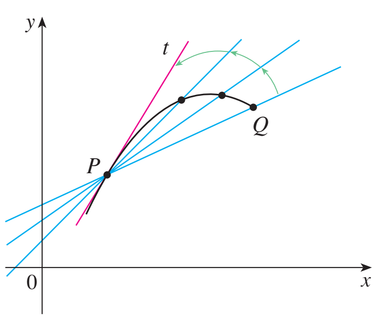


FIGURE 7
Secant lines approaching the tangent line

Specific examples of this procedure will be given in Chapter 1.

The tangent problem has given rise to the branch of calculus called *differential calculus,* which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601–1665) and were developed by the English mathematicians John Wallis (1616–1703), Isaac Barrow (1630–1677), and Isaac Newton (1642–1727) and the German mathematician Gottfried Leibniz (1646–1716).

The two branches of calculus and their chief problems, the area problem and the tangent problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent problem and the area problem are inverse problems in a sense that will be described in Chapter 4.

Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information indicate to us? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at l-second intervals as in the following chart:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	9	24	42	71

As a first step toward finding the velocity after 2 seconds have elapsed, we find the average velocity during the time interval $2 \leqslant t \leqslant 4$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{42 - 9}{4 - 2} = 16.5 \text{ ft/s}$$

Similarly, the average velocity in the time interval $2 \leqslant t \leqslant 3$ is

$$\text{average velocity} = \frac{24 - 9}{3 - 2} = 15 \text{ ft/s}$$

We have the feeling that the velocity at the instant $t = 2$ can't be much different from the average velocity during a short time interval starting at $t = 2$. So let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following chart:

t	2.0	2.1	2.2	2.3	2.4	2.5
d	9.00	10.02	11.16	12.45	13.96	15.80

Then we can compute, for instance, the average velocity over the time interval [2, 2.5]:

$$\text{average velocity} = \frac{15.80 - 9.00}{2.5 - 2} = 13.6 \text{ ft/s}$$

The results of such calculations are shown in the following chart:

Time interval	[2, 3]	[2, 2.5]	[2, 2.4]	[2, 2.3]	[2, 2.2]	[2, 2.1]
Average velocity (ft/s)	15.0	13.6	12.4	11.5	10.8	10.2

The average velocities over successively smaller intervals appear to be getting closer to a number near 10, and so we expect that the velocity at exactly $t = 2$ is about 10 ft/s. In Chapter 2 we will define the instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

In Figure 8 we show a graphical representation of the motion of the car by plotting the distance traveled as a function of time. If we write $d = f(t)$, then $f(t)$ is the number of feet traveled after t seconds. The average velocity in the time interval $[2, t]$ is

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

which is the same as the slope of the secant line PQ in Figure 8. The velocity v when $t = 2$ is the limiting value of this average velocity as t approaches 2; that is,

$$v = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

and we recognize from Equation 2 that this is the same as the slope of the tangent line to the curve at P.

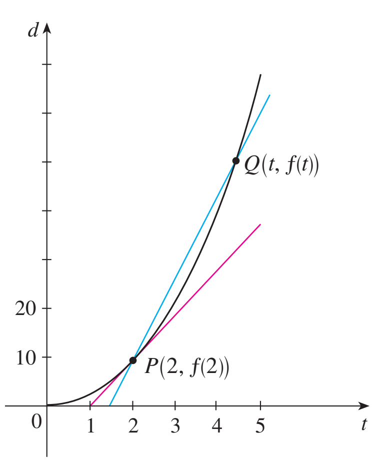


FIGURE 8

Thus, when we solve the tangent problem in differential calculus, we are also solving problems concerning velocities. The same techniques also enable us to solve problems involving rates of change in all of the natural and social sciences.

The Limit of a Sequence

In the fifth century BC the Greek philosopher Zeno of Elea posed four problems, now known as *Zeno's paradoxes*, that were intended to challenge some of the ideas concerning space and time that were held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a tortoise that has been given a head start. Zeno argued, as follows, that Achilles could never pass the tortoise: Suppose that Achilles starts at position a_1 and the tortoise starts at position t_1. (See Figure 9.) When Achilles reaches the point $a_2 = t_1$, the tortoise is farther ahead at position t_2. When Achilles reaches $a_3 = t_2$, the tortoise is at t_3. This process continues indefinitely and so it appears that the tortoise will always be ahead! But this defies common sense.

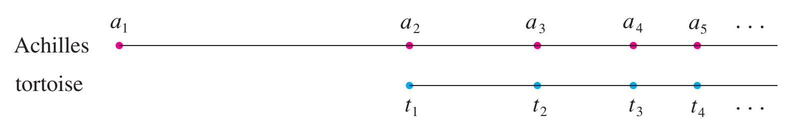


FIGURE 9

One way of explaining this paradox is with the idea of a *sequence*. The successive positions of Achilles $(a_1, a_2, a_3, \ldots)$ or the successive positions of the tortoise $(t_1, t_2, t_3, \ldots)$ form what is known as a sequence.

In general, a sequence $\{a_n\}$ is a set of numbers written in a definite order. For instance, the sequence

$$\left\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots\right\}$$

can be described by giving the following formula for the nth term:

$$a_n = \frac{1}{n}$$

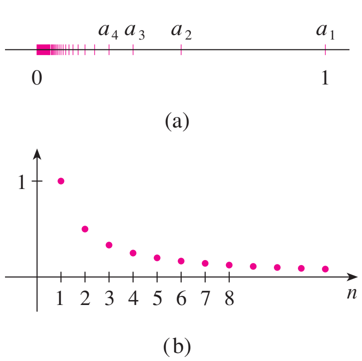


FIGURE 10

We can visualize this sequence by plotting its terms on a number line as in Figure 10(a) or by drawing its graph as in Figure 10(b). Observe from either picture that the terms of the sequence $a_n = 1/n$ are becoming closer and closer to 0 as n increases. In fact, we can find terms as small as we please by making n large enough. We say that the limit of the sequence is 0, and we indicate this by writing

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

is used if the terms a_n approach the number L as n becomes large. This means that the numbers a_n can be made as close as we like to the number L by taking n sufficiently large.

The concept of the limit of a sequence occurs whenever we use the decimal representation of a real number. For instance, if

$$\begin{aligned} a_1 &= 3.1 \\ a_2 &= 3.14 \\ a_3 &= 3.141 \\ a_4 &= 3.1415 \\ a_5 &= 3.14159 \\ a_6 &= 3.141592 \\ a_7 &= 3.1415926 \\ &\vdots \end{aligned}$$

then $$\lim_{n \to \infty} a_n = \pi$$

The terms in this sequence are rational approximations to π.

Let's return to Zeno's paradox. The successive positions of Achilles and the tortoise form sequences $\{a_n\}$ and $\{t_n\}$, where $a_n < t_n$ for all n. It can be shown that both sequences have the same limit:

$$\lim_{n \to \infty} a_n = p = \lim_{n \to \infty} t_n$$

It is precisely at this point p that Achilles overtakes the tortoise.

The Sum of a Series

Another of Zeno's paradoxes, as passed on to us by Aristotle, is the following: "A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains. This process can always be continued and can never be ended." (See Figure 11.)

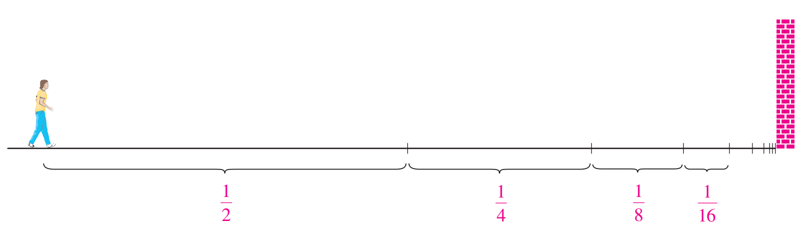


FIGURE 11

Of course, we know that the man can actually reach the wall, so this suggests that perhaps the total distance can be expressed as the sum of infinitely many smaller distances as follows:

3 $$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

Zeno was arguing that it doesn't make sense to add infinitely many numbers together. But there are other situations in which we implicitly use infinite sums. For instance, in decimal notation, the symbol $0.\overline{3} = 0.3333\ldots$ means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$$

and so, in some sense, it must be true that

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots = \frac{1}{3}$$

More generally, if d_n denotes the nth digit in the decimal representation of a number, then

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} + \cdots$$

Therefore some infinite sums, or infinite series as they are called, have a meaning. But we must define carefully what the sum of an infinite series is.

Returning to the series in Equation 3, we denote by s_n the sum of the first n terms of the series. Thus

$$\begin{aligned}
s_1 &= \tfrac{1}{2} = 0.5 \\
s_2 &= \tfrac{1}{2} + \tfrac{1}{4} = 0.75 \\
s_3 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 0.875 \\
s_4 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = 0.9375 \\
s_5 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} = 0.96875 \\
s_6 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} = 0.984375 \\
s_7 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \tfrac{1}{128} = 0.9921875 \\
&\ \vdots \\
s_{10} &= \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \tfrac{1}{1024} \approx 0.99902344 \\
&\ \vdots \\
s_{16} &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{16}} \approx 0.99998474
\end{aligned}$$

Observe that as we add more and more terms, the partial sums become closer and closer to 1. In fact, it can be shown that by taking n large enough (that is, by adding sufficiently many terms of the series), we can make the partial sum s_n as close as we please to the number 1. It therefore seems reasonable to say that the sum of the infinite series is 1 and to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

In other words, the reason the sum of the series is 1 is that

$$\lim_{n \to \infty} s_n = 1$$

In Chapter 11 we will discuss these ideas further. We will then use Newton's idea of combining infinite series with differential and integral calculus.

Summary

We have seen that the concept of a limit arises in trying to find the area of a region, the slope of a tangent to a curve, the velocity of a car, or the sum of an infinite series. In each case the common theme is the calculation of a quantity as the limit of other, easily calculated quantities. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits.

After Sir Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun. Today calculus is used in calculating the orbits of satellites and spacecraft, in predicting population sizes, in estimating how fast oil prices rise or fall, in forecasting weather, in measuring the cardiac output of the heart, in calculating life insurance premiums, and in a great variety of other areas. We will explore some of these uses of calculus in this book.

In order to convey a sense of the power of the subject, we end this preview with a list of some of the questions that you will be able to answer using calculus:

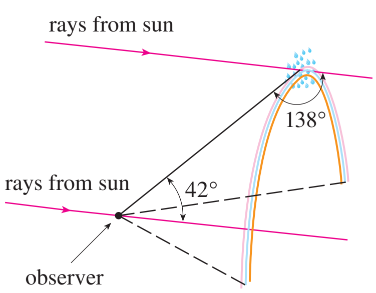


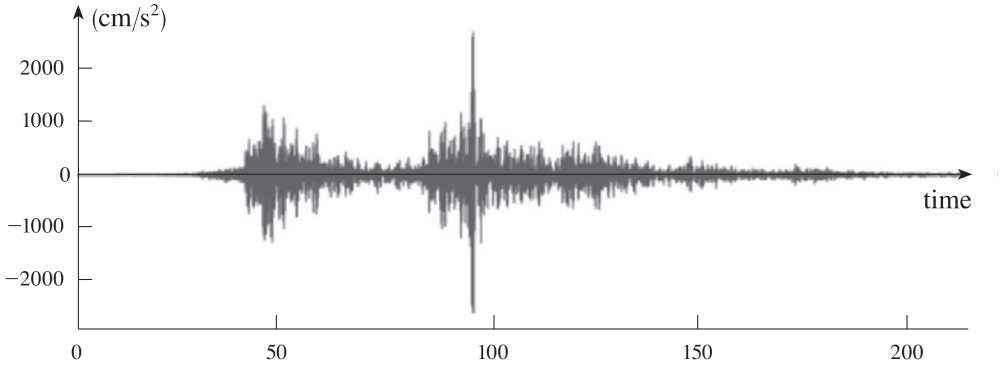
FIGURE 12

1. How can we explain the fact, illustrated in Figure 12, that the angle of elevation from an observer up to the highest point in a rainbow is 42°? (See page 213.)
2. How can we explain the shapes of cans on supermarket shelves? (See page 270.)
3. Where is the best place to sit in a movie theater? (See page 483.)
4. How can we design a roller coaster for a smooth ride? (See page 144.)
5. How far away from an airport should a pilot start descent? (See page 161.)
6. How can we fit curves together to design shapes to represent letters on a laser printer? (See page 697.)
7. How can we estimate the number of workers that were needed to build the Great Pyramid of Khufu in ancient Egypt? (See page 388.)
8. Where should an infielder position himself to catch a baseball thrown by an outfielder and relay it to home plate? (See page 392.)
9. Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height? (See page 649.)
10. How can we explain the fact that planets and satellites move in elliptical orbits? (See page 916.)
11. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production? (See page 1020.)
12. If a marble, a squash ball, a steel bar, and a lead pipe roll down a slope, which of them reaches the bottom first? (See page 1092.)

1 Functions and Limits



Pictura Collectus/Alamy



Seismological Society of America

Often a graph is the best way to represent a function because it conveys so much information at a glance. Shown is a graph of the vertical ground acceleration created by the 2011 earthquake near Tohoku, Japan. The earthquake had a magnitude of 9.0 on the Richter scale and was so powerful that it moved northern Japan 8 feet closer to North America.

THE FUNDAMENTAL OBJECTS THAT WE deal with in calculus are functions. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena.

In *A Preview of Calculus* (page 1) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits of functions and their properties.

1.1 Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

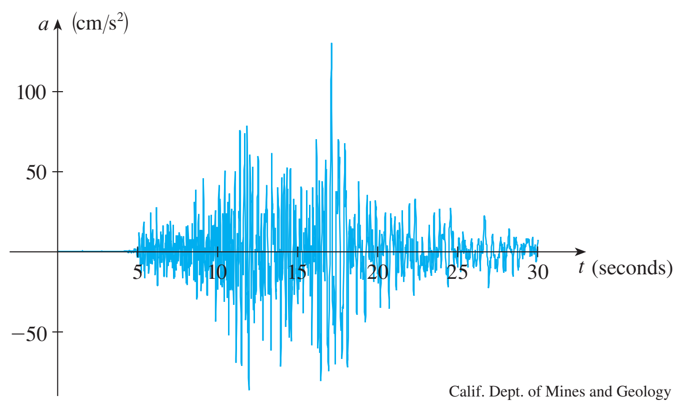


FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

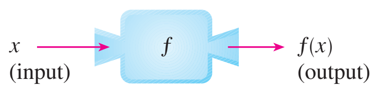


FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{\ }$ (or $\sqrt{x}$) and enter the input x. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geqslant 0$, then an *approximation* to $\sqrt{x}$ will appear in the display. Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

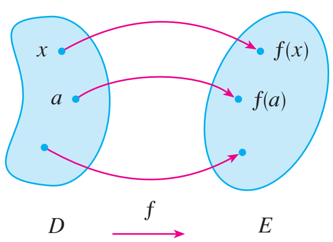


FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

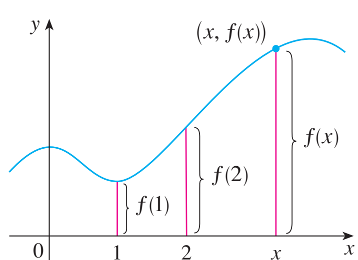


FIGURE 4

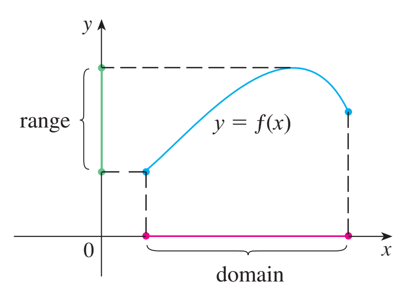


FIGURE 5

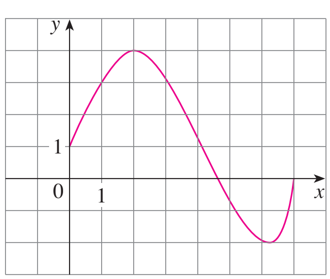


FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 units below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$ ■

The notation for intervals is given in Appendix A.

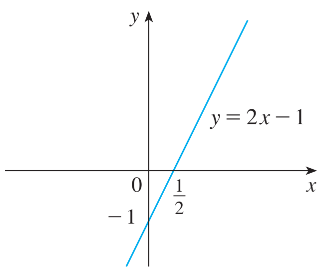


FIGURE 7

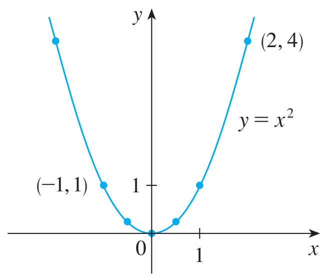


FIGURE 8

EXAMPLE 2 Sketch the graph and find the domain and range of each function.
(a) $f(x) = 2x - 1$ (b) $g(x) = x^2$

SOLUTION
(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1. (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2. But $x^2 \geqslant 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geqslant 0\} = [0, \infty)$. This can also be seen from Figure 8. ■

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a + h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a + h)$ by replacing x by $a + h$ in the expression for $f(x)$:

$$\begin{aligned} f(a + h) &= 2(a + h)^2 - 5(a + h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a + h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

■

The expression

$$\frac{f(a + h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained the graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. Let's measure t so that $t = 0$ corresponds to the year 1900. The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain the approximation

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good "fit." The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

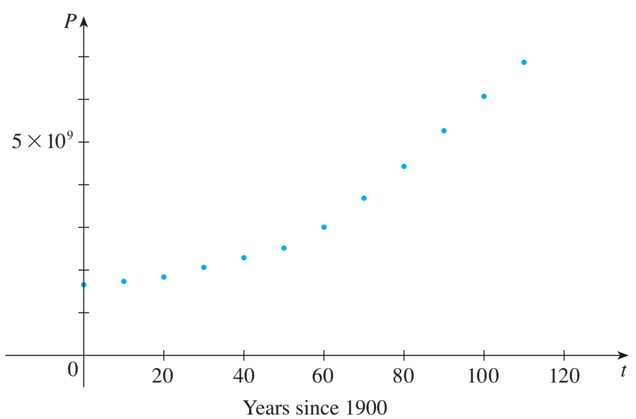


FIGURE 9

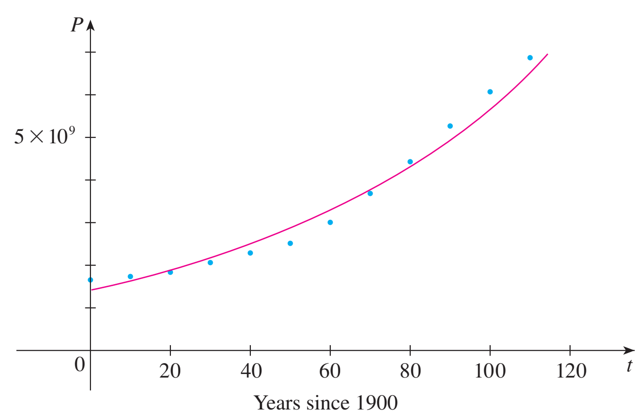


FIGURE 10

The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.

A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leqslant 1$	0.98
$1 < w \leqslant 2$	1.19
$2 < w \leqslant 3$	1.40
$3 < w \leqslant 4$	1.61
$4 < w \leqslant 5$	1.82
⋮	⋮

C. Again the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2015 is as follows: The cost is 98 cents for up to 1 oz, plus 21 cents for each additional ounce (or less) up to 13 oz. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to

know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 4 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

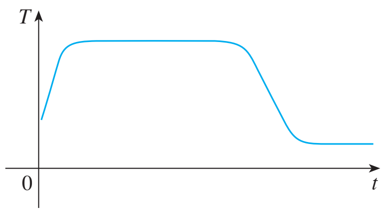


FIGURE 11

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 11. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

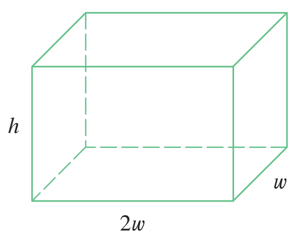


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3. Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w. ■

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 98, particularly *Step 1: Understand the Problem.*

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

Domain Convention
If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

SOLUTION
(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

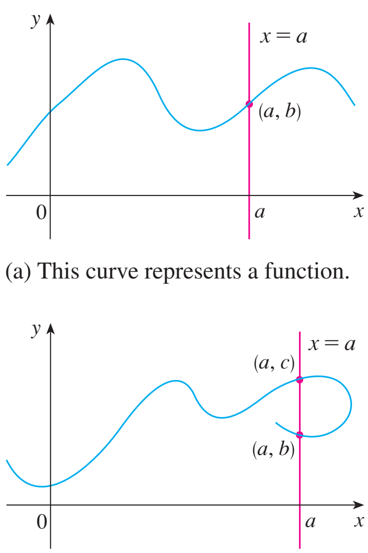
$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$ ■



(a) This curve represents a function.

(b) This curve doesn't represent a function.

FIGURE 13

The graph of a function is a curve in the xy-plane. But the question arises: which curves in the xy-plane are graphs of functions? This is answered by the following test.

> **The Vertical Line Test** A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x. Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).]

We observe that if we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h.

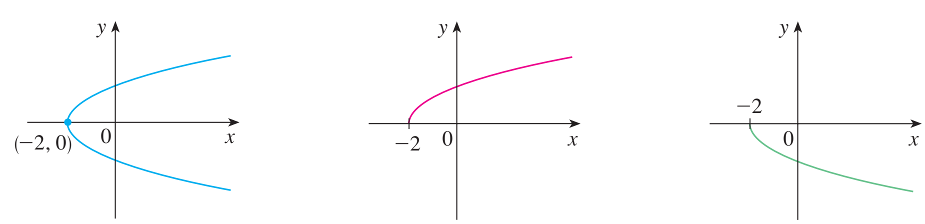


FIGURE 14 (a) $x = y^2 - 2$ (b) $y = \sqrt{x + 2}$ (c) $y = -\sqrt{x + 2}$

Piecewise Defined Functions

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2.

Since $-2 \leqslant -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leqslant -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $0 > -1$, we have $f(0) = 0^2 = 0$.

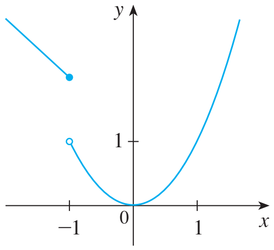


FIGURE 15

How do we draw the graph of f? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

For a more extensive review of absolute values, see Appendix A.

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a & &\text{if } a \geqslant 0 \\ |a| &= -a & &\text{if } a < 0 \end{aligned}$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

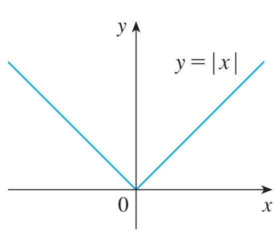


FIGURE 16

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16). ■

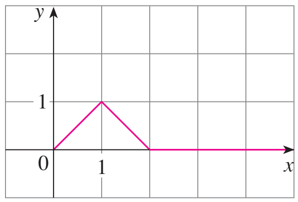


FIGURE 17

Point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

See Appendix B.

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \qquad \text{if } 0 \leqslant x \leqslant 1$$

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \qquad \text{or} \qquad y = 2 - x$$

So we have
$$f(x) = 2 - x \qquad \text{if } 1 < x \leqslant 2$$

We also see that the graph of f coincides with the x-axis for $x > 2$. Putting this information together, we have the following three-piece formula for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

■

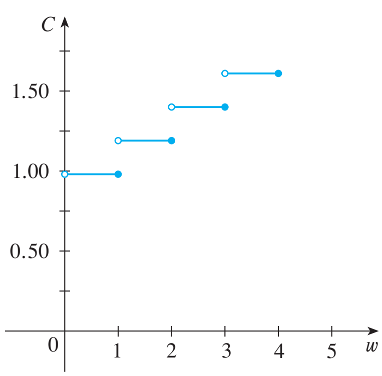


FIGURE 18

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from the table of values on page 13, we have

$$C(w) = \begin{cases} 0.98 & \text{if } 0 < w \leqslant 1 \\ 1.19 & \text{if } 1 < w \leqslant 2 \\ 1.40 & \text{if } 2 < w \leqslant 3 \\ 1.61 & \text{if } 3 < w \leqslant 4 \\ \vdots & \end{cases}$$

The graph is shown in Figure 18. You can see why functions similar to this one are called **step functions**—they jump from one value to the next. Such functions will be studied in Chapter 2. ■

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

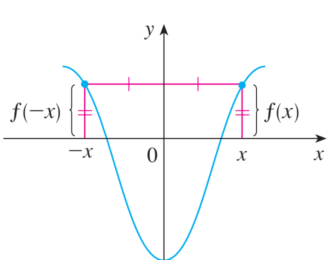


FIGURE 19
An even function

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19). This means that if we have plotted the graph of f for $x \geqslant 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

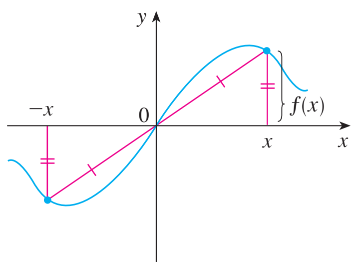


FIGURE 20
An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geqslant 0$, we can obtain the entire graph by rotating this portion through $180°$ about the origin.

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.
(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION
(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

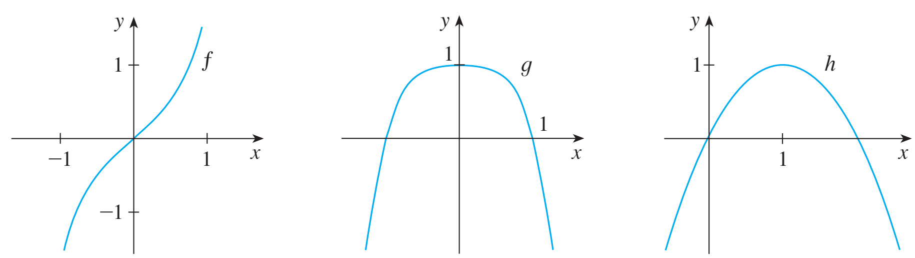


FIGURE 21 (a) (b) (c)

■ Increasing and Decreasing Functions

The graph shown in Figure 22 rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

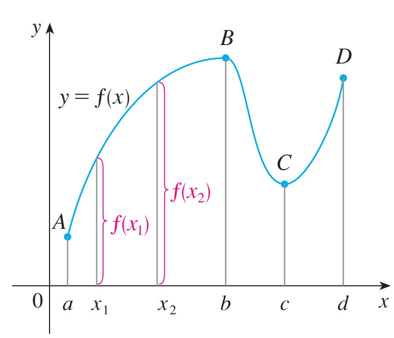


FIGURE 22

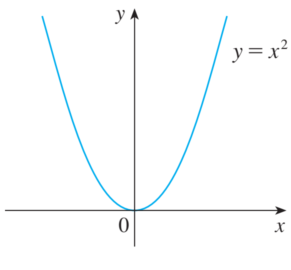


FIGURE 23

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

1.1 EXERCISES

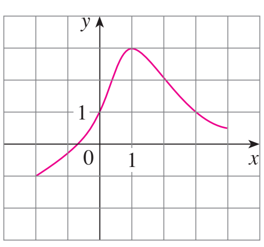
1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

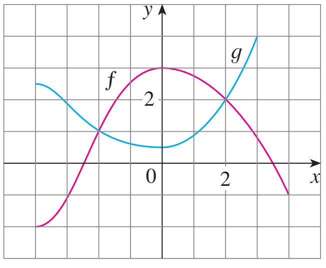
$$f(x) = \frac{x^2 - x}{x - 1} \qquad \text{and} \qquad g(x) = x$$

is it true that $f = g$?

3. The graph of a function f is given.
(a) State the value of $f(1)$.
(b) Estimate the value of $f(-1)$.
(c) For what values of x is $f(x) = 1$?
(d) Estimate the value of x such that $f(x) = 0$.
(e) State the domain and range of f.
(f) On what interval is f increasing?



4. The graphs of f and g are given.



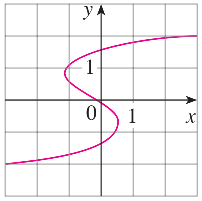
(a) State the values of $f(-4)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) Estimate the solution of the equation $f(x) = -1$.
(d) On what interval is f decreasing?
(e) State the domain and range of f.
(f) State the domain and range of g.

5. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

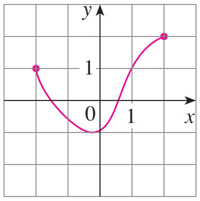
6. In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

7–10 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

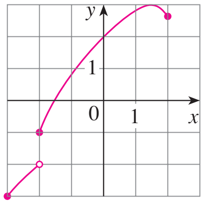
7.



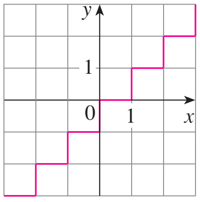
8.



9.



10.



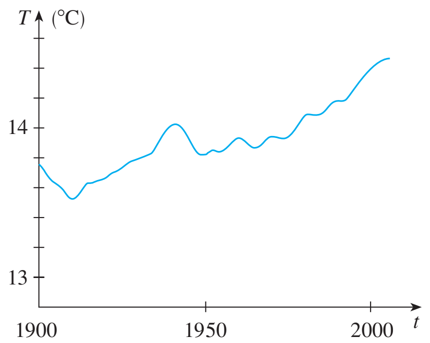
11. Shown is a graph of the global average temperature T during the 20th century. Estimate the following.

(a) The global average temperature in 1950

(b) The year when the average temperature was 14.2°C

(c) The year when the temperature was smallest; the year it was largest

(d) The range of T

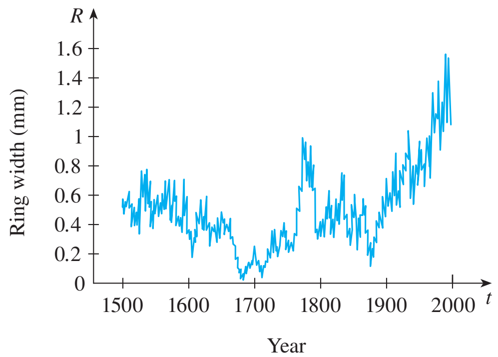


Source: Adapted from *Globe and Mail* [Toronto], 5 Dec. 2009. Print.

12. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of a Siberian pine from 1500 to 2000.

(a) What is the range of the ring width function?

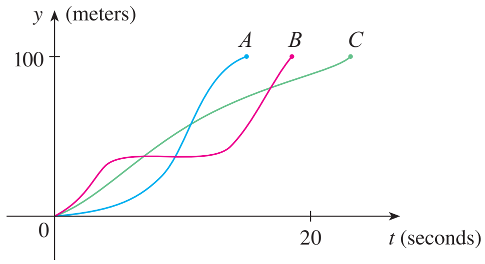
(b) What does the graph tend to say about the temperature of the earth? Does the graph reflect the volcanic eruptions of the mid-19th century?



Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

13. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

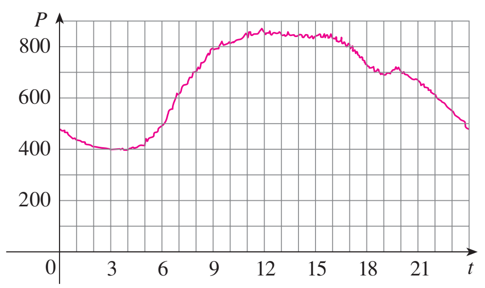
14. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



15. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)

(a) What was the power consumption at 6 AM? At 6 PM?

(b) When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



Pacific Gas & Electric

16. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

17. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

18. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

19. Sketch the graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.

20. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

21. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

22. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.

(a) Sketch a possible graph of $x(t)$.

(b) Sketch a possible graph of $y(t)$.

(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

23. Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Atlanta on June 4, 2013. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	74	69	68	66	70	78	82	86

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use your graph to estimate the temperature at 9:00 AM.

24. Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data they obtained by averaging the BAC (in g/dL) of the eight men.
(a) Use the readings to sketch the graph of the BAC as a function of t.
(b) Use your graph to describe how the effect of alcohol varies with time.

t (hours)	BAC	t (hours)	BAC
0	0	1.75	0.022
0.2	0.025	2.0	0.018
0.5	0.041	2.25	0.015
0.75	0.040	2.5	0.012
1.0	0.033	3.0	0.007
1.25	0.029	3.5	0.003
1.5	0.024	4.0	0.001

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

25. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a+h)$.

26. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

27–30 Evaluate the difference quotient for the given function. Simplify your answer.

27. $f(x) = 4 + 3x - x^2, \quad \dfrac{f(3+h) - f(3)}{h}$

28. $f(x) = x^3, \quad \dfrac{f(a+h) - f(a)}{h}$

29. $f(x) = \dfrac{1}{x}, \quad \dfrac{f(x) - f(a)}{x - a}$

30. $f(x) = \dfrac{x+3}{x+1}, \quad \dfrac{f(x) - f(1)}{x - 1}$

31–37 Find the domain of the function.

31. $f(x) = \dfrac{x+4}{x^2 - 9}$

32. $f(x) = \dfrac{2x^3 - 5}{x^2 + x - 6}$

33. $f(t) = \sqrt[3]{2t - 1}$

34. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

35. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

36. $f(u) = \dfrac{u+1}{1 + \dfrac{1}{u+1}}$

37. $F(p) = \sqrt{2 - \sqrt{p}}$

38. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

39–40 Find the domain and sketch the graph of the function.

39. $f(x) = 1.6x - 2.4$

40. $g(t) = \dfrac{t^2 - 1}{t + 1}$

41–44 Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

41. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geqslant 0 \end{cases}$

42. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geqslant 2 \end{cases}$

43. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

45–50 Sketch the graph of the function.

45. $f(x) = x + |x|$

46. $f(x) = |x + 2|$

47. $g(t) = |1 - 3t|$

48. $h(t) = |t| + |t + 1|$

49. $f(x) = \begin{cases} |x| & \text{if } |x| \leqslant 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

50. $g(x) = \big||x| - 1\big|$

51–56 Find an expression for the function whose graph is the given curve.

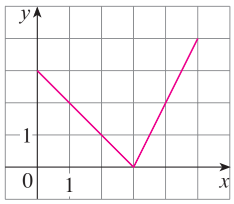
51. The line segment joining the points $(1, -3)$ and $(5, 7)$

52. The line segment joining the points $(-5, 10)$ and $(7, -10)$

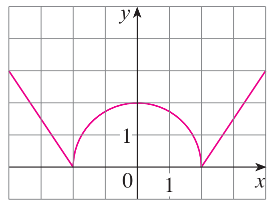
53. The bottom half of the parabola $x + (y - 1)^2 = 0$

54. The top half of the circle $x^2 + (y - 2)^2 = 4$

55.



56.



57–61 Find a formula for the described function and state its domain.

57. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

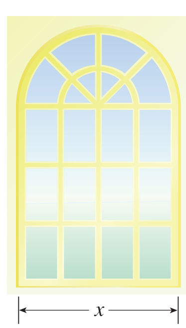
58. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

59. Express the area of an equilateral triangle as a function of the length of a side.

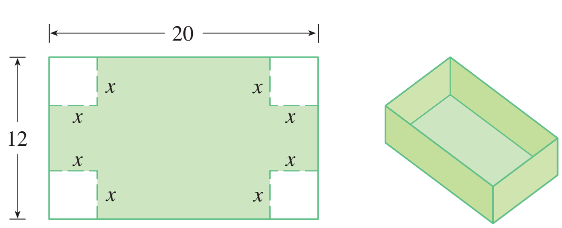
60. A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.

61. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

62. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



63. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



64. A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charges 10 cents for each additional minute of usage. Write the monthly cost C as a function of the number x of minutes used and graph C as a function of x for $0 \leqslant x \leqslant 600$.

65. In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leqslant x \leqslant 100$.

66. An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leqslant x \leqslant 2000$.

67. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

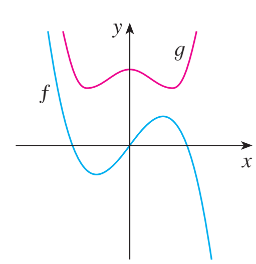
(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

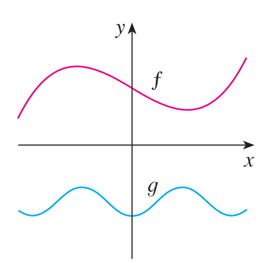
68. The functions in Example 10 and Exercise 67 are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

69–70 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

69.



70.



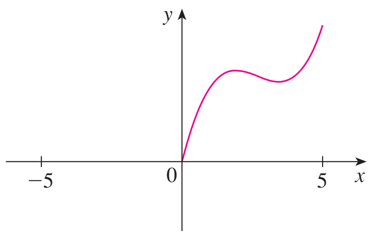
71. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?

(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

72. A function f has domain $[-5, 5]$ and a portion of its graph is shown.

(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.



73–78 Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

73. $f(x) = \dfrac{x}{x^2 + 1}$

74. $f(x) = \dfrac{x^2}{x^4 + 1}$

75. $f(x) = \dfrac{x}{x + 1}$

76. $f(x) = x|x|$

77. $f(x) = 1 + 3x^2 - x^4$

78. $f(x) = 1 + 3x^3 - x^5$

79. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

80. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

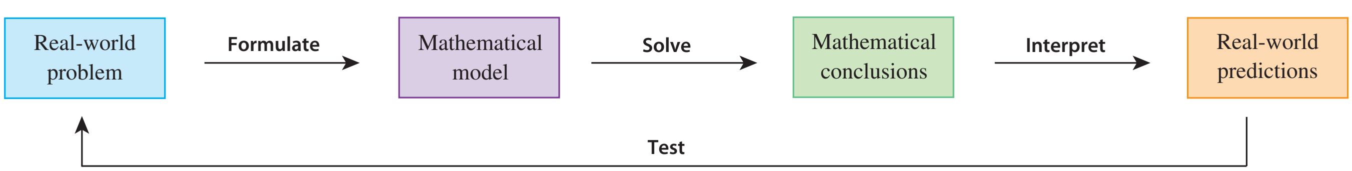


FIGURE 1
The modeling process

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit math-

ematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

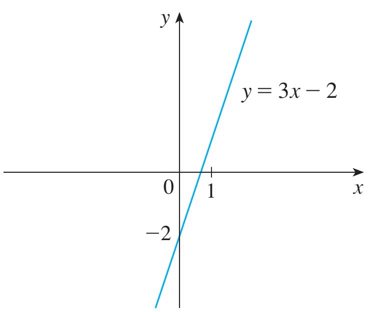
The coordinate geometry of lines is reviewed in Appendix B.

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x. Thus the slope of the graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

FIGURE 2

EXAMPLE 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part (a). What does the slope represent?
(c) What is the temperature at a height of 2.5 km?

SOLUTION

(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

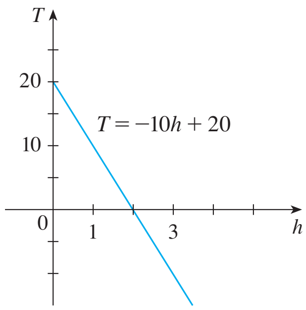


FIGURE 3

(b) The graph is sketched in Figure 3. The slope is $m = -10°C/km$, and this represents the rate of change of temperature with respect to height.

(c) At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = -5°C$$ ■

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the data points.

EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2012. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

Table 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1998	366.5
1982	341.2	2000	369.4
1984	344.4	2002	373.2
1986	347.2	2004	377.5
1988	351.5	2006	381.9
1990	354.2	2008	385.6
1992	356.3	2010	389.9
1994	358.6	2012	393.8
1996	362.4		

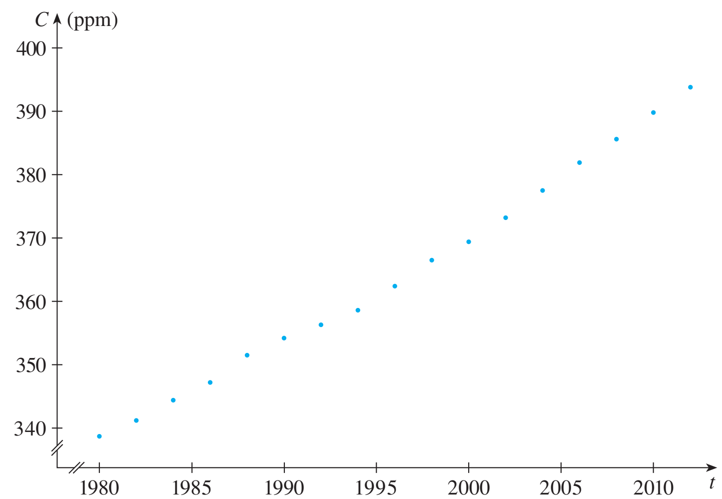


FIGURE 4 Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{393.8 - 338.7}{2012 - 1980} = \frac{55.1}{32} = 1.721875 \approx 1.722$$

We write its equation as

$$C - 338.7 = 1.722(t - 1980)$$

or

1 $$C = 1.722t - 3070.86$$

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5.

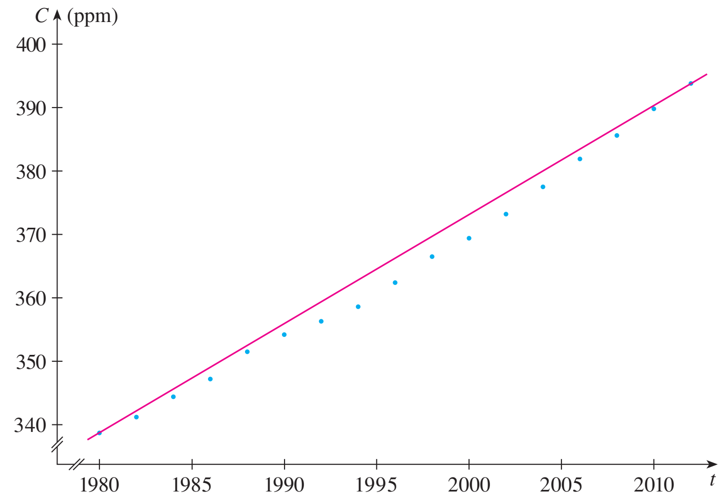


FIGURE 5
Linear model through first and last data points

A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Section 14.7.

Notice that our model gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics called *linear regression*. If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use the fit[leastsquare] command in the stats package; with Mathematica we use the Fit command.) The machine gives the slope and y-intercept of the regression line as

$$m = 1.71262 \qquad b = -3054.14$$

So our least squares model for the CO_2 level is

$$\boxed{2} \qquad C = 1.71262t - 3054.14$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that it gives a better fit than our previous linear model.

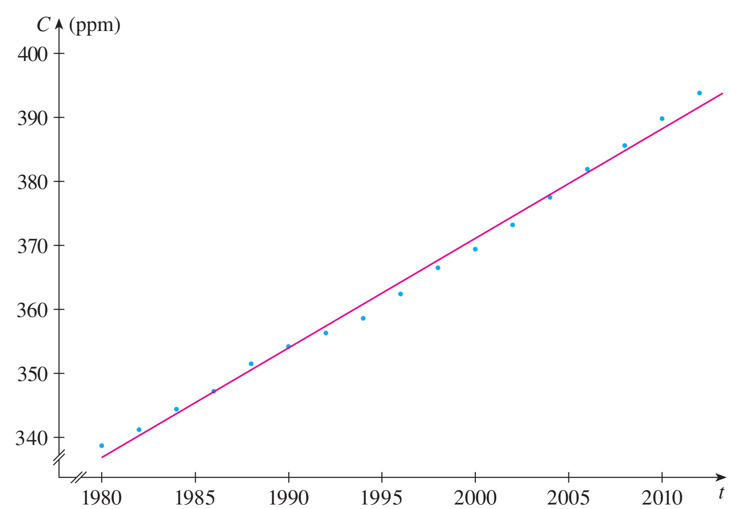


FIGURE 6
The regression line

EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for the year 2020. According to this model, when will the CO_2 level exceed 420 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO_2 level in 1987 was

$$C(1987) = (1.71262)(1987) - 3054.14 \approx 348.84$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2020$, we get

$$C(2020) = (1.71262)(2020) - 3054.14 \approx 405.35$$

So we predict that the average CO_2 level in the year 2020 will be 405.4 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the time frame of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO_2 level exceeds 420 ppm when

$$1.71262t - 3054.14 > 420$$

Solving this inequality, we get

$$t > \frac{3474.14}{1.71262} \approx 2028.55$$

We therefore predict that the CO_2 level will exceed 420 ppm by the year 2029. This prediction is risky because it involves a time quite remote from our observations. In fact, we see from Figure 6 that the trend has been for CO_2 levels to increase rather more rapidly in recent years, so the level might exceed 420 ppm well before 2029. ■

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in the next section. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

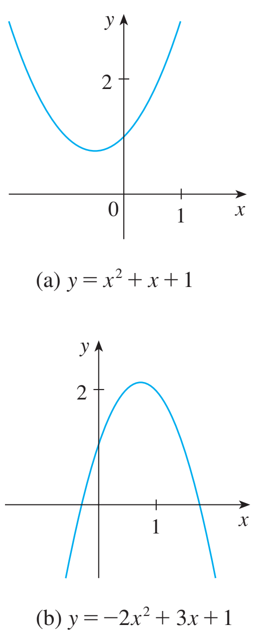


FIGURE 7
The graphs of quadratic functions are parabolas.

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

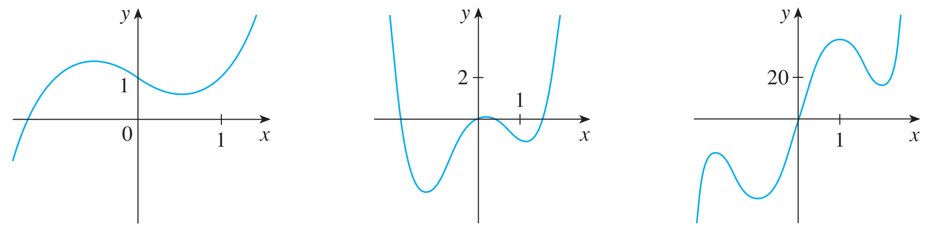


FIGURE 8

(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 2.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

Table 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

3 $$h = 449.36 + 0.96t - 4.90t^2$$

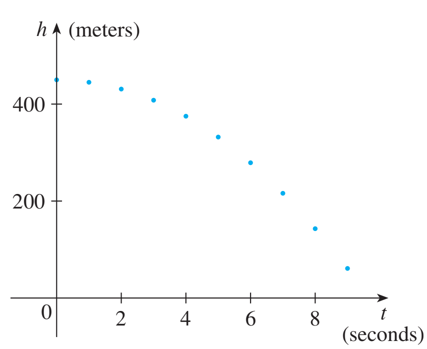


FIGURE 9
Scatter plot for a falling ball

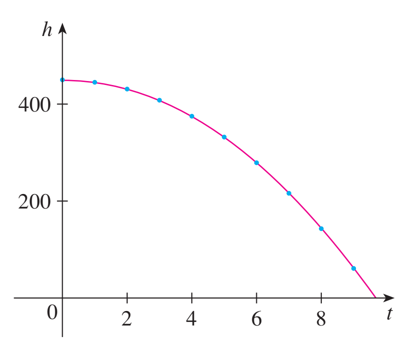


FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds. ■

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 1.1.2(b)].

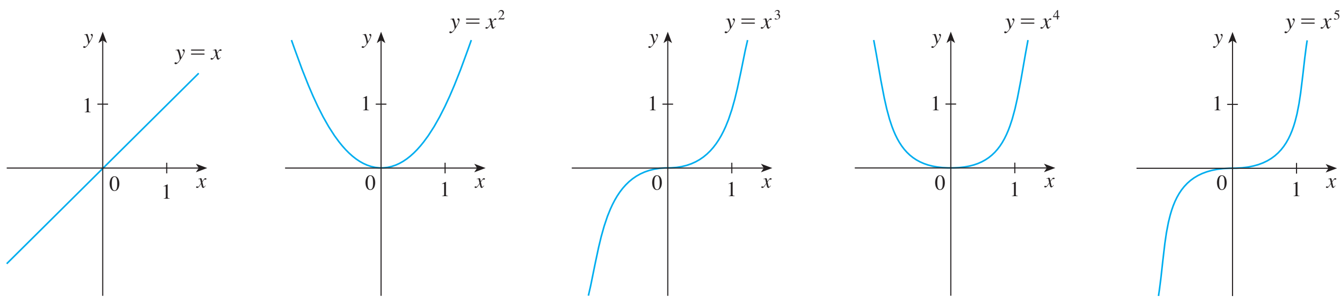


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geqslant 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

A **family of functions** is a collection of functions whose equations are related. Figure 12 shows two families of power functions, one with even powers and one with odd powers.

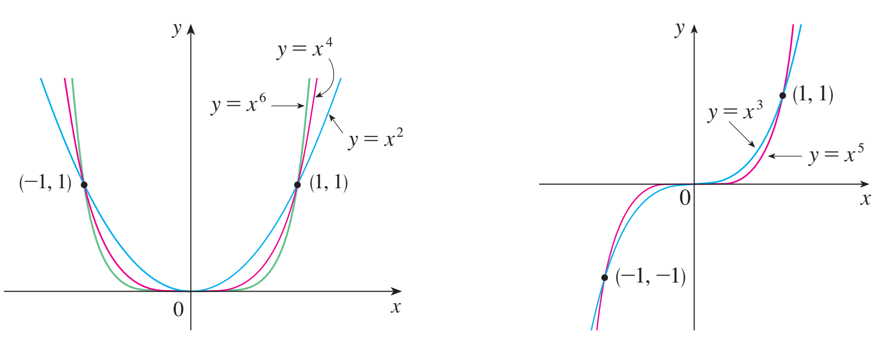
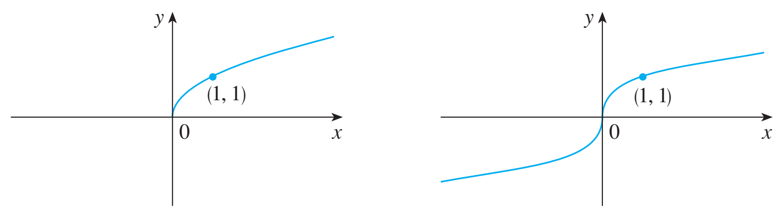


FIGURE 12

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the

parabola $x = y^2$. [See Figure 13(a).] For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.



(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

FIGURE 13
Graphs of root functions

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

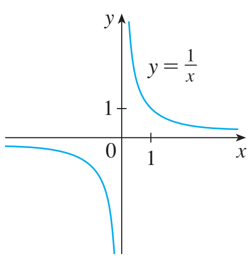


FIGURE 14
The reciprocal function

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

Power functions are also used to model species-area relationships (Exercises 30–31), illumination as a function of distance from a light source (Exercise 29), and the period of revolution of a planet as a function of its distance from the sun (Exercise 32).

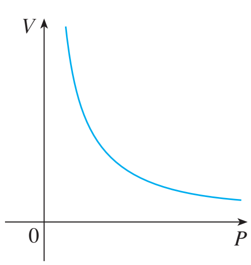


FIGURE 15
Volume as a function of pressure at constant temperature

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.

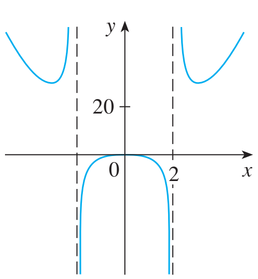


FIGURE 16
$f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 3, we will see that their graphs can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

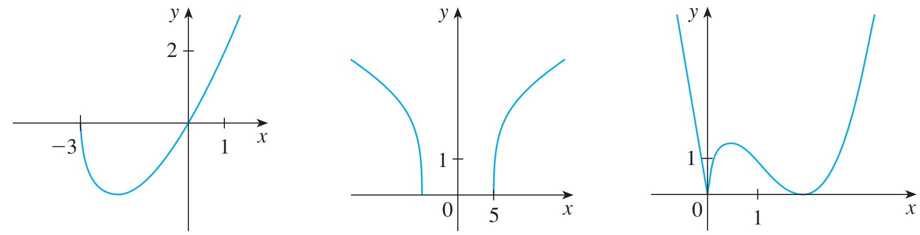


FIGURE 17 (a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

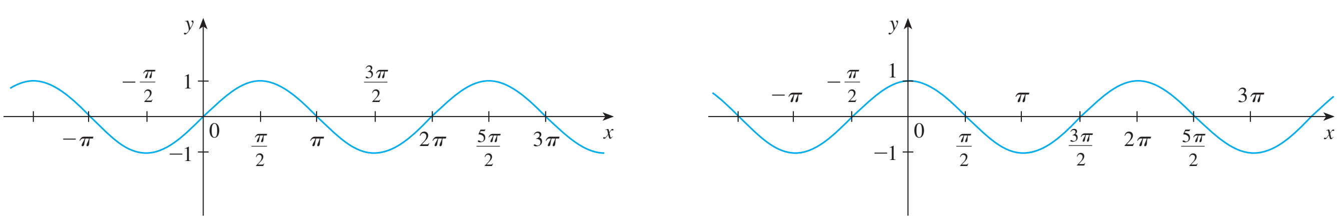
$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

The Reference Pages are located at the back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x. Thus the graphs of the sine and cosine functions are as shown in Figure 18.



(a) $f(x) = \sin x$ (b) $g(x) = \cos x$

FIGURE 18

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

or, in terms of absolute values,

$$|\sin x| \leqslant 1 \qquad |\cos x| \leqslant 1$$

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \qquad \text{when} \qquad x = n\pi \quad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 1.3.4 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

EXAMPLE 5 What is the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$?

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi$$

where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

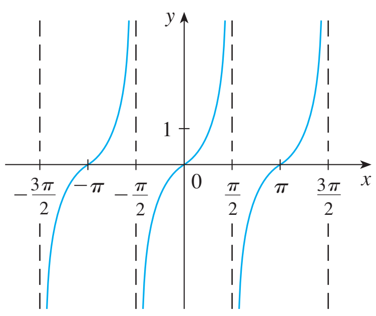


FIGURE 19
$y = \tan x$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

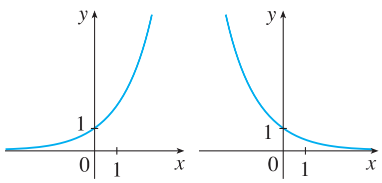
$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Exponential functions will be studied in detail in Chapter 6, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $b > 1$) and radioactive decay (if $b < 1$).



(a) $y = 2^x$ (b) $y = (0.5)^x$

FIGURE 20

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. They will be studied in Chapter 6. Figure

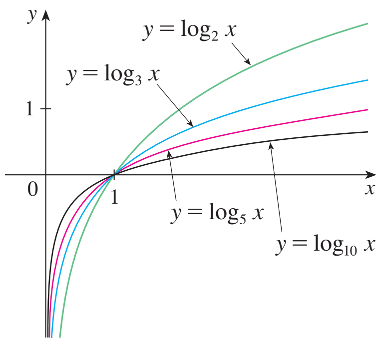


FIGURE 21

21 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

EXAMPLE 6 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$ (b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ (d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

1.2 EXERCISES

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = \log_2 x$ (b) $g(x) = \sqrt[4]{x}$

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ (d) $u(t) = 1 - 1.1t + 2.54t^2$

(e) $v(t) = 5^t$ (f) $w(\theta) = \sin\theta \cos^2\theta$

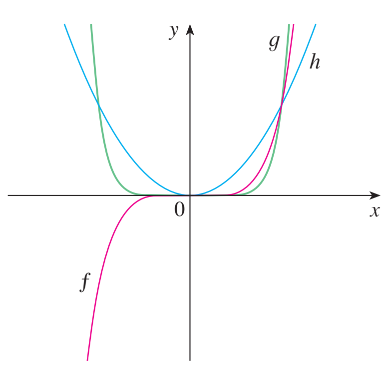
2. (a) $y = \pi^x$ (b) $y = x^\pi$

(c) $y = x^2(2 - x^3)$ (d) $y = \tan t - \cos t$

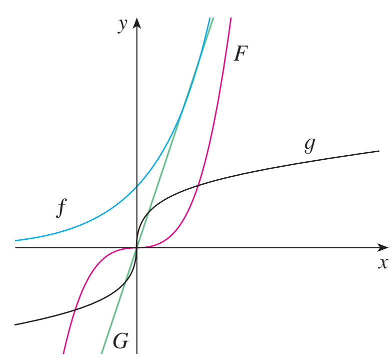
(e) $y = \dfrac{s}{1 + s}$ (f) $y = \dfrac{\sqrt{x^3 - 1}}{1 + \sqrt[3]{x}}$

3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$ (c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5–6 Find the domain of the function.

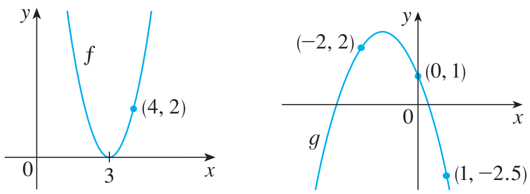
5. $f(x) = \dfrac{\cos x}{1 - \sin x}$ **6.** $g(x) = \dfrac{1}{1 - \tan x}$

7. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
(c) Which function belongs to both families?

8. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

9. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.

10. Find expressions for the quadratic functions whose graphs are shown.



11. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

12. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global surface temperature in 2100.

13. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c. What does it represent?

(b) What is the dosage for a newborn?

14. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

15. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

(a) Sketch a graph of this function.

(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

16. Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-94. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.

(a) Express the distance traveled in terms of the time elapsed.

(b) Draw the graph of the equation in part (a).

(c) What is the slope of this line? What does it represent?

17. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.

(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.

(b) What is the slope of the graph? What does it represent?

(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

18. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

19. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.

(a) Express the water pressure as a function of the depth below the ocean surface.

(b) At what depth is the pressure 100 lb/in^2?

20. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.

(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.

(b) Use part (a) to predict the cost of driving 1500 miles per month.

(c) Draw the graph of the linear function. What does the slope represent?

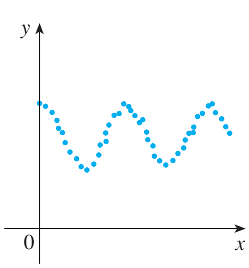
(d) What does the C-intercept represent?

(e) Why does a linear function give a suitable model in this situation?

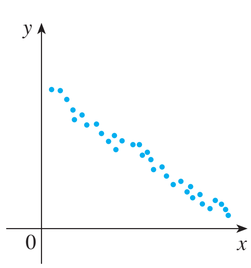
21–22 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.

21.

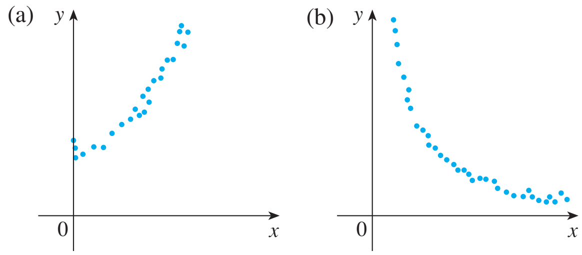
(a)



(b)



22.



23. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.
(b) Find and graph a linear model using the first and last data points.
(c) Find and graph the least squares regression line.
(d) Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
(e) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
(f) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

24. Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.
(a) Make a scatter plot of the data.
(b) Find and graph the regression line.
(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

25. Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Here we find the model by analyzing the data on femur length and height for the eight males given in the following table.
(a) Make a scatter plot of the data.
(b) Find and graph the regression line that models the data.
(c) An anthropologist finds a human femur of length 53 cm. How tall was the person?

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

26. When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.
(a) Find the regression line for the data.
(b) Make a scatter plot and graph the regression line. Does the regression line appear to be a suitable model for the data?
(c) What does the y-intercept of the regression line represent?

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

27. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.
(a) Make a scatter plot and decide whether a linear model is appropriate.
(b) Find and graph the regression line.
(c) Use the linear model to estimate the oil consumption in 2002 and 2012.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302

Source: US Energy Information Administration

28. The table shows average US retail residential prices of electricity from 2000 to 2012, measured in cents per kilowatt hour.
(a) Make a scatter plot. Is a linear model appropriate?
(b) Find and graph the regression line.
(c) Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2013.

Years since 2000	Cents/kWh
0	8.24
2	8.44
4	8.95
6	10.40
8	11.26
10	11.54
12	11.58

Source: US Energy Information Administration

29. Many physical quantities are connected by *inverse square laws*, that is, by power functions of the form $f(x) = kx^{-2}$. In particular, the illumination of an object by a light source is inversely proportional to the square of the distance from the source. Suppose that after dark you are in a room with just one lamp and you are trying to read a book. The light is too dim and so you move halfway to the lamp. How much brighter is the light?

30. It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the species-area relation with a power function and, in particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.
(a) The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60\text{ m}^2$. How many species of bats would you expect to find in that cave?
(b) If you discover that four species of bats live in a cave, estimate the area of the cave.

31. The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.
(a) Use a power function to model N as a function of A.
(b) The Caribbean island of Dominica has area 291 mi^2. How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	4	5
Monserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

32. The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from planet Earth to the sun) and their periods T (time of revolution in years).
(a) Fit a power model to the data.
(b) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun."
Does your model corroborate Kepler's Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). There-

fore the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

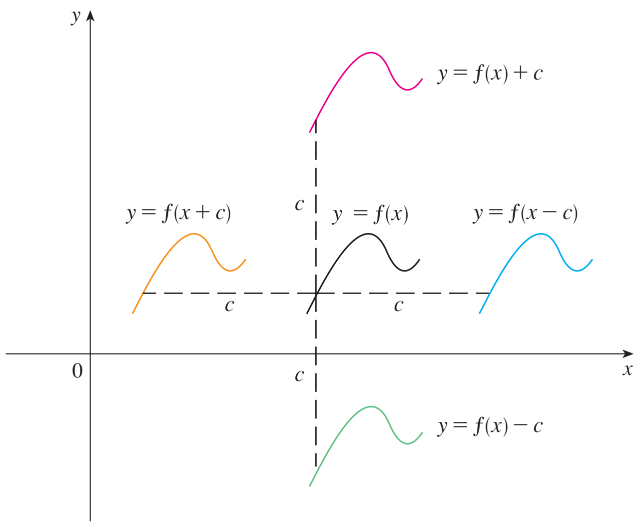


FIGURE 1 Translating the graph of f

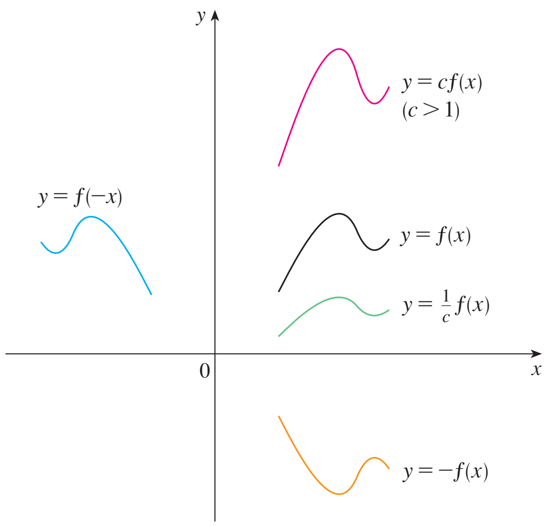


FIGURE 2 Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y-coordi-

nate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

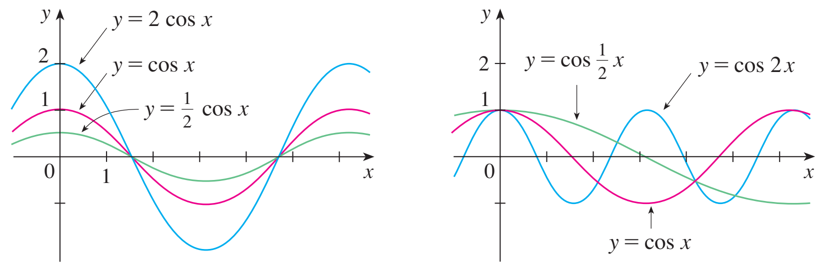
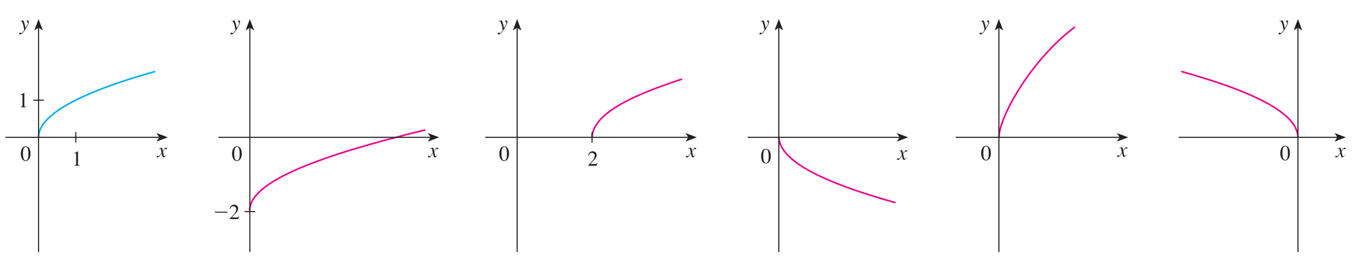


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 1.2.13(a), is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y-axis.



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x - 2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

FIGURE 4

■

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

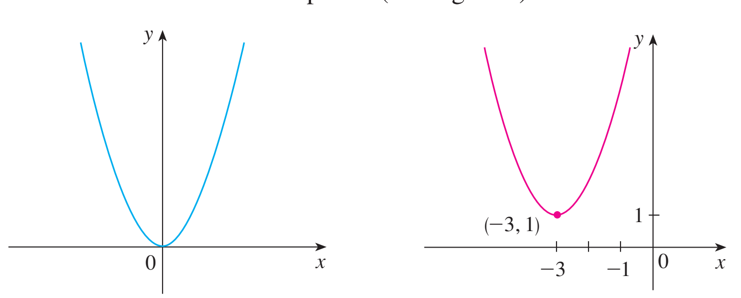


FIGURE 5 (a) $y = x^2$ (b) $y = (x + 3)^2 + 1$

■

EXAMPLE 3 Sketch the graphs of the following functions.

(a) $y = \sin 2x$ (b) $y = 1 - \sin x$

SOLUTION
(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Thus, whereas the period of $y = \sin x$ is 2π, the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

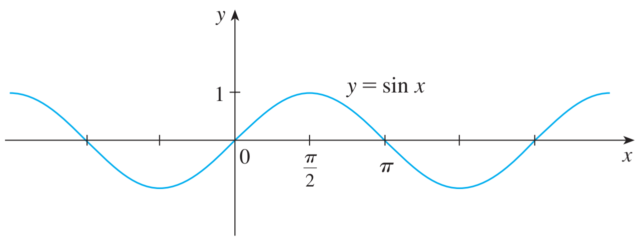


FIGURE 6

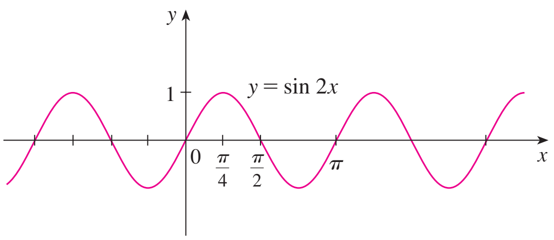


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x-axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

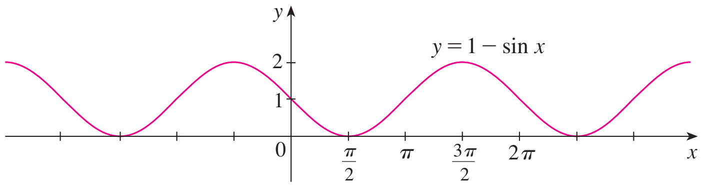


FIGURE 8

■

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

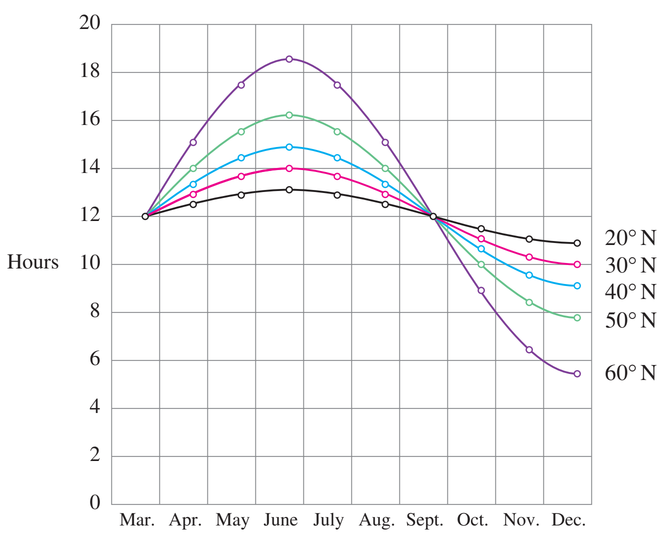


FIGURE 9
Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Adapted from L. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935), 40.

SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is
$\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geqslant 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: the part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

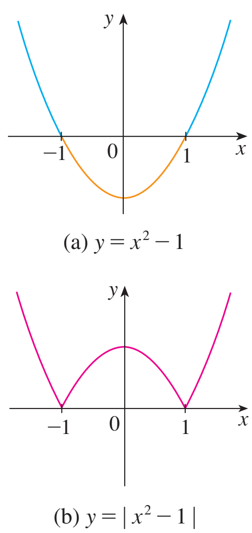


FIGURE 10

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x-axis when $-1 < x < 1$, so we reflect that part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$. Because we can't divide by 0, the domain of f/g is therefore $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

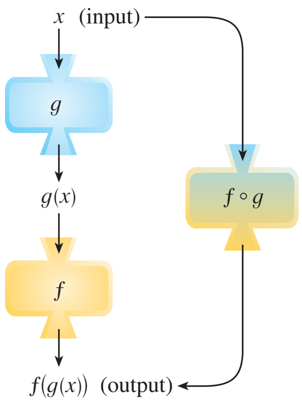


FIGURE 11
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

In general, given any two functions f and g, we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ ("f circle g").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

■

NOTE You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each of the following functions and their domains.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{2 - x}\right) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2 - \sqrt{x}}$$

If $0 \leqslant a \leqslant b$, then $a^2 \leqslant b^2$.

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus we have $0 \leqslant x \leqslant 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) $$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g\left(\sqrt{2 - x}\right) = \sqrt{2 - \sqrt{2 - x}}$$

This expression is defined when both $2 - x \geqslant 0$ and $2 - \sqrt{2 - x} \geqslant 0$. The first inequality means $x \leqslant 2$, and the second is equivalent to $\sqrt{2 - x} \leqslant 2$, or $2 - x \leqslant 4$, or $x \geqslant -2$. Thus $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$. ■

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\ &= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}\end{aligned}$$ ■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

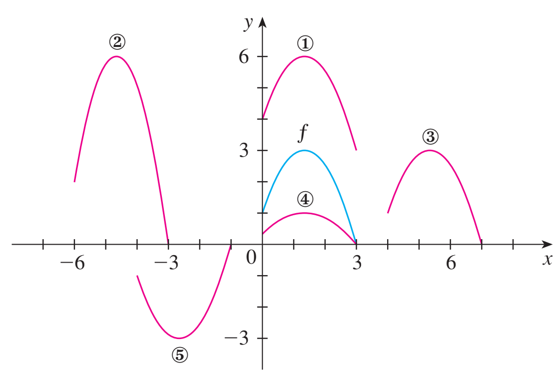
Then
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x)\end{aligned}$$ ■

1.3 EXERCISES

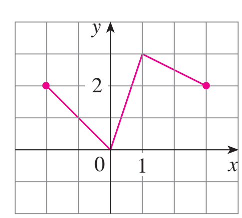
1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.
(a) Shift 3 units upward.
(b) Shift 3 units downward.
(c) Shift 3 units to the right.
(d) Shift 3 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 3.
(h) Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.
(a) $y = f(x) + 8$ (b) $y = f(x + 8)$
(c) $y = 8f(x)$ (d) $y = f(8x)$
(e) $y = -f(x) - 1$ (f) $y = 8f\left(\frac{1}{8}x\right)$

3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
(a) $y = f(x - 4)$ (b) $y = f(x) + 3$
(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$



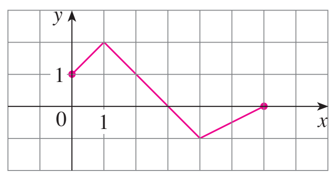
4. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x) - 3$ (b) $y = f(x + 1)$
(c) $y = \frac{1}{2}f(x)$ (d) $y = -f(x)$



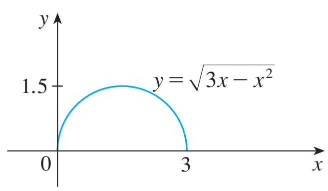
5. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$

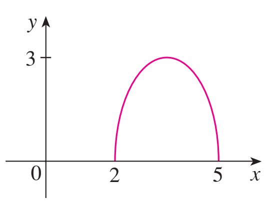
(c) $y = f(-x)$ (d) $y = -f(-x)$



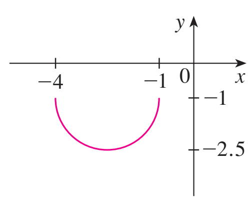
6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



6.



7.



8. (a) How is the graph of $y = 2\sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2\sin x$.

(b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

9–24 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = -x^2$

10. $y = (x - 3)^2$

11. $y = x^3 + 1$

12. $y = 1 - \dfrac{1}{x}$

13. $y = 2\cos 3x$

14. $y = 2\sqrt{x + 1}$

15. $y = x^2 - 4x + 5$

16. $y = 1 + \sin \pi x$

17. $y = 2 - \sqrt{x}$

18. $y = 3 - 2\cos x$

19. $y = \sin\left(\frac{1}{2}x\right)$

20. $y = |x| - 2$

21. $y = |x - 2|$

22. $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$

23. $y = \left|\sqrt{x} - 1\right|$

24. $y = |\cos \pi x|$

25. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

27. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on June 30, 2009, high tide occurred at 6:45 AM. Find a function involving the cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

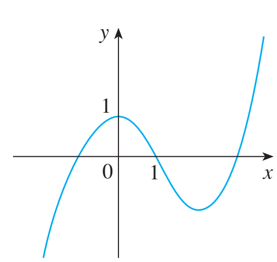
28. In a normal respiratory cycle the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL and a single respiratory cycle for an average human takes about 4 seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

29. (a) How is the graph of $y = f(|x|)$ related to the graph of f?

(b) Sketch the graph of $y = \sin|x|$.

(c) Sketch the graph of $y = \sqrt{|x|}$.

30. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



31–32 Find (a) $f + g$, (b) $f - g$, (c) fg, and (d) f/g and state their domains.

31. $f(x) = x^3 + 2x^2, \quad g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{3 - x}, \quad g(x) = \sqrt{x^2 - 1}$

33–38 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

33. $f(x) = 3x + 5, \quad g(x) = x^2 + x$

34. $f(x) = x^3 - 2, \quad g(x) = 1 - 4x$

35. $f(x) = \sqrt{x + 1}, \quad g(x) = 4x - 3$

36. $f(x) = \sin x, \quad g(x) = x^2 + 1$

37. $f(x) = x + \dfrac{1}{x}, \quad g(x) = \dfrac{x + 1}{x + 2}$

38. $f(x) = \dfrac{x}{1 + x}, \quad g(x) = \sin 2x$

39–42 Find $f \circ g \circ h$.

39. $f(x) = 3x - 2, \quad g(x) = \sin x, \quad h(x) = x^2$

40. $f(x) = |x - 4|, \quad g(x) = 2^x, \quad h(x) = \sqrt{x}$

41. $f(x) = \sqrt{x - 3}, \quad g(x) = x^2, \quad h(x) = x^3 + 2$

42. $f(x) = \tan x, \quad g(x) = \dfrac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

43–48 Express the function in the form $f \circ g$.

43. $F(x) = (2x + x^2)^4$

44. $F(x) = \cos^2 x$

45. $F(x) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

46. $G(x) = \sqrt[3]{\dfrac{x}{1 + x}}$

47. $v(t) = \sec(t^2)\tan(t^2)$

48. $u(t) = \dfrac{\tan t}{1 + \tan t}$

49–51 Express the function in the form $f \circ g \circ h$.

49. $R(x) = \sqrt{\sqrt{x} - 1}$

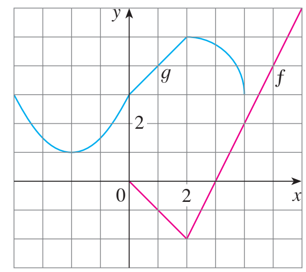
50. $H(x) = \sqrt[8]{2 + |x|}$

51. $S(t) = \sin^2(\cos t)$

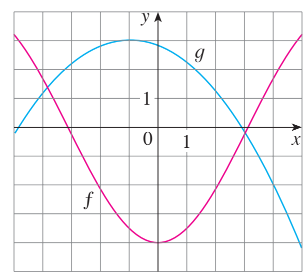
52. Use the table to evaluate each expression.
(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

53. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



54. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



55. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

56. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

57. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.
(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.
(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.
(c) Find $f \circ g$. What does this function represent?

58. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.
(b) Express the distance s between the plane and the radar station as a function of d.
(c) Use composition to express s as a function of t.

59. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

60. The Heaviside function defined in Exercise 59 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

(a) Sketch the graph of the ramp function $y = tH(t)$.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

61. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

62. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A.

63. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

64. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

65. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

66. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.4 The Tangent and Velocity Problems

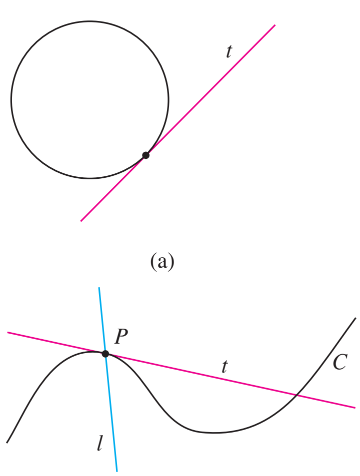
In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.



(a)

(b)

FIGURE 1

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m

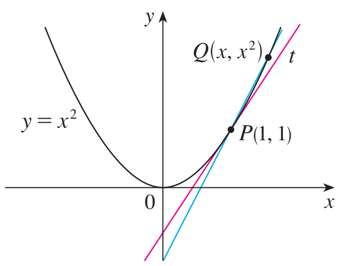


FIGURE 2

by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

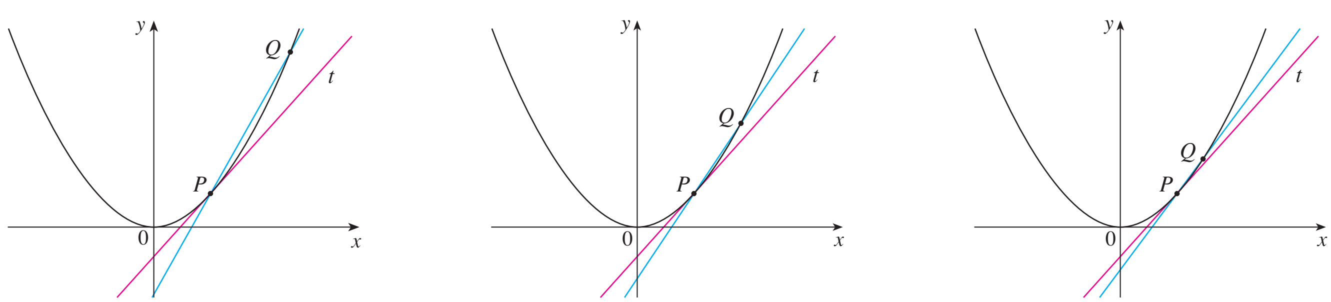
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

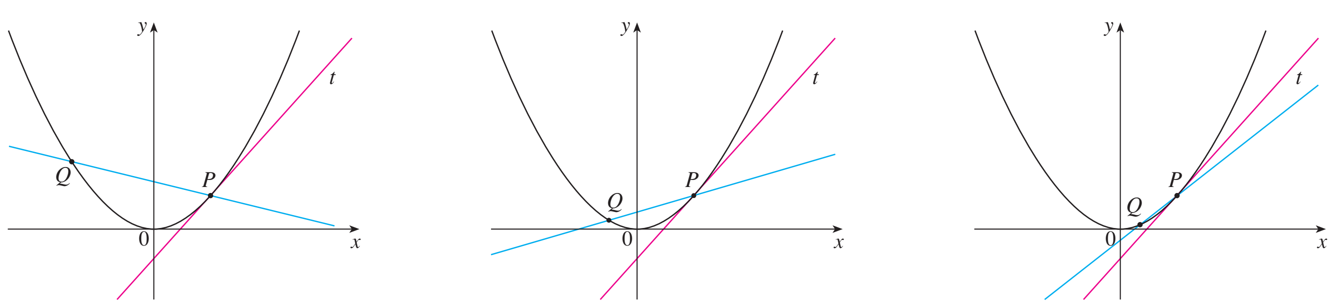
Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.



Q approaches *P* from the right



Q approaches *P* from the left

FIGURE 3

TEC In Visual 1.4 you can see how the process in Figure 3 works for additional functions.

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

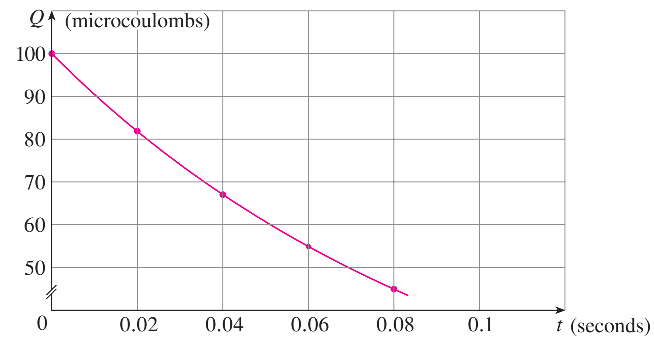


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be about -675.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 5.

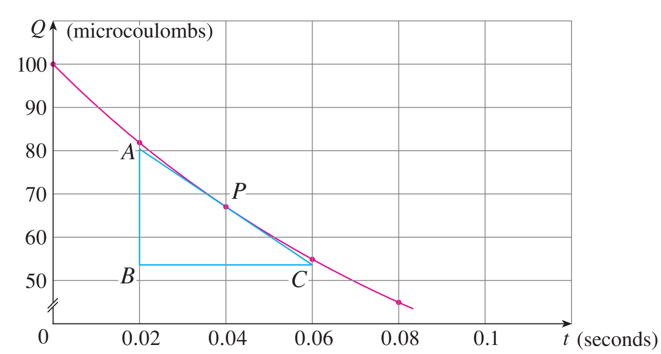


FIGURE 5

This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 seconds is about −670 microamperes.

The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the speed doesn't stay the same for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.



Steve Allen / Stockbyte / Getty Images

The CN Tower in Toronto was the tallest freestanding building in the world for 32 years.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

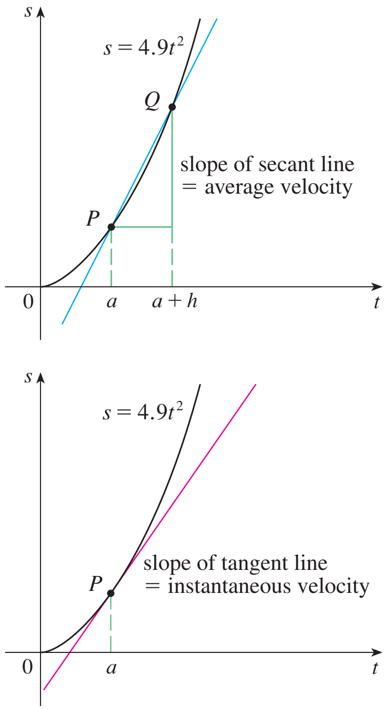


FIGURE 6

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is

$$v = 49 \text{ m/s}$$

■

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 6) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next four sections, we will return to the problems of finding tangents and velocities in Chapter 2.

1.4 EXERCISES

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5$, 10, 20, 25, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$

What are your conclusions?

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.

(a) If Q is the point $(x, 1/(1 - x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.
(a) If Q is the point $(x, \cos \pi x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0 (ii) 0.4 (iii) 0.49
(iv) 0.499 (v) 1 (vi) 0.6
(vii) 0.51 (viii) 0.501
(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.
(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.
(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 0.5 seconds (ii) 0.1 seconds
(iii) 0.05 seconds (iv) 0.01 seconds
(b) Estimate the instantaneous velocity when $t = 2$.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.
(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]
(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a motorcyclist after accelerating from rest.

t (seconds)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.2	124.8	176.7

(a) Find the average velocity for each time period:
(i) [2, 4] (ii) [3, 4]
(iii) [4, 5] (iv) [4, 6]
(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.
(a) Find the average velocity during each time period:
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]
(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9. Do the slopes appear to be approaching a limit?
(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

1.5 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

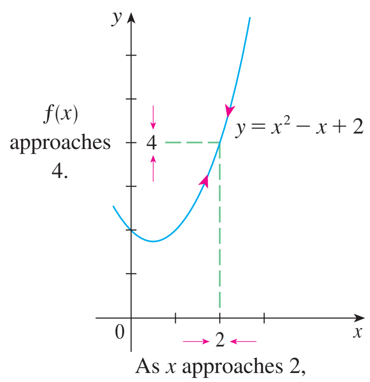


FIGURE 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that the closer x is to 2 (on either side of 2), the closer $f(x)$ is to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 1.7.)

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.

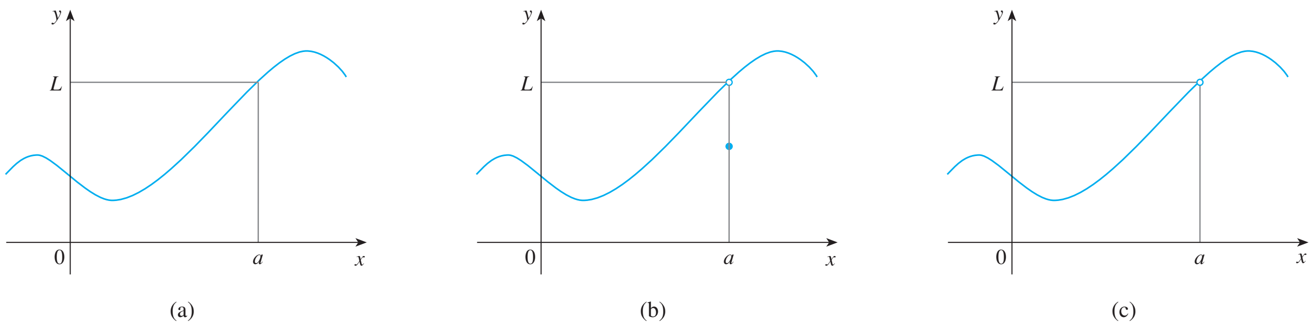
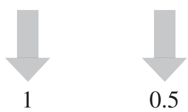


FIGURE 2 $\lim_{x \to a} f(x) = L$ in all three cases

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975



EXAMPLE 1 Guess the value of $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-1}$.

SOLUTION Notice that the function $f(x) = (x-1)/(x^2-1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x\to a} f(x)$ says that we consider values of x that are close to a but not equal to a.

The tables at the left give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the tables, we make the guess that

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5$$

■

Example 1 is illustrated by the graph of f in Figure 3. Now let's change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1. (See Figure 4.)

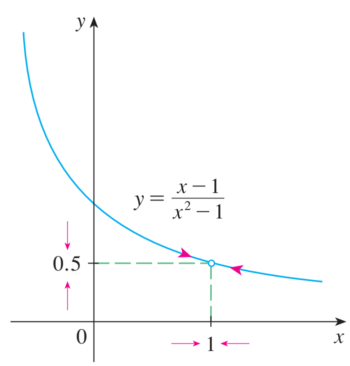


FIGURE 3

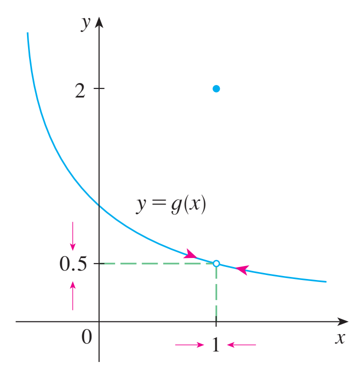


FIGURE 4

EXAMPLE 2 Estimate the value of $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . .

As t approaches 0, the values of the function seem to approach 0.1666666 . . . and so we guess that

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$

■

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

www.stewartcalculus.com
For a further explanation of why calculators sometimes give false values, click on *Lies My Calculator and Computer Told Me*. In particular, see the section called *The Perils of Subtraction*.

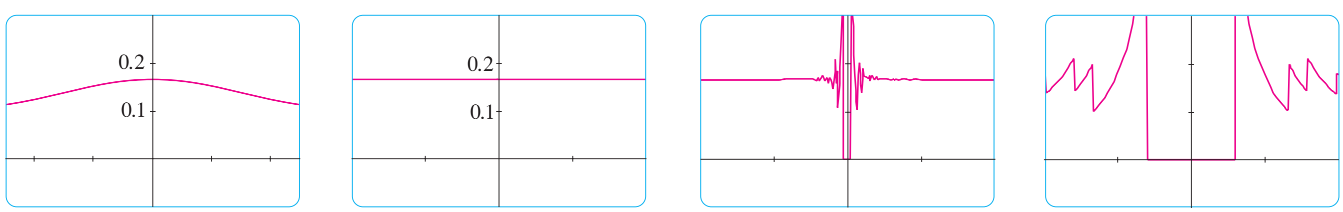
In Example 2 what would have happened if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2+9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2+9}$ is 3.000. . . to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$$

of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f, and when we use the trace mode (if available) we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too much, as in parts (c) and (d), then we get inaccurate graphs, again because of rounding errors from the subtraction.



(a) $-5 \leqslant t \leqslant 5$ (b) $-0.1 \leqslant t \leqslant 0.1$ (c) $-10^{-6} \leqslant t \leqslant 10^{-6}$ (d) $-10^{-7} \leqslant t \leqslant 10^{-7}$

FIGURE 5

EXAMPLE 3 Guess the value of $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places. From the table at the left and the graph in Figure 6 we guess that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 2 using a geometric argument.

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

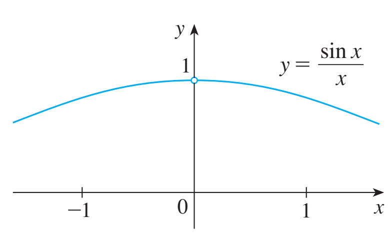


FIGURE 6 ■

EXAMPLE 4 Investigate $\displaystyle\lim_{x\to 0} \sin\frac{\pi}{x}$.

Computer Algebra Systems

Computer algebra systems (CAS) have commands that compute limits. In order to avoid the types of pitfalls demonstrated in Examples 2, 4, and 5, they don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

SOLUTION Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$

$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$

$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x \to 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x (such as $2/5$ or $2/101$) that approach 0. You can see this from the graph of f shown in Figure 7.

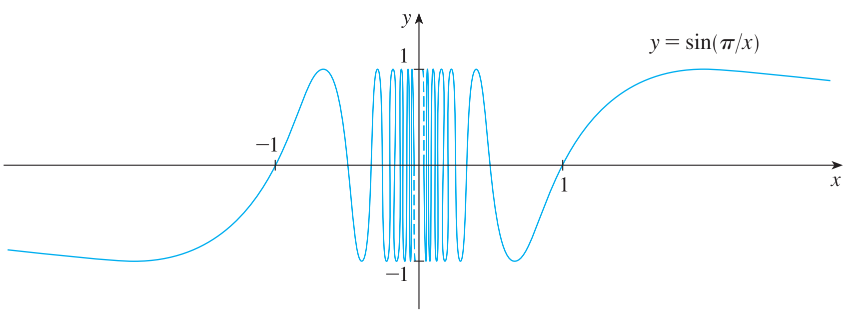


FIGURE 7

The dashed lines near the y-axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. (See Exercise 45.)

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

EXAMPLE 5 Find $\displaystyle\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right)$.

SOLUTION As before, we construct a table of values. From the first table in the margin it appears that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0$$

But if we persevere with smaller values of x, the second table suggests that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.000100 = \frac{1}{10{,}000}$$

In Section 1.8 we will see that $\lim_{x \to 0} \cos 5x = 1$; then it follows that the limit is 0.0001.

Examples 4 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to

know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits.

One-Sided Limits

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

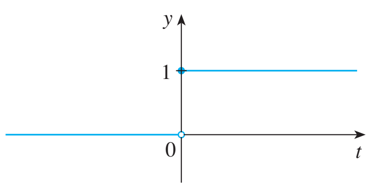


FIGURE 8
The Heaviside function

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 8.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore $\lim_{t \to 0} H(t)$ does not exist. ■

We noticed in Example 6 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

The notation $t \to 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t \to 0^+$ indicates that we consider only values of t that are greater than 0.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus the notation $x \to a^+$ means that we consider only x *greater than* a. These definitions are illustrated in Figure 9.

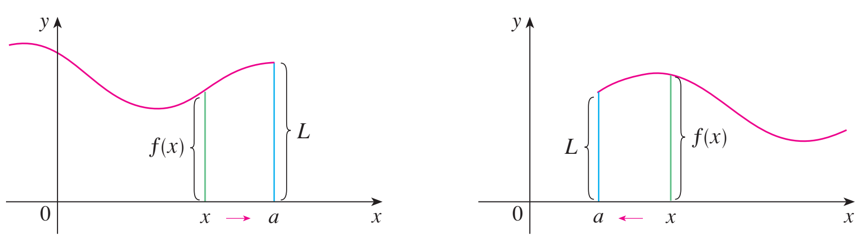


FIGURE 9 (a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

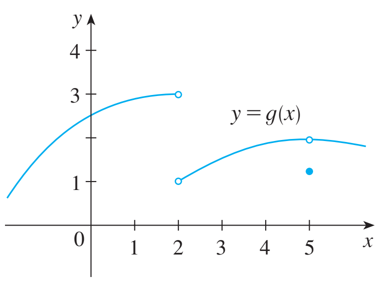


FIGURE 10

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)} \lim_{x \to 2^-} g(x) = 3 \quad \text{and} \quad \text{(b)} \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \to 2} g(x)$ does not exist.

The graph also shows that

$$\text{(d)} \lim_{x \to 5^-} g(x) = 2 \quad \text{and} \quad \text{(e)} \lim_{x \to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

Infinite Limits

x	$\dfrac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000

EXAMPLE 8 Find $\lim_{x \to 0} \dfrac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table in the margin.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 11 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} (1/x^2)$ does not exist. ■

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

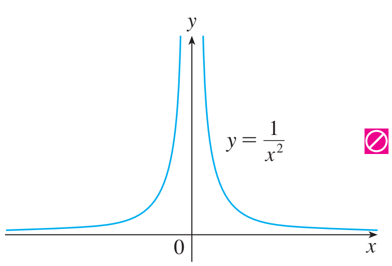


FIGURE 11

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or "increase without bound") as x becomes closer and closer to a.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

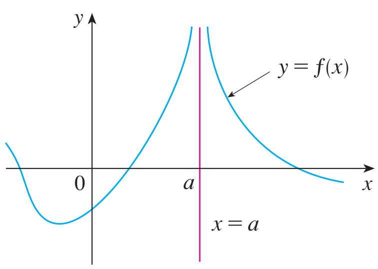


FIGURE 12
$\lim_{x \to a} f(x) = \infty$

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \qquad \text{as} \qquad x \to a$$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 12.

A similar sort of limit, for functions that become large negative as x gets close to a, is defined in Definition 5 and is illustrated in Figure 13.

When we say a number is "large negative," we mean that it is negative but its magnitude (absolute value) is large.

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

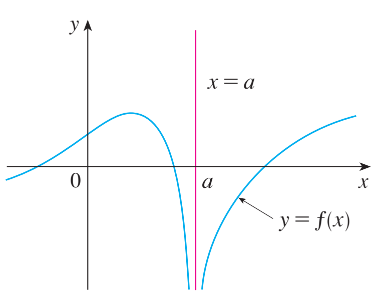


FIGURE 13
$\lim_{x \to a} f(x) = -\infty$

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that $x \to a^-$ means that we consider only values of x that are less than a,

and similarly $x \to a^+$ means that we consider only $x > a$. Illustrations of these four cases are given in Figure 14.

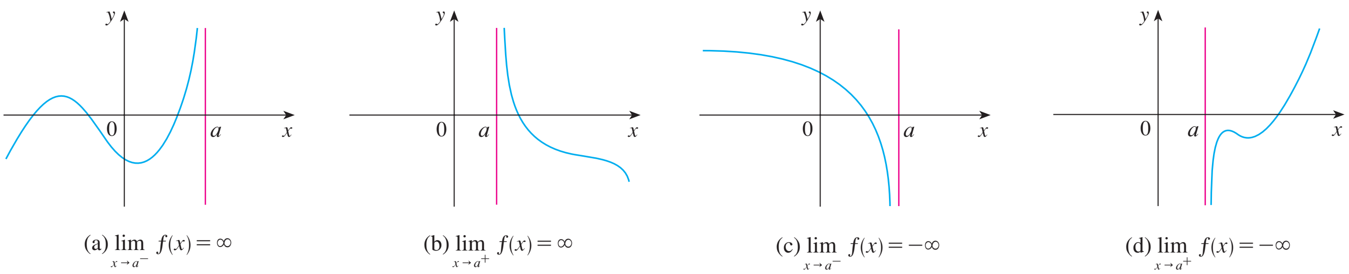


FIGURE 14

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 14 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 9 Find $\lim_{x \to 3^+} \dfrac{2x}{x - 3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x - 3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. [For instance, if $x = 3.01$ then $2x/(x - 3) = 6.02/0.01 = 602$.] Thus, intuitively, we see that

$$\lim_{x \to 3^+} \frac{2x}{x - 3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty$$

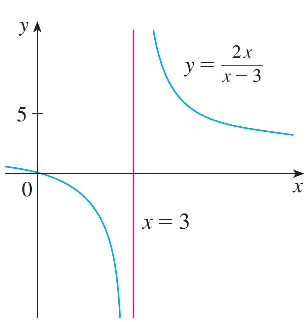


FIGURE 15

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote. ■

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

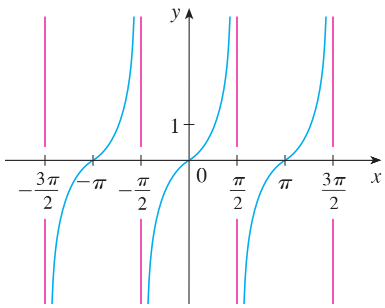


FIGURE 16
$y = \tan x$

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive (near 1) when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = \pi/2 + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this. ■

1.5 EXERCISES

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

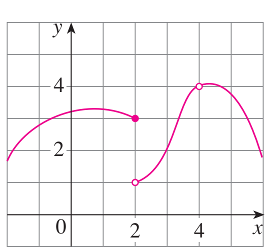
3. Explain the meaning of each of the following.

(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

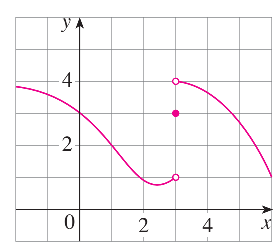
(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$



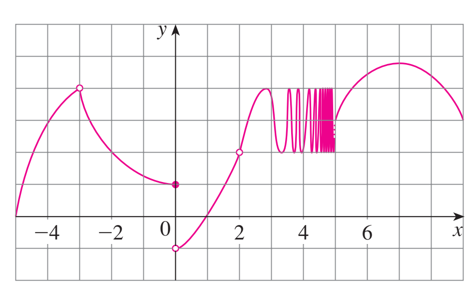
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$

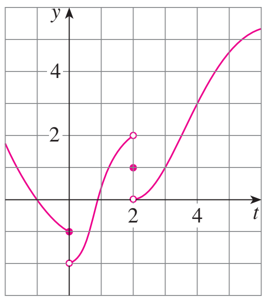


7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$

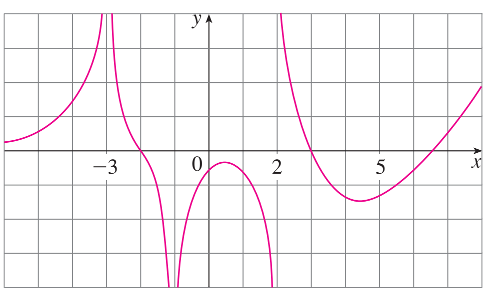


8. For the function A whose graph is shown, state the following.

(a) $\lim_{x \to -3} A(x)$ (b) $\lim_{x \to 2^-} A(x)$

(c) $\lim_{x \to 2^+} A(x)$ (d) $\lim_{x \to -1} A(x)$

(e) The equations of the vertical asymptotes

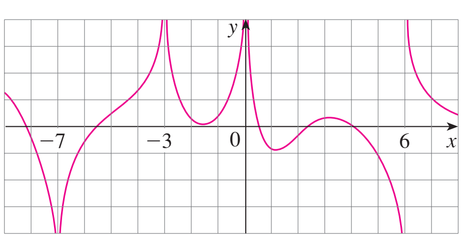


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

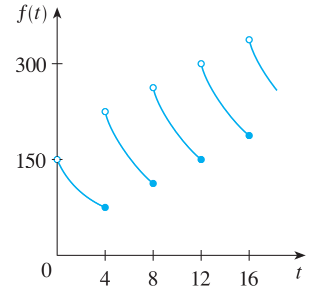
(f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \to 12^-} f(t) \quad \text{and} \quad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

11. $f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leqslant x < 1 \\ 2 - x & \text{if } x \geqslant 1 \end{cases}$

12. $f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leqslant x \leqslant \pi \\ \sin x & \text{if } x > \pi \end{cases}$

13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

13. $f(x) = \dfrac{1}{1 + 2^{1/x}}$ **14.** $f(x) = \dfrac{x^2 + x}{\sqrt{x^3 + x^2}}$

15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x \to 0^-} f(x) = -1, \quad \lim_{x \to 0^+} f(x) = 2, \quad f(0) = 1$

16. $\lim_{x \to 0} f(x) = 1, \quad \lim_{x \to 3^-} f(x) = -2, \quad \lim_{x \to 3^+} f(x) = 2,$
$f(0) = -1, \quad f(3) = 1$

17. $\lim_{x \to 3^+} f(x) = 4, \quad \lim_{x \to 3^-} f(x) = 2, \quad \lim_{x \to -2} f(x) = 2,$
$f(3) = 3, \quad f(-2) = 1$

18. $\lim_{x \to 0^-} f(x) = 2, \quad \lim_{x \to 0^+} f(x) = 0, \quad \lim_{x \to 4^-} f(x) = 3,$
$\lim_{x \to 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1$

19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\lim_{x \to 3} \dfrac{x^2 - 3x}{x^2 - 9}$,

$x = 3.1, 3.05, 3.01, 3.001, 3.0001,$
$2.9, 2.95, 2.99, 2.999, 2.9999$

20. $\lim_{x \to -3} \dfrac{x^2 - 3x}{x^2 - 9}$,
$x = -2.5, -2.9, -2.95, -2.99, -2.999, -2.9999,$
$-3.5, -3.1, -3.05, -3.01, -3.001, -3.0001$

21. $\lim_{x \to 0} \dfrac{\sin x}{x + \tan x}$, $x = \pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01$

22. $\lim_{h \to 0} \dfrac{(2 + h)^5 - 32}{h}$,
$h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

23–26 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\lim_{\theta \to 0} \dfrac{\sin 3\theta}{\tan 2\theta}$

24. $\lim_{p \to -1} \dfrac{1 + p^9}{1 + p^{15}}$

25. $\lim_{x \to 0^+} x^x$

26. $\lim_{t \to 0} \dfrac{5^t - 1}{t}$

27. (a) By graphing the function $f(x) = (\cos 2x - \cos x)/x^2$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

28. (a) Estimate the value of

$$\lim_{x \to 0} \frac{\sin x}{\sin \pi x}$$

by graphing the function $f(x) = (\sin x)/(\sin \pi x)$. State your answer correct to two decimal places.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

29–39 Determine the infinite limit.

29. $\lim_{x \to 5^+} \dfrac{x + 1}{x - 5}$

30. $\lim_{x \to 5^-} \dfrac{x + 1}{x - 5}$

31. $\lim_{x \to 1} \dfrac{2 - x}{(x - 1)^2}$

32. $\lim_{x \to 3^-} \dfrac{\sqrt{x}}{(x - 3)^5}$

33. $\lim_{x \to -2^+} \dfrac{x - 1}{x^2(x + 2)}$

34. $\lim_{x \to 0} \dfrac{x - 1}{x^2(x + 2)}$

35. $\lim_{x \to (\pi/2)^+} \dfrac{1}{x} \sec x$

36. $\lim_{x \to \pi^-} \cot x$

37. $\lim_{x \to 2\pi^-} x \csc x$

38. $\lim_{x \to 2^-} \dfrac{x^2 - 2x}{x^2 - 4x + 4}$

39. $\lim_{x \to 2^+} \dfrac{x^2 - 2x - 8}{x^2 - 5x + 6}$

40. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

(b) Confirm your answer to part (a) by graphing the function.

41. Determine $\lim_{x \to 1^-} \dfrac{1}{x^3 - 1}$ and $\lim_{x \to 1^+} \dfrac{1}{x^3 - 1}$
(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
(b) by reasoning as in Example 9, and
(c) from a graph of f.

42. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.
(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

43. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1$, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05, and guess the value of

$$\lim_{x \to 0} \left(x^2 - \frac{2^x}{1000} \right)$$

(b) Evaluate $f(x)$ for $x = 0.04$, 0.02, 0.01, 0.005, 0.003, and 0.001. Guess again.

44. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1$, 0.5, 0.1, 0.05, 0.01, and 0.005.
(b) Guess the value of $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$.
(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained values of 0 for $h(x)$. (In Section 6.8 a method for evaluating this limit will be explained.)
(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

45. Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

46. Consider the function $f(x) = \tan \dfrac{1}{x}$.
(a) Show that $f(x) = 0$ for $x = \dfrac{1}{\pi}, \dfrac{1}{2\pi}, \dfrac{1}{3\pi}, \ldots$
(b) Show that $f(x) = 1$ for $x = \dfrac{4}{\pi}, \dfrac{4}{5\pi}, \dfrac{4}{9\pi}, \ldots$
(c) What can you conclude about $\lim_{x \to 0^+} \tan \dfrac{1}{x}$?

47. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \qquad -\pi \leqslant x \leqslant \pi$$

Then find the exact equations of these asymptotes.

48. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

49. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

(b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

1.6 Calculating Limits Using the Limit Laws

In Section 1.5 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law **1.** The limit of a sum is the sum of the limits.

Difference Law **2.** The limit of a difference is the difference of the limits.

Constant Multiple Law **3.** The limit of a constant times a function is the constant times the limit of the function.

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 1.7 we give a

precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

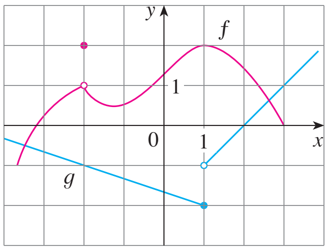


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\begin{aligned} \lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] && \text{(by Limit Law 1)} \\ &= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) && \text{(by Limit Law 3)} \\ &= 1 + 5(-1) = -4 \end{aligned}$$

(b) We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \to 1^-} [f(x)g(x)] = \lim_{x \to 1^-} f(x) \cdot \lim_{x \to 1^-} g(x) = 2 \cdot (-2) = -4$$

$$\lim_{x \to 1^+} [f(x)g(x)] = \lim_{x \to 1^+} f(x) \cdot \lim_{x \to 1^+} g(x) = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

(c) The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$ **8.** $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 1.7.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\displaystyle\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 1.7.37.)

10. $\displaystyle\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law, which is proved in Section 1.8 as a consequence of Law 10.

Root Law

11. $\displaystyle\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \displaystyle\lim_{x \to a} f(x) > 0.\right]$

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\displaystyle\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)

$$\begin{aligned}
\lim_{x \to 5} (2x^2 - 3x + 4) &= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 && \text{(by Laws 2 and 1)} \\
&= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{(by 3)} \\
&= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)} \\
&= 39
\end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned}
\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\displaystyle\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\displaystyle\lim_{x \to -2} (5 - 3x)} && \text{(by Law 5)} \\
&= \frac{\displaystyle\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\displaystyle\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x} && \text{(by 1, 2, and 3)} \\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} && \text{(by 9, 8, and 7)} \\
&= -\frac{1}{11}
\end{aligned}$$

■

Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

NOTE If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 57 and 58). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 1.8. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\displaystyle \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor and then compute the limit by direct substitution as follows:

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

The limit in this example arose in Example 1.4.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point (1, 1). ■

NOTE 1 In Example 3 we do not have an infinite limit even though the denominator approaches 0 as $x \to 1$. When both numerator and denominator approach 0, the limit may be infinite or it may be some finite value.

NOTE 2 In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x *approaches* 1 does not depend on the value of the function *at* 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$ ■

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

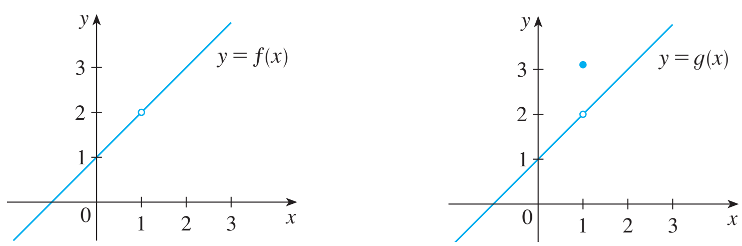


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

EXAMPLE 5 Evaluate $\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = \frac{h(6 + h)}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$ ■

EXAMPLE 6 Find $\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\\ &= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)}\\ &= \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}\\ &= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}\\ &= \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3}\\ &= \frac{1}{3+3}=\frac{1}{6}\end{aligned}$$

Here we use several properties of limits (5, 1, 10, 7, 9).

This calculation confirms the guess that we made in Example 1.5.2. ■

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 1.5. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x\to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 3.

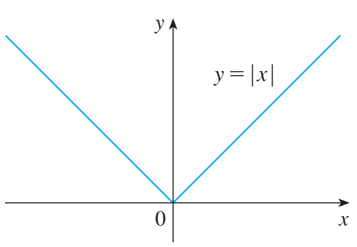


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x\to 0} |x| = 0$$ ■

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTION Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found. ■

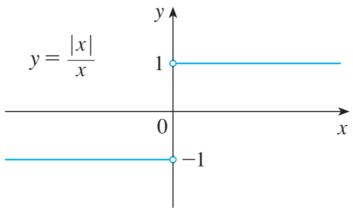


FIGURE 4

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x - 4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

It is shown in Example 1.7.3 that $\lim_{x \to 0^+} \sqrt{x} = 0$.

SOLUTION Since $f(x) = \sqrt{x - 4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 5. ■

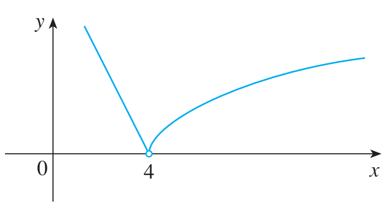


FIGURE 5

EXAMPLE 10 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1. ■

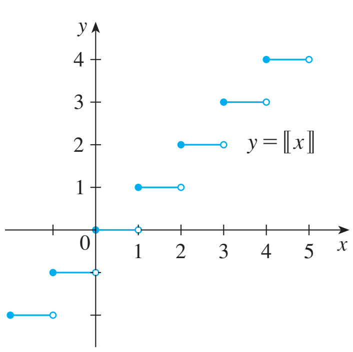


FIGURE 6
Greatest integer function

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

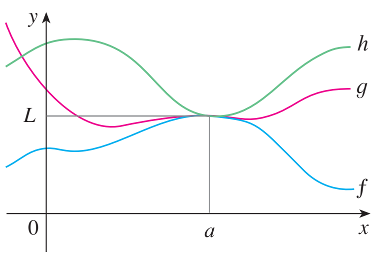


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 11 Show that $\displaystyle\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 1.5.4).

Instead we apply the Squeeze Theorem, and so we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write.

4 $$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2, we get

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

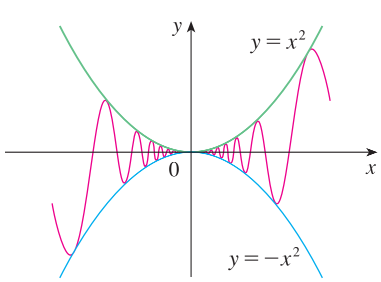


FIGURE 8
$y = x^2 \sin(1/x)$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$ ■

1.6 EXERCISES

1. Given that

$$\lim_{x\to 2} f(x) = 4 \qquad \lim_{x\to 2} g(x) = -2 \qquad \lim_{x\to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}\,[f(x) + 5g(x)]$ (b) $\lim_{x\to 2}\,[g(x)]^3$

(c) $\lim_{x\to 2}\sqrt{f(x)}$ (d) $\lim_{x\to 2}\dfrac{3f(x)}{g(x)}$

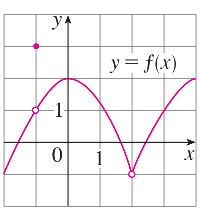
(e) $\lim_{x\to 2}\dfrac{g(x)}{h(x)}$ (f) $\lim_{x\to 2}\dfrac{g(x)h(x)}{f(x)}$

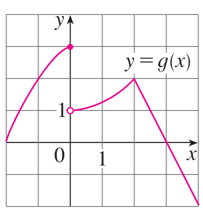
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}\,[f(x) + g(x)]$ (b) $\lim_{x\to 0}\,[f(x) - g(x)]$

(c) $\lim_{x\to -1}\,[f(x)g(x)]$ (d) $\lim_{x\to 3}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}\,[x^2 f(x)]$ (f) $f(-1) + \lim_{x\to -1} g(x)$





3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to 3}\,(5x^3 - 3x^2 + x - 6)$

4. $\lim_{x\to -1}\,(x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t\to -2}\dfrac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{u\to -2}\sqrt{u^4 + 3u + 6}$

7. $\lim_{x\to 8}\left(1 + \sqrt[3]{x}\right)(2 - 6x^2 + x^3)$

8. $\lim_{t\to 2}\left(\dfrac{t^2 - 2}{t^3 - 3t + 5}\right)^2$

9. $\lim_{x\to 2}\sqrt{\dfrac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}\,(x + 3)$$

is correct.

11–32 Evaluate the limit, if it exists.

11. $\lim_{x\to 5}\dfrac{x^2 - 6x + 5}{x - 5}$

12. $\lim_{x\to -3}\dfrac{x^2 + 3x}{x^2 - x - 12}$

13. $\lim_{x\to 5}\dfrac{x^2 - 5x + 6}{x - 5}$

14. $\lim_{x\to 4}\dfrac{x^2 + 3x}{x^2 - x - 12}$

15. $\lim_{t\to -3}\dfrac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x\to -1}\dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

17. $\lim_{h\to 0}\dfrac{(-5 + h)^2 - 25}{h}$

18. $\lim_{h\to 0}\dfrac{(2 + h)^3 - 8}{h}$

19. $\lim_{x\to -2}\dfrac{x + 2}{x^3 + 8}$

20. $\lim_{t\to 1}\dfrac{t^4 - 1}{t^3 - 1}$

21. $\lim_{h\to 0}\dfrac{\sqrt{9 + h} - 3}{h}$

22. $\lim_{u\to 2}\dfrac{\sqrt{4u + 1} - 3}{u - 2}$

23. $\lim_{x\to 3}\dfrac{\dfrac{1}{x} - \dfrac{1}{3}}{x - 3}$

24. $\lim_{h\to 0}\dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

25. $\lim_{t\to 0}\dfrac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

26. $\lim_{t\to 0}\left(\dfrac{1}{t} - \dfrac{1}{t^2 + t}\right)$

27. $\lim_{x\to 16}\dfrac{4 - \sqrt{x}}{16x - x^2}$

28. $\lim_{x\to 2}\dfrac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

29. $\lim_{t\to 0}\left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t}\right)$

30. $\lim_{x\to -4}\dfrac{\sqrt{x^2 + 9} - 5}{x + 4}$

31. $\lim_{h\to 0}\dfrac{(x + h)^3 - x^3}{h}$

32. $\lim_{h\to 0}\dfrac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}$

33. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

35. Use the Squeeze Theorem to show that $\lim_{x \to 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

36. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 + x^2}\, \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x \to 4} f(x)$.

38. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

39. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

40. Prove that $\lim_{x \to 0^+} \sqrt{x}\, [1 + \sin^2(2\pi/x)] = 0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x \to 3} \left(2x + |x - 3|\right)$

42. $\lim_{x \to -6} \dfrac{2x + 12}{|x + 6|}$

43. $\lim_{x \to 0.5^-} \dfrac{2x - 1}{|2x^3 - x^2|}$

44. $\lim_{x \to -2} \dfrac{2 - |x|}{2 + x}$

45. $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

46. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

---

47. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$
(iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

48. Let $g(x) = \operatorname{sgn}(\sin x)$.
(a) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x \to 0^+} g(x)$ (ii) $\lim_{x \to 0^-} g(x)$ (iii) $\lim_{x \to 0} g(x)$
(iv) $\lim_{x \to \pi^+} g(x)$ (v) $\lim_{x \to \pi^-} g(x)$ (vi) $\lim_{x \to \pi} g(x)$
(b) For which values of a does $\lim_{x \to a} g(x)$ not exist?
(c) Sketch a graph of g.

49. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.
(a) Find
(i) $\lim_{x \to 2^+} g(x)$ (ii) $\lim_{x \to 2^-} g(x)$
(b) Does $\lim_{x \to 2} g(x)$ exist?
(c) Sketch the graph of g.

50. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

51. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t \to 2} B(t)$ exists.

52. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.
(i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$
(iv) $\lim_{x \to 2^-} g(x)$ (v) $\lim_{x \to 2^+} g(x)$ (vi) $\lim_{x \to 2} g(x)$
(b) Sketch the graph of g.

53. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$
(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

54. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.
(a) Sketch the graph of f.
(b) Evaluate each limit, if it exists.
(i) $\lim_{x \to 0} f(x)$ (ii) $\lim_{x \to (\pi/2)^-} f(x)$
(iii) $\lim_{x \to (\pi/2)^+} f(x)$ (iv) $\lim_{x \to \pi/2} f(x)$
(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

55. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

56. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

57. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

58. If r is a rational function, use Exercise 57 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

59. If $\lim_{x\to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

60. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \dfrac{f(x)}{x}$

61. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

62. Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

63. Show by means of an example that $\lim_{x\to a} [f(x)\,g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

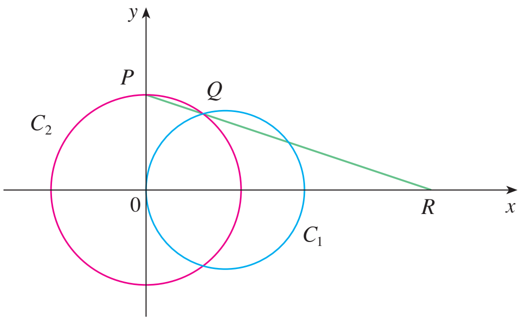
64. Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

65. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

66. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



1.7 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 1.5 is inadequate for some purposes because such phrases as "x is close to 2" and "$f(x)$ gets closer and closer to L" are vague. In order to be able to prove conclusively that

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0.0001 \qquad \text{or} \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x\to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

It is traditional to use the Greek letter δ (delta) in this situation.

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad |x - 3| < \delta \quad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < \delta$$

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$, then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2(0.05) = 0.1$$

that is, $$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01, and 0.001 that we have considered are *error tolerances* that we might allow. For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

1 $$|f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because (1) says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by restricting the values of x to be within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that (1) can be rewritten as follows:

$$\text{if} \quad 3 - \delta < x < 3 + \delta \quad (x \neq 3) \qquad \text{then} \qquad 5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1. By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using (1) as a model, we give a precise definition of a limit.

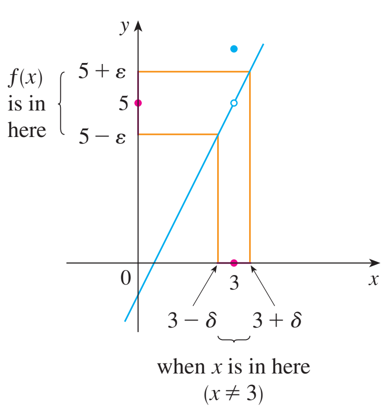


FIGURE 1

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

> $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by requiring that the distance from x to a be sufficiently small (but not 0).

Alternatively,

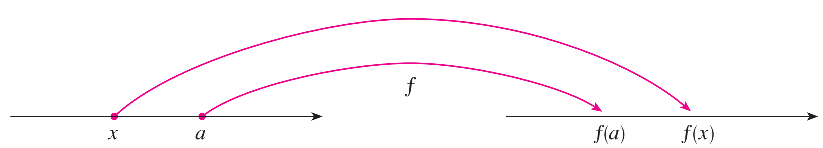
> $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by requiring x to be close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

> $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

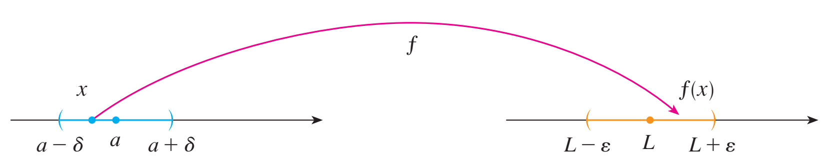
We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where f maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

FIGURE 2



The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

FIGURE 3



Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.) If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

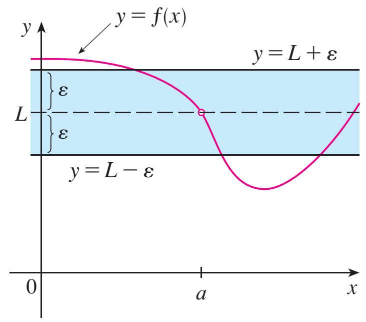


FIGURE 4

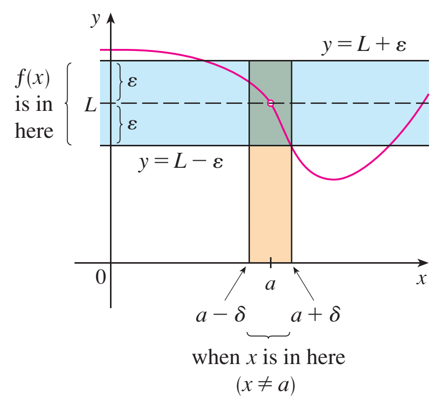


FIGURE 5

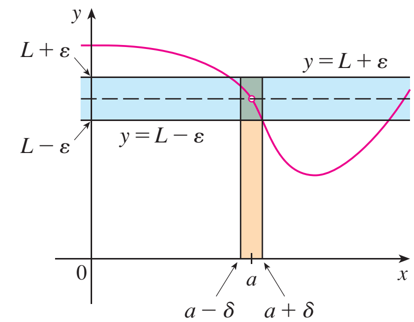


FIGURE 6

EXAMPLE 1 Since $f(x) = x^3 - 5x + 6$ is a polynomial, we know from the Direct Substitution Property that $\lim_{x \to 1} f(x) = f(1) = 1^3 - 5(1) + 6 = 2$. Use a graph to find a number δ such that if x is within δ of 1, then y is within 0.2 of 2, that is,

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

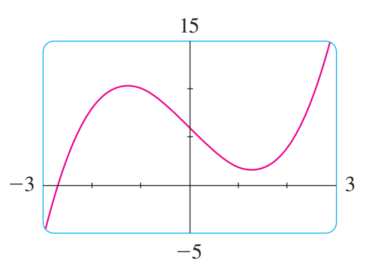


FIGURE 7

SOLUTION A graph of f is shown in Figure 7; we are interested in the region near the point (1, 2). Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as

$$-0.2 < (x^3 - 5x + 6) - 2 < 0.2$$

or equivalently

$$1.8 < x^3 - 5x + 6 < 2.2$$

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point (1, 2) in Figure 8. Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding toward 1 to be safe, we can say that

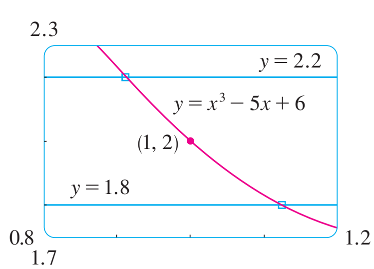


FIGURE 8

$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval (0.92, 1.12) is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \quad |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked. ■

The graphical procedure in Example 1 gives an illustration of the definition for $\varepsilon = 0.2$, but it does not *prove* that the limit is equal to 2. A proof has to provide a δ for *every* ε.

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number ε. Then you must be able to produce a suitable δ. You have to be able to do this for *every* $\varepsilon > 0$, not just a particular ε.

TEC In Module 1.7/3.4 you can explore the precise definition of a limit both graphically and numerically.

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a corresponding δ. Usually the smaller the value of ε, the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε, then $\lim_{x \to a} f(x) = L$.

EXAMPLE 2 Prove that $\displaystyle\lim_{x \to 3} (4x - 5) = 7$.

SOLUTION

1. *Preliminary analysis of the problem (guessing a value for δ).* Let ε be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$. Therefore we want δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad 4|x - 3| < \varepsilon$$

$$\text{that is,} \quad \text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x - 3| < \frac{\varepsilon}{4}$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. *Proof (showing that this δ works).* Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

Cauchy and Limits

After the invention of calculus in the 17th century, there followed a period of free development of the subject in the 18th century. Mathematicians like the Bernoulli brothers and Euler were eager to exploit the power of calculus and boldly explored the consequences of this new and wonderful mathematical theory without worrying too much about whether their proofs were completely correct.

The 19th century, by contrast, was the Age of Rigor in mathematics. There was a movement to go back to the foundations of the subject—to provide careful definitions and rigorous proofs. At the forefront of this movement was the French mathematician Augustin-Louis Cauchy (1789–1857), who started out as a military engineer before becoming a mathematics professor in Paris. Cauchy took Newton's idea of a limit, which was kept alive in the 18th century by the French mathematician Jean d'Alembert, and made it more precise. His definition of a limit reads as follows: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the *limit* of all the others." But when Cauchy used this definition in examples and proofs, he often employed delta-epsilon inequalities similar to the ones in this section. A typical Cauchy proof starts with: "Designate by δ and ε two very small numbers; . . ." He used ε because of the correspondence between epsilon and the French word *erreur* and δ because delta corresponds to *différence*. Later, the German mathematician Karl Weierstrass (1815–1897) stated the definition of a limit exactly as in our Definition 2.

This example is illustrated by Figure 9.

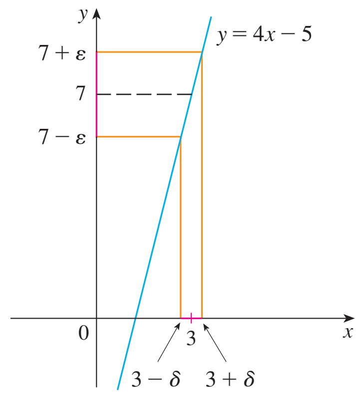


FIGURE 9

■

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ. But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

The intuitive definitions of one-sided limits that were given in Section 1.5 can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Notice that Definition 3 is the same as Definition 2 except that x is restricted to lie in the *left* half $(a - \delta, a)$ of the interval $(a - \delta, a + \delta)$. In Definition 4, x is restricted to lie in the *right* half $(a, a + \delta)$ of the interval $(a - \delta, a + \delta)$.

EXAMPLE 3 Use Definition 4 to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

SOLUTION

1. *Guessing a value for* δ. Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon$$

that is, if $0 < x < \delta$ then $\sqrt{x} < \varepsilon$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so

$$|\sqrt{x} - 0| < \varepsilon$$

According to Definition 4, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$. ■

EXAMPLE 4 Prove that $\lim\limits_{x \to 3} x^2 = 9$.

SOLUTION

1. *Guessing a value for δ.* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x^2 - 9| < \varepsilon$$

To connect $|x^2 - 9|$ with $|x - 3|$ we write $|x^2 - 9| = |(x + 3)(x - 3)|$. Then we want

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x + 3||x - 3| < \varepsilon$$

Notice that if we can find a positive constant C such that $|x + 3| < C$, then

$$|x + 3||x - 3| < C|x - 3|$$

and we can make $C|x - 3| < \varepsilon$ by taking $|x - 3| < \varepsilon/C$, so we could choose $\delta = \varepsilon/C$.

We can find such a number C if we restrict x to lie in some interval centered at 3. In fact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance 1 from 3, that is, $|x - 3| < 1$. Then $2 < x < 4$, so $5 < x + 3 < 7$. Thus we have $|x + 3| < 7$, and so $C = 7$ is a suitable choice for the constant.

But now there are two restrictions on $|x - 3|$, namely

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

To make sure that both of these inequalities are satisfied, we take δ to be the smaller of the two numbers 1 and $\varepsilon/7$. The notation for this is $\delta = \min\{1, \varepsilon/7\}$.

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x + 3| < 7$ (as in part 1).

We also have $|x - 3| < \varepsilon/7$, so

$$|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x \to 3} x^2 = 9$. ■

As Example 4 shows, it is not always easy to prove that limit statements are true using the ε, δ definition. In fact, if we had been given a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately this is unnecessary because the Limit Laws stated in Section 1.6 can be proved using Definition 2, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

For instance, we prove the Sum Law: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

The remaining laws are proved in the exercises and in Appendix F.

PROOF OF THE SUM LAW Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Triangle Inequality:

$$|a + b| \leqslant |a| + |b|$$

(See Appendix A).

Using the Triangle Inequality we can write

5
$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leqslant |f(x) - L| + |g(x) - M| \end{aligned}$$

We make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x \to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of the numbers δ_1 and δ_2. Notice that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

and so

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by (5),

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\leqslant |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite limits can also be defined in a precise way. The following is a precise version of Definition 1.5.4.

6 Precise Definition of an Infinite Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

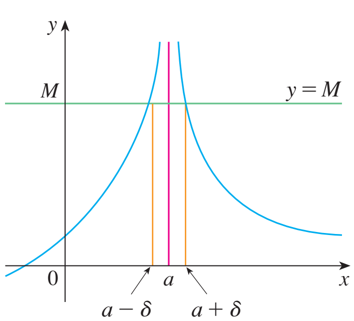


FIGURE 10

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by requiring x to be close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller δ may be required.

EXAMPLE 5 Use Definition 6 to prove that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

SOLUTION Let M be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x - 0| < \delta \quad \text{then} \quad 1/x^2 > M$$

But

$$\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that $1/x^2 \to \infty$ as $x \to 0$.

Similarly, the following is a precise version of Definition 1.5.5. It is illustrated by Figure 11.

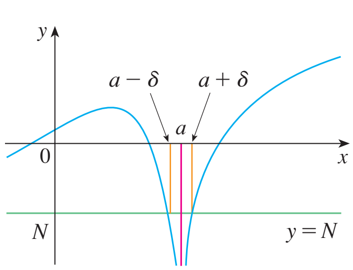


FIGURE 11

7 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

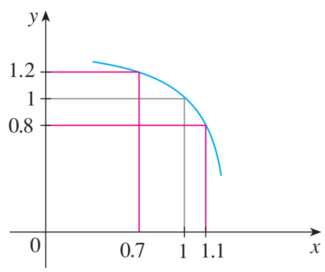
means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

1.7 EXERCISES

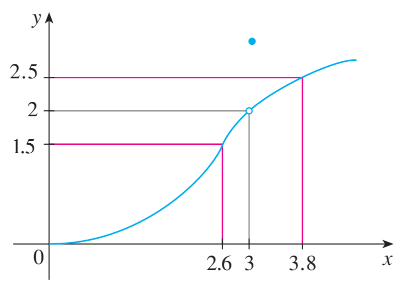
1. Use the given graph of f to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |f(x) - 1| < 0.2$$



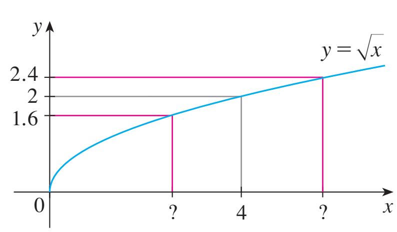
2. Use the given graph of f to find a number δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |f(x) - 2| < 0.5$$



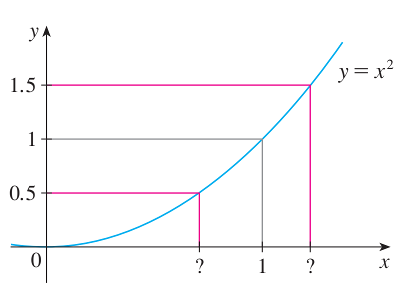
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if} \quad \left| x - \frac{\pi}{4} \right| < \delta \quad \text{then} \quad |\tan x - 1| < 0.2$$

6. Use a graph to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad \left| \frac{2x}{x^2 + 4} - 0.4 \right| < 0.1$$

7. For the limit

$$\lim_{x \to 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 2} \frac{4x + 1}{3x - 4} = 4.5$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. (a) Use a graph to find a number δ such that

$$\text{if} \quad 4 < x < 4 + \delta \quad \text{then} \quad \frac{x^2 + 4}{\sqrt{x - 4}} > 100$$

(b) What limit does part (a) suggest is true?

10. Given that $\lim_{x \to \pi} \csc^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 500$ and (b) $M = 1000$.

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.
(a) How much power is needed to maintain the temperature at 200°C?

(b) If the temperature is allowed to vary from 200°C by up to ±1°C, what range of wattage is allowed for the input power?

(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x \to 2}(5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\displaystyle\lim_{x \to 3} \left(1 + \tfrac{1}{3}x\right) = 2$

16. $\displaystyle\lim_{x \to 4} (2x - 5) = 3$

17. $\displaystyle\lim_{x \to -3} (1 - 4x) = 13$

18. $\displaystyle\lim_{x \to -2} (3x + 5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\displaystyle\lim_{x \to 1} \frac{2 + 4x}{3} = 2$

20. $\displaystyle\lim_{x \to 10} \left(3 - \tfrac{4}{5}x\right) = -5$

21. $\displaystyle\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4} = 6$

22. $\displaystyle\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$

23. $\displaystyle\lim_{x \to a} x = a$

24. $\displaystyle\lim_{x \to a} c = c$

25. $\displaystyle\lim_{x \to 0} x^2 = 0$

26. $\displaystyle\lim_{x \to 0} x^3 = 0$

27. $\displaystyle\lim_{x \to 0} |x| = 0$

28. $\displaystyle\lim_{x \to -6^+} \sqrt[8]{6 + x} = 0$

29. $\displaystyle\lim_{x \to 2} (x^2 - 4x + 5) = 1$

30. $\displaystyle\lim_{x \to 2} (x^2 + 2x - 7) = 1$

31. $\displaystyle\lim_{x \to -2} (x^2 - 1) = 3$

32. $\displaystyle\lim_{x \to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x \to 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x \to 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

CAS **35.** (a) For the limit $\lim_{x \to 1} (x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
(b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\displaystyle\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

37. Prove that $\displaystyle\lim_{x \to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

$\left[\textit{Hint: } \text{Use } \left|\sqrt{x} - \sqrt{a}\right| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}}.\right]$

38. If H is the Heaviside function defined in Example 1.5.6, prove, using Definition 2, that $\lim_{t \to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1.6.1.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\displaystyle\lim_{x \to -3} \frac{1}{(x + 3)^4} = \infty$.

43. Prove that $\displaystyle\lim_{x \to -1^-} \frac{5}{(x + 1)^3} = -\infty$.

44. Suppose that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c$, where c is a real number. Prove each statement.
(a) $\displaystyle\lim_{x \to a} [f(x) + g(x)] = \infty$
(b) $\displaystyle\lim_{x \to a} [f(x)g(x)] = \infty$ if $c > 0$
(c) $\displaystyle\lim_{x \to a} [f(x)g(x)] = -\infty$ if $c < 0$

1.8 Continuity

We noticed in Section 1.6 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

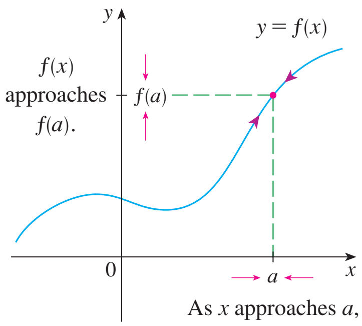


FIGURE 1

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 1.5.6, where the Heaviside function is discontinuous at 0 because $\lim_{t \to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

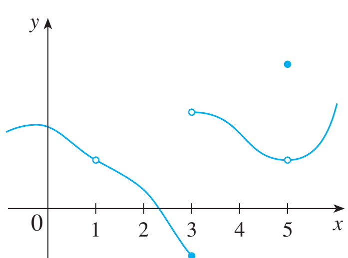


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 1.5.8.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

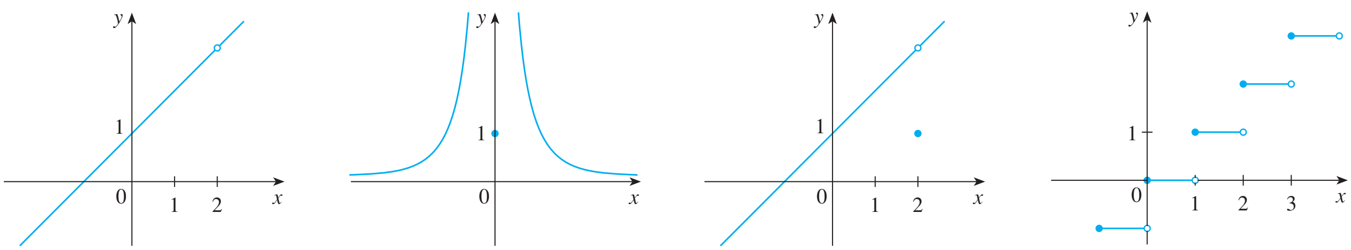
exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer. (See Example 1.6.10 and Exercise 1.6.53.) ■

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

FIGURE 3
Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n, the function $f(x) = [\![x]\!]$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but $$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$ ■

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 7)} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{(by 11)} \\
&= 1 - \sqrt{1 - a^2} && \text{(by 2, 7, and 9)} \\
&= f(a)
\end{aligned}$$

Thus, by Definition l, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$ ■

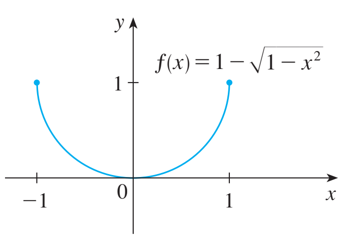


FIGURE 4

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

PROOF Each of the five parts of this theorem follows from the corresponding Limit Law in Section 1.6. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)]\\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{(by Law 1)}\\ &= f(a)+g(a)\\ &= (f+g)(a)\end{aligned}$$

This shows that $f + g$ is continuous at a. ■

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 1.6 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

PROOF

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x\to a} c_0 = c_0 \qquad \text{(by Law 7)}$$

and

$$\lim_{x\to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 9)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D. ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time, as we might expect.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 1.6.2(b).

EXAMPLE 5 Find $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

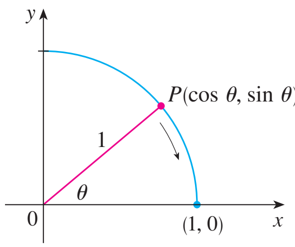


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 64) is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 1.2.18), we would certainly guess that they are continuous. We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$. Thus

6 $$\lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ (for $\theta > 0$), which is proved in Section 2.4.

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 64 and 65).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).

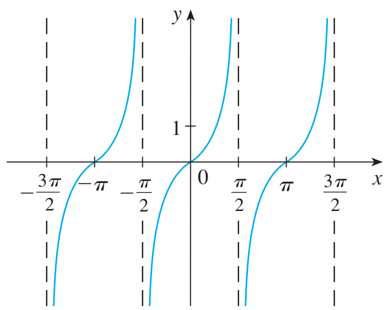


FIGURE 6
$y = \tan x$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

EXAMPLE 6 On what intervals is each function continuous?

(a) $f(x) = x^{100} - 2x^{37} + 75$ (b) $g(x) = \dfrac{x^2 + 2x + 17}{x^2 - 1}$

(c) $h(x) = \sqrt{x} + \dfrac{x + 1}{x - 1} - \dfrac{x + 1}{x^2 + 1}$

SOLUTION

(a) f is a polynomial, so it is continuous on $(-\infty, \infty)$ by Theorem 5(a).

(b) g is a rational function, so by Theorem 5(b), it is continuous on its domain, which is $D = \{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$. Thus g is continuous on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

(c) We can write $h(x) = F(x) + G(x) - H(x)$, where

$$F(x) = \sqrt{x} \qquad G(x) = \frac{x + 1}{x - 1} \qquad H(x) = \frac{x + 1}{x^2 + 1}$$

F is continuous on $[0, \infty)$ by Theorem 7. G is a rational function, so it is continuous everywhere except when $x - 1 = 0$, that is, $x = 1$. H is also a rational function, but its denominator is never 0, so H is continuous everywhere. Thus, by parts 1 and 2 of Theorem 4, h is continuous on the intervals $[0, 1)$ and $(1, \infty)$. ■

EXAMPLE 7 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geqslant -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \to \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$ ■

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\displaystyle\lim_{x \to a} g(x) = b$, then $\displaystyle\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and

$$f\Big(\lim_{x \to a} g(x)\Big) = \sqrt[n]{\lim_{x \to a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$$

and so Limit Law 11 has now been proved. (We assume that the roots exist.)

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

PROOF Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a. ■

EXAMPLE 8 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \dfrac{1}{\sqrt{x^2 + 7} - 4}$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

Now g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) Notice that F can be broken up as the composition of four continuous functions:

$$F = f \circ g \circ h \circ k \qquad \text{or} \qquad F(x) = f(g(h(k(x))))$$

where $f(x) = \dfrac{1}{x}$ $\quad g(x) = x - 4 \quad h(x) = \sqrt{x} \quad k(x) = x^2 + 7$

We know that each of these functions is continuous on its domain (by Theorems 5 and 7), so by Theorem 9, F is continuous on its domain, which is

$$\{x \in \mathbb{R} \mid \sqrt{x^2 + 7} \neq 4\} = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$ ■

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 7. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

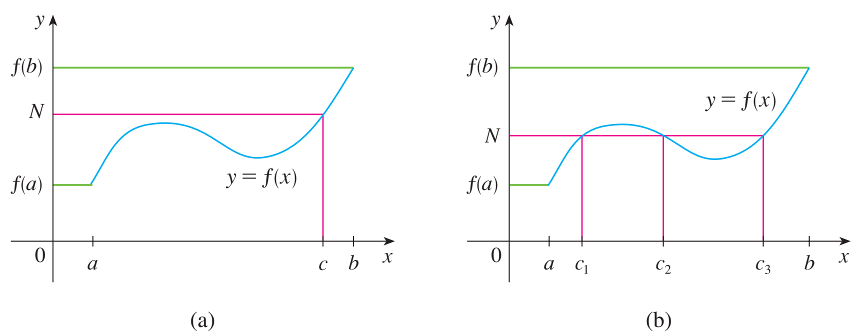


FIGURE 7

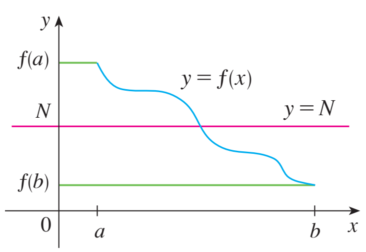


FIGURE 8

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 8, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 50).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 9 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \quad \text{and} \quad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$. ■

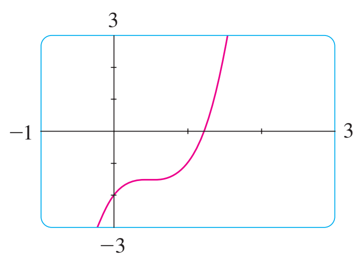


FIGURE 9

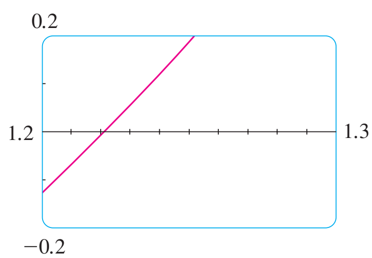


FIGURE 10

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 9 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 10 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

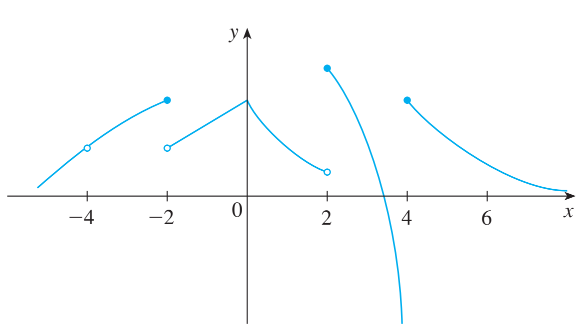
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore "connects the dots" by turning on the intermediate pixels.

1.8 EXERCISES

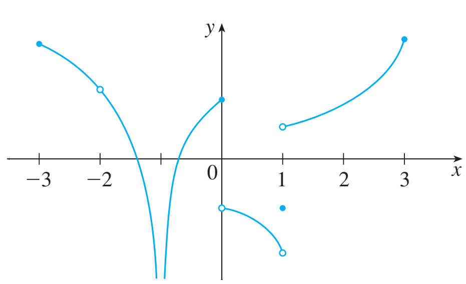
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5–8 Sketch the graph of a function f that is continuous except for the stated discontinuity.

5. Discontinuous at 2, but continuous from the right there

6. Discontinuities at -1 and 4, but continuous from the left at -1 and from the right at 4

7. Removable discontinuity at 3, jump discontinuity at 5

8. Neither left nor right continuous at -2, continuous only from the left at 2

9. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.

(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.

(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

10. Explain why each function is continuous or discontinuous.

(a) The temperature at a specific location as a function of time

(b) The temperature at a specific time as a function of the distance due west from New York City

(c) The altitude above sea level as a function of the distance due west from New York City

(d) The cost of a taxi ride as a function of the distance traveled

(e) The current in the circuit for the lights in a room as a function of time

11–14 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

13. $p(v) = 2\sqrt{3v^2 + 1}, \quad a = 1$

14. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}, \quad a = 2$

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = x + \sqrt{x - 4}, \quad [4, \infty)$

16. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$

17–22 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

17. $f(x) = \dfrac{1}{x + 2} \qquad a = -2$

18. $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \qquad a = -2$

19. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases} \qquad a = 1$

20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \qquad a = 3$

23–24 How would you "remove the discontinuity" of f? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$

26. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$

27. $Q(x) = \dfrac{\sqrt[3]{x} - 2}{x^3 - 2}$

28. $h(x) = \dfrac{\sin x}{x + 1}$

29. $h(x) = \cos(1 - x^2)$

30. $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$

31. $M(x) = \sqrt{1 + \dfrac{1}{x}}$

32. $F(x) = \sin(\cos(\sin x))$

33–34 Locate the discontinuities of the function and illustrate by graphing.

33. $y = \dfrac{1}{1 + \sin x}$

34. $y = \tan\sqrt{x}$

35–38 Use continuity to evaluate the limit.

35. $\displaystyle\lim_{x \to 2} x\sqrt{20 - x^2}$

36. $\displaystyle\lim_{x \to \pi} \sin(x + \sin x)$

37. $\displaystyle\lim_{x \to \pi/4} x^2 \tan x$

38. $\displaystyle\lim_{x \to 2} x^3/\sqrt{x^2 + x - 2}$

39–40 Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \sqrt{x - 1} & \text{if } x > 1 \end{cases}$

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

41–43 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

41. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

42. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

46. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

47. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

48. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.
(a) Find $(f \circ g)(x)$.
(b) Is $f \circ g$ continuous everywhere? Explain.

49. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

50. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

51. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

52. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0, \quad (1, 2)$

54. $2/x = x - \sqrt{x}, \quad (2, 3)$

55. $\cos x = x, \quad (0, 1)$

56. $\sin x = x^2 - x, \quad (1, 2)$

57–58 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

57. $\cos x = x^3$

58. $x^5 - x^2 + 2x + 3 = 0$

59–60 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

59. $x^5 - x^2 - 4 = 0$

60. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

61–62 Prove, without graphing, that the graph of the function has at least two x-intercepts in the specified interval.

61. $y = \sin x^3, \quad (1, 2)$

62. $y = x^2 - 3 + 1/x, \quad (0, 2)$

63. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

64. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 63 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

65. Prove that cosine is a continuous function.

66. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

67. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

68. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

69. Is there a number that is exactly 1 more than its cube?

70. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

71. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

72. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is $|f|$.
(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

73. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

1 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.
(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$
(d) $y = 1/x$ (e) $y = |x|$ (f) $y = \sqrt{x}$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward.
(b) Shift 2 units downward.
(c) Shift 2 units to the right.
(d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. Explain what each of the following means and illustrate with a sketch.
(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$ (c) $\lim_{x \to a^-} f(x) = L$
(d) $\lim_{x \to a} f(x) = \infty$ (e) $\lim_{x \to a} f(x) = -\infty$

13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

15. State the following Limit Laws.
(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

16. What does the Squeeze Theorem say?

17. (a) What does it mean for f to be continuous at a?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

18. (a) Give examples of functions that are continuous on $[-1, 1]$.
(b) Give an example of a function that is not continuous on $[0, 1]$.

19. What does the Intermediate Value Theorem say?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is a function, then $f(s + t) = f(s) + f(t)$.

2. If $f(s) = f(t)$, then $s = t$.

3. If f is a function, then $f(3x) = 3f(x)$.

4. If $x_1 < x_2$ and f is a decreasing function, then $f(x_1) > f(x_2)$.

5. A vertical line intersects the graph of a function at most once.

6. If x is any real number, then $\sqrt{x^2} = x$.

7. $\lim_{x \to 4}\left(\frac{2x}{x - 4} - \frac{8}{x - 4}\right) = \lim_{x \to 4}\frac{2x}{x - 4} - \lim_{x \to 4}\frac{8}{x - 4}$

8. $\lim_{x \to 1}\frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \to 1}(x^2 + 6x - 7)}{\lim_{x \to 1}(x^2 + 5x - 6)}$

9. $\lim_{x \to 1}\frac{x - 3}{x^2 + 2x - 4} = \frac{\lim_{x \to 1}(x - 3)}{\lim_{x \to 1}(x^2 + 2x - 4)}$

10. $\frac{x^2 - 9}{x - 3} = x + 3$

11. $\lim_{x \to 3}\frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3)$

12. If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5}[f(x)/g(x)]$ does not exist.

13. If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5}[f(x)/g(x)]$ does not exist.

14. If neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists, then $\lim_{x \to a}[f(x) + g(x)]$ does not exist.

15. If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} g(x)$ does not exist, then $\lim_{x \to a}[f(x) + g(x)]$ does not exist.

16. If $\lim_{x \to 6}[f(x)\,g(x)]$ exists, then the limit must be $f(6)\,g(6)$.

17. If p is a polynomial, then $\lim_{x \to b} p(x) = p(b)$.

18. If $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0} g(x) = \infty$, then $\lim_{x \to 0}[f(x) - g(x)] = 0$.

19. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

20. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

21. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x \to 2} f(4x^2 - 11) = 2$.

22. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

23. Let f be a function such that $\lim_{x \to 0} f(x) = 6$. Then there exists a positive number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.

24. If $f(x) > 1$ for all x and $\lim_{x \to 0} f(x)$ exists, then $\lim_{x \to 0} f(x) > 1$.

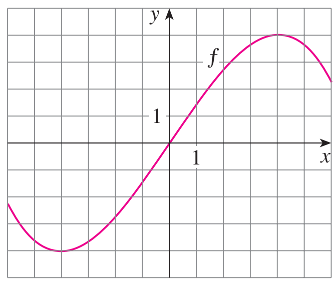
25. The equation $x^{10} - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.

26. If f is continuous at a, so is $|f|$.

27. If $|f|$ is continuous at a, so is f.

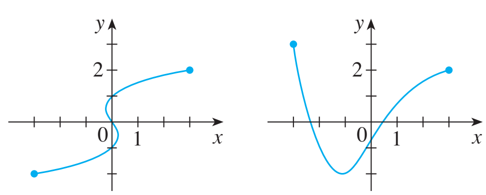
EXERCISES

1. Let f be the function whose graph is given.



(a) Estimate the value of $f(2)$.
(b) Estimate the values of x such that $f(x) = 3$.
(c) State the domain of f.
(d) State the range of f.
(e) On what interval is f increasing?
(f) Is f even, odd, or neither even nor odd? Explain.

2. Determine whether each curve is the graph of a function of x. If it is, state the domain and range of the function.



3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

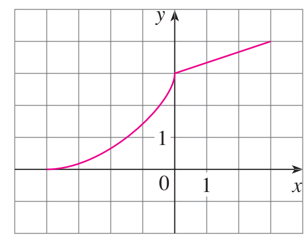
5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$ **6.** $g(x) = \sqrt{16 - x^4}$

7. $y = 1 + \sin x$ **8.** $F(t) = 3 + \cos 2t$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 8$ (b) $y = f(x + 8)$
(c) $y = 1 + 2f(x)$ (d) $y = f(x - 2) - 2$
(e) $y = -f(x)$ (f) $y = 3 - f(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$ (b) $y = -f(x)$
(c) $y = 2 - f(x)$ (d) $y = \frac{1}{2}f(x) - 1$



11–16 Use transformations to sketch the graph of the function.

11. $y = (x - 2)^3$ **12.** $y = 2\sqrt{x}$

13. $y = x^2 - 2x + 2$ **14.** $y = \dfrac{1}{x - 1}$

15. $f(x) = -\cos 2x$ **16.** $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ 1 + x^2 & \text{if } x \geq 0 \end{cases}$

17. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$
(b) $f(x) = x^3 - x^7$
(c) $f(x) = \cos(x^2)$ (d) $f(x) = 1 + \sin x$

18. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

19. If $f(x) = \sqrt{x}$ and $g(x) = \sin x$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

20. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

21. Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2010.

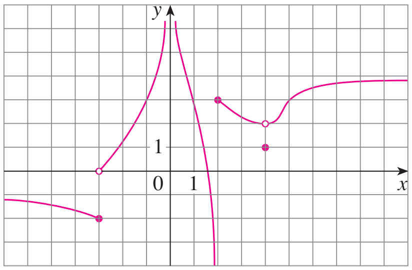
Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6		

22. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.
(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

23. The graph of f is given.



(a) Find each limit, or explain why it does not exist.

(i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to -3^+} f(x)$ (iii) $\lim_{x\to -3} f(x)$

(iv) $\lim_{x\to 4} f(x)$ (v) $\lim_{x\to 0} f(x)$ (vi) $\lim_{x\to 2^-} f(x)$

(b) State the equations of the vertical asymptotes.

(c) At what numbers is f discontinuous? Explain.

24. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$\lim_{x\to 0^+} f(x) = -2$, $\lim_{x\to 0^-} f(x) = 1$, $f(0) = -1$,

$\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$

25–38 Find the limit.

25. $\lim_{x\to 0} \cos(x + \sin x)$

26. $\lim_{x\to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

27. $\lim_{x\to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

28. $\lim_{x\to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

29. $\lim_{h\to 0} \dfrac{(h - 1)^3 + 1}{h}$

30. $\lim_{t\to 2} \dfrac{t^2 - 4}{t^3 - 8}$

31. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r - 9)^4}$

32. $\lim_{v\to 4^+} \dfrac{4 - v}{|4 - v|}$

33. $\lim_{u\to 1} \dfrac{u^4 - 1}{u^3 + 5u^2 - 6u}$

34. $\lim_{x\to 3} \dfrac{\sqrt{x + 6} - x}{x^3 - 3x^2}$

35. $\lim_{s\to 16} \dfrac{4 - \sqrt{s}}{s - 16}$

36. $\lim_{v\to 2} \dfrac{v^2 + 2v - 8}{v^4 - 16}$

37. $\lim_{x\to 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$

38. $\lim_{x\to 1} \left(\dfrac{1}{x - 1} + \dfrac{1}{x^2 - 3x + 2}\right)$

39. If $2x - 1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x\to 1} f(x)$.

40. Prove that $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$.

41–44 Prove the statement using the precise definition of a limit.

41. $\lim_{x\to 2} (14 - 5x) = 4$

42. $\lim_{x\to 0} \sqrt[3]{x} = 0$

43. $\lim_{x\to 2} (x^2 - 3x) = -2$

44. $\lim_{x\to 4^+} \dfrac{2}{\sqrt{x - 4}} = \infty$

45. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$ (iii) $\lim_{x\to 0} f(x)$

(iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$ (vi) $\lim_{x\to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

46. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2 - x & \text{if } 2 < x \leqslant 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geqslant 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g.

47–48 Show that the function is continuous on its domain. State the domain.

47. $h(x) = \sqrt[4]{x} + x^3 \cos x$

48. $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$

49–50 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

49. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

50. $2 \sin x = 3 - 2x$, $(0, 1)$

51. Suppose that $|f(x)| \leqslant g(x)$ for all x, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

52. Let $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) For what values of a does $\lim_{x\to a} f(x)$ exist?

(b) At what numbers is f discontinuous?

Principles of Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

1 UNDERSTAND THE PROBLEM

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q, we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n, it is frequently helpful to use the following principle.

Principle of Mathematical Induction Let S_n be a statement about the positive integer n. Suppose that

1. S_1 is true.

2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

As the first example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 1 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geqslant 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases}$$

$$= \begin{cases} x-3 & \text{if } x \geqslant 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

Similarly

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geqslant 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases}$$

$$= \begin{cases} x+2 & \text{if } x \geqslant -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

PS Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leqslant x < 3 \qquad x \geqslant 3$$

CASE I If $x < -2$, we have

$$|x-3| + |x+2| < 11$$

$$-x+3-x-2 < 11$$

$$-2x < 10$$

$$x > -5$$

CASE II If $-2 \leqslant x < 3$, the given inequality becomes

$$-x+3+x+2 < 11$$

$$5 < 11 \qquad \text{(always true)}$$

CASE III If $x \geqslant 3$, the inequality becomes

$$x-3+x+2 < 11$$

$$2x < 12$$

$$x < 6$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$. ■

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1 Prove that S_n is true when $n = 1$.

Step 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

PS Analogy: Try a similar, simpler problem

EXAMPLE 2 If $f_0(x) = x/(x+1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1$, 2, and 3.

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1} = \frac{\dfrac{x}{x+1}}{\dfrac{2x+1}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\dfrac{x}{2x+1}}{\dfrac{x}{2x+1}+1} = \frac{\dfrac{x}{2x+1}}{\dfrac{3x+1}{2x+1}} = \frac{x}{3x+1}$$

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

PS Look for a pattern

$$= \frac{\dfrac{x}{3x+1}}{\dfrac{x}{3x+1}+1} = \frac{\dfrac{x}{3x+1}}{\dfrac{4x+1}{3x+1}} = \frac{x}{4x+1}$$

We notice a pattern: The coefficient of x in the denominator of $f_n(x)$ is $n + 1$ in the three cases we have computed. So we make the guess that, in general,

$$\boxed{1} \qquad f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (1) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then $$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{x}{(k+1)x+1}+1} = \frac{\dfrac{x}{(k+1)x+1}}{\dfrac{(k+2)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (1) is true for $n = k + 1$. Therefore, by mathematical induction, it is true for all positive integers n. ■

In the following example we show how the problem-solving strategy of *introducing something extra* is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 4.5, we will make more extensive use of this general idea.

EXAMPLE 3 Evaluate $\lim_{x \to 0} \dfrac{\sqrt[3]{1 + cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 1.6 we evaluated several limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1 + cx}$$

We also need to express x in terms of t, so we solve this equation:

$$t^3 = 1 + cx \qquad x = \frac{t^3 - 1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x \to 0$ is equivalent to $t \to 1$. This allows us to convert the given limit into one involving the variable t:

$$\lim_{x \to 0} \frac{\sqrt[3]{1 + cx} - 1}{x} = \lim_{t \to 1} \frac{t - 1}{(t^3 - 1)/c} = \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} = \lim_{t \to 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} = \lim_{t \to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3}$$

In making the change of variable we had to rule out the case $c = 0$. But if $c = 0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$. ■

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving.

PROBLEMS

1. Solve the equation $|2x - 1| - |x + 5| = 3$.
2. Solve the inequality $|x - 1| - |x - 3| \geqslant 5$.
3. Sketch the graph of the function $f(x) = |x^2 - 4|x| + 3|$.

4. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.

5. Draw the graph of the equation $x + |x| = y + |y|$.

6. Sketch the region in the plane consisting of all points (x, y) such that

$$|x - y| + |x| - |y| \leqslant 2$$

7. The notation $\max\{a, b, \ldots\}$ means the largest of the numbers $a, b, \ldots$. Sketch the graph of each function.
   (a) $f(x) = \max\{x, 1/x\}$
   (b) $f(x) = \max\{\sin x, \cos x\}$
   (c) $f(x) = \max\{x^2, 2 + x, 2 - x\}$

8. Sketch the region in the plane defined by each of the following equations or inequalities.
   (a) $\max\{x, 2y\} = 1$
   (b) $-1 \leqslant \max\{x, 2y\} \leqslant 1$
   (c) $\max\{x, y^2\} = 1$

9. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; she drives the second half at 60 mi/h. What is her average speed on this trip?

10. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?

11. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

12. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

13. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$, find a formula for $f_n(x)$.

14. (a) If $f_0(x) = \dfrac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.
   (b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.

15. Evaluate $\displaystyle\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

16. Find numbers a and b such that $\displaystyle\lim_{x \to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$.

17. Evaluate $\displaystyle\lim_{x \to 0} \frac{|2x - 1| - |2x + 1|}{x}$.

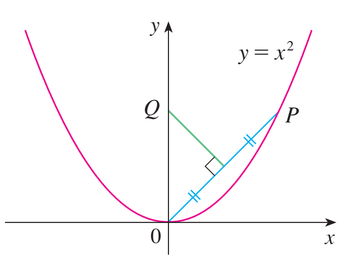


FIGURE FOR PROBLEM 18

18. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

19. Evaluate the following limits, if they exist, where $[\![x]\!]$ denotes the greatest integer function.
   (a) $\displaystyle\lim_{x \to 0} \frac{[\![x]\!]}{x}$
   (b) $\displaystyle\lim_{x \to 0} x[\![1/x]\!]$

20. Sketch the region in the plane defined by each of the following equations.
   (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$
   (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$
   (c) $[\![x + y]\!]^2 = 1$
   (d) $[\![x]\!] + [\![y]\!] = 1$

21. Find all values of a such that f is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x + 1 & \text{if } x \leqslant a \\ x^2 & \text{if } x > a \end{cases}$$

22. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

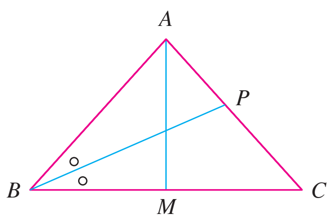
(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

23. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, find $\lim_{x \to a} [f(x)\, g(x)]$.

24. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.



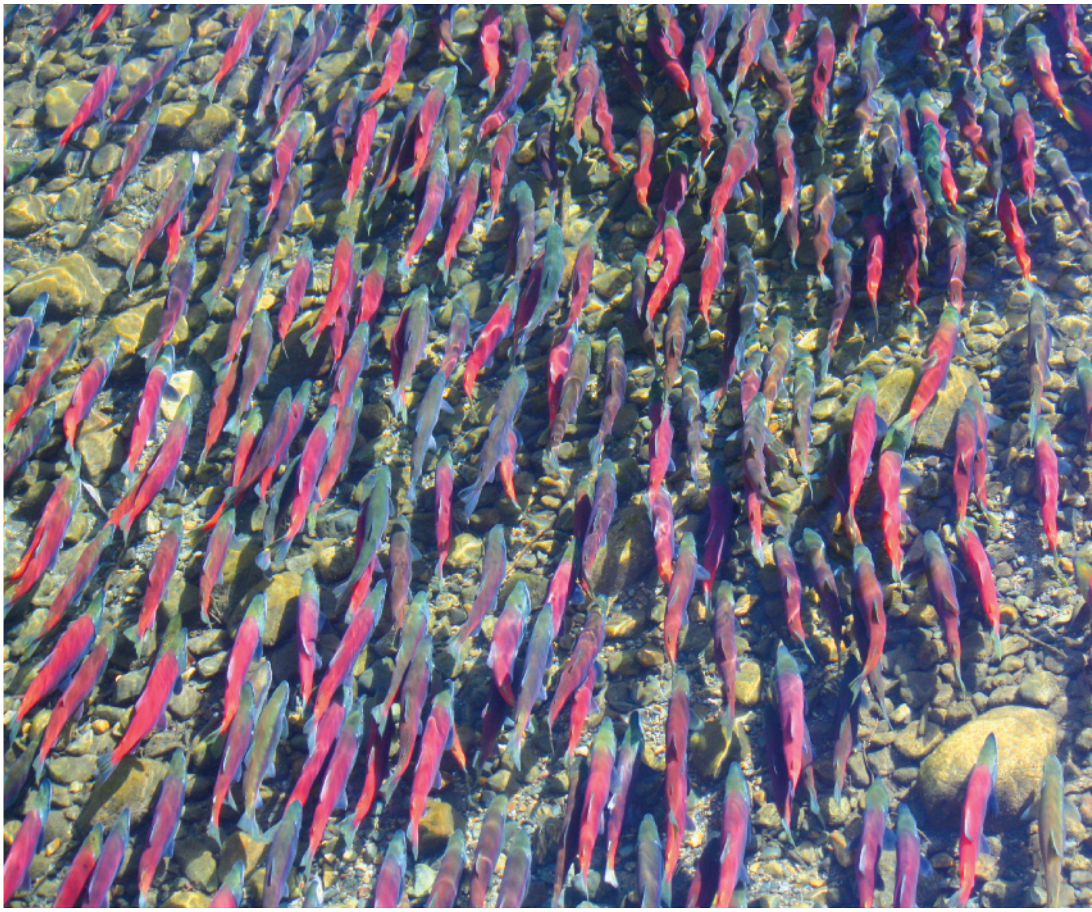
(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

25. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.

(b) Does the result in part (a) hold for points lying on any circle on the earth's surface?

(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

2 Derivatives



The maximum sustainable swimming speed *S* of salmon depends on the water temperature *T*. Exercise 58 in Section 2.1 asks you to analyze how *S* varies as *T* changes by estimating the derivative of *S* with respect to *T*.

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IN THIS CHAPTER WE BEGIN our study of differential calculus, which is concerned with how one quantity changes in relation to another quantity. The central concept of differential calculus is the *derivative*, which is an outgrowth of the velocities and slopes of tangents that we considered in Chapter 1. After learning how to calculate derivatives, we use them to solve problems involving rates of change and the approximation of functions.

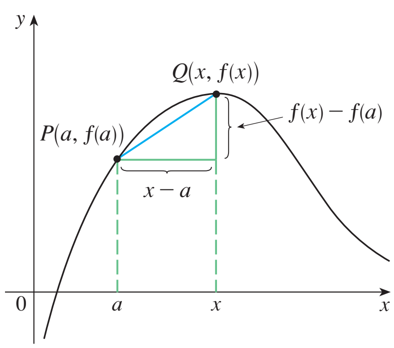
2.1 Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in Section 1.4. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

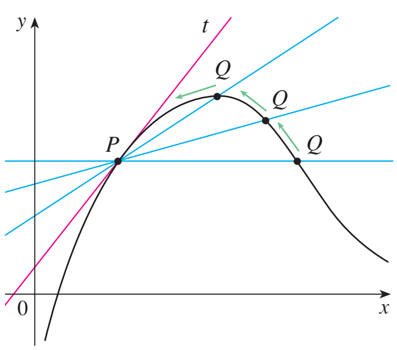


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 1.4.1.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Point-slope form for a line through the point (x_1, y_1) with slope m:

$$y - y_1 = m(x - x_1)$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

TEC Visual 2.1 shows an animation of Figure 2.

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

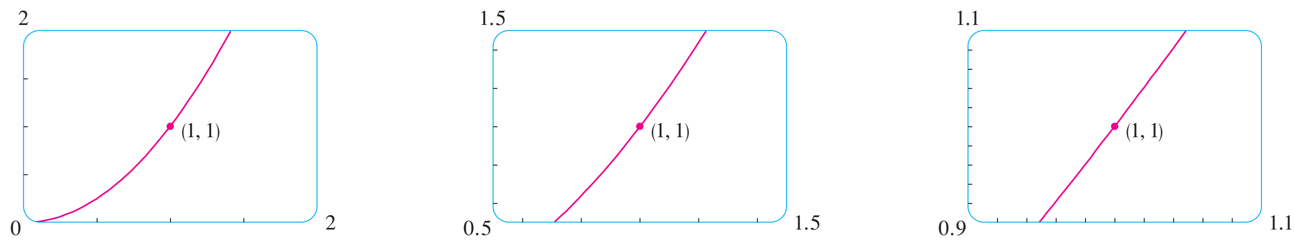


FIGURE 2 Zooming in toward the point (1, 1) on the parabola $y = x^2$

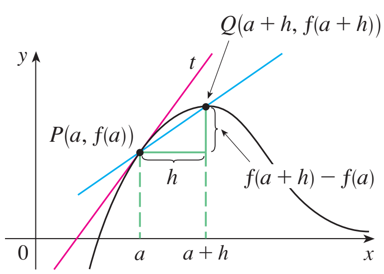


FIGURE 3

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2
$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1).

SOLUTION Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at (3, 1) is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

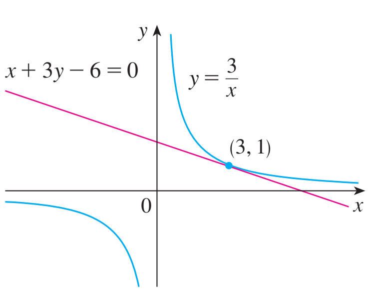


FIGURE 4

The hyperbola and its tangent are shown in Figure 4. ■

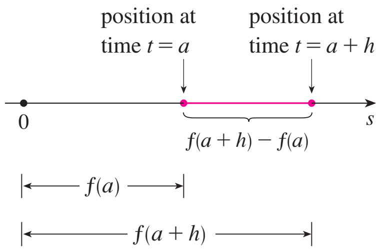


FIGURE 5

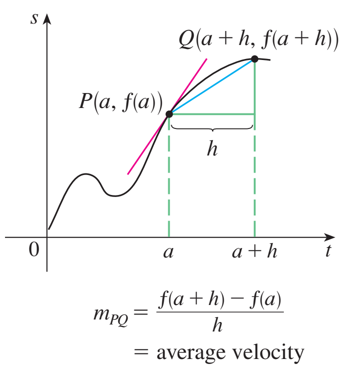


FIGURE 6

Velocities

In Section 1.4 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. (See Figure 5.)

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3
$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P (compare Equations 2 and 3).

Now that we know how to compute limits, let's reconsider the problem of the falling ball.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.
(a) What is the velocity of the ball after 5 seconds?
(b) How fast is the ball traveling when it hits the ground?

Recall from Section 1.4: The distance (in meters) fallen after t seconds is $4.9t^2$.

SOLUTION We will need to find the velocity both when $t = 5$ and when the ball hits the ground, so it's efficient to start by finding the velocity at a general time t. Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$\begin{aligned} v(t) &= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{4.9(t + h)^2 - 4.9t^2}{h} \\ &= \lim_{h \to 0} \frac{4.9(t^2 + 2th + h^2 - t^2)}{h} = \lim_{h \to 0} \frac{4.9(2th + h^2)}{h} \\ &= \lim_{h \to 0} \frac{4.9h(2t + h)}{h} = \lim_{h \to 0} 4.9(2t + h) = 9.8t \end{aligned}$$

(a) The velocity after 5 seconds is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t when $s(t) = 450$, that is,

$$4.9t^2 = 450$$

This gives

$$t^2 = \frac{450}{4.9} \qquad \text{and} \qquad t = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v\left(\sqrt{\frac{450}{4.9}}\right) = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$ ■

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3). In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

$f'(a)$ is read "f prime of a."

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

EXAMPLE 4 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

SOLUTION From Definition 4 we have

Definitions 4 and 5 are equivalent, so we can use either one to compute the derivative. In practice, Definition 4 often leads to simpler computations.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$ ■

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2. Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

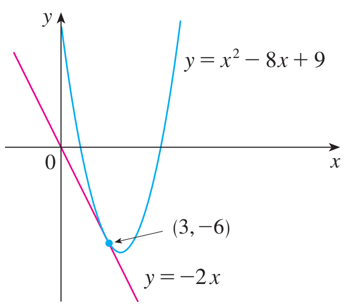


FIGURE 7

EXAMPLE 5 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4 we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$ ■

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

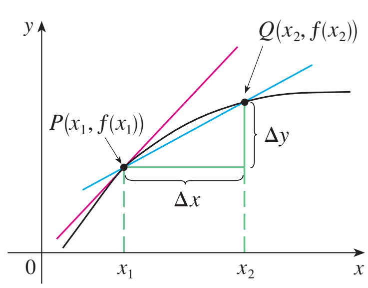
$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



average rate of change $= m_{PQ}$

instantaneous rate of change = slope of tangent at P

FIGURE 8

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change of *y* with respect to *x*** at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

> **6** $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

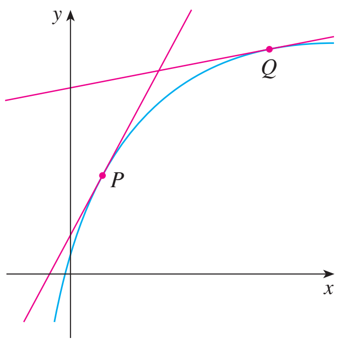


FIGURE 9
The y-values are changing rapidly at P and slowly at Q.

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 9), the y-values change rapidly. When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t. In other words, *$f'(a)$ is the velocity of the particle at time $t = a$*. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

In the next example we discuss the meaning of the derivative of a function that is defined verbally.

EXAMPLE 6 A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) In practical terms, what does it mean to say that $f'(1000) = 9$?
(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION
(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.* This idea is discussed in more detail in Sections 2.7 and 3.7.)

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more

efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ ■

In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

t	$D(t)$
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	14,025.2

Source: US Dept. of the Treasury

EXAMPLE 7 Let $D(t)$ be the US national debt at time t. The table in the margin gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1985 to 2010. Interpret and estimate the value of $D'(2000)$.

SOLUTION The derivative $D'(2000)$ means the rate of change of D with respect to t when $t = 2000$, that is, the rate of increase of the national debt in 2000.

According to Equation 5,

$$D'(2000) = \lim_{t \to 2000} \frac{D(t) - D(2000)}{t - 2000}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as follows.

t	Time interval	Average rate of change $= \dfrac{D(t) - D(2000)}{t - 2000}$
1985	[1985, 2000]	247.75
1990	[1990, 2000]	229.74
1995	[1995, 2000]	134.70
2005	[2000, 2005]	501.64
2010	[2000, 2010]	836.30

From this table we see that $D'(2000)$ lies somewhere between 134.70 and 501.64 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1995 and 2005.] We estimate that the rate of increase of the national debt of the United States in 2000 was the average of these two numbers, namely,

$$D'(2000) \approx 318 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 2000$. ■

A Note on Units

The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt, namely, billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*).

A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 2.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.1 EXERCISES

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Graph the curve $y = \sin x$ in the viewing rectangles $[-2, 2]$ by $[-2, 2]$, $[-1, 1]$ by $[-1, 1]$, and $[-0.5, 0.5]$ by $[-0.5, 0.5]$. What do you notice about the curve as you zoom in toward the origin?

3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 4x - 3x^2$, $(2, -4)$

6. $y = x^3 - 3x + 1$, $(2, 3)$

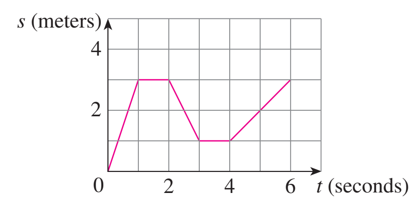
7. $y = \sqrt{x}$, $(1, 1)$

8. $y = \dfrac{2x + 1}{x + 2}$, $(1, 1)$

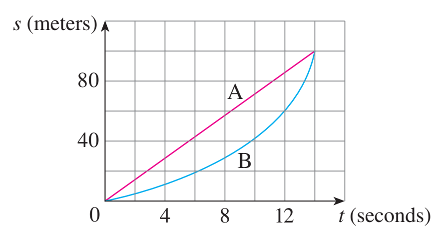
9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.
(c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 1)$ and $\left(4, \frac{1}{2}\right)$.
(c) Graph the curve and both tangents on a common screen.

11. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown in the figure. When is the particle moving to the right? Moving to the left? Standing still?
(b) Draw a graph of the velocity function.



12. Shown are graphs of the position functions of two runners, A and B, who run a 100-meter race and finish in a tie.



(a) Describe and compare how the runners run the race.
(b) At what time is the distance between the runners the greatest?
(c) At what time do they have the same velocity?

13. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

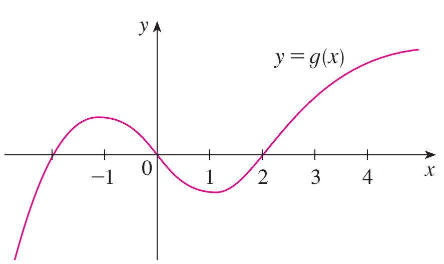
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

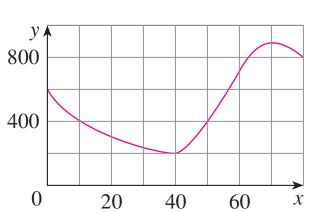
16. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
(i) [4, 8] (ii) [6, 8]
(iii) [8, 10] (iv) [8, 12]
(b) Find the instantaneous velocity when $t = 8$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



18. The graph of a function f is shown.
(a) Find the average rate of change of f on the interval [20, 60].
(b) Identify an interval on which the average rate of change of f is 0.
(c) Which interval gives a larger average rate of change, [40, 60] or [40, 70]?
(d) Compute $\dfrac{f(40) - f(10)}{40 - 10}$. What does this value represent geometrically?



19. For the function f graphed in Exercise 18:
(a) Estimate the value of $f'(50)$.
(b) Is $f'(10) > f'(30)$?
(c) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.

20. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

21. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

22. If the tangent line to $y = f(x)$ at (4, 3) passes through the point (0, 2), find $f(4)$ and $f'(4)$.

23. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

24. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x \to 5^-} g(x) = \infty$, and $\lim_{x \to -1^+} g(x) = -\infty$.

25. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x \to -5^+} g(x) = \infty$, and $\lim_{x \to 5^-} g(x) = 3$.

26. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x \to 2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ± 1, and f is odd.

27. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

28. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

29. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point (2, 2).
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

30. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points (2, 8) and (3, 9).
(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

31–36 Find $f'(a)$.

31. $f(x) = 3x^2 - 4x + 1$

32. $f(t) = 2t^3 + t$

33. $f(t) = \dfrac{2t + 1}{t + 3}$

34. $f(x) = x^{-2}$

35. $f(x) = \sqrt{1 - 2x}$

36. $f(x) = \dfrac{4}{\sqrt{1 - x}}$

37–42 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

37. $\displaystyle\lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$

38. $\displaystyle\lim_{h \to 0} \frac{2^{3+h} - 8}{h}$

39. $\lim_{x \to 2} \frac{x^6 - 64}{x - 2}$

40. $\lim_{x \to 1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

41. $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

42. $\lim_{\theta \to \pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \pi/6}$

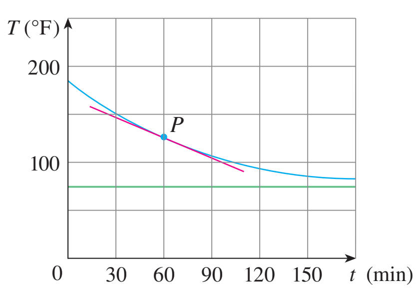
43–44 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

43. $f(t) = 80t - 6t^2$

44. $f(t) = 10 + \frac{45}{t + 1}$

45. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

46. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



47. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

(a) Find the average rate of change of C with respect to t over each time interval:
(i) [1.0, 2.0] (ii) [1.5, 2.0]
(iii) [2.0, 2.5] (iv) [2.0, 3.0]
In each case, include the units.
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

48. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	2004	2006	2008	2010	2012
N	8569	12,440	16,680	16,858	18,066

(a) Find the average rate of growth
(i) from 2006 to 2008
(ii) from 2008 to 2010
In each case, include the units. What can you conclude?
(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.

49. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.
(a) Compute and interpret the average rate of change from 1990 to 2005. What are the units?
(b) Estimate the instantaneous rate of change in 2000 by taking the average of two average rates of change. What are its units?

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302

Source: US Energy Information Administration

50. The table shows values of the viral load $V(t)$ in HIV patient 303, measured in RNA copies/mL, t days after ABT-538 treatment was begun.

t	4	8	11	15	22
$V(t)$	53	18	9.4	5.2	3.6

(a) Find the average rate of change of V with respect to t over each time interval:
(i) [4, 11] (ii) [8, 11]
(iii) [11, 15] (iv) [11, 22]
What are the units?
(b) Estimate and interpret the value of the derivative $V'(11)$.

Source: Adapted from D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

51. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
  (i) from $x = 100$ to $x = 105$
  (ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 2.7.)

52. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \tfrac{1}{60}t\right)^2 \qquad 0 \leqslant t \leqslant 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0, 10, 20, 30, 40, 50$, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

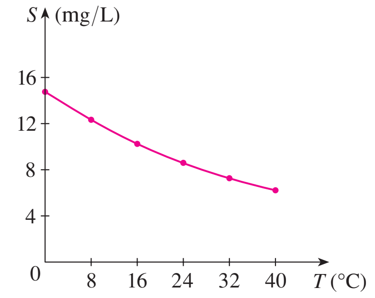
53. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

54. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
(a) What is the meaning of the derivative $f'(5)$? What are its units?
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

55. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Fahrenheit.
(a) What is the meaning of $H'(58)$? What are its units?
(b) Would you expect $H'(58)$ to be positive or negative? Explain.

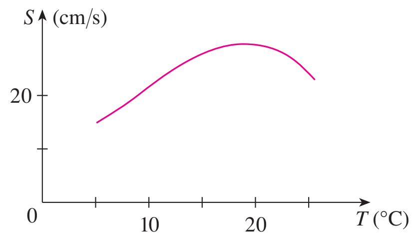
56. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

57. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Source: C. Kupchella et al., *Environmental Science: Living Within the System of Nature*, 2d ed. (Boston: Allyn and Bacon, 1989).

58. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



59–60 Determine whether $f'(0)$ exists.

59. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

60. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

---

61. (a) Graph the function $f(x) = \sin x - \frac{1}{1000}\sin(1000x)$ in the viewing rectangle $[-2\pi, 2\pi]$ by $[-4, 4]$. What slope does the graph appear to have at the origin?
(b) Zoom in to the viewing window $[-0.4, 0.4]$ by $[-0.25, 0.25]$ and estimate the value of $f'(0)$. Does this agree with your answer from part (a)?
(c) Now zoom in to the viewing window $[-0.008, 0.008]$ by $[-0.005, 0.005]$. Do you wish to revise your estimate for $f'(0)$?

WRITING PROJECT EARLY METHODS FOR FINDING TANGENTS

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.1 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989), pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

2.2 The Derivative as a Function

In the preceding section we considered the derivative of a function f at a fixed number a:

1
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

2
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of *f*** and defined by Equation 2. We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

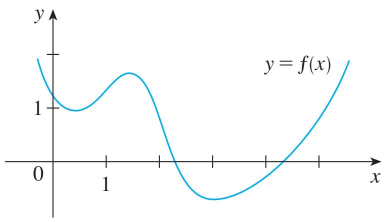


FIGURE 1

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$. This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P. (The slope of the graph of f becomes the y-value on the graph of f'.) Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis (where $y = 0$) at the points A', B', and C', directly beneath A, B, and C. Between A and B the tangents have positive slope, so $f'(x)$ is positive there. (The graph is above the x-axis.) But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

TEC Visual 2.2 shows an animation of Figure 2 for several functions.

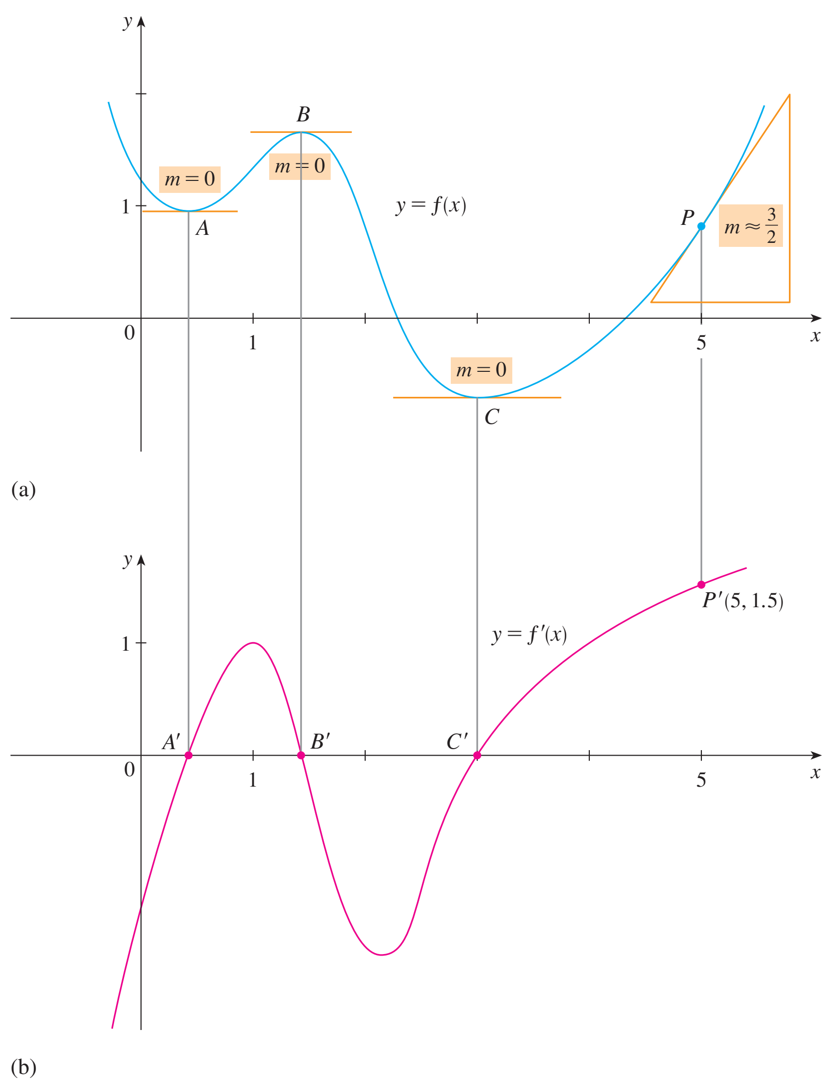


FIGURE 2

■

EXAMPLE 2
(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate this formula by comparing the graphs of f and f'.

SOLUTION
(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

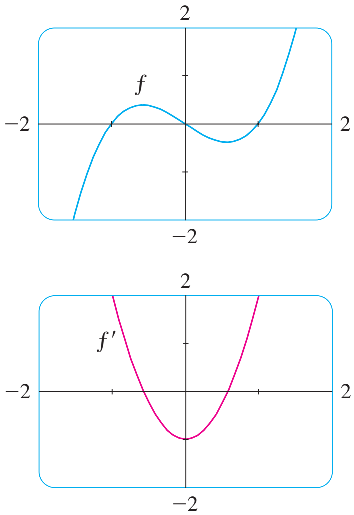


FIGURE 3

(b) We use a graphing device to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a). ■

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

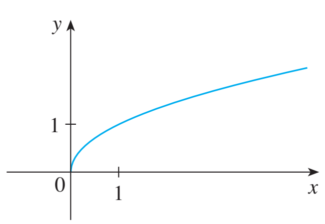
$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \qquad \text{(Rationalize the numerator.)}$$

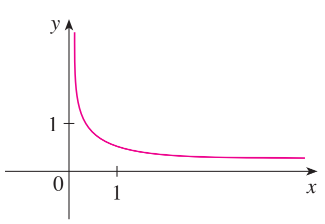
$$= \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$. ■



(a) $f(x) = \sqrt{x}$



(b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

FIGURE 4

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/\left(2\sqrt{x}\right)$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$$

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2+x+h)(2+x)} = \lim_{h \to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}$$ ■

Leibniz

Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published.

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.1.6, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

3 Definition A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{-(x + h) - (-x)}{h}$$

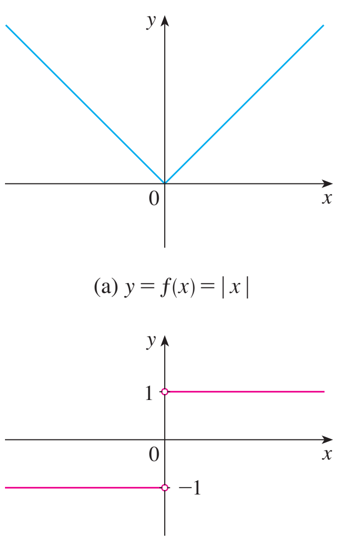
$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \qquad \text{(if it exists)}$$



(a) $y = f(x) = |x|$

(b) $y = f'(x)$

FIGURE 5

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 5(a).] ■

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

PROOF To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$. We do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

PS An important aspect of problem solving is trying to find a connection between the given and the unknown. See Step 2 (Think of a Plan) in *Principles of Problem Solving* on page 98.

exists (see Equation 2.1.5). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Thus, using the Product Law and Equation 2.1.5, we can write

$$\begin{aligned} \lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + (f(x) - f(a))] \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a) \end{aligned}$$

Therefore f is continuous at a. ■

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

(See Example 1.6.7.) But in Example 5 we showed that f is not differentiable at 0.

How Can a Function Fail To Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

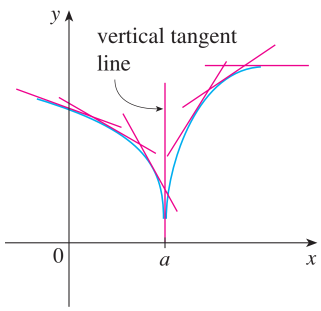


FIGURE 6

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another. Figure 7 illustrates the three possibilities that we have discussed.

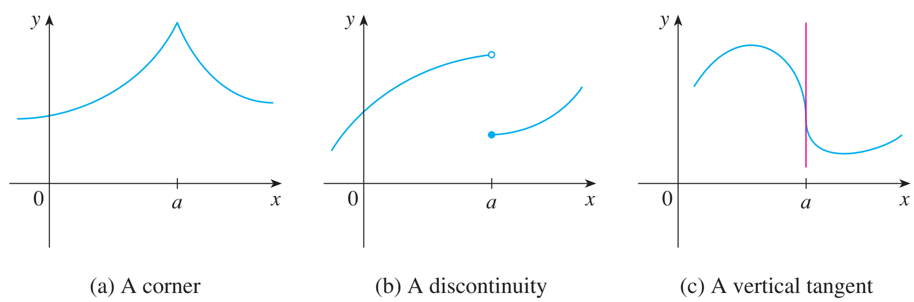


FIGURE 7
Three ways for f not to be differentiable at a

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point $(a, f(a))$ the graph straightens out and appears more and more like a line. (See Figure 8. We saw a specific example of this in Figure 2.1.2.) But no matter how much we zoom in toward a point like the ones in Figures 6 and 7(a), we can't eliminate the sharp point or corner (see Figure 9).

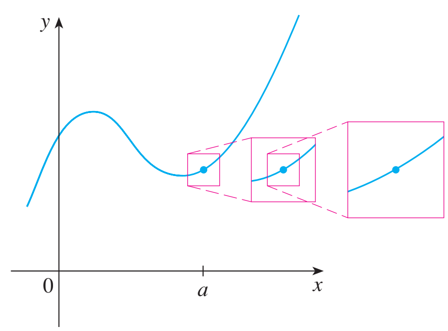


FIGURE 8
f is differentiable at a.

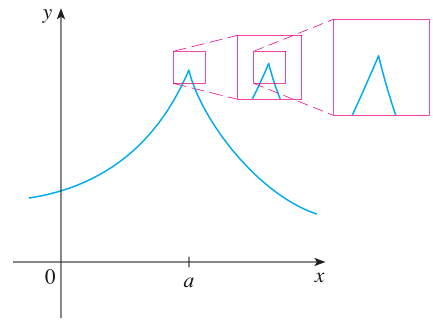


FIGURE 9
f is not differentiable at a.

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$f''(x) = (f')'(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h) = 6x$$

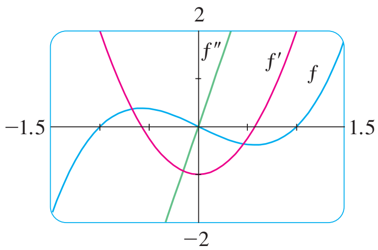


FIGURE 10

TEC In Module 2.2 you can see how changing the coefficients of a polynomial f affects the appearance of the graphs of f, f', and f''.

The graphs of f, f', and f'' are shown in Figure 10.

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Acceleration is the change in velocity you would feel when your car is speeding up or slowing down.

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

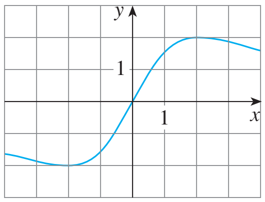
$$f^{(4)}(x) = 0$$ ■

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 3.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

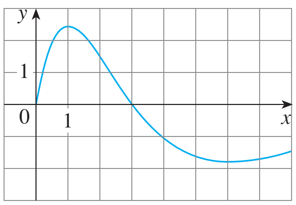
2.2 EXERCISES

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

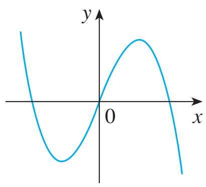
1. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$ (d) $f'(0)$
(e) $f'(1)$ (f) $f'(2)$ (g) $f'(3)$

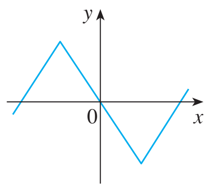


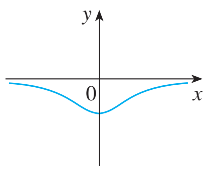
2. (a) $f'(0)$ (b) $f'(1)$ (c) $f'(2)$ (d) $f'(3)$
(e) $f'(4)$ (f) $f'(5)$ (g) $f'(6)$ (h) $f'(7)$

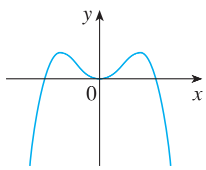


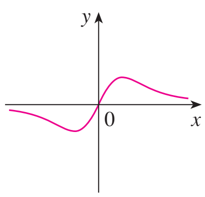
3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

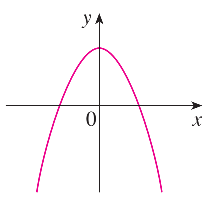
(a)


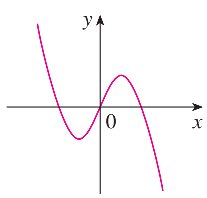
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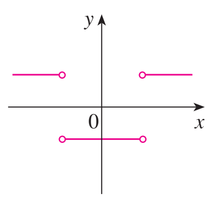
(c)


(d)


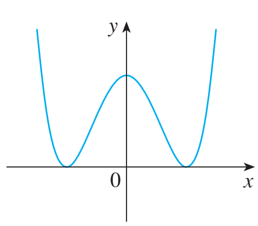
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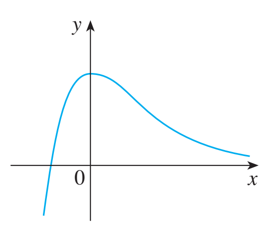
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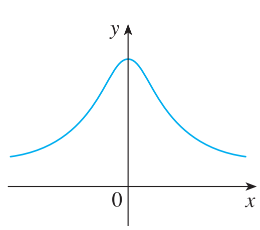
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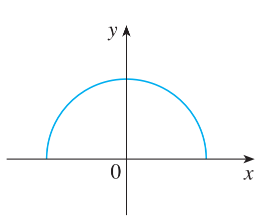
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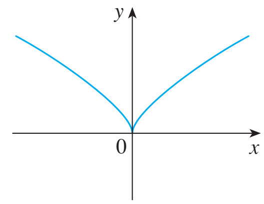
4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

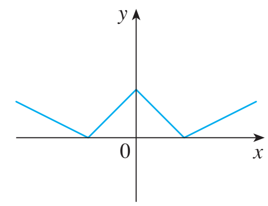
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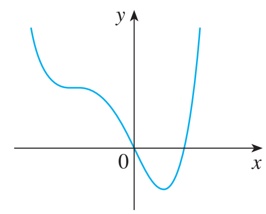
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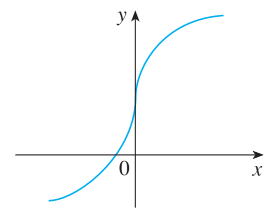
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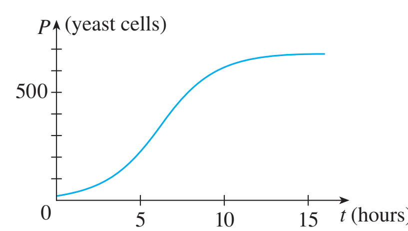
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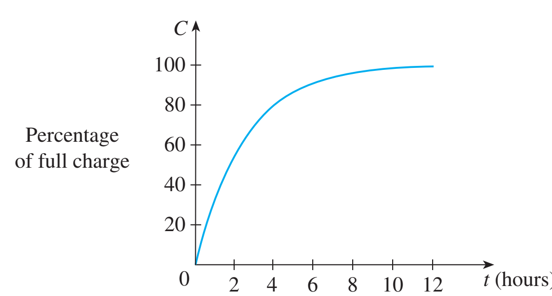
12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?

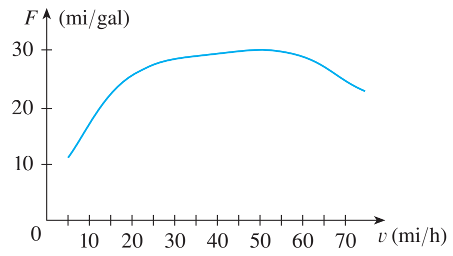


14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.

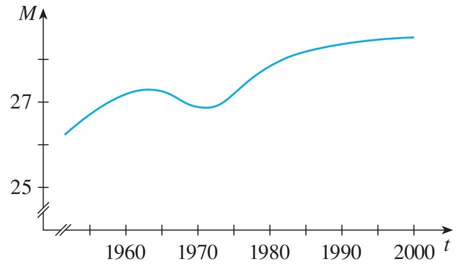
(a) What is the meaning of the derivative $F'(v)$?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average age M of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16. Make a careful sketch of the graph of the sine function and below it sketch the graph of its derivative in the same manner as in Example 1. Can you guess what the derivative of the sine function is from its graph?

17. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of derivative to prove that your guess in part (c) is correct.

18. Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

(d) Guess a formula for $f'(x)$.

(e) Use the definition of derivative to prove that your guess in part (d) is correct.

19–29 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

19. $f(x) = 3x - 8$

20. $f(x) = mx + b$

21. $f(t) = 2.5t^2 + 6t$

22. $f(x) = 4 + 8x - 5x^2$

23. $f(x) = x^2 - 2x^3$

24. $g(t) = \dfrac{1}{\sqrt{t}}$

25. $g(x) = \sqrt{9 - x}$

26. $f(x) = \dfrac{x^2 - 1}{2x - 3}$

27. $G(t) = \dfrac{1 - 2t}{3 + t}$

28. $f(x) = x^{3/2}$

29. $f(x) = x^4$

30. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

31. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

32. (a) If $f(x) = x + 1/x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

33. The unemployment rate $U(t)$ varies with time. The table gives the percentage of unemployed in the US labor force from 2003 to 2012.

(a) What is the meaning of $U'(t)$? What are its units?

(b) Construct a table of estimated values for $U'(t)$.

t	$U(t)$	t	$U(t)$
2003	6.0	2008	5.8
2004	5.5	2009	9.3
2005	5.1	2010	9.6
2006	4.6	2011	8.9
2007	4.6	2012	8.1

Source: US Bureau of Labor Statistics

34. The table gives the number $N(t)$, measured in thousands, of minimally invasive cosmetic surgery procedures performed in the United States for various years t.

t	$N(t)$ (thousands)
2000	5,500
2002	4,897
2004	7,470
2006	9,138
2008	10,897
2010	11,561
2012	13,035

Source: American Society of Plastic Surgeons

(a) What is the meaning of $N'(t)$? What are its units?
(b) Construct a table of estimated values for $N'(t)$.
(c) Graph N and N'.
(d) How would it be possible to get more accurate values for $N'(t)$?

35. The table gives the height as time passes of a typical pine tree grown for lumber at a managed site.

Tree age (years)	14	21	28	35	42	49
Height (feet)	41	54	64	72	78	83

Source: Arkansas Forestry Commission

If $H(t)$ is the height of the tree after t years, construct a table of estimated values for H' and sketch its graph.

36. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W' and sketch its graph. What are the units for $W'(x)$?

Source: Adapted from J. Chadwick Jr., "Temperature Effects on Growth and Stress Physiology of Brook Trout: Implications for Climate Change Impacts on an Iconic Cold-Water Fish." *Masters Theses.* Paper 897. 2012. scholarworks.umass.edu/theses/897.

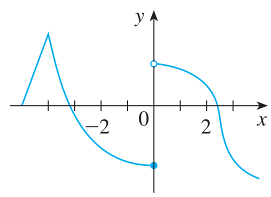
37. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2000.
(a) What does dP/dt represent in this context?
(b) Interpret the statement

$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

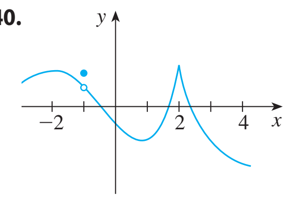
38. Suppose N is the number of people in the United States who travel by car to another state for a vacation this year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain.

39–42 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

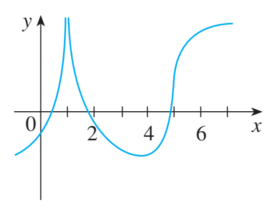
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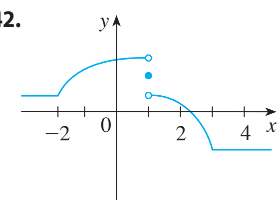
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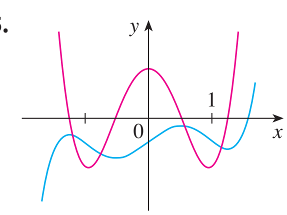


43. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

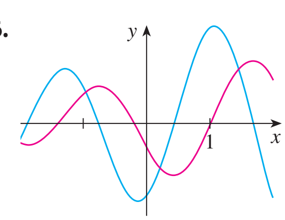
44. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

45–46 The graphs of a function f and its derivative f' are shown. Which is bigger, $f'(-1)$ or $f''(1)$?

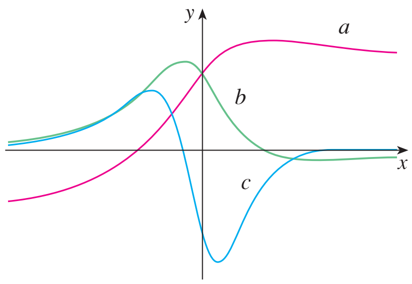
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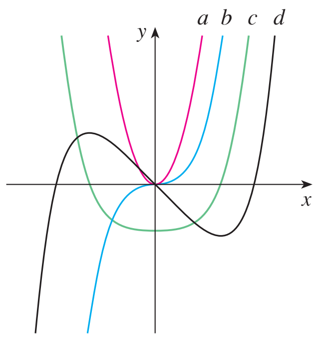
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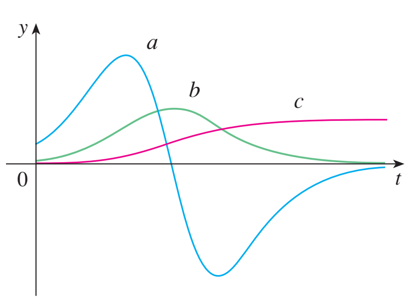
47. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



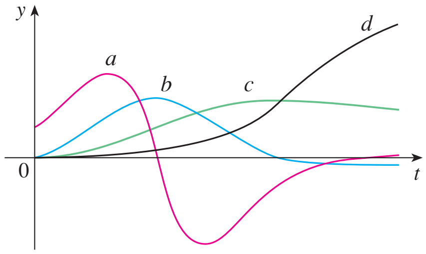
48. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



49. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



50. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.

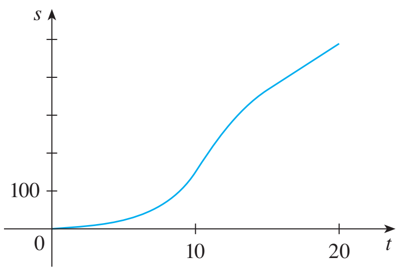


51–52 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

51. $f(x) = 3x^2 + 2x + 1$ **52.** $f(x) = x^3 - 3x$

53. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

54. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

55. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.1.5 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

56. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

57. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

58. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

59. (a) Sketch the graph of the function $f(x) = x|x|$.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

60. (a) Sketch the graph of the function $g(x) = x + |x|$.
(b) For what values of x is g differentiable?
(c) Find a formula for g'.

61. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

62. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_{-}(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

and
$$f'_{+}(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

(a) Find $f'_{-}(4)$ and $f'_{+}(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \geqslant 4 \end{cases}$$

(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

63. Nick starts jogging and runs faster and faster for 3 mintues, then he walks for 5 minutes. He stops at an intersection for 2 minutes, runs fairly quickly for 5 minutes, then walks for 4 minutes.
(a) Sketch a possible graph of the distance s Nick has covered after t minutes.
(b) Sketch a graph of ds/dt.

64. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
(a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(b) Describe how the rate of change of T with respect to t varies as t increases.
(c) Sketch a graph of the derivative of T.

65. Let ℓ be the tangent line to the parabola $y = x^2$ at the point (1, 1). The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2.3 Differentiation Formulas

If it were always necessary to compute derivatives directly from the definition, as we did in the preceding section, such computations would be tedious and the evaluation of some limits would require ingenuity. Fortunately, several rules have been developed for finding derivatives without having to use the definition directly. These formulas greatly simplify the task of differentiation.

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

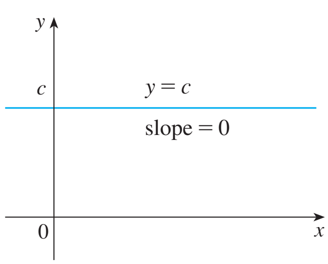


FIGURE 1
The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

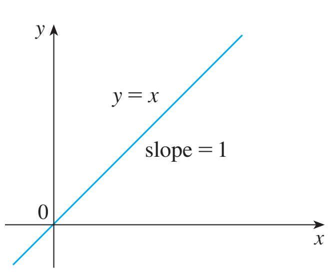


FIGURE 2
The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

1
$$\frac{d}{dx}(x) = 1$$

(You can also verify Equation 1 from the definition of a derivative.) We have already

investigated the cases $n = 2$ and $n = 3$. In fact, in Section 2.2 (Exercises 17 and 18) we found that

2 $$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Thus

3 $$\frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in (1), (2), and (3), we see a pattern emerging. It seems to be a reasonable guess that, when n is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true. We prove it in two ways; the second proof uses the Binomial Theorem.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

FIRST PROOF The formula

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x) = x^n$, we can use Equation 2.1.5 for $f'(a)$ and the equation above to write

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

SECOND PROOF

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

The Binomial Theorem is given on Reference Page 1.

In finding the derivative of x^4 we had to expand $(x + h)^4$. Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$f'(x) = \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1}$$

because every term except the first has h as a factor and therefore approaches 0. ■

We illustrate the Power Rule using various notations in Example 1.

EXAMPLE 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

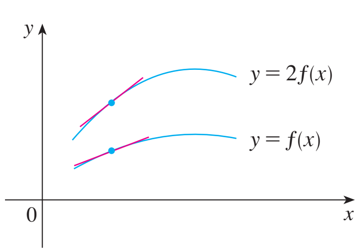
(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$.

(d) $\frac{d}{dr}(r^3) = 3r^2$ ■

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions. In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function.*

Geometric Interpretation of the Constant Multiple Rule



Multiplying by $c = 2$ stretches the graph vertically by a factor of 2. All the rises have been doubled but the runs stay the same. So the slopes are doubled too.

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

PROOF Let $g(x) = cf(x)$. Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h \to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(by Limit Law 3)}$$

$$= cf'(x)$$ ■

EXAMPLE 2

(a) $\dfrac{d}{dx}(3x^4) = 3\dfrac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

(b) $\dfrac{d}{dx}(-x) = \dfrac{d}{dx}[(-1)x] = (-1)\dfrac{d}{dx}(x) = -1(1) = -1$ ■

The next rule tells us that *the derivative of a sum of functions is the sum of the derivatives*.

Using prime notation, we can write the Sum Rule as

$$(f + g)' = f' + g'$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

PROOF Let $F(x) = f(x) + g(x)$. Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \qquad \text{(by Limit Law 1)}$$

$$= f'(x) + g'(x)$$ ■

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

EXAMPLE 3

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$ ■

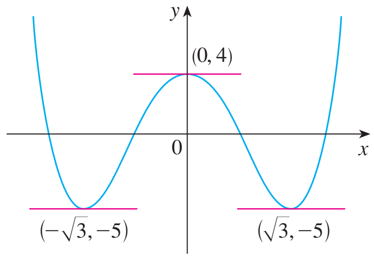


FIGURE 3
The curve $y = x^4 - 6x^2 + 4$ and its horizontal tangents

EXAMPLE 4 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)$$

$$= 4x^3 - 12x + 0 = 4x(x^2 - 3)$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 3.) ■

EXAMPLE 5 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

SOLUTION The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 12(2) - 10 = 14 \text{ cm/s}^2$. ■

Next we need a formula for the derivative of a product of two functions. By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$. The correct formula was discovered by Leibniz (soon after his false start) and is called the Product Rule.

In prime notation:

$$(fg)' = fg' + gf'$$

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

PROOF Let $F(x) = f(x)g(x)$. Then

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$

In order to evaluate this limit, we would like to separate the functions f and g as in the proof of the Sum Rule. We can achieve this separation by subtracting and adding the term $f(x + h)g(x)$ in the numerator:

$$F'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x + h)g(x) + f(x + h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x + h) \frac{g(x + h) - g(x)}{h} + g(x) \frac{f(x + h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} f(x + h) \cdot \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x)$$

Note that $\lim_{h \to 0} g(x) = g(x)$ because $g(x)$ is a constant with respect to the variable h. Also, since f is differentiable at x, it is continuous at x by Theorem 2.2.4, and so $\lim_{h \to 0} f(x + h) = f(x)$. (See Exercise 1.8.63.) ■

In words, the Product Rule says that the *derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

EXAMPLE 6 Find $F'(x)$ if $F(x) = (6x^3)(7x^4)$.

SOLUTION By the Product Rule, we have

$$F'(x) = (6x^3) \frac{d}{dx}(7x^4) + (7x^4) \frac{d}{dx}(6x^3)$$

$$= (6x^3)(28x^3) + (7x^4)(18x^2)$$

$$= 168x^6 + 126x^6 = 294x^6$$ ■

Notice that we could verify the answer to Example 6 directly by first multiplying the factors:

$$F(x) = (6x^3)(7x^4) = 42x^7 \qquad \Rightarrow \qquad F'(x) = 42(7x^6) = 294x^6$$

But later we will meet functions, such as $y = x^2 \sin x$, for which the Product Rule is the only possible method.

EXAMPLE 7 If $h(x) = xg(x)$ and it is known that $g(3) = 5$ and $g'(3) = 2$, find $h'(3)$.

SOLUTION Applying the Product Rule, we get

$$h'(x) = \frac{d}{dx}[xg(x)] = x\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[x]$$

$$= x \cdot g'(x) + g(x) \cdot (1)$$

Therefore $$h'(3) = 3g'(3) + g(3) = 3 \cdot 2 + 5 = 11$$ ■

In prime notation:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

PROOF Let $F(x) = f(x)/g(x)$. Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

We can separate f and g in this expression by subtracting and adding the term $f(x)g(x)$ in the numerator:

$$F'(x) = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{g(x)\dfrac{f(x+h) - f(x)}{h} - f(x)\dfrac{g(x+h) - g(x)}{h}}{g(x+h)g(x)}$$

$$= \frac{\lim\limits_{h\to 0} g(x) \cdot \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h} - \lim\limits_{h\to 0} f(x) \cdot \lim\limits_{h\to 0}\dfrac{g(x+h) - g(x)}{h}}{\lim\limits_{h\to 0} g(x+h) \cdot \lim\limits_{h\to 0} g(x)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Again g is continuous by Theorem 2.2.4, so $\lim_{h\to 0} g(x+h) = g(x)$. ■

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The theorems of this section show that any polynomial is differentiable on $\mathbb{R}$ and any rational function is differentiable on its domain. Furthermore, the Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

We can use a graphing device to check that the answer to Example 8 is plausible. Figure 4 shows the graphs of the function of Example 8 and its derivative. Notice that when y grows rapidly (near -2), y' is large. And when y grows slowly, y' is near 0.

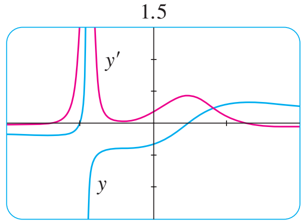


FIGURE 4

EXAMPLE 8 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to first rewrite a quotient to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

General Power Functions

The Quotient Rule can be used to extend the Power Rule to the case where the exponent is a negative integer.

> If n is a positive integer, then
>
> $$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

PROOF

$$\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right)$$

$$= \frac{x^n \dfrac{d}{dx}(1) - 1 \cdot \dfrac{d}{dx}(x^n)}{(x^n)^2} = \frac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{x^{2n}}$$

$$= \frac{-nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$$

EXAMPLE 9

(a) If $y = \dfrac{1}{x}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{-1}) = -x^{-2} = -\dfrac{1}{x^2}$

(b) $\dfrac{d}{dt}\left(\dfrac{6}{t^3}\right) = 6\dfrac{d}{dt}(t^{-3}) = 6(-3)t^{-4} = -\dfrac{18}{t^4}$ ■

So far we know that the Power Rule holds if the exponent n is a positive or negative integer. If $n = 0$, then $x^0 = 1$, which we know has a derivative of 0. Thus the Power Rule holds for any integer n. What if the exponent is a fraction? In Example 2.2.3 we found that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

which can be written as

$$\frac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{-1/2}$$

This shows that the Power Rule is true even when $n = \frac{1}{2}$. In fact, it also holds for *any real number n*, as we will prove in Chapter 6. (A proof for rational values of n is indicated in Exercise 2.6.48.) In the meantime we state the general version and use it in the examples and exercises.

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

EXAMPLE 10

(a) If $f(x) = x^{\pi}$, then $f'(x) = \pi x^{\pi-1}$.

(b) Let
$$y = \frac{1}{\sqrt[3]{x^2}}$$

Then
$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2/3}) = -\tfrac{2}{3}x^{-(2/3)-1}$$
$$= -\tfrac{2}{3}x^{-5/3}$$ ■

In Example 11, a and b are constants. It is customary in mathematics to use letters near the beginning of the alphabet to represent constants and letters near the end of the alphabet to represent variables.

EXAMPLE 11 Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}\left(\sqrt{t}\right)$$
$$= \sqrt{t}\cdot b + (a + bt)\cdot \tfrac{1}{2}t^{-1/2}$$
$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

which is equivalent to the answer given in Solution 1. ■

The differentiation rules enable us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*. The **normal line** to a curve C at a point P is the line through P that is perpendicular to the tangent line at P. (In the study of optics, one needs to consider the angle between a light ray and the normal line to a lens.)

EXAMPLE 12 Find equations of the tangent line and normal line to the curve $y = \sqrt{x}/(1 + x^2)$ at the point $\left(1, \tfrac{1}{2}\right)$.

SOLUTION According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1 + x^2)\dfrac{d}{dx}\left(\sqrt{x}\right) - \sqrt{x}\dfrac{d}{dx}(1 + x^2)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2)\dfrac{1}{2\sqrt{x}} - \sqrt{x}(2x)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2) - 4x^2}{2\sqrt{x}(1 + x^2)^2} = \frac{1 - 3x^2}{2\sqrt{x}(1 + x^2)^2}$$

So the slope of the tangent line at $\left(1, \tfrac{1}{2}\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1 - 3 \cdot 1^2}{2\sqrt{1}(1 + 1^2)^2} = -\frac{1}{4}$$

We use the point-slope form to write an equation of the tangent line at $\left(1, \tfrac{1}{2}\right)$:

$$y - \tfrac{1}{2} = -\tfrac{1}{4}(x - 1) \qquad \text{or} \qquad y = -\tfrac{1}{4}x + \tfrac{3}{4}$$

The slope of the normal line at $\left(1, \tfrac{1}{2}\right)$ is the negative reciprocal of $-\tfrac{1}{4}$, namely 4, so an equation is

$$y - \tfrac{1}{2} = 4(x - 1) \qquad \text{or} \qquad y = 4x - \tfrac{7}{2}$$

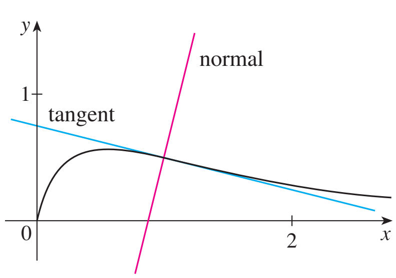


FIGURE 5

The curve and its tangent and normal lines are graphed in Figure 5. ■

EXAMPLE 13 At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

SOLUTION Since $xy = 12$ can be written as $y = 12/x$, we have

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{-1}) = 12(-x^{-2}) = -\frac{12}{x^2}$$

Let the x-coordinate of one of the points in question be a. Then the slope of the tangent

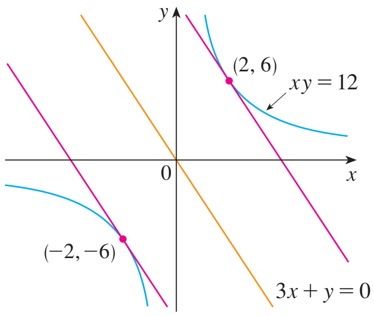


FIGURE 6

line at that point is $-12/a^2$. This tangent line will be parallel to the line $3x + y = 0$, or $y = -3x$, if it has the same slope, that is, -3. Equating slopes, we get

$$-\frac{12}{a^2} = -3 \qquad \text{or} \qquad a^2 = 4 \qquad \text{or} \qquad a = \pm 2$$

Therefore the required points are $(2, 6)$ and $(-2, -6)$. The hyperbola and the tangents are shown in Figure 6. ■

We summarize the differentiation formulas we have learned so far as follows.

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

2.3 EXERCISES

1–22 Differentiate the function.

1. $f(x) = 2^{40}$
2. $f(x) = \pi^2$
3. $f(x) = 5.2x + 2.3$
4. $g(x) = \frac{7}{4}x^2 - 3x + 12$
5. $f(t) = 2t^3 - 3t^2 - 4t$
6. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$
7. $g(x) = x^2(1 - 2x)$
8. $H(u) = (3u - 1)(u + 2)$
9. $g(t) = 2t^{-3/4}$
10. $B(y) = cy^{-6}$
11. $F(r) = \dfrac{5}{r^3}$
12. $y = x^{5/3} - x^{2/3}$
13. $S(p) = \sqrt{p} - p$
14. $y = \sqrt[3]{x}(2 + x)$
15. $R(a) = (3a + 1)^2$
16. $S(R) = 4\pi R^2$
17. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
18. $y = \dfrac{\sqrt{x} + x}{x^2}$
19. $G(q) = (1 + q^{-1})^2$
20. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$
21. $u = \left(\dfrac{1}{t} - \dfrac{1}{\sqrt{t}}\right)^2$
22. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

23. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

24. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

25–44 Differentiate.

25. $f(x) = (5x^2 - 2)(x^3 + 3x)$
26. $B(u) = (u^3 + 1)(2u^2 - 4u - 1)$
27. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$
28. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$
29. $g(x) = \dfrac{1 + 2x}{3 - 4x}$
30. $h(t) = \dfrac{6t + 1}{6t - 1}$
31. $y = \dfrac{x^2 + 1}{x^3 - 1}$
32. $y = \dfrac{1}{t^3 + 2t^2 - 1}$
33. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3}$
34. $y = \dfrac{(u + 2)^2}{1 - u}$
35. $y = \dfrac{s - \sqrt{s}}{s^2}$
36. $y = \dfrac{\sqrt{x}}{2 + x}$

37. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3}$

38. $y = \dfrac{cx}{1 + cx}$

39. $F(x) = \dfrac{2x^5 + x^4 - 6x}{x^3}$

40. $A(v) = v^{2/3}(2v^2 + 1 - v^{-2})$

41. $G(y) = \dfrac{B}{Ay^3 + B}$

42. $F(t) = \dfrac{At}{Bt^2 + Ct^3}$

43. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

44. $f(x) = \dfrac{ax + b}{cx + d}$

45. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Find the derivative of P.

46–48 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

46. $f(x) = x/(x^2 - 1)$

47. $f(x) = 3x^{15} - 5x^3 + 3$

48. $f(x) = x + \dfrac{1}{x}$

49. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 2.2.1.)

(c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

50. (a) Graph the function $g(x) = x^2/(x^2 + 1)$ in the viewing rectangle $[-4, 4]$ by $[-1, 1.5]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 2.2.1.)

(c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

51–52 Find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{2x}{x + 1}$, $(1, 1)$

52. $y = 2x^3 - x^2 + 2$, $(1, 3)$

53. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

54. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

55–58 Find equations of the tangent line and normal line to the curve at the given point.

55. $y = x + \sqrt{x}$, $(1, 2)$

56. $y^2 = x^3$, $(1, 1)$

57. $y = \dfrac{3x + 1}{x^2 + 1}$, $(1, 2)$

58. $y = \dfrac{\sqrt{x}}{x + 1}$, $(4, 0.4)$

59–62 Find the first and second derivatives of the function.

59. $f(x) = 0.001x^5 - 0.02x^3$

60. $G(r) = \sqrt{r} + \sqrt[3]{r}$

61. $f(x) = \dfrac{x^2}{1 + 2x}$

62. $f(x) = \dfrac{1}{3 - x}$

63. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

(a) the velocity and acceleration as functions of t,

(b) the acceleration after 2 s, and

(c) the acceleration when the velocity is 0.

64. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.

(a) Find the velocity and acceleration as functions of t.

(b) Find the acceleration after 1 s.

(c) Graph the position, velocity, and acceleration functions on the same screen.

65. Biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Calculate

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and interpret your answer.

66. The number of tree species S in a given area A in the Pasoh Forest Reserve in Malaysia has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

Source: Adapted from K. Kochummen et al., "Floristic Composition of Pasoh Forest Reserve, A Lowland Rain Forest in Peninsular Malaysia," *Journal of Tropical Forest Science* 3 (1991):1–13.

67. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
(a) Suppose that the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Write V as a function of P.
(b) Calculate dV/dP when $P = 50$ kPa. What is the meaning of the derivative? What are its units?

68. Car tires need to be inflated properly because overinflation or underinflation can cause premature tread wear. The data in the table show tire life L (in thousands of miles) for a certain type of tire at various pressures P (in lb/in^2).

P	26	28	31	35	38	42	45
L	50	66	78	81	74	70	59

(a) Use a calculator to model tire life with a quadratic function of the pressure.
(b) Use the model to estimate dL/dP when $P = 30$ and when $P = 40$. What is the meaning of the derivative? What are the units? What is the significance of the signs of the derivatives?

69. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

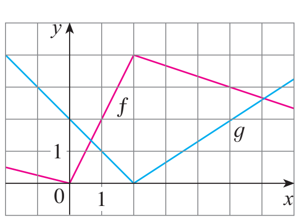
70. Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$.
(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{f(x) + g(x)}$

71. If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $f'(4)$.

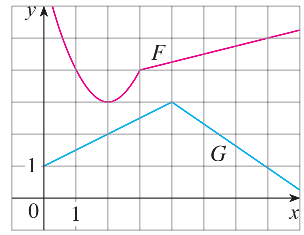
72. If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}$$

73. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
(a) Find $u'(1)$. (b) Find $v'(5)$.



74. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.
(a) Find $P'(2)$. (b) Find $Q'(7)$.



75. If g is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$
(c) $y = \dfrac{g(x)}{x}$

76. If f is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = x^2 f(x)$ (b) $y = \dfrac{f(x)}{x^2}$
(c) $y = \dfrac{x^2}{f(x)}$ (d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

77. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

78. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

79. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

80. Find an equation of the tangent line to the curve $y = x^4 + 1$ that is parallel to the line $32x - y = 15$.

81. Find equations of both lines that are tangent to the curve $y = x^3 - 3x^2 + 3x - 3$ and are parallel to the line $3x - y = 15$.

82. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

83. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

84. Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

85. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

86. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

87. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = (x^4 + 3x^3 + 17x + 82)^3$.

88. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$ (b) $f(x) = 1/x$

89. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

90. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 9.)

91. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

92. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

93. In this exercise we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population of this area was 961,400, and the population was increasing at roughly 9200 people per year. The average annual income was \$30,593 per capita, and this average was increasing at about \$1400 per year (a little above the national average of about \$1225 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in the Richmond-Petersburg area in 1999. Explain the meaning of each term in the Product Rule.

94. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.
(a) What does it mean to say that $f(20) = 10{,}000$ and $f'(20) = -350$?
(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

95. The Michaelis-Menten equation for the enzyme chymotrypsin is

$$v = \frac{0.14[\text{S}]}{0.015 + [\text{S}]}$$

where v is the rate of an enzymatic reaction and [S] is the concentration of a substrate S. Calculate $dv/d[\text{S}]$ and interpret it.

96. The *biomass* $B(t)$ of a fish population is the total mass of the members of the population at time t. It is the product of the number of individuals $N(t)$ in the population and the average mass $M(t)$ of a fish at time t. In the case of guppies, breeding occurs continually. Suppose that at time $t = 4$ weeks the population is 820 guppies and is growing at a rate of 50 guppies per week, while the average mass is 1.2 g and is increasing at a rate of 0.14 g/week. At what rate is the biomass increasing when $t = 4$?

97. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geqslant 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f'.

98. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leqslant 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g'.

99. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f'.
(b) Sketch the graphs of f and f'.

100. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h'.

101. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

102. (a) If $F(x) = f(x)\,g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.
(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Guess a formula for $F^{(n)}$.

103. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

104. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

105. An easy proof of the Quotient Rule can be given if we make the prior assumption that $F'(x)$ exists, where $F = f/g$. Write $f = Fg$; then differentiate using the Product Rule and solve the resulting equation for F'.

106. A tangent line is drawn to the hyperbola $xy = c$ at a point P.
(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.
(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

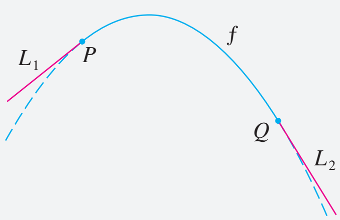
107. Evaluate $\lim_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

108. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

109. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leqslant \frac{1}{2}$?

110. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

APPLIED PROJECT BUILDING A BETTER ROLLER COASTER





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Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. (See the figure.) To simplify the equations, you decide to place the origin at P.

1. (a) Suppose the horizontal distance between P and Q is 100 ft. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.
(b) Solve the equations in part (a) for a, b, and c to find a formula for $f(x)$.
(c) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.
(d) Find the difference in elevation between P and Q.

2. The solution in Problem 1 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leqslant x \leqslant 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. So you decide to improve the design by using a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $10 \leqslant x \leqslant 90$ and connecting it to the linear functions by means of two cubic functions:

$$g(x) = kx^3 + lx^2 + mx + n \qquad 0 \leqslant x < 10$$

$$h(x) = px^3 + qx^2 + rx + s \qquad 90 < x \leqslant 100$$

(a) Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.
CAS (b) Solve the equations in part (a) with a computer algebra system to find formulas for $q(x)$, $g(x)$, and $h(x)$.
(c) Plot L_1, g, q, h, and L_2, and compare with the plot in Problem 1(c).

2.4 Derivatives of Trigonometric Functions

A review of the trigonometric functions is given in Appendix D.

Before starting this section, you might need to review the trigonometric functions. In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot. Recall from Section 1.8 that all of the trigonometric functions are continuous at every number in their domains.

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' (see Exercise 2.2.16), then it looks as if the graph of f' may be the same as the cosine curve (see Figure 1).

TEC Visual 2.4 shows an animation of Figure 1.

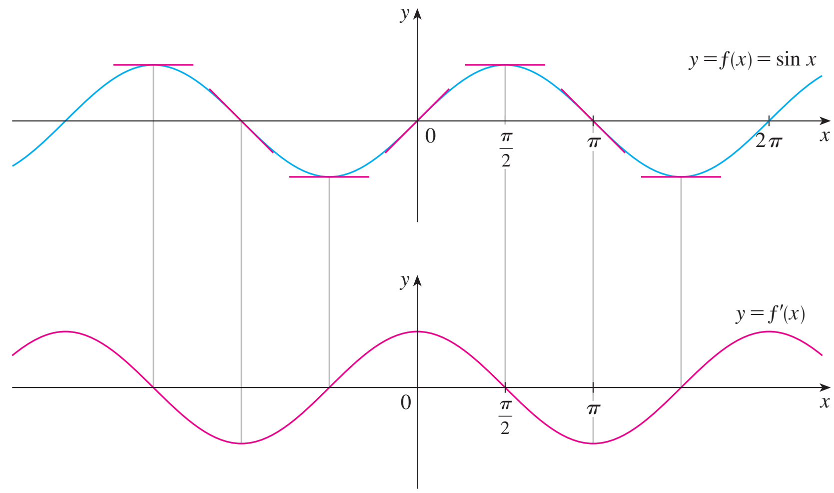


FIGURE 1

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

We have used the addition formula for sine. See Appendix D.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\ &= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \end{aligned}$$

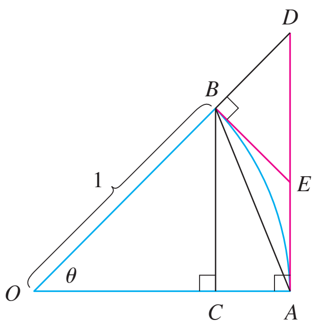
1 $$= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \to 0$, we have

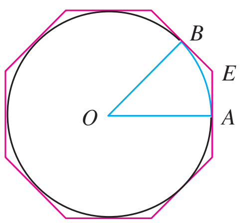
$$\lim_{h \to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h \to 0} \cos x = \cos x$$

The limit of $(\sin h)/h$ is not so obvious. In Example 1.5.3 we made the guess, on the basis of numerical and graphical evidence, that

2 $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



(a)



(b)

FIGURE 2

We now use a geometric argument to prove Equation 2. Assume first that θ lies between 0 and $\pi/2$. Figure 2(a) shows a sector of a circle with center O, central angle θ, and radius 1. BC is drawn perpendicular to OA. By the definition of radian measure, we have arc $AB = \theta$. Also $|BC| = |OB| \sin \theta = \sin \theta$. From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore
$$\sin \theta < \theta \qquad \text{so} \qquad \frac{\sin \theta}{\theta} < 1$$

Let the tangent lines at A and B intersect at E. You can see from Figure 2(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$. Thus

$$\begin{aligned} \theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan \theta \\ &= \tan \theta \end{aligned}$$

(In Appendix F the inequality $\theta \leqslant \tan \theta$ is proved directly from the definition of the length of an arc without resorting to geometric intuition as we did here.) Therefore we have

$$\theta < \frac{\sin \theta}{\cos \theta}$$

so
$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos \theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

But the function $(\sin \theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 2.

We can deduce the value of the remaining limit in (1) as follows:

We multiply numerator and denominator by $\cos \theta + 1$ in order to put the function in a form in which we can use the limits we know.

$$\begin{aligned} \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)} \\ &= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right) \\ &= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1} \\ &= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0 \qquad \text{(by Equation 2)} \end{aligned}$$

3 $$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

If we now put the limits (2) and (3) in (1), we get

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

4 $$\frac{d}{dx}(\sin x) = \cos x$$

EXAMPLE 1 Differentiate $y = x^2 \sin x$.

Figure 3 shows the graphs of the function of Example 1 and its derivative. Notice that $y' = 0$ whenever y has a horizontal tangent.

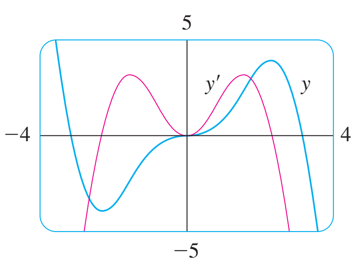


FIGURE 3

SOLUTION Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$ ■

Using the same methods as in the proof of Formula 4, one can prove (see Exercise 20) that

5 $$\frac{d}{dx}(\cos x) = -\sin x$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

6 $$\frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule (see Exercises 17–19). We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when x is measured in radians.

When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2}$$

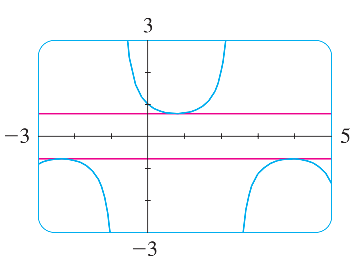


FIGURE 4
The horizontal tangents in Example 2

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure 4). ■

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

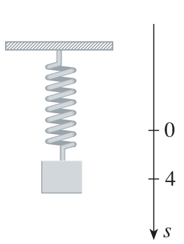


FIGURE 5

EXAMPLE 3 An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 5 and note that the downward direction is positive.) Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

SOLUTION The velocity and acceleration are

$$v = \frac{ds}{dt} = \frac{d}{dt}(4\cos t) = 4\frac{d}{dt}(\cos t) = -4\sin t$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-4\sin t) = -4\frac{d}{dt}(\sin t) = -4\cos t$$

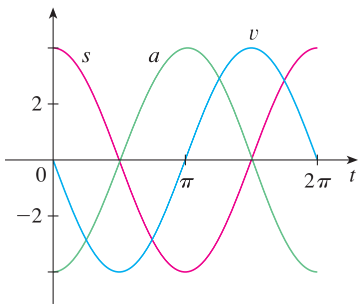


FIGURE 6

The object oscillates from the lowest point ($s = 4$ cm) to the highest point ($s = -4$ cm). The period of the oscillation is 2π, the period of $\cos t$.

The speed is $|v| = 4|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$. So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the high and low points.

The acceleration $a = -4\cos t = 0$ when $s = 0$. It has greatest magnitude at the high and low points. See the graphs in Figure 6. ■

EXAMPLE 4 Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

PS Look for a pattern.

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$ ■

Our main use for the limit in Equation 2 has been to prove the differentiation formula for the sine function. But this limit is also useful in finding certain other trigonometric limits, as the following two examples show.

EXAMPLE 5 Find $\lim_{x \to 0} \dfrac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7\sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

If we let $\theta = 7x$, then $\theta \to 0$ as $x \to 0$, so by Equation 2 we have

$$\lim_{x\to 0} \frac{\sin 7x}{4x} = \frac{7}{4} \lim_{x\to 0} \left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$ ■

EXAMPLE 6 Calculate $\lim_{x\to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x\cos x}{\sin x}$$

$$= \lim_{x\to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 2)}$$

$$= 1$$ ■

2.4 EXERCISES

1–16 Differentiate.

1. $f(x) = x^2 \sin x$

2. $f(x) = x \cos x + 2 \tan x$

3. $f(x) = 3 \cot x - 2 \cos x$

4. $y = 2 \sec x - \csc x$

5. $y = \sec\theta \tan\theta$

6. $g(t) = 4 \sec t + \tan t$

7. $y = c \cos t + t^2 \sin t$

8. $y = u(a \cos u + b \cot u)$

9. $y = \dfrac{x}{2 - \tan x}$

10. $y = \sin\theta \cos\theta$

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$

12. $y = \dfrac{\cos x}{1 - \sin x}$

13. $y = \dfrac{t \sin t}{1 + t}$

14. $y = \dfrac{\sin t}{1 + \tan t}$

15. $f(\theta) = \theta \cos\theta \sin\theta$

16. $y = x^2 \sin x \tan x$

17. Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

18. Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

19. Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

20. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 Find an equation of the tangent line to the curve at the given point.

21. $y = \sin x + \cos x$, $(0, 1)$

22. $y = (1 + x)\cos x$, $(0, 1)$

23. $y = \cos x - \sin x$, $(\pi, -1)$

24. $y = x + \tan x$, (π, π)

25. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

27. (a) If $f(x) = \sec x - x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

28. (a) If $f(x) = \sqrt{x} \sin x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 \leqslant x \leqslant 2\pi$.

29. If $H(\theta) = \theta \sin\theta$, find $H'(\theta)$ and $H''(\theta)$.

30. If $f(t) = \sec t$, find $f''(\pi/4)$.

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

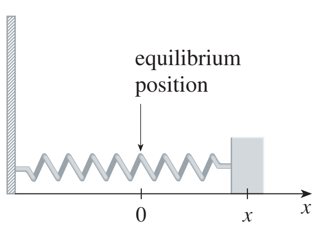
(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers to parts (a) and (b) are equivalent.

32. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

33. For what values of x does the graph of $f(x) = x + 2 \sin x$ have a horizontal tangent?

34. Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

35. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity and acceleration at time t.
(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



36. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geqslant 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
(a) Find the velocity and acceleration at time t.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

37. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

38. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

39–50 Find the limit.

39. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$

40. $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sin \pi x}$

41. $\displaystyle\lim_{t\to 0} \frac{\tan 6t}{\sin 2t}$

42. $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

43. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{5x^3 - 4x}$

44. $\displaystyle\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2}$

45. $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta}$

46. $\displaystyle\lim_{x\to 0} \csc x \sin(\sin x)$

47. $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2}$

48. $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x}$

49. $\displaystyle\lim_{x\to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

50. $\displaystyle\lim_{x\to 1} \frac{\sin(x - 1)}{x^2 + x - 2}$

51–52 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

51. $\dfrac{d^{99}}{dx^{99}}(\sin x)$

52. $\dfrac{d^{35}}{dx^{35}}(x \sin x)$

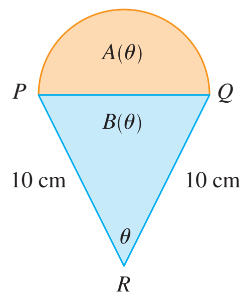
53. Find constants A and B such that the function $y = A \sin x + B \cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

54. Evaluate $\displaystyle\lim_{x\to 0} x \sin\frac{1}{x}$ and illustrate by graphing $y = x \sin(1/x)$.

55. Differentiate each trigonometric identity to obtain a new (or familiar) identity.
(a) $\tan x = \dfrac{\sin x}{\cos x}$ (b) $\sec x = \dfrac{1}{\cos x}$
(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

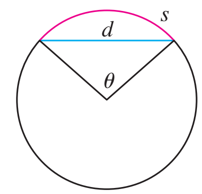
56. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$



57. The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find

$$\lim_{\theta \to 0^+} \frac{s}{d}$$



58. Let $f(x) = \dfrac{x}{\sqrt{1 - \cos 2x}}$.

(a) Graph f. What type of discontinuity does it appear to have at 0?

(b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

2.5 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

See Section 1.3 for a review of composite functions.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

James Gregory

The first person to formulate the Chain Rule was the Scottish mathematician James Gregory (1638–1675), who also designed the first practical reflecting telescope. Gregory discovered the basic ideas of calculus at about the same time as Newton. He became the first Professor of Mathematics at the University of St. Andrews and later held the same position at the University of Edinburgh. But one year after accepting that position he died at the age of 36.

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

1
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

(Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous.)

$$= \frac{dy}{du}\frac{du}{dx}$$

The only flaw in this reasoning is that in (1) it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we can't divide by 0. Nonetheless, this reasoning does at least *suggest* that the Chain Rule is true. A full proof of the Chain Rule is given at the end of this section. ■

The Chain Rule can be written either in the prime notation

2
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du. Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

EXAMPLE 1 Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$ ■

When using Formula 3 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (called the *derivative of y with respect to x*), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x $\left(y = \sqrt{x^2+1}\right)$ and also as a function of u $\left(y = \sqrt{u}\right)$. Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2+1}} \qquad \text{whereas} \qquad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

NOTE In using the Chain Rule we work from the outside to the inside. Formula 2 says that *we differentiate the outer function f [at the inner function g(x)] and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\ \text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\ \text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)}_{\substack{\text{inner}\\ \text{function}}}{}^2 = \underbrace{2}_{\substack{\text{derivative}\\ \text{of outer}\\ \text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated}\\ \text{at inner}\\ \text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\ \text{of inner}\\ \text{function}}}$$

The answer can be left as $2\sin x\,\cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula). ■

See Reference Page 2 or Appendix D.

In Example 2(a) we combined the Chain Rule with the rule for differentiating the sine function. In general, if $y = \sin u$, where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Notice that the derivative in Example 1 could be calculated by taking $n = \frac{1}{2}$ in Rule 4.

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f: $\quad f(x) = (x^2 + x + 1)^{-1/3}$

Thus

$$f'(x) = -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}\frac{d}{dx}(x^2 + x + 1)$$

$$= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

■

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1 - 2(t-2)}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$ ■

EXAMPLE 6 Differentiate $y = (2x+1)^5(x^3 - x + 1)^4$.

SOLUTION In this example we must use the Product Rule before using the Chain Rule:

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}(x^3 - x + 1)^4 + (x^3 - x + 1)^4 \frac{d}{dx}(2x+1)^5$$

$$= (2x+1)^5 \cdot 4(x^3 - x + 1)^3 \frac{d}{dx}(x^3 - x + 1)$$

$$+ (x^3 - x + 1)^4 \cdot 5(2x+1)^4 \frac{d}{dx}(2x+1)$$

$$= 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1) + 5(x^3 - x + 1)^4(2x+1)^4 \cdot 2$$

Noticing that each term has the common factor $2(2x+1)^4(x^3 - x + 1)^3$, we could factor it out and write the answer as

$$\frac{dy}{dx} = 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$ ■

The graphs of the functions y and y' in Example 6 are shown in Figure 1. Notice that y' is large when y increases rapidly and $y' = 0$ when y has a horizontal tangent. So our answer appears to be reasonable.

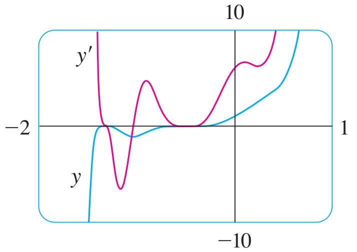


FIGURE 1

The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link. Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions. Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

EXAMPLE 7 If $f(x) = \sin(\cos(\tan x))$, then

$$f'(x) = \cos(\cos(\tan x)) \frac{d}{dx}\cos(\tan x)$$

$$= \cos(\cos(\tan x))[-\sin(\tan x)] \frac{d}{dx}(\tan x)$$

$$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$$

Notice that we used the Chain Rule twice. ■

EXAMPLE 8 Differentiate $y = \sqrt{\sec x^3}$.

SOLUTION Here the outer function is the square root function, the middle function is the secant function, and the inner function is the cubing function. So we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{\sec x^3}} \frac{d}{dx}(\sec x^3) \\ &= \frac{1}{2\sqrt{\sec x^3}} \sec x^3 \tan x^3 \frac{d}{dx}(x^3) \\ &= \frac{3x^2 \sec x^3 \tan x^3}{2\sqrt{\sec x^3}} \end{aligned}$$

■

How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But
$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \qquad \Rightarrow \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

5
$$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$$

and ε is a continuous function of Δx. This property of differentiable functions is what enables us to prove the Chain Rule.

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use Equation 5 to write

6
$$\Delta u = g'(a)\,\Delta x + \varepsilon_1\,\Delta x = [g'(a) + \varepsilon_1]\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

7
$$\Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u = [f'(b) + \varepsilon_2]\,\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. If we now substitute the expression for Δu from Equation 6 into Equation 7, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\,\Delta x$$

so $$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 6 shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$
$$= f'(b)\,g'(a) = f'(g(a))\,g'(a)$$

This proves the Chain Rule. ■

2.5 EXERCISES

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sqrt[3]{1 + 4x}$

2. $y = (2x^3 + 5)^4$

3. $y = \tan \pi x$

4. $y = \sin(\cot x)$

5. $y = \sqrt{\sin x}$

6. $y = \sin \sqrt{x}$

7–46 Find the derivative of the function.

7. $F(x) = (5x^6 + 2x^3)^4$

8. $F(x) = (1 + x + x^2)^{99}$

9. $f(x) = \sqrt{5x + 1}$

10. $g(x) = (2 - \sin x)^{3/2}$

11. $A(t) = \dfrac{1}{(\cos t + \tan t)^2}$

12. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}}$

13. $f(\theta) = \cos(\theta^2)$

14. $g(\theta) = \cos^2 \theta$

15. $h(v) = v\sqrt[3]{1 + v^2}$

16. $f(t) = t \sin \pi t$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5$

18. $g(x) = (x^2 + 1)^3(x^2 + 2)^6$

19. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3$

20. $F(t) = (3t - 1)^4(2t + 1)^{-3}$

21. $g(u) = \left(\dfrac{u^3 - 1}{u^3 + 1}\right)^8$

22. $y = \left(x + \dfrac{1}{x}\right)^5$

23. $y = \sqrt{\dfrac{x}{x + 1}}$

24. $U(y) = \left(\dfrac{y^4 + 1}{y^2 + 1}\right)^5$

25. $h(\theta) = \tan(\theta^2 \sin \theta)$

26. $f(t) = \sqrt{\dfrac{t}{t^2 + 4}}$

27. $y = \dfrac{\cos x}{\sqrt{1 + \sin x}}$

28. $F(t) = \dfrac{t^2}{\sqrt{t^3 + 1}}$

29. $H(r) = \dfrac{(r^2 - 1)^3}{(2r + 1)^5}$

30. $s(t) = \sqrt{\dfrac{1 + \sin t}{1 + \cos t}}$

31. $y = \cos(\sec 4x)$

32. $J(\theta) = \tan^2(n\theta)$

33. $y = \sin\sqrt{1 + x^2}$

34. $y = \sqrt{\sin(1 + x^2)}$

35. $y = \left(\dfrac{1 - \cos 2x}{1 + \cos 2x}\right)^4$

36. $y = x \sin \dfrac{1}{x}$

37. $y = \cot^2(\sin \theta)$

38. $y = \sin\left(t + \cos \sqrt{t}\right)$

39. $f(t) = \tan(\sec(\cos t))$

40. $g(u) = [(u^2 - 1)^6 - 3u]^4$

41. $y = \sqrt{x + \sqrt{x}}$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

43. $g(x) = (2r \sin rx + n)^p$

44. $y = \cos^4(\sin^3 x)$

45. $y = \cos\sqrt{\sin(\tan \pi x)}$

46. $y = [x + (x + \sin^2 x)^3]^4$

47–50 Find y' and y''.

47. $y = \cos(\sin 3\theta)$

48. $y = \dfrac{1}{(1 + \tan x)^2}$

49. $y = \sqrt{1 - \sec t}$

50. $y = \dfrac{4x}{\sqrt{x + 1}}$

51–54 Find an equation of the tangent line to the curve at the given point.

51. $y = (3x - 1)^{-6}$, $(0, 1)$

52. $y = \sqrt{1 + x^3}$, $(2, 3)$

53. $y = \sin(\sin x)$, $(\pi, 0)$

54. $y = \sin^2 x \cos x$, $(\pi/2, 0)$

55. (a) Find an equation of the tangent line to the curve $y = \tan(\pi x^2/4)$ at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

56. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point (1, 1).
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

57. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

58. The function $f(x) = \sin(x + \sin 2x)$, $0 \leqslant x \leqslant \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f'.
(b) Calculate $f'(x)$ and use this expression, with a calculator, to graph f'. Compare with your sketch in part (a).

59. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

60. At what point on the curve $y = \sqrt{1 + 2x}$ is the tangent line perpendicular to the line $6x + 2y = 1$?

61. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

62. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

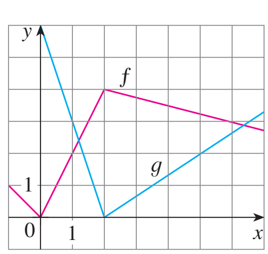
63. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

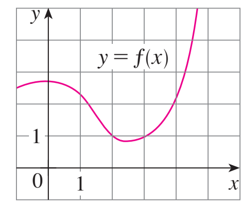
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

64. Let f and g be the functions in Exercise 63.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

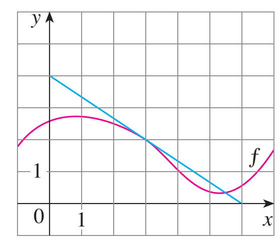
65. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



66. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



67. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



68. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

69. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

70. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

71. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

72. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

73–74 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

73. $D^{103} \cos 2x$ **74.** $D^{35}\, x \sin \pi x$

---

75. The displacement of a particle on a vibrating string is given by the equation $s(t) = 10 + \frac{1}{4}\sin(10\pi t)$ where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

76. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

77. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is

Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

(a) Find the rate of change of the brightness after t days.
(b) Find, correct to two decimal places, the rate of increase after one day.

78. In Example 1.3.4 we arrived at a model for the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 and May 21.

79. A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t)\frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds.

80. Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.
(a) What do the derivatives dV/dr and dV/dt represent?
(b) Express dV/dt in terms of dr/dt.

CAS **81.** Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.
(a) Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.
(b) Use a CAS to find the derivative in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?

CAS **82.** (a) Use a CAS to differentiate the function

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

and to simplify the result.
(b) Where does the graph of f have horizontal tangents?
(c) Graph f and f' on the same screen. Are the graphs consistent with your answer to part (b)?

83. Use the Chain Rule to prove the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

84. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.
[*Hint:* Write $f(x)/g(x) = f(x)[g(x)]^{-1}$.]

85. (a) If n is a positive integer, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1}x \cos(n + 1)x$$

(b) Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the one in part (a).

86. Suppose $y = f(x)$ is a curve that always lies above the x-axis and never has a horizontal tangent, where f is differentiable everywhere. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x?

87. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: the differentiation formulas would not be as simple if we used degree measure.)

88. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

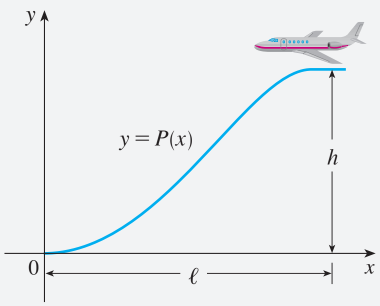
(b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f'. Where is f not differentiable?
(c) If $g(x) = \sin|x|$, find $g'(x)$ and sketch the graphs of g and g'. Where is g not differentiable?

89. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

90. If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for d^3y/dx^3 similar to the one given in Exercise 89.

APPLIED PROJECT WHERE SHOULD A PILOT START DESCENT?



An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:

(i) The cruising altitude is h when descent starts at a horizontal distance ℓ from touchdown at the origin.

(ii) The pilot must maintain a constant horizontal speed v throughout descent.

(iii) The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

1. Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.

2. Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \leqslant k$$

3. Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 860 \text{ mi/h}^2$. If the cruising altitude of a plane is 35,000 ft and the speed is 300 mi/h, how far away from the airport should the pilot start descent?

4. Graph the approach path if the conditions stated in Problem 3 are satisfied.

2.6 Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \qquad \text{or} \qquad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25 \tag{1}$$

or

$$x^3 + y^3 = 6xy \tag{2}$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

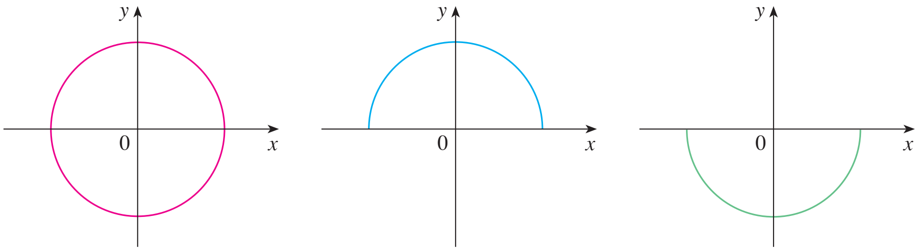


FIGURE 1 (a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x. The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f.

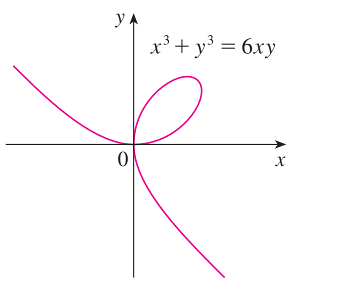


FIGURE 2 The folium of Descartes

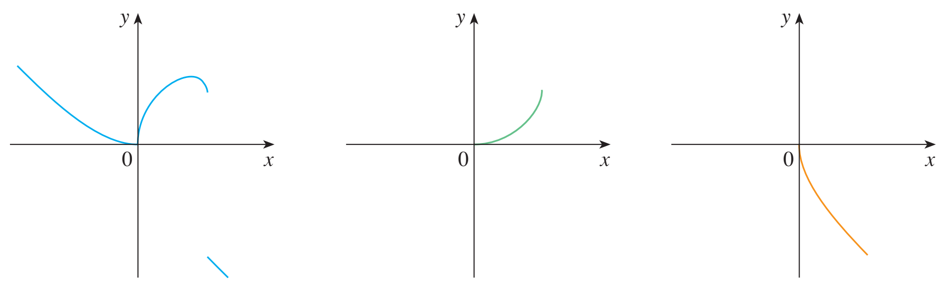


FIGURE 3 Graphs of three functions defined by the folium of Descartes

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

EXAMPLE 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

SOLUTION 1

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus $$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

SOLUTION 2

(b) Solving the equation $x^2 + y^2 = 25$ for y, we get $y = \pm\sqrt{25 - x^2}$. The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2) \\ &= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

Example 1 illustrates that even when it is possible to solve an equation explicitly for y in terms of x, it may be easier to use implicit differentiation.

So
$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$. ■

NOTE 1 The formula $dy/dx = -x/y$ in Solution 1 gives the derivative in terms of both x and y. It is correct no matter which function y is determined by the given equation. For instance, for $y = f(x) = \sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{25 - x^2}}$$

whereas for $y = g(x) = -\sqrt{25 - x^2}$ we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{-\sqrt{25 - x^2}} = \frac{x}{\sqrt{25 - x^2}}$$

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).
(c) At what point in the first quadrant is the tangent line horizontal?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or
$$x^2 + y^2y' = 2xy' + 2y$$

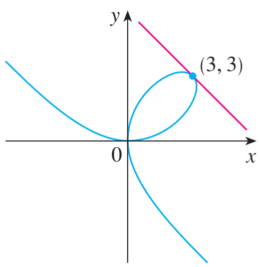


FIGURE 4

We now solve for y':

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at (3, 3). So an equation of the tangent to the folium at (3, 3) is

$$y - 3 = -1(x - 3) \qquad \text{or} \qquad x + y = 6$$

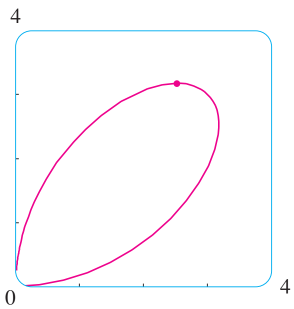


FIGURE 5

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately (2.5198, 3.1748). Looking at Figure 5, we see that our answer is reasonable. ■

NOTE 2 There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If we use this formula (or a computer algebra system) to solve the equation $x^3 + y^3 = 6xy$ for y in terms of x, we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}$$

and

$$y = \tfrac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\tfrac{1}{2}x^3 + \sqrt{\tfrac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\tfrac{1}{2}x^3 - \sqrt{\tfrac{1}{4}x^6 - 8x^3}}\right)\right]$$

(These are the three functions whose graphs are shown in Figure 3.) You can see that the method of implicit differentiation saves an enormous amount of work in cases such as this. Moreover, implicit differentiation works just as easily for equations such as

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

for which it is *impossible* to find a similar expression for y in terms of x.

Abel and Galois

The Norwegian mathematician Niels Abel proved in 1824 that no general formula can be given for the roots of a fifth-degree equation in terms of radicals. Later the French mathematician Evariste Galois proved that it is impossible to find a general formula for the roots of an nth-degree equation (in terms of algebraic operations on the coefficients) if n is any integer larger than 4.

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

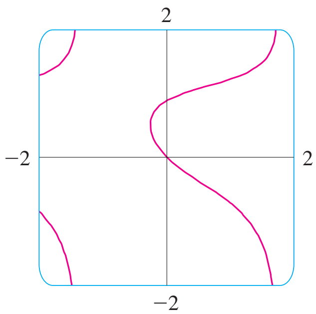


FIGURE 6

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So
$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Figure 6, drawn with the implicit-plotting command of a computer algebra system, shows part of the curve $\sin(x + y) = y^2 \cos x$. As a check on our calculation, notice that $y' = -1$ when $x = y = 0$ and it appears from the graph that the slope is approximately -1 at the origin. ■

Figures 7, 8, and 9 show three more curves produced by a computer algebra system with an implicit-plotting command. In Exercises 41–42 you will have an opportunity to create and examine unusual curves of this nature.

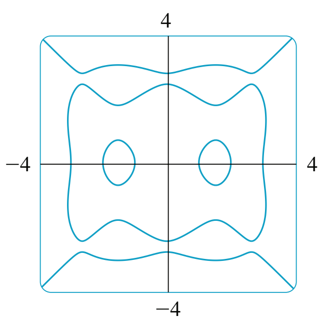


FIGURE 7
$(x^2 - 1)(x^2 - 4)(x^2 - 9)$
$= y^2(y^2 - 4)(y^2 - 9)$

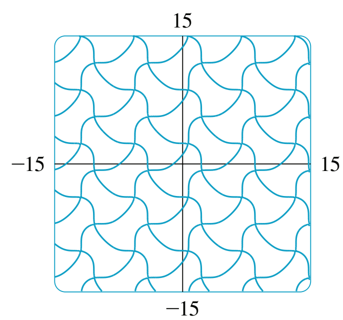


FIGURE 8
$\cos(x - \sin y) = \sin(y - \sin x)$

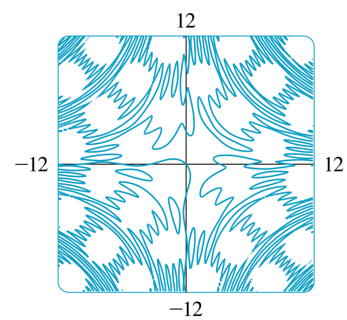


FIGURE 9
$\sin(xy) = \sin x + \sin y$

The following example shows how to find the second derivative of a function that is defined implicitly.

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

3
$$y' = -\frac{x^3}{y^3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$ ■

Figure 10 shows the graph of the curve $x^4 + y^4 = 16$ of Example 4. Notice that it's a stretched and flattened version of the circle $x^2 + y^2 = 4$. For this reason it's sometimes called a *fat circle*. It starts out very steep on the left but quickly becomes very flat. This can be seen from the expression

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3$$

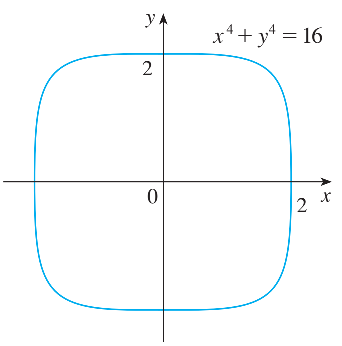


FIGURE 10

2.6 EXERCISES

1–4
(a) Find y' by implicit differentiation.
(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $9x^2 - y^2 = 1$

2. $2x^2 + x + xy = 1$

3. $\sqrt{x} + \sqrt{y} = 1$

4. $\dfrac{2}{x} - \dfrac{1}{y} = 4$

5–20 Find dy/dx by implicit differentiation.

5. $x^2 - 4xy + y^2 = 4$

6. $2x^2 + xy - y^2 = 2$

7. $x^4 + x^2y^2 + y^3 = 5$

8. $x^3 - xy^2 + y^3 = 1$

9. $\dfrac{x^2}{x + y} = y^2 + 1$

10. $y^5 + x^2y^3 = 1 + x^4y$

11. $y \cos x = x^2 + y^2$

12. $\cos(xy) = 1 + \sin y$

13. $\sqrt{x + y} = x^4 + y^4$

14. $y \sin(x^2) = x \sin(y^2)$

15. $\tan(x/y) = x + y$

16. $xy = \sqrt{x^2 + y^2}$

17. $\sqrt{xy} = 1 + x^2y$

18. $x \sin y + y \sin x = 1$

19. $\sin(xy) = \cos(x + y)$

20. $\tan(x - y) = \dfrac{y}{1 + x^2}$

21. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

22. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

23–24 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

23. $x^4y^2 - x^3y + 2xy^3 = 0$

24. $y \sec x = x \tan y$

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

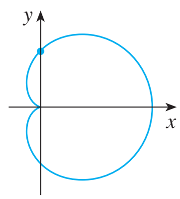
25. $y \sin 2x = x \cos 2y$, $(\pi/2, \pi/4)$

26. $\sin(x + y) = 2x - 2y$, (π, π)

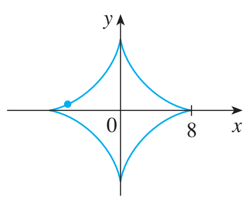
27. $x^2 - xy - y^2 = 1$, $(2, 1)$ (hyperbola)

28. $x^2 + 2xy + 4y^2 = 12$, $(2, 1)$ (ellipse)

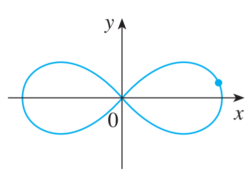
29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $\left(0, \frac{1}{2}\right)$ (cardioid)



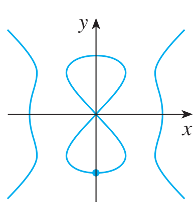
30. $x^{2/3} + y^{2/3} = 4$, $\left(-3\sqrt{3}, 1\right)$ (astroid)



31. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$ (lemniscate)



32. $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$ (devil's curve)



33. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)

34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

35–38 Find y'' by implicit differentiation.

35. $x^2 + 4y^2 = 4$

36. $x^2 + xy + y^2 = 3$

37. $\sin y + \cos x = 1$

38. $x^3 - y^3 = 7$

39. If $xy + y^3 = 1$, find the value of y'' at the point where $x = 0$.

40. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

CAS **41.** Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.

(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.

(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation in part (a).

CAS **42.** (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.

(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

43. Find the points on the lemniscate in Exercise 31 where the tangent is horizontal.

44. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

45. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

46. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

47. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

48. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable

function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

49–52 Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

49. $x^2 + y^2 = r^2, \quad ax + by = 0$

50. $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

51. $y = cx^2, \quad x^2 + 2y^2 = k$

52. $y = ax^3, \quad x^2 + 3y^2 = b$

---

53. Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

54. Find the value of the number a such that the families of curves $y = (x + c)^{-1}$ and $y = a(x + k)^{1/3}$ are orthogonal trajectories.

55. (a) The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. If T remains constant, use implicit differentiation to find dV/dP.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592\ \text{L}^2\text{-atm/mole}^2$ and $b = 0.04267$ L/mole.

56. (a) Use implicit differentiation to find y' if

$$x^2 + xy + y^2 + 1 = 0$$

(b) Plot the curve in part (a). What do you see? Prove that what you see is correct.

(c) In view of part (b), what can you say about the expression for y' that you found in part (a)?

57. The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

58. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

59. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

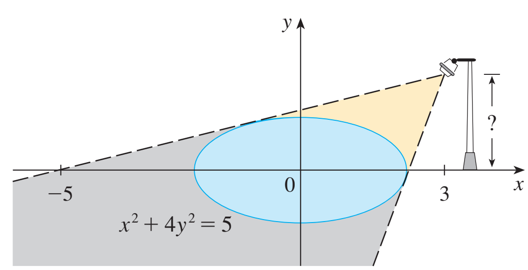
60. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

61. The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

62. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leqslant 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



LABORATORY PROJECT FAMILIES OF IMPLICIT CURVES

In this project you will explore the changing shapes of implicitly defined curves as you vary the constants in a family, and determine which features are common to all members of the family.

1. Consider the family of curves

$$y^2 - 2x^2(x + 8) = c[(y + 1)^2(y + 9) - x^2]$$

(a) By graphing the curves with $c = 0$ and $c = 2$, determine how many points of intersection there are. (You might have to zoom in to find all of them.)

(b) Now add the curves with $c = 5$ and $c = 10$ to your graphs in part (a). What do you notice? What about other values of c?

2. (a) Graph several members of the family of curves

$$x^2 + y^2 + cx^2y^2 = 1$$

Describe how the graph changes as you change the value of c.

(b) What happens to the curve when $c = -1$? Describe what appears on the screen. Can you prove it algebraically?

(c) Find y' by implicit differentiation. For the case $c = -1$, is your expression for y' consistent with what you discovered in part (b)?

2.7 Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.1 the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

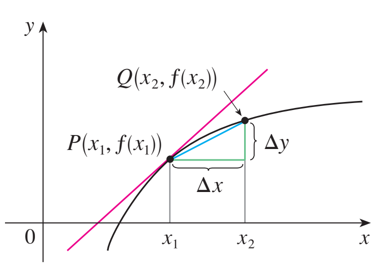
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is **the average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of y with respect to x** or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

FIGURE 1

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. (As we discussed in Section 2.1, the units for dy/dx are the units for y divided by the units for x.) We now look at some of these interpretations in the natural and social sciences.

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$. This was discussed in Sections 2.1 and 2.2, but now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

EXAMPLE 1 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.
(a) Find the velocity at time t.
(b) What is the velocity after 2 s? After 4 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration at time t and after 4 s.
(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 5$.
(i) When is the particle speeding up? When is it slowing down?

SOLUTION
(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

(b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

(c) The particle is at rest when $v(t) = 0$, that is,

$$3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) = 0$$

and this is true when $t = 1$ or $t = 3$. Thus the particle is at rest after 1 s and after 3 s.
(d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$). Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$. It moves backward (in the negative direction) when $1 < t < 3$.

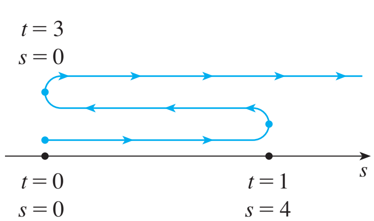


FIGURE 2

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the s-axis).

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is $4 + 4 + 20 = 28$ m.

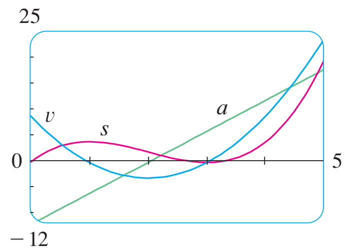


FIGURE 3

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$$

$$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$$

(h) Figure 3 shows the graphs of s, v, and a.

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.) From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$. The particle slows down when v and a have opposite signs, that is, when $0 \leqslant t < 1$ and when $2 < t < 3$. Figure 4 summarizes the motion of the particle.

TEC In Module 2.7 you can see an animation of Figure 4 with an expression for s that you can choose yourself.

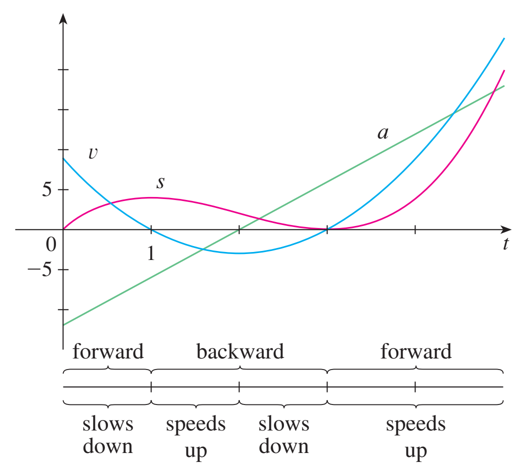


FIGURE 4

■

EXAMPLE 2 If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.

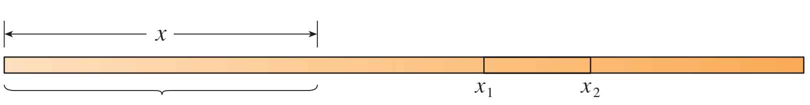


FIGURE 5

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals. The **linear density** ρ at x_1 is the limit of these average

densities as $\Delta x \to 0$; that is, the linear density is the rate of change of mass with respect to length. Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \leqslant x \leqslant 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx 0.48 \text{ kg/m}$$

while the density right at $x = 1$ is

$$\rho = \frac{dm}{dx}\bigg|_{x=1} = \frac{1}{2\sqrt{x}}\bigg|_{x=1} = 0.50 \text{ kg/m}$$ ■

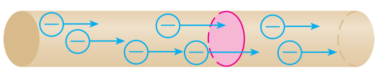


FIGURE 6

EXAMPLE 3 A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red. If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current** I at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes). ■

Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

■ Chemistry

EXAMPLE 4 A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the "equation"

$$2H_2 + O_2 \to 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water. Let's consider the reaction

$$A + B \to C$$

where A and B are the reactants and C is the product. The **concentration** of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by [A]. The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t). The average rate of reaction of the product C over a time interval

$t_1 \leqslant t \leqslant t_2$ is

$$\frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{[\mathrm{C}](t_2) - [\mathrm{C}](t_1)}{t_2 - t_1}$$

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{d[\mathrm{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[\mathrm{C}]/dt$ will be positive, and so the rate of reaction of C is positive. The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[\mathrm{A}]/dt$ and $d[\mathrm{B}]/dt$. Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = -\frac{d[\mathrm{A}]}{dt} = -\frac{d[\mathrm{B}]}{dt}$$

More generally, it turns out that for a reaction of the form

$$a\mathrm{A} + b\mathrm{B} \to c\mathrm{C} + d\mathrm{D}$$

we have

$$-\frac{1}{a}\frac{d[\mathrm{A}]}{dt} = -\frac{1}{b}\frac{d[\mathrm{B}]}{dt} = \frac{1}{c}\frac{d[\mathrm{C}]}{dt} = \frac{1}{d}\frac{d[\mathrm{D}]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction (see Exercise 24). ■

EXAMPLE 5 One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

The rate of change of V with respect to P when $P = 50$ kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50}$$

$$= -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}$$

The compressibility at that pressure is

$$\beta = -\frac{1}{V}\frac{dV}{dP}\bigg|_{P=50} = \frac{0.00212}{\frac{5.3}{50}} = 0.02\ (\text{m}^3/\text{kPa})/\text{m}^3$$ ■

Biology

EXAMPLE 6 Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leqslant t \leqslant t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.

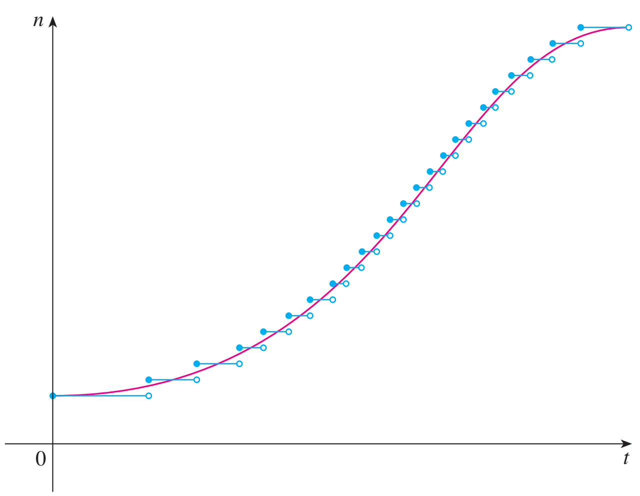


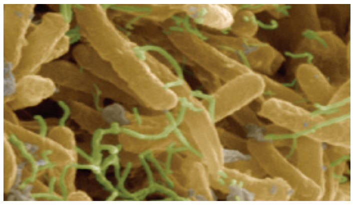
FIGURE 7
A smooth curve approximating a growth function

To be more specific, consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

$$f(3) = 2f(2) = 2^3 n_0$$



E. coli bacteria are about 2 micrometers (μm) long and 0.75 μm wide. The image was produced with a scanning electron microscope.

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

This is an example of an exponential function. In Chapter 6 we will discuss exponential functions in general; at that time we will be able to compute their derivatives and thereby determine the rate of growth of the bacteria population. ■

EXAMPLE 7 When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in Figure 8.

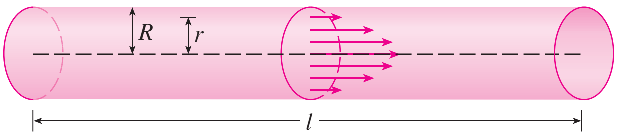


FIGURE 8
Blood flow in an artery

Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube. The velocity decreases as the distance r from the axis increases, until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

$$\boxed{1} \qquad v = \frac{P}{4\eta l}(R^2 - r^2)$$

For more detailed information, see W. Nichols and M. O'Rourke (eds.), *McDonald's Blood Flow in Arteries: Theoretical, Experimental, and Clinical Principles*, 5th ed. (New York, 2005).

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r) = -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$
$$\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$$

At $r = 0.002$ cm the blood is flowing at a speed of

$$\begin{aligned} v(0.002) &\approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6}) \\ &= 1.11 \text{ cm/s} \end{aligned}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74 \text{ (cm/s)/cm}$$

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm. The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of $r = 20$ μm. The fact that $dv/dr = -74$ (μm/s)/μm means that, when $r = 20$ μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center. ■

Economics

EXAMPLE 8 Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10{,}000 + 5(501) + 0.01(501)^2] \\ &\qquad - [10{,}000 + 5(500) + 0.01(500)^2] \\ &= \$15.01 \end{aligned}$$

Notice that $C'(500) \approx C(501) - C(500)$. ■

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions. These will be considered in Chapter 3 after we have developed techniques for finding the maximum and minimum values of functions.

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 6.5.19).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, dP/dt.

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t, then the derivative dp/dt represents the rate of spread of the rumor (see Exercise 6.2.65).

A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them."

2.7 EXERCISES

1–4 A particle moves according to a law of motion $s = f(t)$, $t \geqslant 0$, where t is measured in seconds and s in feet.

(a) Find the velocity at time t.
(b) What is the velocity after 1 second?
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 6 seconds.
(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.
(g) Find the acceleration at time t and after 1 second.
(h) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 6$.
(i) When is the particle speeding up? When is it slowing down?

1. $f(t) = t^3 - 9t^2 + 24t$

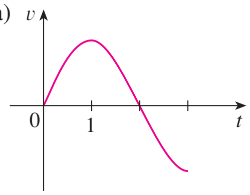
2. $f(t) = 0.01t^4 - 0.04t^3$

3. $f(t) = \sin(\pi t/2)$

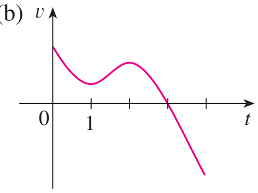
4. $f(t) = \dfrac{9t}{t^2 + 9}$

5. Graphs of the *velocity* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

(a)

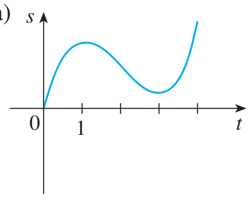


(b)

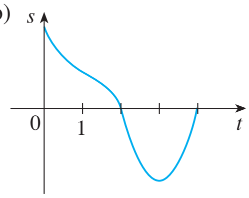


6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

(a)



(b)



7. The height (in meters) of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds.
(a) Find the velocity after 2 s and after 4 s.
(b) When does the projectile reach its maximum height?
(c) What is the maximum height?
(d) When does it hit the ground?
(e) With what velocity does it hit the ground?

8. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.
(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

9. If a rock is thrown vertically upward from the surface of Mars with velocity 15 m/s, its height after t seconds is $h = 15t - 1.86t^2$.
(a) What is the velocity of the rock after 2 s?
(b) What is the velocity of the rock when its height is 25 m on its way up? On its way down?

10. A particle moves with position function

$$s = t^4 - 4t^3 - 20t^2 + 20t \qquad t \geqslant 0$$

(a) At what time does the particle have a velocity of 20 m/s?
(b) At what time is the acceleration 0? What is the significance of this value of t?

11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.
(b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?

12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.
(b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).

13. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1
(b) Find the instantaneous rate of change when $r = 2$.
(c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
(i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
(b) Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu\text{m}$.
(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000\left(1 - \tfrac{1}{40}t\right)^2 \qquad 0 \leqslant t \leqslant 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. [See Example 3. The unit of current is an ampere (1 A = 1 C/s).] At what time is the current lowest?

20. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.
(a) Find dF/dr and explain its meaning. What does the minus sign indicate?
(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

21. The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

22. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is a little more than 12 hours and on June 30, 2009, high tide occurred at 6:45 AM. This helps explain the following model for the water depth D (in meters) as a function of the time t (in hours after midnight) on that day:

$$D(t) = 7 + 5\cos[0.503(t - 6.75)]$$

How fast was the tide rising (or falling) at the following times?
(a) 3:00 AM (b) 6:00 AM
(c) 9:00 AM (d) Noon

23. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.
(a) Find the rate of change of volume with respect to pressure.
(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
(c) Prove that the isothermal compressibility (see Example 5) is given by $\beta = 1/P$.

24. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then

$$[C] = a^2kt/(akt + 1)$$

where k is a constant.
(a) Find the rate of reaction at time t.
(b) Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

25. The table gives the population of the world $P(t)$, in millions, where t is measured in years and $t = 0$ corresponds to the year 1900.

t	Population (millions)	t	Population (millions)
0	1650	60	3040
10	1750	70	3710
20	1860	80	4450
30	2070	90	5280
40	2300	100	6080
50	2560	110	6870

(a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
(b) Use a graphing device to find a cubic function (a third-degree polynomial) that models the data.

(c) Use your model in part (b) to find a model for the rate of population growth.
(d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).
(e) Estimate the rate of growth in 1985.

26. The table shows how the average age of first marriage of Japanese women has varied since 1950.

t	$A(t)$	t	$A(t)$
1950	23.0	1985	25.5
1955	23.8	1990	25.9
1960	24.4	1995	26.3
1965	24.5	2000	27.0
1970	24.2	2005	28.0
1975	24.7	2010	28.8
1980	25.2		

(a) Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
(b) Use part (a) to find a model for $A'(t)$.
(c) Estimate the rate of change of marriage age for women in 1990.
(d) Graph the data points and the models for A and A'.

27. Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm^2, and viscosity $\eta = 0.027$.
(a) Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.
(b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.
(c) Where is the velocity the greatest? Where is the velocity changing most?

28. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3rd ed. (Pacific Grove, CA: Brooks/Cole, 2002).]
(a) Find the rate of change of the frequency with respect to
  (i) the length (when T and ρ are constant),
  (ii) the tension (when L and ρ are constant), and
  (iii) the linear density (when L and T are constant).
(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
  (i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
  (ii) when the tension is increased by turning a tuning peg,
  (iii) when the linear density is increased by switching to another string.

29. Suppose that the cost (in dollars) for a company to produce x pairs of a new line of jeans is

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3$$

(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning. What does it predict?
(c) Compare $C'(100)$ with the cost of manufacturing the 101st pair of jeans.

30. The cost function for a certain commodity is

$$C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3$$

(a) Find and interpret $C'(100)$.
(b) Compare $C'(100)$ with the cost of producing the 101st item.

31. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

(a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

32. If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.
(a) Find the sensitivity.
(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

33. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres, atm), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ moles.

34. Invasive species often display a wave of advance as they colonize new areas. Mathematical models based on ran-

dom dispersal and reproduction have demonstrated that the speed with which such waves move is given by the function $f(r) = 2\sqrt{Dr}$, where r is the reproductive rate of individuals and D is a parameter quantifying dispersal. Calculate the derivative of the wave speed with respect to the reproductive rate r and explain its meaning.

35. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

(a) What values of dC/dt and dW/dt correspond to stable populations?

(b) How would the statement "The caribou go extinct" be represented mathematically?

(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

36. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

(a) What value of dP/dt corresponds to a stable population?

(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

(c) What happens if β is raised to 5%?

2.8 Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

PS According to the Principles of Problem Solving discussed on page 98, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

SOLUTION We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown:*

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation:*

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius

is dr/dt. We can therefore restate the given and the unknown as follows:

$$\textit{Given:} \qquad \frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

$$\textit{Unknown:} \qquad \frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

PS The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

Notice that, although dV/dt is constant, dr/dt is *not* constant.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s. ■

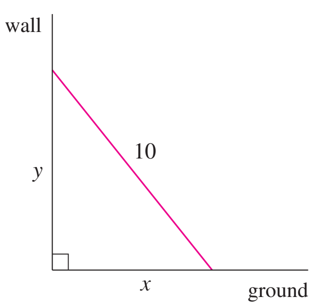


FIGURE 1

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

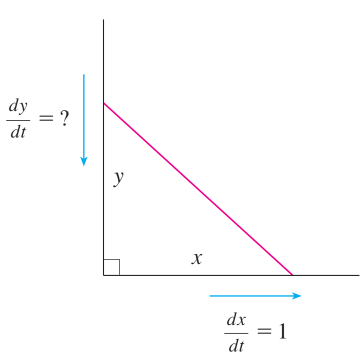


FIGURE 2

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s. ■

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

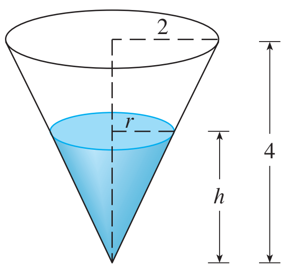


FIGURE 3

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2$ m^3/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min. ■

PS Look back: What have we learned from Examples 1–3 that will help us solve future problems?

Problem Solving Strategy It is useful to recall some of the problem-solving principles from page 98 and adapt them to related rates in light of our experience in Examples 1–3:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 3 we dealt with general values of h until we finally substituted $h = 3$ at the last stage. (If we had put $h = 3$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).

6. Use the Chain Rule to differentiate both sides of the equation with respect to t.

7. Substitute the given information into the resulting equation and solve for the unknown rate.

The following examples are further illustrations of the strategy.

EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

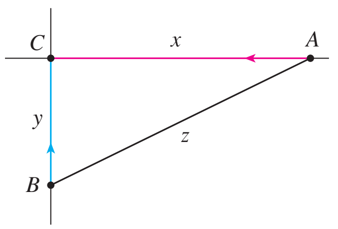


FIGURE 4

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that $dx/dt = -50$ mi/h and $dy/dt = -60$ mi/h. (The derivatives are negative because x and y are decreasing as t increases.) We are asked to find dz/dt. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{0.5}[0.3(-50) + 0.4(-60)]\\ &= -78 \text{ mi/h}\end{aligned}$$

The cars are approaching each other at a rate of 78 mi/h. ■

EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

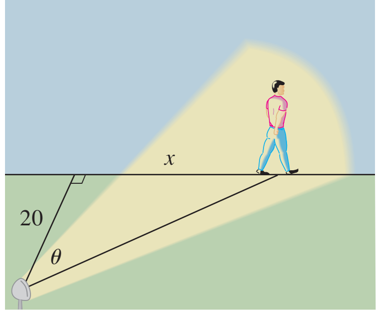


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20 \sec^2\theta \frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20} \cos^2\theta \frac{dx}{dt}$$

$$= \frac{1}{20} \cos^2\theta\,(4) = \frac{1}{5} \cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

2.8 EXERCISES

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

5. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

7. The radius of a spherical ball is increasing at a rate of 2 cm/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

8. The area of a triangle with sides of lengths a and b and contained angle θ is

$$A = \tfrac{1}{2}ab \sin\theta$$

(a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?
(b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?
(c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm, and $\theta = \pi/3$?

9. Suppose $y = \sqrt{2x + 1}$, where x and y are functions of t.
(a) If $dx/dt = 3$, find dy/dt when $x = 4$.
(b) If $dy/dt = 5$, find dx/dt when $x = 12$.

10. Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t.
(a) If $dy/dt = \frac{1}{3}$, find dx/dt when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.
(b) If $dx/dt = 3$, find dy/dt when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

11. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

12. A particle is moving along a hyperbola $xy = 8$. As it reaches the point (4, 2), the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

13–16
(a) What quantities are given in the problem?
(b) What is the unknown?
(c) Draw a picture of the situation for any time t.
(d) Write an equation that relates the quantities.
(e) Finish solving the problem.

13. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

15. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

16. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

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17. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

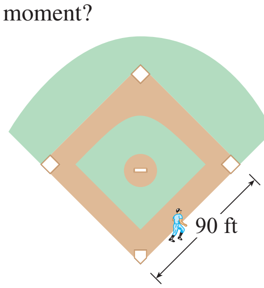
18. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

19. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after the woman starts walking?

20. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.

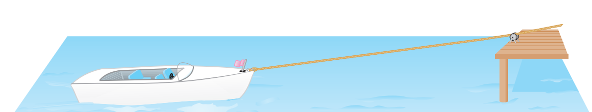
(a) At what rate is his distance from second base decreasing when he is halfway to first base?

(b) At what rate is his distance from third base increasing at the same moment?



21. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

22. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



23. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

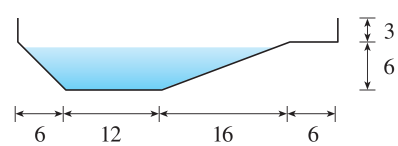
24. A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

25. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

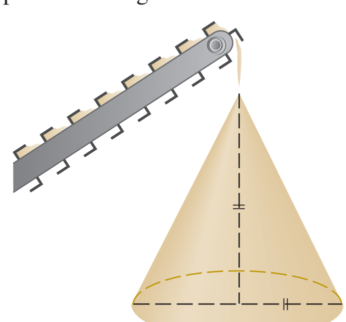
26. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?

27. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

28. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.8 ft^3/min, how fast is the water level rising when the depth at the deepest point is 5 ft?



29. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



30. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

31. The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long?

32. How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 6 ft from the wall?

33. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?

34. According to the model we used to solve Example 2, what happens as the top of the ladder approaches the ground? Is the model appropriate for small values of y?

35. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

36. A faucet is filling a hemispherical basin of diameter 60 cm with water at a rate of 2 L/min. Find the rate at which the water is rising in the basin when it is half full. [Use the following facts: 1 L is 1000 cm^3. The volume of the portion of a sphere with radius r from the bottom to a height h is $V = \pi\left(rh^2 - \frac{1}{3}h^3\right)$, as we will show in Chapter 5.]

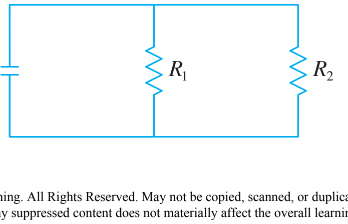
37. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

38. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

39. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R, measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

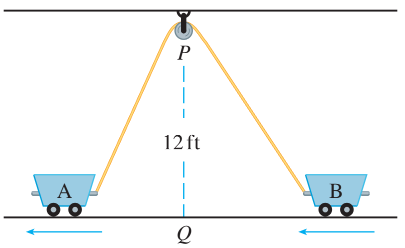
If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$?



40. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

41. Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°?

42. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



43. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

44. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

45. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ radians per minute. How fast is the plane traveling at that time?

46. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

47. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs

at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?

48. Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

49. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

50. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

2.9 Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2.1.2.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

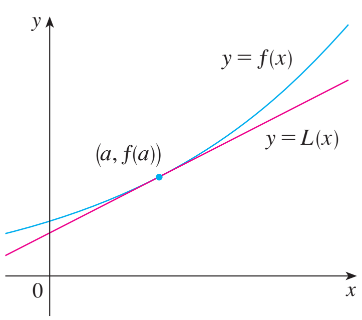


FIGURE 1

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$\boxed{1} \qquad f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$\boxed{2} \qquad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \tfrac{1}{4}$. Putting these values into Equation 2, we see

that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \tfrac{7}{4} + \tfrac{0.98}{4} = 1.995 \qquad \text{and} \qquad \sqrt{4.05} \approx \tfrac{7}{4} + \tfrac{1.05}{4} = 2.0125$$

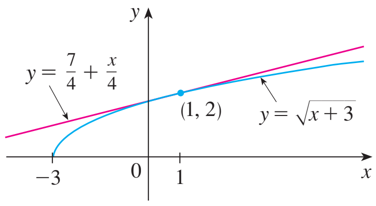


FIGURE 2

The linear approximation is illustrated in Figure 2. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near l. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval.* ■

In the following table we compare the estimates from the linear approximation in Example 1 with the true values. Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when x is close to 1 but the accuracy of the approximation deteriorates when x is farther away from 1.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176. . .
$\sqrt{3.98}$	0.98	1.995	1.99499373. . .
$\sqrt{4}$	1	2	2.00000000. . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117. . .
$\sqrt{4.1}$	1.1	2.025	2.02484567. . .
$\sqrt{5}$	2	2.25	2.23606797. . .
$\sqrt{6}$	3	2.5	2.44948974. . .

How good is the approximation that we obtained in Example 1? The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

EXAMPLE 2 For what values of x is the linear approximation

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x + 3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Equivalently, we could write

$$\sqrt{x + 3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x + 3} + 0.5$$

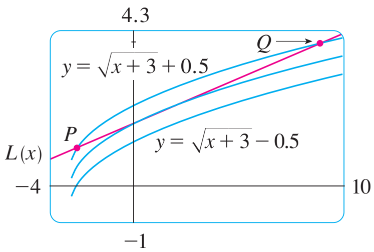


FIGURE 3

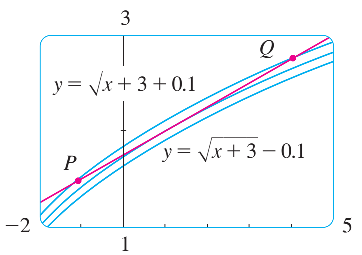


FIGURE 4

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x + 3}$ upward and downward by an amount 0.5. Figure 3 shows the tangent line $y = (7 + x)/4$ intersecting the upper curve $y = \sqrt{x + 3} + 0.5$ at P and Q. Zooming in and using the cursor, we estimate that the x-coordinate of P is about -2.66 and the x-coordinate of Q is about 8.66. Thus we see from the graph that the approximation

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded -2.66 up and 8.66 down to be safe.)

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$. ■

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation. For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large. [See, for example, *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), p. 431.] You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

(see Exercise 40). So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays*. In paraxial (or Gaussian) optics, both $\sin \theta$ and $\cos \theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin \theta \approx \theta \qquad \text{and} \qquad \cos \theta \approx 1$$

are used because θ is close to 0. The results of calculations made with these approximations became the basic theoretical tool used to design lenses. [See *Optics,* 4th ed., by Eugene Hecht (San Francisco, 2002), p. 154.]

In Section 11.11 we will present several other applications of the idea of linear approximations to physics and engineering.

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials*. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

3 $$dy = f'(x)\, dx$$

If $dx \neq 0$, we can divide both sides of Equation 3 by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

We have seen similar equations before, but now the left side can genuinely be interpreted as a ratio of differentials.

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

The geometric meaning of differentials is shown in Figure 5. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

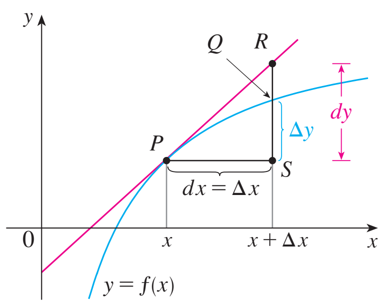


FIGURE 5

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\,dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization) when x changes by an amount dx, whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount $dx = \Delta x$.

EXAMPLE 3 Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

SOLUTION

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

Figure 6 shows the function in Example 3 and a comparison of dy and Δy when $a = 2$. The viewing rectangle is [1.8, 2.5] by [6, 18].

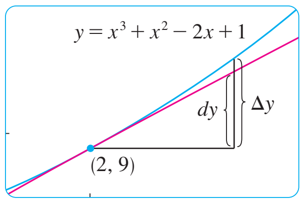


FIGURE 6

In general,
$$dy = f'(x)\,dx = (3x^2 + 2x - 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

(b)
$$f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701$$

$$\Delta y = f(2.01) - f(2) = 0.140701$$

When $dx = \Delta x = 0.01$,

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$

■

Notice that the approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 3. Notice also that dy was easier to compute than Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

In the notation of differentials, the linear approximation (1) can be written as

$$f(a + dx) \approx f(a) + dy$$

For instance, for the function $f(x) = \sqrt{x + 3}$ in Example 1, we have

$$dy = f'(x)\,dx = \frac{dx}{2\sqrt{x + 3}}$$

If $a = 1$ and $dx = \Delta x = 0.05$, then

$$dy = \frac{0.05}{2\sqrt{1 + 3}} = 0.0125$$

and $$\sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.0125$$

just as we found in Example 1.

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2\, dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3. ■

NOTE Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = 3\,\frac{dr}{r}$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.

2.9 EXERCISES

1–4 Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^3 - x^2 + 3, \quad a = -2$

2. $f(x) = \sin x, \quad a = \pi/6$

3. $f(x) = \sqrt{x}, \quad a = 4$

4. $f(x) = 2/\sqrt{x^2 - 5}, \quad a = 3$

5. Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

6. Find the linear approximation of the function $g(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\sqrt[4]{1 + 2x} \approx 1 + \frac{1}{2}x$

8. $(1 + x)^{-3} \approx 1 - 3x$

9. $1/(1 + 2x)^4 \approx 1 - 8x$

10. $\tan x \approx x$

11–14 Find the differential dy of each function.

11. (a) $y = (x^2 - 3)^{-2}$ (b) $y = \sqrt{1 - t^4}$

12. (a) $y = \dfrac{1 + 2u}{1 + 3u}$ (b) $y = \theta^2 \sin 2\theta$

13. (a) $y = \tan \sqrt{t}$ (b) $y = \dfrac{1 - v^2}{1 + v^2}$

14. (a) $y = \sqrt{t - \cos t}$ (b) $y = \dfrac{1}{x} \sin x$

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

15. $y = \tan x$, $x = \pi/4$, $dx = -0.1$

16. $y = \cos \pi x$, $x = \frac{1}{3}$, $dx = -0.02$

17. $y = \sqrt{3 + x^2}$, $x = 1$, $dx = -0.1$

18. $y = \dfrac{x + 1}{x - 1}$, $x = 2$, $dx = 0.05$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx, dy, and Δy.

19. $y = x^2 - 4x$, $x = 3$, $\Delta x = 0.5$

20. $y = x - x^3$, $x = 0$, $\Delta x = -0.3$

21. $y = \sqrt{x - 2}$, $x = 3$, $\Delta x = 0.8$

22. $y = x^3$, $x = 1$, $\Delta x = 0.5$

23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(1.999)^4$ **24.** $1/4.002$

25. $\sqrt[3]{1001}$ **26.** $\sqrt{100.5}$

27. $\tan 2°$ **28.** $\cos 29°$

29–30 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

29. $\sec 0.08 \approx 1$ **30.** $\sqrt{4.02} \approx 2.005$

31. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

32. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
(a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error? What is the percentage error?

33. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.
(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

34. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

35. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r, and thickness Δr.
(b) What is the error involved in using the formula from part (a)?

36. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of ±1°.
(a) Use differentials to estimate the error in computing the length of the hypotenuse.
(b) What is the percentage error?

37. If a current I passes through a resistor with resistance R, Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

38. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

39. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).
(a) $dc = 0$ (b) $d(cu) = c\,du$
(c) $d(u + v) = du + dv$ (d) $d(uv) = u\,dv + v\,du$
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$ (f) $d(x^n) = nx^{n-1}\,dx$

40. On page 431 of *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L, the author obtains the equation $a_T = -g \sin \theta$ for the tangential acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in

radians is very nearly the value of sin θ; they differ by less than 2% out to about 20°."

(a) Verify the linear approximation at 0 for the sine function:

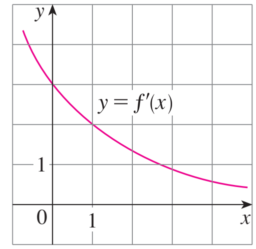
$$\sin x \approx x$$

(b) Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

41. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.

(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small? Explain.



42. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x.

(a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

(b) Are your estimates in part (a) too large or too small? Explain.

LABORATORY PROJECT TAYLOR POLYNOMIALS

The tangent line approximation $L(x)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same rate of change (derivative) at a. For a better approximation than a linear one, let's try a second-degree (quadratic) approximation $P(x)$. In other words, we approximate a curve by a parabola instead of by a straight line. To make sure that the approximation is a good one, we stipulate the following:

(i) $P(a) = f(a)$ (P and f should have the same value at a.)

(ii) $P'(a) = f'(a)$ (P and f should have the same rate of change at a.)

(iii) $P''(a) = f''(a)$ (The slopes of P and f should change at the same rate at a.)

1. Find the quadratic approximation $P(x) = A + Bx + Cx^2$ to the function $f(x) = \cos x$ that satisfies conditions (i), (ii), and (iii) with $a = 0$. Graph P, f, and the linear approximation $L(x) = 1$ on a common screen. Comment on how well the functions P and L approximate f.

2. Determine the values of x for which the quadratic approximation $f(x) \approx P(x)$ in Problem 1 is accurate to within 0.1. [*Hint:* Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]

3. To approximate a function f by a quadratic function P near a number a, it is best to write P in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

4. Find the quadratic approximation to $f(x) = \sqrt{x + 3}$ near $a = 1$. Graph f, the quadratic approximation, and the linear approximation from Example 2.9.2 on a common screen. What do you conclude?

5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let's try to find better approximations with higher-degree polynomials. We look for an nth-degree polynomial

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots + c_n(x - a)^n$$

such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the ***n*th-degree Taylor polynomial of *f* centered at *a***.

6. Find the 8th-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f.

2 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

2. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.
(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
(b) The instantaneous rate of change of y with respect to x at $x = x_1$.

4. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

5. (a) What does it mean for f to be differentiable at a?
(b) What is the relation between the differentiability and continuity of a function?
(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

7. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

8. State each differentiation rule both in symbols and in words.
(a) The Power Rule (b) The Constant Multiple Rule
(c) The Sum Rule (d) The Difference Rule
(e) The Product Rule (f) The Quotient Rule
(g) The Chain Rule

9. State the derivative of each function.
(a) $y = x^n$ (b) $y = \sin x$ (c) $y = \cos x$
(d) $y = \tan x$ (e) $y = \csc x$ (f) $y = \sec x$
(g) $y = \cot x$

10. Explain how implicit differentiation works.

11. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

12. (a) Write an expression for the linearization of f at a.
(b) If $y = f(x)$, write an expression for the differential dy.
(c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is continuous at a, then f is differentiable at a.

2. If f and g are differentiable, then
$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

3. If f and g are differentiable, then
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

4. If f and g are differentiable, then
$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

5. If f is differentiable, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

6. If f is differentiable, then $\dfrac{d}{dx}f\left(\sqrt{x}\right) = \dfrac{f'(x)}{2\sqrt{x}}$.

7. $\dfrac{d}{dx}\left|x^2 + x\right| = \left|2x + 1\right|$

8. If $f'(r)$ exists, then $\lim_{x\to r} f(x) = f(r)$.

9. If $g(x) = x^5$, then $\displaystyle\lim_{x\to 2}\frac{g(x) - g(2)}{x - 2} = 80$

10. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

11. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

12. $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$

13. The derivative of a polynomial is a polynomial.

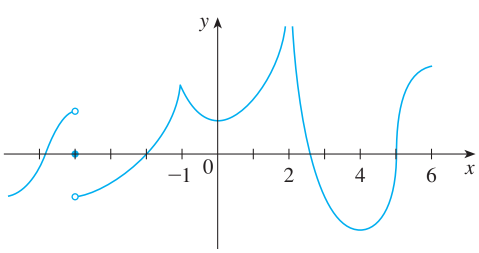
14. The derivative of a rational function is a rational function.

15. If $f(x) = (x^6 - x^4)^5$, then $f^{(31)}(x) = 0$.

EXERCISES

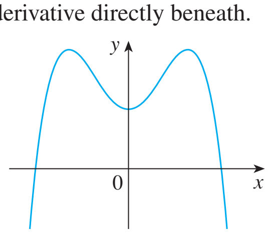
1. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.
   (a) Find the average velocity over each time period.
   (i) $[1, 3]$ (ii) $[1, 2]$ (iii) $[1, 1.5]$ (iv) $[1, 1.1]$
   (b) Find the instantaneous velocity when $t = 1$.

2. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.

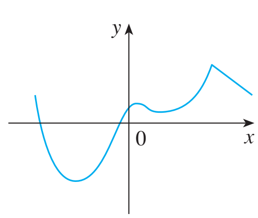


3–4 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

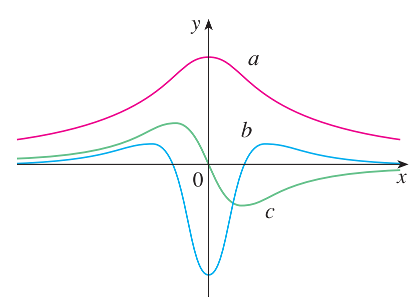
3.



4.



5. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



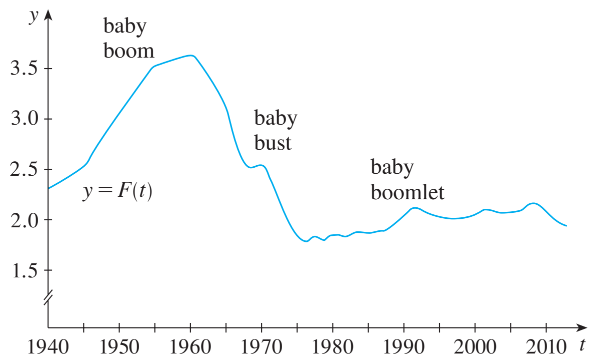
6. Find a function f and a number a such that
$$\lim_{h\to 0}\frac{(2 + h)^6 - 64}{h} = f'(a)$$

7. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
   (a) What is the meaning of the derivative $f'(r)$? What are its units?
   (b) What does the statement $f'(10) = 1200$ mean?
   (c) Is $f'(r)$ always positive or does it change sign?

8. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph

of the total fertility rate in the United States shows the fluctuations from 1940 to 2010.

(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
(b) What are the meanings of these derivatives?
(c) Can you suggest reasons for the values of these derivatives?



9. Let $P(t)$ be the percentage of Americans under the age of 18 at time t. The table gives values of this function in census years from 1950 to 2010.

t	$P(t)$	t	$P(t)$
1950	31.1	1990	25.7
1960	35.7	2000	25.7
1970	34.0	2010	24.0
1980	28.0		

(a) What is the meaning of $P'(t)$? What are its units?
(b) Construct a table of estimated values for $P'(t)$.
(c) Graph P and P'.
(d) How would it be possible to get more accurate values for $P'(t)$?

10–11 Find $f'(x)$ from first principles, that is, directly from the definition of a derivative.

10. $f(x) = \dfrac{4 - x}{3 + x}$ **11.** $f(x) = x^3 + 5x + 4$

12. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

13–40 Calculate y'.

13. $y = (x^2 + x^3)^4$ **14.** $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$

15. $y = \dfrac{x^2 - x + 2}{\sqrt{x}}$ **16.** $y = \dfrac{\tan x}{1 + \cos x}$

17. $y = x^2 \sin \pi x$ **18.** $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$

19. $y = \dfrac{t^4 - 1}{t^4 + 1}$ **20.** $y = \sin(\cos x)$

21. $y = \tan \sqrt{1 - x}$ **22.** $y = \dfrac{1}{\sin(x - \sin x)}$

23. $xy^4 + x^2y = x + 3y$ **24.** $y = \sec(1 + x^2)$

25. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta}$ **26.** $x^2 \cos y + \sin 2y = xy$

27. $y = (1 - x^{-1})^{-1}$ **28.** $y = 1/\sqrt[3]{x + \sqrt{x}}$

29. $\sin(xy) = x^2 - y$ **30.** $y = \sqrt{\sin \sqrt{x}}$

31. $y = \cot(3x^2 + 5)$ **32.** $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4}$

33. $y = \sqrt{x} \cos \sqrt{x}$ **34.** $y = \dfrac{\sin mx}{x}$

35. $y = \tan^2(\sin \theta)$ **36.** $x \tan y = y - 1$

37. $y = \sqrt[5]{x \tan x}$ **38.** $y = \dfrac{(x - 1)(x - 4)}{(x - 2)(x - 3)}$

39. $y = \sin\left(\tan \sqrt{1 + x^3}\right)$ **40.** $y = \sin^2\left(\cos \sqrt{\sin \pi x}\right)$

41. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

42. If $g(\theta) = \theta \sin \theta$, find $g''(\pi/6)$.

43. Find y'' if $x^6 + y^6 = 1$.

44. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

45–46 Find the limit.

45. $\displaystyle\lim_{x \to 0} \frac{\sec x}{1 - \sin x}$ **46.** $\displaystyle\lim_{t \to 0} \frac{t^3}{\tan^3 2t}$

47–48 Find an equation of the tangent to the curve at the given point.

47. $y = 4 \sin^2 x$, $(\pi/6, 1)$ **48.** $y = \dfrac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

49–50 Find equations of the tangent line and normal line to the curve at the given point.

49. $y = \sqrt{1 + 4 \sin x}$, $(0, 1)$

50. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

51. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.
(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.

(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.

(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

52. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f''.

(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

53. At what points on the curve $y = \sin x + \cos x$, $0 \leqslant x \leqslant 2\pi$, is the tangent line horizontal?

54. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

55. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

56. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

57. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

58. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.

(b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

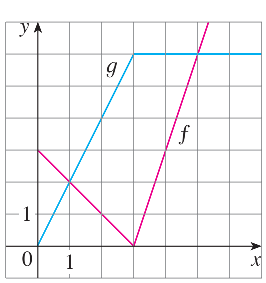
obtain the addition formula for the cosine function.

59. Suppose that

$$f(1) = 2 \quad f'(1) = 3 \quad f(2) = 1 \quad f'(2) = 2$$
$$g(1) = 3 \quad g'(1) = 1 \quad g(2) = 1 \quad g'(2) = 4$$

(a) If $S(x) = f(x) + g(x)$, find $S'(1)$.
(b) If $P(x) = f(x)g(x)$, find $P'(2)$.
(c) If $Q(x) = f(x)/g(x)$, find $Q'(1)$.
(d) If $C(x) = f(g(x))$, find $C'(2)$.

60. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



61–68 Find f' in terms of g'.

61. $f(x) = x^2 g(x)$

62. $f(x) = g(x^2)$

63. $f(x) = [g(x)]^2$

64. $f(x) = x^a g(x^b)$

65. $f(x) = g(g(x))$

66. $f(x) = \sin(g(x))$

67. $f(x) = g(\sin x)$

68. $f(x) = g\left(\tan \sqrt{x}\right)$

69–71 Find h' in terms of f' and g'.

69. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

70. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}}$

71. $h(x) = f(g(\sin 4x))$

72. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geqslant 0$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

73. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geqslant 0$.
(a) Find the velocity and acceleration functions.
(b) When is the particle moving upward and when is it moving downward?
(c) Find the distance that the particle travels in the time interval $0 \leqslant t \leqslant 3$.
(d) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 3$.
(e) When is the particle speeding up? When is it slowing down?

74. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

75. The mass of part of a wire is $x\left(1 + \sqrt{x}\right)$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

76. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

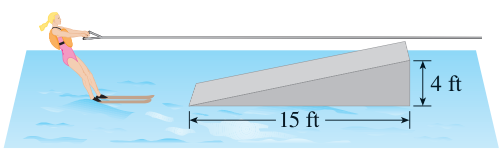
(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning.
(c) Compare $C'(100)$ with the cost of producing the 101st item.

77. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?

78. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

79. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

80. A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?



81. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?

82. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
(b) Illustrate part (a) by graphing f and the linear approximation.
(c) For what values of x is the linear approximation accurate to within 0.1?

83. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

84. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

85. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

86–88 Express the limit as a derivative and evaluate.

86. $\lim_{x \to 1} \frac{x^{17} - 1}{x - 1}$

87. $\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

88. $\lim_{\theta \to \pi/3} \frac{\cos\theta - 0.5}{\theta - \pi/3}$

89. Evaluate $\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

90. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

91. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

92. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Problems Plus

Before you look at the example, cover up the solution and try it yourself first.

EXAMPLE How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

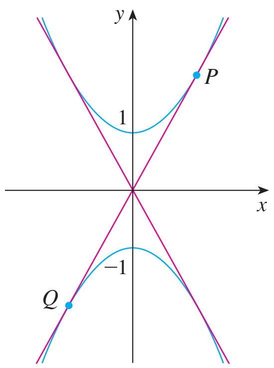


FIGURE 1

SOLUTION To gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x-axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let P be a point at which one of these tangents touches the upper parabola and let a be its x-coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a. However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y-coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P. We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. ■

PROBLEMS

1. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle. (See the figure.)

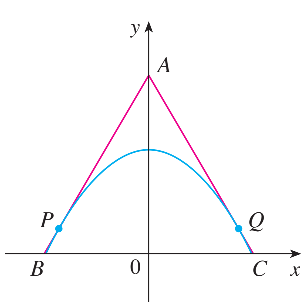


FIGURE FOR PROBLEM 1

2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.

3. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x-coordinates p and q must intersect at a point whose x-coordinate is halfway between p and q.

4. Show that

$$\frac{d}{dx}\left(\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x}\right) = -\cos 2x$$

5. If $f(x) = \lim_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, find the value of $f'(\pi/4)$.

6. Find the values of the constants a and b such that

$$\lim_{x \to 0} \frac{\sqrt[3]{ax + b} - 2}{x} = \frac{5}{12}$$

7. Prove that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

8. If f is differentiable at a, where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

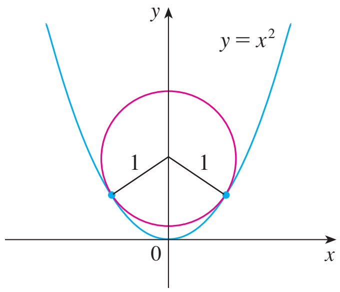


FIGURE FOR PROBLEM 9

9. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.

10. Find all values of c such that the parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles.

11. How many lines are tangent to both of the circles $x^2 + y^2 = 4$ and $x^2 + (y - 3)^2 = 1$? At what points do these tangent lines touch the circles?

12. If $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x}$, calculate $f^{(46)}(3)$. Express your answer using factorial notation: $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n - 1) \cdot n$.

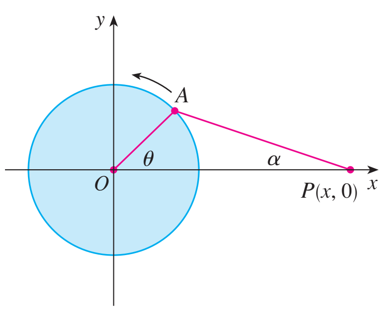


FIGURE FOR PROBLEM 13

13. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.
   (a) Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.
   (b) Express the distance $x = |OP|$ in terms of θ.
   (c) Find an expression for the velocity of the pin P in terms of θ.

14. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P. Another tangent line T is drawn at a point between P_1 and P_2; it intersects T_1 at Q_1 and T_2 at Q_2. Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

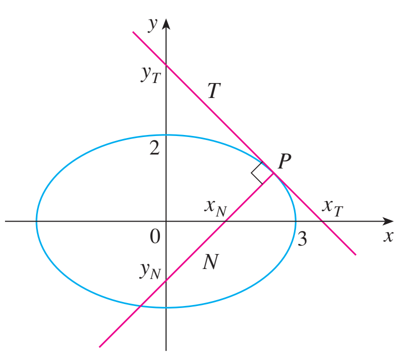


FIGURE FOR PROBLEM 15

15. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x- and y-intercepts of T and x_N and y_N be the intercepts of N. As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T, y_T, x_N, and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.

16. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin(3 + x)^2 - \sin 9}{x}$.

17. (a) Use the identity for $\tan(x - y)$ (see Equation 14b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle α, then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

   where m_1 and m_2 are the slopes of L_1 and L_2, respectively.
   (b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.
   (i) $y = x^2$ and $y = (x - 2)^2$
   (ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$

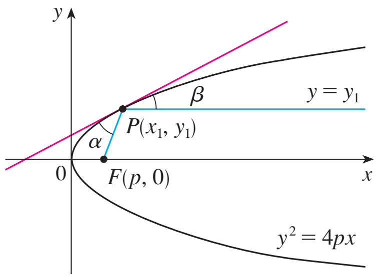


FIGURE FOR PROBLEM 18

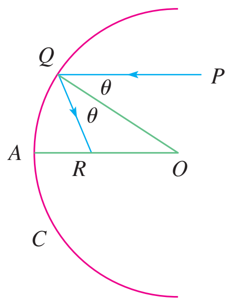


FIGURE FOR PROBLEM 19

18. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP, and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the x-axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)

19. Suppose that we replace the parabolic mirror of Problem 18 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O. A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

20. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

21. Evaluate $\displaystyle \lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$.

22. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.

23. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

24. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

25. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{5}$ intersects some of these circles.

26. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

27. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is $\pi r l$, where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of 2 cm^3/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

CAS **28.** (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?

(b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a, b, and c. Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.

3 Applications of Differentiation



When flying, some small birds—like the finch pictured here—alternate between flapping their wings and keeping them folded while gliding. In the project on page 271, we will investigate how frequently a bird should flap its wings in order to minimize the energy required.

WE HAVE ALREADY INVESTIGATED SOME of the applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. Here we learn how derivatives affect the shape of a graph of a function and, in particular, how they help us locate maximum and minimum values of functions. Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation. In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.

3.1 Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values.

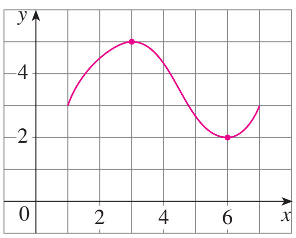


FIGURE 1

We see that the highest point on the graph of the function f shown in Figure 1 is the point $(3, 5)$. In other words, the largest value of f is $f(3) = 5$. Likewise, the smallest value is $f(6) = 2$. We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

> **1 Definition** Let c be a number in the domain D of a function f. Then $f(c)$ is the
>
> - **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
> - **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

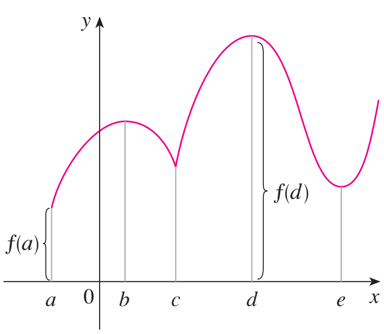


FIGURE 2
Abs min $f(a)$, abs max $f(d)$,
loc min $f(c), f(e)$, loc max $f(b), f(d)$

An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Figure 2 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point. In Figure 2, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of f. Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \leqslant f(x)$ for x near c [in the interval (b, d), for instance]. The function f also has a local minimum at e. In general, we have the following definition.

> **2 Definition** The number $f(c)$ is a
>
> - **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
> - **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

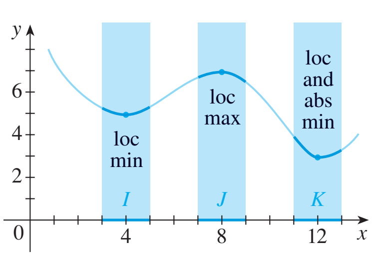


FIGURE 3

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c. For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I. It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance). In fact $f(12) = 3$ is both a local minimum and the absolute minimum. Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

EXAMPLE 1 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leqslant \cos x \leqslant 1$ for all x. (See Figure 4.) Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.

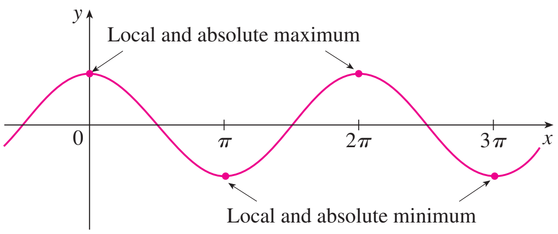


FIGURE 4
$y = \cos x$

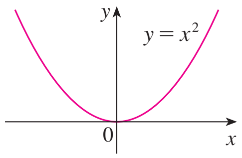


FIGURE 5
Mimimum value 0, no maximum

EXAMPLE 2 If $f(x) = x^2$, then $f(x) \geqslant f(0)$ because $x^2 \geqslant 0$ for all x. Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$. (See Figure 5.) However, there is no highest point on the parabola and so this function has no maximum value. ■

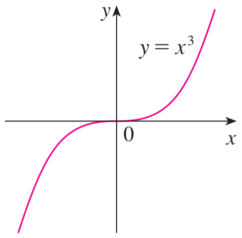


FIGURE 6
No mimimum, no maximum

EXAMPLE 3 From the graph of the function $f(x) = x^3$, shown in Figure 6, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either. ■

EXAMPLE 4 The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

is shown in Figure 7. You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$.

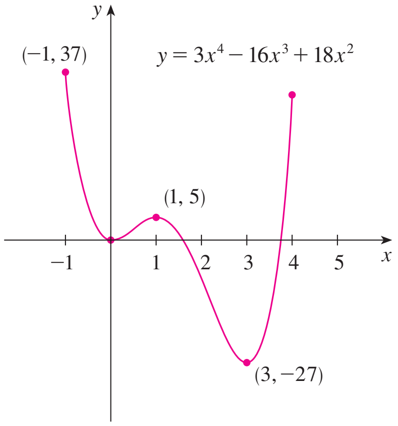


FIGURE 7

■

We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 8. Note that a function can attain an extreme value more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

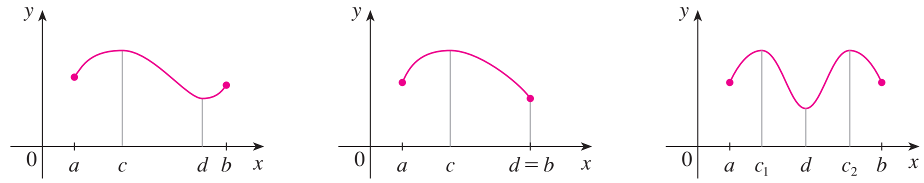


FIGURE 8
Functions continuous on a closed interval always attain extreme values.

Figures 9 and 10 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

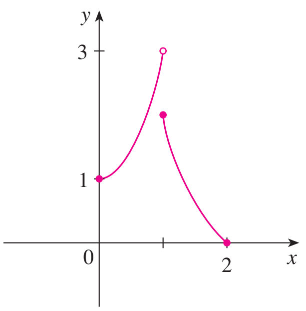


FIGURE 9
This function has minimum value $f(2) = 0$, but no maximum value.

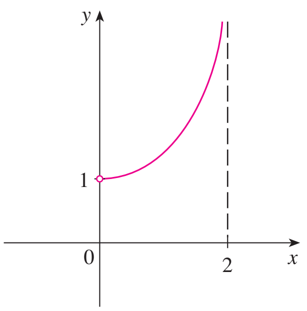


FIGURE 10
This continuous function g has no maximum or minimum.

The function f whose graph is shown in Figure 9 is defined on the closed interval $[0, 2]$ but has no maximum value. (Notice that the range of f is $[0, 3)$. The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem because f is not continuous. [Nonetheless, a discontinuous function *could* have maximum and minimum values. See Exercise 13(b).]

The function g shown in Figure 10 is continuous on the open interval $(0, 2)$ but has neither a maximum nor a minimum value. [The range of g is $(1, \infty)$. The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval $(0, 2)$ is not closed.

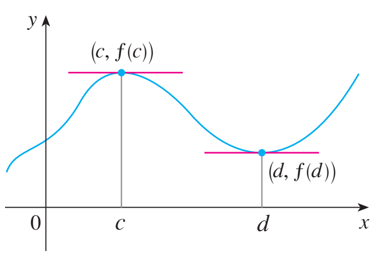


FIGURE 11

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. Notice in Figure 8 that the absolute maximum and minimum values that are *between* a and b occur at local maximum or minimum values, so we start by looking for local extreme values.

Figure 11 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. We know that the derivative is the slope of the

tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

Fermat

Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

PROOF Suppose, for the sake of definiteness, that f has a local maximum at c. Then, according to Definition 2, $f(c) \geqslant f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geqslant f(c + h)$$

and therefore

$$f(c + h) - f(c) \leqslant 0 \tag{5}$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c + h) - f(c)}{h} \leqslant 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 1.6.2), we get

$$\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \leqslant \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h}$$

and so we have shown that $f'(c) \leqslant 0$.

If $h < 0$, then the direction of the inequality (5) is reversed when we divide by h:

$$\frac{f(c + h) - f(c)}{h} \geqslant 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \geqslant 0$$

We have shown that $f'(c) \geqslant 0$ and also that $f'(c) \leqslant 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner, or we could use Exercise 70 to deduce it from the case we have just proved (see Exercise 71). ■

The following examples caution us against reading too much into Fermat's Theorem: We can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

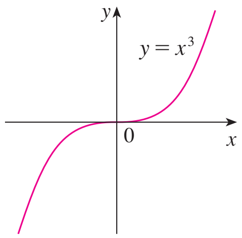


FIGURE 12
If $f(x) = x^3$, then $f'(0) = 0$, but f has no maximum or minimum.

EXAMPLE 5 If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in Figure 12. (Or observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0). Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there. ■

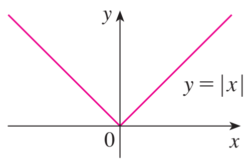


FIGURE 13
If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

EXAMPLE 6 The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, as was shown in Example 2.2.5, $f'(0)$ does not exist. (See Figure 13.) ■

WARNING Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$ there need not be a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.) Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

Figure 14 shows a graph of the function f in Example 7. It supports our answer because there is a horizontal tangent when $x = 1.5$ [where $f'(x) = 0$] and a vertical tangent when $x = 0$ [where $f'(x)$ is undefined].

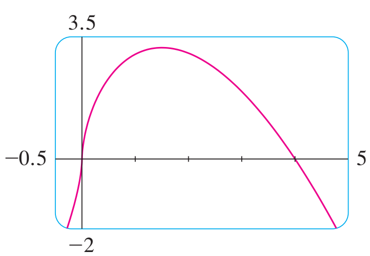


FIGURE 14

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. ■

In terms of critical numbers, Fermat's Theorem can be rephrased as follows (compare Definition 6 with Theorem 4):

7 If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number by (7)] or it occurs at an endpoint of the interval, as we see from the examples in Figure 8. Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 15. ■

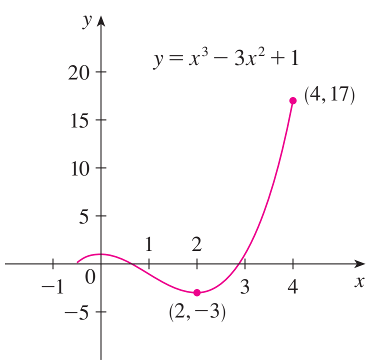


FIGURE 15

If you have a graphing calculator or a computer with graphing software, it is possible to estimate maximum and minimum values very easily. But, as the next example shows, calculus is needed to find the *exact* values.

EXAMPLE 9

(a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \leqslant x \leqslant 2\pi$.

(b) Use calculus to find the exact minimum and maximum values.

SOLUTION

(a) Figure 16 shows a graph of f in the viewing rectangle $[0, 2\pi]$ by $[-1, 8]$. By moving the cursor close to the maximum point, we see that the y-coordinates don't change very much in the vicinity of the maximum. The absolute maximum value is about 6.97 and it occurs when $x \approx 5.2$. Similarly, by moving the cursor close to the minimum point, we see that the absolute minimum value is about -0.68 and it occurs when $x \approx 1.0$. It is possible to get more accurate estimates by zooming in toward the

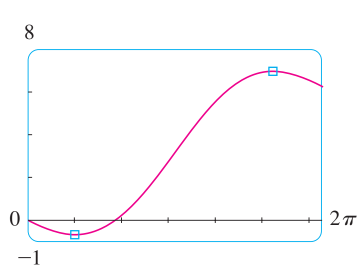


FIGURE 16

maximum and minimum points (or using a built-in maximum or minimum feature), but instead let's use calculus.

(b) The function $f(x) = x - 2\sin x$ is continuous on $[0, 2\pi]$. Since $f'(x) = 1 - 2\cos x$, we have $f'(x) = 0$ when $\cos x = \frac{1}{2}$ and this occurs when $x = \pi/3$ or $5\pi/3$. The values of f at these critical numbers are

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.684853$$

and
$$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$$

The values of f at the endpoints are

$$f(0) = 0 \qquad \text{and} \qquad f(2\pi) = 2\pi \approx 6.28$$

Comparing these four numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(\pi/3) = \pi/3 - \sqrt{3}$ and the absolute maximum value is $f(5\pi/3) = 5\pi/3 + \sqrt{3}$. The values from part (a) serve as a check on our work. ■


NASA

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ seconds, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$a(t) = v'(t) = \frac{d}{dt}(0.001302t^3 - 0.09029t^2 + 23.61t - 3.083)$$
$$= 0.003906t^2 - 0.18058t + 23.61$$

We now apply the Closed Interval Method to the continuous function a on the interval $0 \leqslant t \leqslant 126$. Its derivative is

$$a'(t) = 0.007812t - 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = 23.61 \qquad a(t_1) \approx 21.52 \qquad a(126) \approx 62.87$$

So the maximum acceleration is about 62.87 ft/s^2 and the minimum acceleration is about 21.52 ft/s^2. ■

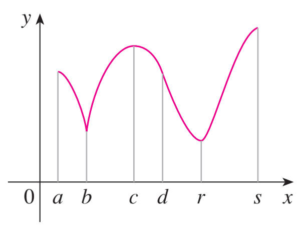
3.1 EXERCISES

1. Explain the difference between an absolute minimum and a local minimum.

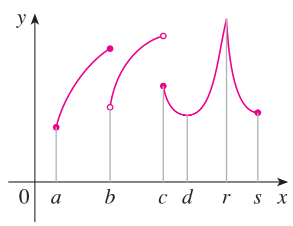
2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
(a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
(b) What steps would you take to find those maximum and minimum values?

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.

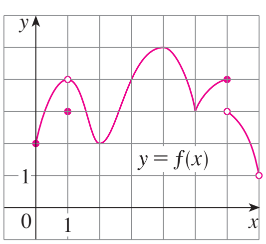


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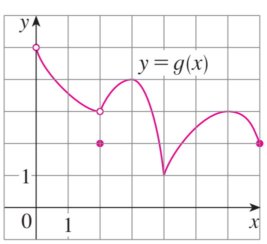


5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

5.



6.



7–10 Sketch the graph of a function f that is continuous on [1, 5] and has the given properties.

7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3

9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2

10. Absolute maximum at 2, absolute minimum at 5, 4 is a critical number but there is no local maximum or minimum there.

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = \frac{1}{2}(3x - 1), \quad x \leqslant 3$

16. $f(x) = 2 - \frac{1}{3}x, \quad x \geqslant -2$

17. $f(x) = 1/x, \quad x \geqslant 1$

18. $f(x) = 1/x, \quad 1 < x < 3$

19. $f(x) = \sin x, \quad 0 \leqslant x < \pi/2$

20. $f(x) = \sin x, \quad 0 < x \leqslant \pi/2$

21. $f(x) = \sin x, \quad -\pi/2 \leqslant x \leqslant \pi/2$

22. $f(t) = \cos t, \quad -3\pi/2 \leqslant t \leqslant 3\pi/2$

23. $f(x) = 1 + (x + 1)^2, \quad -2 \leqslant x < 5$

24. $f(x) = |x|$

25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = 1 - x^3$

27. $f(x) = \begin{cases} x^2 & \text{if } -1 \leqslant x \leqslant 0 \\ 2 - 3x & \text{if } 0 < x \leqslant 1 \end{cases}$

28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leqslant x < 1 \\ 4 - 2x & \text{if } 1 \leqslant x \leqslant 3 \end{cases}$

29–42 Find the critical numbers of the function.

29. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

30. $f(x) = x^3 + 6x^2 - 15x$

31. $f(x) = 2x^3 - 3x^2 - 36x$

32. $f(x) = 2x^3 + x^2 + 2x$

33. $g(t) = t^4 + t^3 + t^2 + 1$

34. $g(t) = |3t - 4|$

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

36. $h(p) = \dfrac{p - 1}{p^2 + 4}$

37. $h(t) = t^{3/4} - 2t^{1/4}$

38. $g(x) = \sqrt[3]{4 - x^2}$

39. $F(x) = x^{4/5}(x - 4)^2$

40. $g(\theta) = 4\theta - \tan\theta$

41. $f(\theta) = 2\cos\theta + \sin^2\theta$

42. $g(x) = \sqrt{1 - x^2}$

43–44 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

43. $f'(x) = 1 + \dfrac{210 \sin x}{x^2 - 6x + 10}$

44. $f'(x) = \dfrac{100 \cos^2 x}{10 + x^2} - 1$

45–56 Find the absolute maximum and absolute minimum values of f on the given interval.

45. $f(x) = 12 + 4x - x^2$, $[0, 5]$

46. $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

47. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

48. $f(x) = x^3 - 6x^2 + 5$, $[-3, 5]$

49. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, $[-2, 3]$

50. $f(t) = (t^2 - 4)^3$, $[-2, 3]$

51. $f(x) = x + \dfrac{1}{x}$, $[0.2, 4]$

52. $f(x) = \dfrac{x}{x^2 - x + 1}$, $[0, 3]$

53. $f(t) = t - \sqrt[3]{t}$, $[-1, 4]$

54. $f(t) = \dfrac{\sqrt{t}}{1 + t^2}$, $[0, 2]$

55. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$

56. $f(t) = t + \cot(t/2)$, $[\pi/4, 7\pi/4]$

57. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leqslant x \leqslant 1$.

58. Use a graph to estimate the critical numbers of $f(x) = |1 + 5x - x^3|$ correct to one decimal place.

59–62

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

59. $f(x) = x^5 - x^3 + 2$, $-1 \leqslant x \leqslant 1$

60. $f(x) = x^4 - 3x^3 + 3x^2 - x$, $0 \leqslant x \leqslant 2$

61. $f(x) = x\sqrt{x - x^2}$

62. $f(x) = x - 2\cos x$, $-2 \leqslant x \leqslant 0$

63. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

64. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

65. The water level, measured in feet above mean sea level, of Lake Lanier in Georgia, USA, during 2012 can be modeled by the function

$$L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1$$

where t is measured in months since January 1, 2012. Estimate when the water level was highest during 2012.

66. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

67. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \tfrac{1}{2}r_0 \leqslant r \leqslant r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0, r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

68. Show that 5 is a critical number of the function

$$g(x) = 2 + (x - 5)^3$$

but g does not have a local extreme value at 5.

69. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

70. If f has a local minimum value at c, show that the function $g(x) = -f(x)$ has a local maximum value at c.

71. Prove Fermat's Theorem for the case in which f has a local minimum at c.

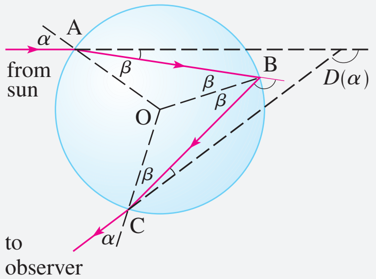
72. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.

(b) How many local extreme values can a cubic function have?

APPLIED PROJECT THE CALCULUS OF RAINBOWS

Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.

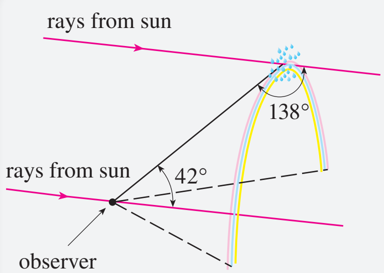


Formation of the primary rainbow

1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that $\sin \alpha = k \sin \beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The *angle of deviation* $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

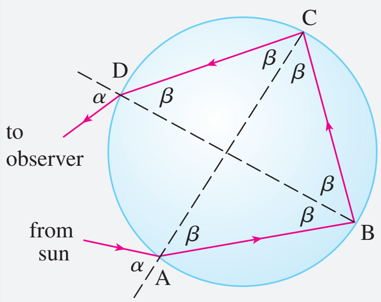
$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha \approx 59.4°$.



The significance of the minimum deviation is that when $\alpha \approx 59.4°$ we have $D'(\alpha) \approx 0$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure at the left shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. (This angle is called the *rainbow angle.*)

2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion.*) For red light the refractive index is $k \approx 1.3318$, whereas for violet light it is $k \approx 1.3435$. By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.



Formation of the secondary rainbow

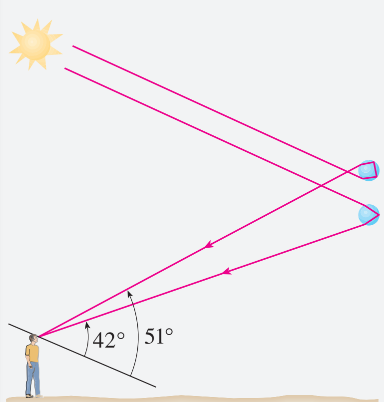
3. Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see the figure at the left). This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

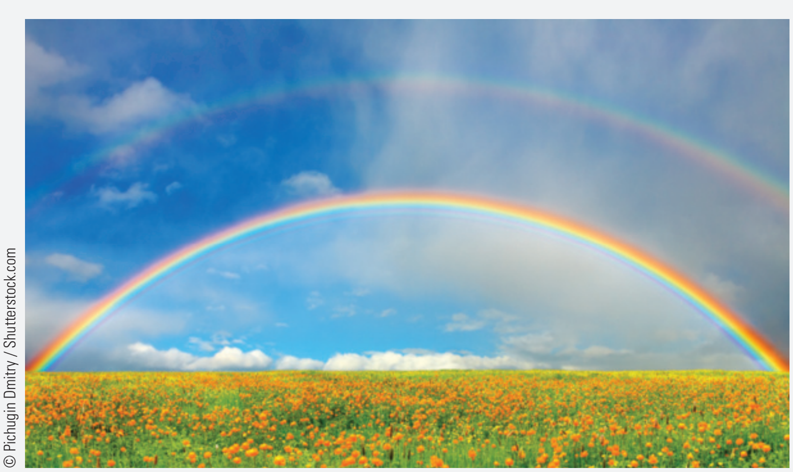
and $D(\alpha)$ has a minimum value when

$$\cos\alpha = \sqrt{\frac{k^2 - 1}{8}}$$

Taking $k = \frac{4}{3}$, show that the minimum deviation is about 129° and so the rainbow angle for the secondary rainbow is about 51°, as shown in the figure at the left.



4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.



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3.2 The Mean Value Theorem

We will see that many of the results of this chapter depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

Rolle

Rolle's Theorem was first published in 1691 by the French mathematician Michel Rolle (1652–1719) in a book entitled *Méthode pour resoudre les Egalitez*. He was a vocal critic of the methods of his day and attacked calculus as being a "collection of ingenious fallacies." Later, however, he became convinced of the essential correctness of the methods of calculus.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus Rolle's Theorem is plausible.

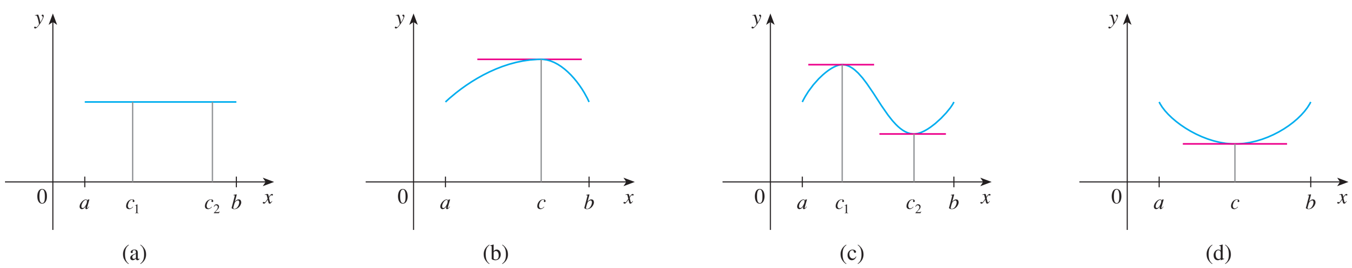


FIGURE 1

PROOF There are three cases:

PS Take cases

CASE I $f(x) = k$, **a constant**
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

CASE II $f(x) > f(a)$ **for some** x **in** (a, b) [as in Figure 1(b) or (c)]
By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III $f(x) < f(a)$ **for some** x **in** (a, b) [as in Figure 1(c) or (d)]
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem. ■

EXAMPLE 1 Let's apply Rolle's Theorem to the position function $s = f(t)$ of a moving object. If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$. Rolle's Theorem says that there is some instant of time $t = c$ between a and b when $f'(c) = 0$; that is, the velocity is 0. (In particular, you can see that this is true when a ball is thrown directly upward.) ■

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

SOLUTION First we use the Intermediate Value Theorem (1.8.10) to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a

Figure 2 shows a graph of the function $f(x) = x^3 + x - 1$ discussed in Example 2. Rolle's Theorem shows that, no matter how much we enlarge the viewing rectangle, we can never find a second x-intercept.

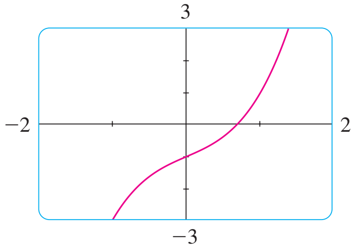


FIGURE 2

polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geqslant 1 \qquad \text{for all } x$$

(since $x^2 \geqslant 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore the equation can't have two real roots. ■

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem is an example of what is called an existence theorem. Like the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem, it guarantees that there *exists* a number with a certain property, but it doesn't tell us how to find the number.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

3 $$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB. (Imagine a line far away that stays parallel to AB while moving toward AB until it touches the graph for the first time.)

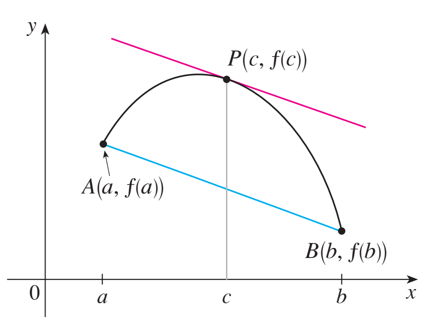


FIGURE 3

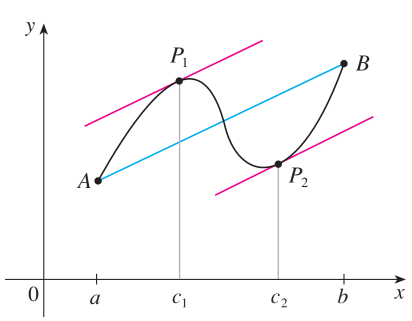


FIGURE 4

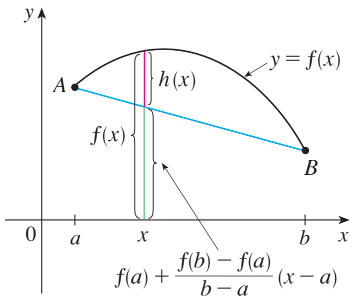


FIGURE 5

PROOF We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using Equation 3 and the point-slope equation of a line, we see that the equation of the line AB can be written as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or as

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

So, as shown in Figure 5,

$$\boxed{4} \qquad h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.

2. The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact, we can compute h' directly from Equation 4:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

(Note that $f(a)$ and $[f(b) - f(a)]/(b - a)$ are constants.)

3.
$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a)$$

$$= f(b) - f(a) - [f(b) - f(a)] = 0$$

Therefore $h(a) = h(b)$.

Since h satisfies the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ ■

Lagrange and the Mean Value Theorem

The Mean Value Theorem was first formulated by Joseph-Louis Lagrange (1736–1813), born in Italy of a French father and an Italian mother. He was a child prodigy and became a professor in Turin at the tender age of 19. Lagrange made great contributions to number theory, theory of functions, theory of equations, and analytical and celestial mechanics. In particular, he applied calculus to the analysis of the stability of the solar system. At the invitation of Frederick the Great, he succeeded Euler at the Berlin Academy and, when Frederick died, Lagrange accepted King Louis XVI's invitation to Paris, where he was given apartments in the Louvre and became a professor at the Ecole Polytechnique. Despite all the trappings of luxury and fame, he was a kind and quiet man, living only for science.

EXAMPLE 3 To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

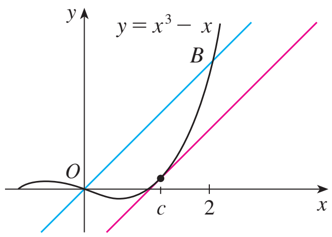


FIGURE 6

Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$. Figure 6 illustrates this calculation: The tangent line at this value of c is parallel to the secant line OB. ■

EXAMPLE 4 If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem (in the form of Equation 1) tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

In general, the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval. ■

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative. The next example provides an instance of this principle.

EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leqslant 10$, so

$$f(2) = -3 + 2f'(c) \leqslant -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7. ■

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus. One of these basic facts is the following theorem. Others will be found in the following sections.

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

PROOF Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By

applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

6 $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \qquad \text{or} \qquad f(x_2) = f(x_1)$$

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b). ■

Corollary 7 says that if two functions have the same derivatives on an interval, then their graphs must be vertical translations of each other there. In other words, the graphs have the same shape, but could be shifted up or down.

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by Theorem 5, F is constant; that is, $f - g$ is constant. ■

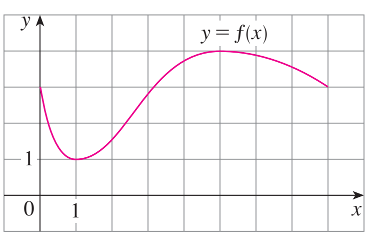
NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

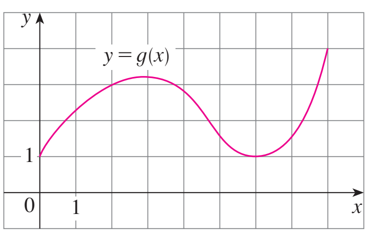
3.2 EXERCISES

1. The graph of a function f is shown. Verify that f satisfies the hypotheses of Rolle's Theorem on the interval $[0, 8]$. Then estimate the value(s) of c that satisfy the conclusion of Rolle's Theorem on that interval.



2. Draw the graph of a function defined on $[0, 8]$ such that $f(0) = f(8) = 3$ and the function does not satisfy the conclusion of Rolle's Theorem on $[0, 8]$.

3. The graph of a function g is shown.



(a) Verify that g satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 8]$.

(b) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 8]$.

(c) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[2, 6]$.

4. Draw the graph of a function that is continuous on $[0, 8]$ where $f(0) = 1$ and $f(8) = 4$ and that does not satisfy the conclusion of the Mean Value Theorem on $[0, 8]$.

5–8 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

5. $f(x) = 2x^2 - 4x + 5, \quad [-1, 3]$

6. $f(x) = x^3 - 2x^2 - 4x + 2, \quad [-2, 2]$

7. $f(x) = \sin(x/2), \quad [\pi/2, 3\pi/2]$

8. $f(x) = x + 1/x, \quad \left[\frac{1}{2}, 2\right]$

9. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

10. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 2x^2 - 3x + 1, \quad [0, 2]$

12. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$

13. $f(x) = \sqrt[3]{x}, \quad [0, 1]$

14. $f(x) = 1/x, \quad [1, 3]$

15–16 Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?

15. $f(x) = \sqrt{x}, \quad [0, 4]$

16. $f(x) = x^3 - 2x, \quad [-2, 2]$

17. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

18. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

19–20 Show that the equation has exactly one real root.

19. $2x + \cos x = 0$

20. $2x - 1 - \sin x = 0$

21. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

22. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

23. (a) Show that a polynomial of degree 3 has at most three real roots.
(b) Show that a polynomial of degree n has at most n real roots.

24. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.
(c) Can you generalize parts (a) and (b)?

25. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

26. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

27. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

28. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint:* Apply the Mean Value Theorem to the function $h = f - g$.]

29. Show that $\sin x < x$ if $0 < x < 2\pi$.

30. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

31. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

32. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

33. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

34. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

35. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

36. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

3.3 How Derivatives Affect the Shape of a Graph

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

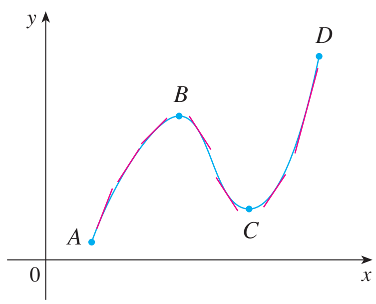


FIGURE 1

What Does f' Say About f?

To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1. (Increasing functions and decreasing functions were defined in Section 1.1.) Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$. Between B and C, the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative. To prove that this is always the case, we use the Mean Value Theorem.

Let's abbreviate the name of this test to the I/D Test.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF

(a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. According to the definition of an increasing function (page 19), we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. So, by the Mean Value Theorem, there is a number c between x_1 and x_2 such that

$$\boxed{1} \qquad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$. Thus the right side of Equation 1 is positive, and so

$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_1) < f(x_2)$$

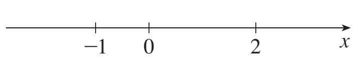
This shows that f is increasing.

Part (b) is proved similarly. ■

EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION We start by differentiating f:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$



To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. To solve these inequalities we first find where $f'(x) = 0$, namely, at $x = 0$, 2, and -1. These are the critical numbers of f, and they divide the domain into four intervals (see the number line at the left). Within each interval, $f'(x)$ must be always positive or always negative. (See Examples 3 and 4 in Appendix A.) We can determine which is the case for each interval from the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$, as shown in the following chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last col-

umn of the chart gives the conclusion based on the I/D Test. For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

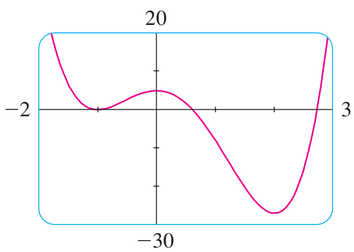


FIGURE 2

The graph of f shown in Figure 2 confirms the information in the chart. ■

Local Extreme Values

Recall from Section 3.1 that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem), but not every critical number gives rise to a maximum or a minimum. We therefore need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

You can see from Figure 2 that $f(0) = 5$ is a local maximum value of f because f increases on $(-1, 0)$ and decreases on $(0, 2)$. Or, in terms of derivatives, $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$. In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.

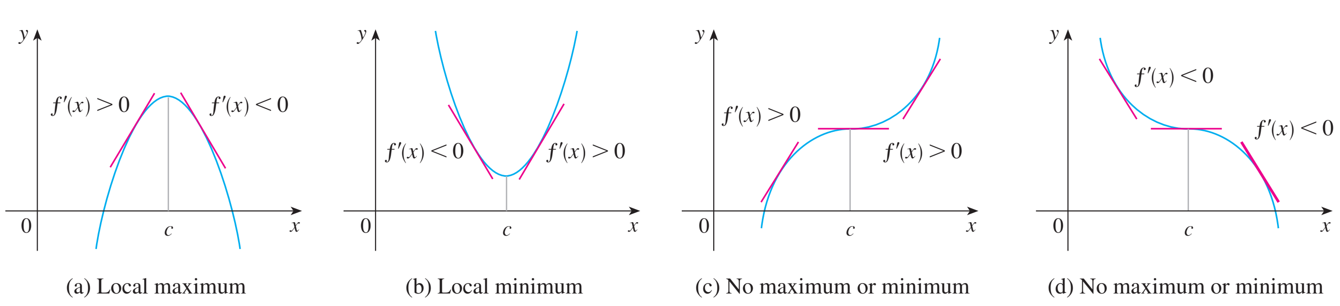


FIGURE 3

EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

SOLUTION From the chart in the solution to Example 1 we see that $f'(x)$ changes from negative to positive at -1, so $f(-1) = 0$ is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ is also a local minimum value. As noted previously, $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0. ■

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \leqslant x \leqslant 2\pi$$

SOLUTION As in Example 1, we start by finding the critical numbers. The derivative is

$$g'(x) = 1 + 2 \cos x$$

so $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$. Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$. We split the domain into intervals according to the critical numbers. Within each interval, $g'(x)$ is either always positive or always negative and so we analyze g in the following chart.

The $+$ signs in the chart come from the fact that $g'(x) > 0$ when $\cos x > -\frac{1}{2}$. From the graph of $y = \cos x$, this is true in the indicated intervals.

Interval	$g'(x) = 1 + 2 \cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

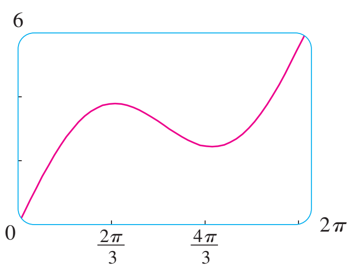


FIGURE 4
$g(x) = x + 2 \sin x$

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2 \sin \frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2 \sin \frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$$

is a local minimum value. The graph of g in Figure 4 supports our conclusion. ■

What Does f'' Say About f?

Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior?

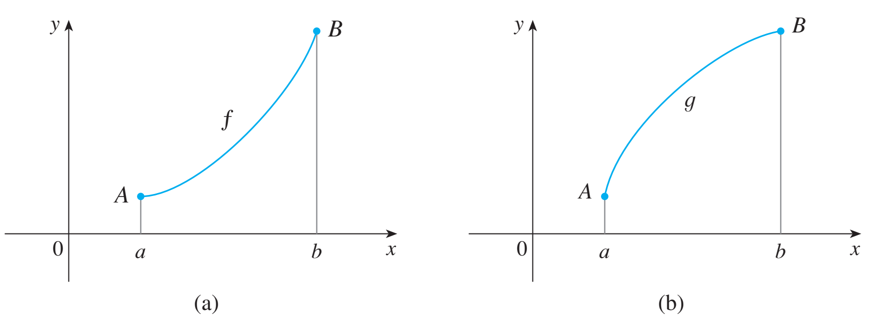


FIGURE 5

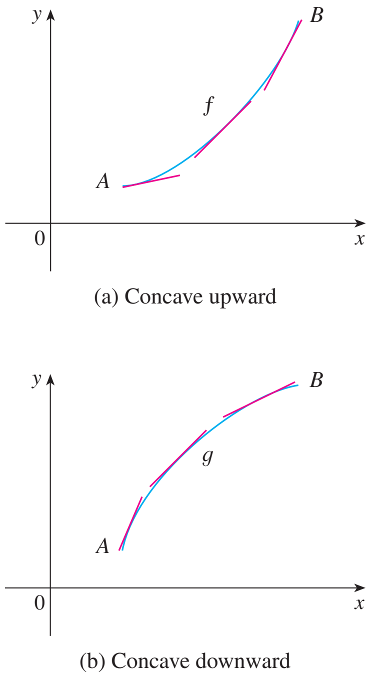


FIGURE 6

In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).

> **Definition** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

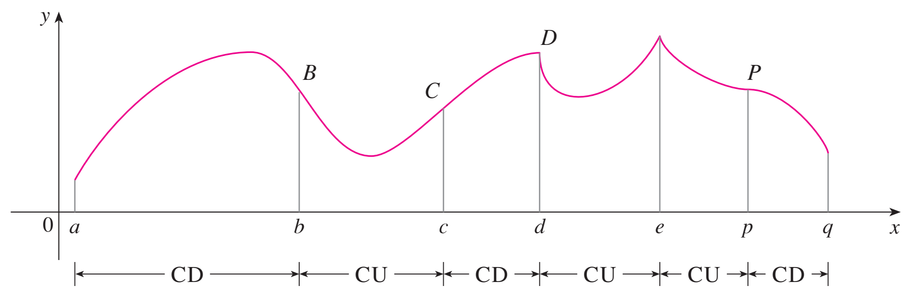


FIGURE 7

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 6(a), you can see that, going from left to right, the slope of the tangent increases. This means that the derivative f' is an increasing function and therefore its derivative f'' is positive. Likewise, in Figure 6(b) the slope of the tangent decreases from left to right, so f' decreases and therefore f'' is negative. This reasoning can be reversed and suggests that the following theorem is true. A proof is given in Appendix F with the help of the Mean Value Theorem.

> **Concavity Test**
>
> (a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
>
> (b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

EXAMPLE 4 Figure 8 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

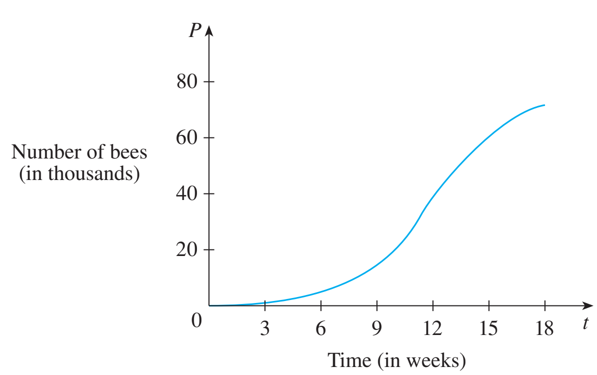


FIGURE 8

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upward on (0, 12) and concave downward on (12, 18). ■

In Example 4, the population curve changed from concave upward to concave downward at approximately the point (12, 38,000). This point is called an *inflection point* of the curve. The significance of this point is that the rate of population increase has its maximum value there. In general, an inflection point is a point where a curve changes its direction of concavity.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

For instance, in Figure 7, B, C, D, and P are the points of inflection. Notice that if a curve has a tangent at a point of inflection, then the curve crosses its tangent there.

In view of the Concavity Test, there is a point of inflection at any point where the second derivative changes sign.

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f(0) = 0, \quad f(2) = 3, \quad f(4) = 6, \quad f'(0) = f'(4) = 0$

(ii) $f'(x) > 0$ for $0 < x < 4, \quad f'(x) < 0$ for $x < 0$ and for $x > 4$

(iii) $f''(x) > 0$ for $x < 2, \quad f''(x) < 0$ for $x > 2$

SOLUTION Condition (i) tells us that the graph has horizontal tangents at the points (0, 0) and (4, 6). Condition (ii) says that f is increasing on the interval (0, 4) and decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$. It follows from the I/D Test that $f(0) = 0$ is a local minimum and $f(4) = 6$ is a local maximum.

Condition (iii) says that the graph is concave upward on the interval $(-\infty, 2)$ and concave downward on $(2, \infty)$. Because the curve changes from concave upward to concave downward when $x = 2$, the point (2, 3) is an inflection point.

We use this information to sketch the graph of f in Figure 9. Notice that we made the curve bend upward when $x < 2$ and bend downward when $x > 2$. ■

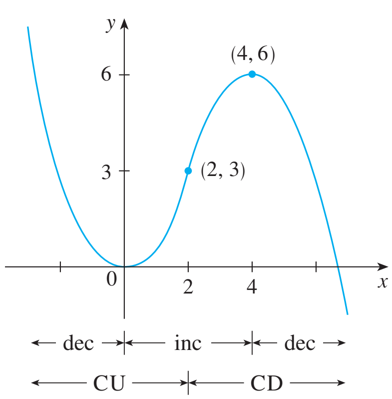


FIGURE 9

Another application of the second derivative is the following test for identifying local maximum and minimum values. It is a consequence of the Concavity Test and serves as an alternative to the First Derivative Test.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

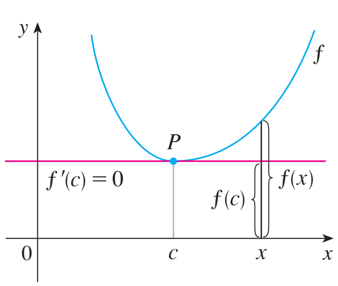


FIGURE 10
$f''(c) > 0$, f is concave upward

For instance, part (a) is true because $f''(x) > 0$ near c and so f is concave upward near c. This means that the graph of f lies *above* its horizontal tangent at c and so f has a local minimum at c. (See Figure 10.)

EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. (Note that f' is a polynomial and hence defined everywhere.) To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.] Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

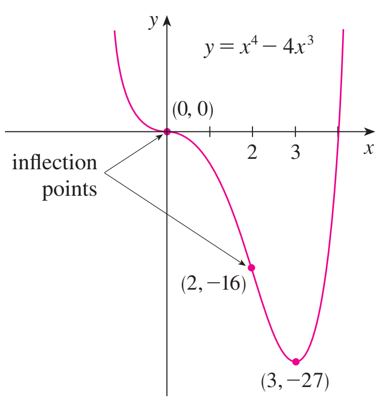


FIGURE 11

The point $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11. ■

NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 6). This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

SOLUTION Calculation of the first two derivatives gives

Use the differentiation rules to check these calculations.

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

TEC In Module 3.3 you can practice using information about f', f'', and asymptotes to determine the shape of the graph of f.

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.

Try reproducing the graph in Figure 12 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the y-axis, and some produce only the portion between $x = 0$ and $x = 6$. For an explanation and cure, see Example 7 in "Graphing Calculators and Computers" at www.stewartcalculus.com. An equivalent expression that gives the correct graph is

$$y = (x^2)^{1/3} \cdot \frac{6 - x}{|6 - x|} |6 - x|^{1/3}$$

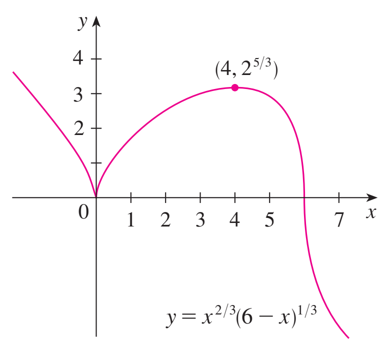


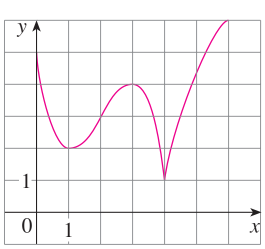
FIGURE 12

3.3 EXERCISES

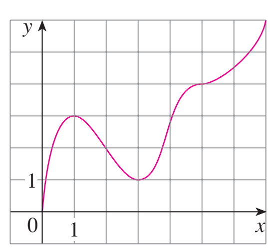
1–2 Use the given graph of f to find the following.

(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



2.

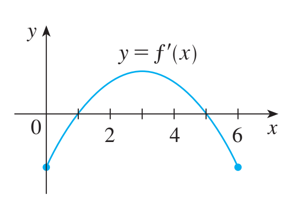


3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

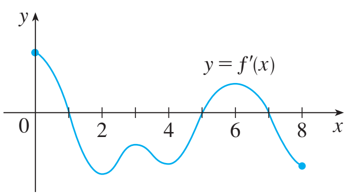
4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

5–6 The graph of the *derivative* f' of a function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?

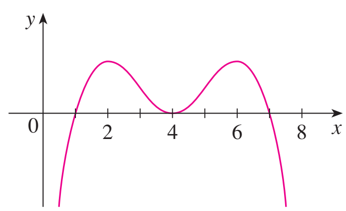
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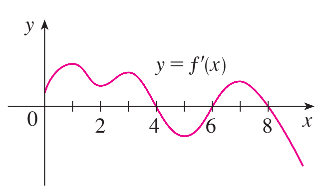
6.



7. In each part state the x-coordinates of the inflection points of f. Give reasons for your answers.
(a) The curve is the graph of f.
(b) The curve is the graph of f'.
(c) The curve is the graph of f''.



8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9–14
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = x^3 - 3x^2 - 9x + 4$

10. $f(x) = 2x^3 - 9x^2 + 12x - 3$

11. $f(x) = x^4 - 2x^2 + 3$

12. $f(x) = \dfrac{x}{x^2 + 1}$

13. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

14. $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leqslant x \leqslant 2\pi$

15–17 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

15. $f(x) = 1 + 3x^2 - 2x^3$

16. $f(x) = \dfrac{x^2}{x - 1}$

17. $f(x) = \sqrt{x} - \sqrt[4]{x}$

18. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

19. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

20–27 Sketch the graph of a function that satisfies all of the given conditions.

20. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

21. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x
(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

22. Vertical asymptote $x = 0$, $\quad f'(x) > 0$ if $x < -2$,
$f'(x) < 0$ if $x > -2 \ (x \neq 0)$,
$f''(x) < 0$ if $x < 0$, $\quad f''(x) > 0$ if $x > 0$

23. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $\quad f''(x) < 0$ if $x < 1$ or $x > 3$

24. $f'(x) > 0$ for all $x \neq 1$, $\quad$ vertical asymptote $x = 1$,
$f''(x) > 0$ if $x < 1$ or $x > 3$, $\quad f''(x) < 0$ if $1 < x < 3$

25. $f'(5) = 0$, $\quad f'(x) < 0$ when $x < 5$,
$f'(x) > 0$ when $x > 5$, $\quad f''(2) = 0$, $\quad f''(8) = 0$,
$f''(x) < 0$ when $x < 2$ or $x > 8$,
$f''(x) > 0$ for $2 < x < 8$

26. $f'(0) = f'(4) = 0$, $\quad f'(x) = 1$ if $x < -1$,
$f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4$,
$\lim_{x \to 2^-} f'(x) = \infty$, $\quad \lim_{x \to 2^+} f'(x) = -\infty$,
$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4$,
$f''(x) < 0$ if $x > 4$

27. $f(0) = f'(0) = f'(2) = f'(4) = f'(6) = 0$,
$f'(x) > 0$ if $0 < x < 2$ or $4 < x < 6$,
$f'(x) < 0$ if $2 < x < 4$ or $x > 6$,
$f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5$,
$f''(x) < 0$ if $1 < x < 3$ or $x > 5$, $\quad f(-x) = f(x)$

28. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
(a) Sketch a possible graph for f.
(b) How many solutions does the equation $f(x) = 0$ have? Why?
(c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

29. Suppose f is a continuous function where $f(x) > 0$ for all x, $f(0) = 4$, $\quad f'(x) > 0$ if $x < 0$ or $x > 2$, $\quad f'(x) < 0$ if $0 < x < 2$, $\quad f''(-1) = f''(1) = 0$, $\quad f''(x) > 0$ if $x < -1$ or $x > 1$, $\quad f''(x) < 0$ if $-1 < x < 1$.
(a) Can f have an absolute maximum? If so, sketch a possible graph of f. If not, explain why.

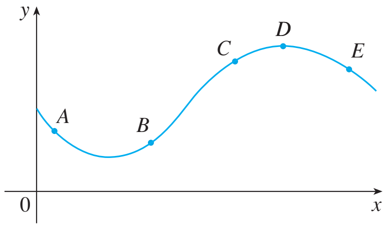
(b) Can f have an absolute minimum? If so, sketch a possible graph of f. If not, explain why.
(c) Sketch a possible graph for f that does *not* achieve an absolute minimum.

30. The graph of a function $y = f(x)$ is shown. At which point(s) are the following true?

(a) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are both positive.

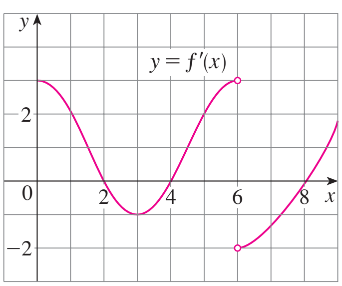
(b) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are both negative.

(c) $\dfrac{dy}{dx}$ is negative but $\dfrac{d^2y}{dx^2}$ is positive.

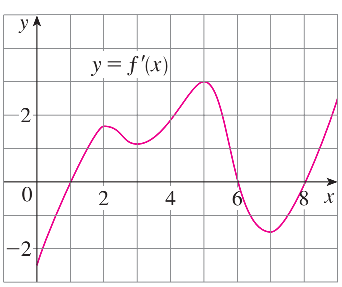


31–32 The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing? Decreasing?
(b) At what values of x does f have a local maximum? Local minimum?
(c) On what intervals is f concave upward? Concave downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

31.



32.



33–44
(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work with a graphing device if you have one.

33. $f(x) = x^3 - 12x + 2$

34. $f(x) = 36x + 3x^2 - 2x^3$

35. $f(x) = \frac{1}{2}x^4 - 4x^2 + 3$

36. $g(x) = 200 + 8x^3 + x^4$

37. $h(x) = (x + 1)^5 - 5x - 2$

38. $h(x) = 5x^3 - 3x^5$

39. $F(x) = x\sqrt{6 - x}$

40. $G(x) = 5x^{2/3} - 2x^{5/3}$

41. $C(x) = x^{1/3}(x + 4)$

42. $f(x) = 2\sqrt{x} - 4x^2$

43. $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

44. $S(x) = x - \sin x, \quad 0 \leqslant x \leqslant 4\pi$

45. Suppose the derivative of a function f is $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$. On what interval is f increasing?

46. Use the methods of this section to sketch the curve $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ from each other?

47–48
(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.
(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

47. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

48. $f(x) = x + 2\cos x, \quad 0 \leqslant x \leqslant 2\pi$

49–50
(a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

49. $f(x) = \sin 2x + \sin 4x, \quad 0 \leqslant x \leqslant \pi$

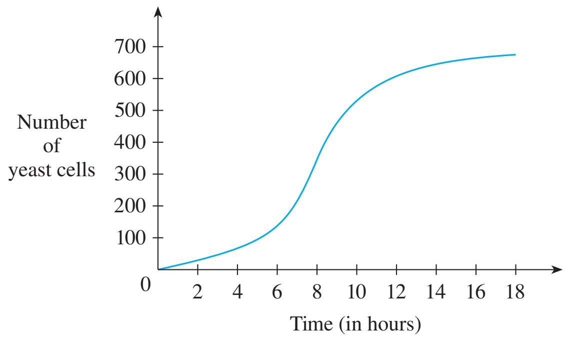
50. $f(x) = (x - 1)^2(x + 1)^3$

CAS **51–52** Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph f''.

51. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$

52. $f(x) = \dfrac{(x + 1)^3(x^2 + 5)}{(x^3 + 1)(x^2 + 4)}$

53. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



54. In an episode of *The Simpsons* television show, Homer reads from a newspaper and announces "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate." Interpret Homer's statement in terms of a function and its first and second derivatives.

55. The president announces that the national deficit is increasing, but at a decreasing rate. Interpret this statement in terms of a function and its first and second derivatives.

56. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$
(b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$
(d) $f'(3) = -2, \quad f''(3) = -4$

57. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

58. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?



59. Find a cubic function

$$f(x) = ax^3 + bx^2 + cx + d$$

that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

60. Show that the curve

$$y = \frac{1 + x}{1 + x^2}$$

has three points of inflection and they all lie on one straight line.

61. (a) If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = 1/\sqrt{3}$, what are the values of a and b?
(b) Which of the tangent lines to the curve in part (a) has the smallest slope?

62. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2y + ax + by = 0$? What additional inflection points does the curve have?

63. Show that the inflection points of the curve $y = x \sin x$ lie on the curve $y^2(x^2 + 4) = 4x^2$.

64–66 Assume that all of the functions are twice differentiable and the second derivatives are never 0.

64. (a) If f and g are concave upward on I, show that $f + g$ is concave upward on I.
(b) If f is positive and concave upward on I, show that the function $g(x) = [f(x)]^2$ is concave upward on I.

65. (a) If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.
(b) Show that part (a) remains true if f and g are both decreasing.
(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

66. Suppose f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

---

67. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

68. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}$$

69. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

70. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

71. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [*Hint:* Apply the First Derivative Test and Fermat's Theorem to the function $g = f'$.]

72. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

73. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

74. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

75. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

76. For what values of c is the function

$$f(x) = cx + \frac{1}{x^2 + 3}$$

increasing on $(-\infty, \infty)$?

77. The three cases in the First Derivative Test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x} \qquad g(x) = x^4\left(2 + \sin\frac{1}{x}\right)$$

$$h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

3.4 Limits at Infinity; Horizontal Asymptotes

In Sections 1.5 and 1.7 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

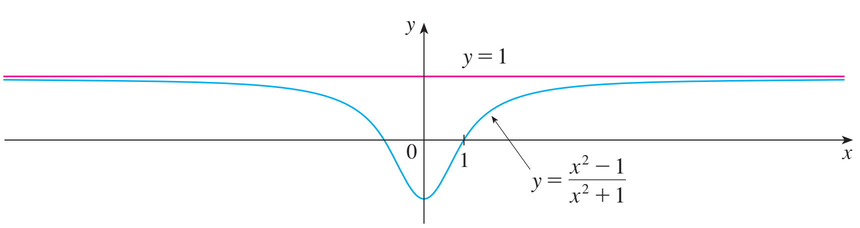


FIGURE 1

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. (The graph of f approaches the horizontal line $y = 1$ as we look to the right.) In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim\limits_{x \to \infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 1.7, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

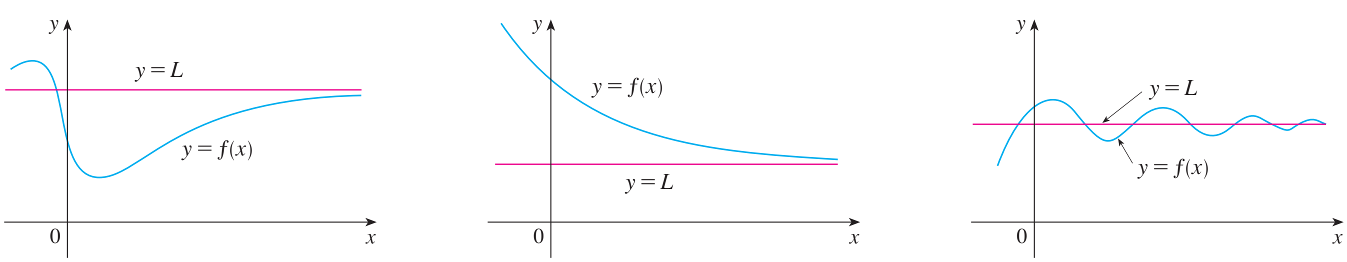


FIGURE 2
Examples illustrating $\lim\limits_{x \to \infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x \to -\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

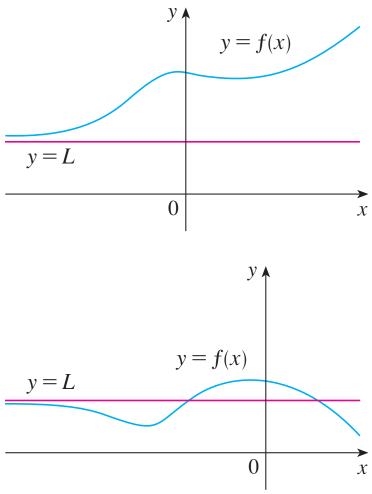


FIGURE 3
Examples illustrating $\lim_{x \to -\infty} f(x) = L$

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The curve $y = f(x)$ sketched in Figure 4 has both $y = -1$ and $y = 2$ as horizontal asymptotes because

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

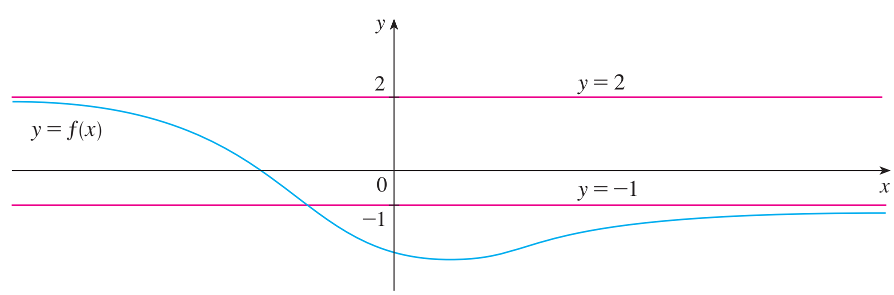


FIGURE 4

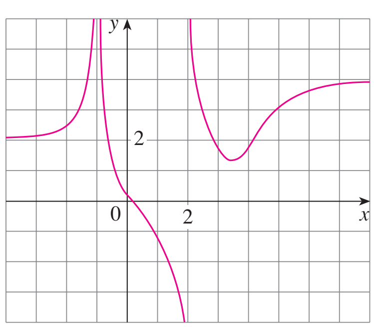


FIGURE 5

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■

EXAMPLE 2 Find $\displaystyle\lim_{x \to \infty} \frac{1}{x}$ and $\displaystyle\lim_{x \to -\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.) ■

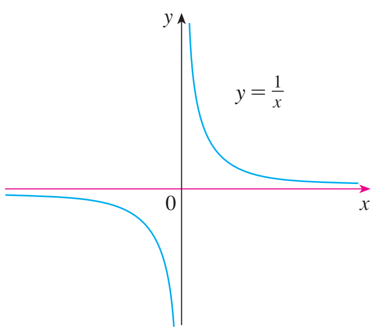


FIGURE 6

$\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0, \ \lim_{x \to -\infty} \frac{1}{x} = 0$

Most of the Limit Laws that were given in Section 1.6 also hold for limits at infinity. It can be proved that *the Limit Laws listed in Section 1.6 (with the exception of Laws 9 and 10) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Laws 6 and 11 with the results of Example 2, we obtain the following important rule for calculating limits.

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)} \qquad \text{(by Limit Law 5)}$$

$$= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + 4\lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \dfrac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{(by 7 and Theorem 4)}$$

$$= \frac{3}{5}$$

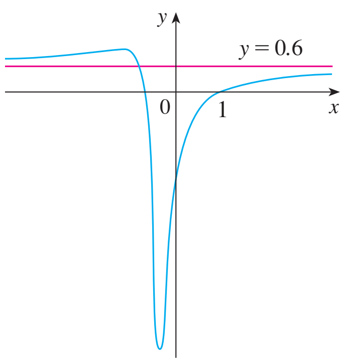


FIGURE 7
$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5} = 0.6$. ■

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{2x^2 + 1}}{x}}{\dfrac{3x - 5}{x}} = \lim_{x\to\infty} \frac{\sqrt{\dfrac{2x^2 + 1}{x^2}}}{\dfrac{3x - 5}{x}} \qquad \text{(since } \sqrt{x^2} = x \text{ for } x > 0\text{)}$$

$$= \frac{\lim\limits_{x\to\infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty}\left(3 - \dfrac{5}{x}\right)} = \frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - 5\lim\limits_{x\to\infty} \dfrac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}} = -\sqrt{\frac{2x^2+1}{x^2}} = -\sqrt{2+\frac{1}{x^2}}$$

Therefore

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}} = \frac{-\sqrt{2+\lim_{x\to-\infty}\dfrac{1}{x^2}}}{3-5\lim_{x\to-\infty}\dfrac{1}{x}} = -\frac{\sqrt{2}}{3}$$

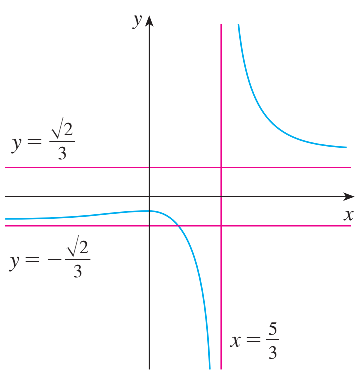


FIGURE 8

$y = \dfrac{\sqrt{2x^2+1}}{3x-5}$

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2+1}$ is always positive, so $f(x)$ is positive. Therefore

$$\lim_{x\to(5/3)^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

(Notice that the numerator does *not* approach 0 as $x \to 5/3$.)

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x\to(5/3)^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8. ■

EXAMPLE 5 Compute $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$.

SOLUTION Because both $\sqrt{x^2+1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

We can think of the given function as having a denominator of 1.

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$

Notice that the denominator of this last expression $\left(\sqrt{x^2+1}+x\right)$ becomes large as $x \to \infty$ (it's bigger than x). So

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$$

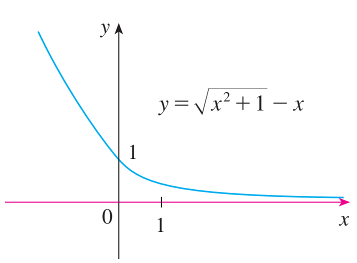


FIGURE 9

Figure 9 illustrates this result. ■

EXAMPLE 6 Evaluate $\lim_{x\to\infty} \sin \frac{1}{x}$.

PS The problem-solving strategy for Example 6 is *introducing something extra* (see page 98). Here, the something extra, the auxiliary aid, is the new variable t.

SOLUTION If we let $t = 1/x$, then $t \to 0^+$ as $x \to \infty$. Therefore

$$\lim_{x\to\infty} \sin \frac{1}{x} = \lim_{t\to 0^+} \sin t = 0$$

(See Exercise 73.) ■

EXAMPLE 7 Evaluate $\lim_{x\to\infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x\to\infty} \sin x$ does not exist. ■

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

EXAMPLE 8 Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by requiring x to be large enough. Therefore we can write

$$\lim_{x\to\infty} x^3 = \infty$$

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x\to-\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 10. ■

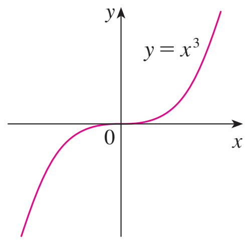


FIGURE 10
$\lim_{x\to\infty} x^3 = \infty,\ \lim_{x\to-\infty} x^3 = -\infty$

EXAMPLE 9 Find $\lim_{x\to\infty} (x^2 - x)$.

SOLUTION It would be **wrong** to write

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too. ■

EXAMPLE 10 Find $\lim_{x \to \infty} \dfrac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is just x:

$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x} = \lim_{x \to \infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to 0 - 1 = -1$ as $x \to \infty$. ■

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without computing derivatives.

EXAMPLE 11 Sketch the graph of $y = (x - 2)^4(x + 1)^3(x - 1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The y-intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x-intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x - 2)^4$ is never negative, the function doesn't change sign at 2; thus the graph doesn't cross the x-axis at 2. The graph crosses the axis at -1 and 1.

When x is large positive, all three factors are large, so

$$\lim_{x \to \infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x \to -\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

Combining this information, we give a rough sketch of the graph in Figure 11. ■

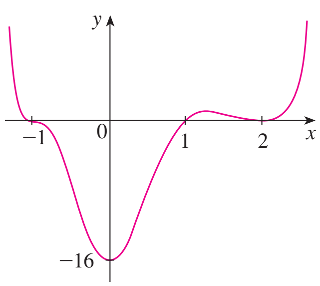


FIGURE 11
$y = (x - 2)^4(x + 1)^3(x - 1)$

Precise Definitions

Definition 1 can be stated precisely as follows.

5 Precise Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by requiring x to be sufficiently large (larger than N, where N depends on ε). Graphically it says that by keeping x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 12. This must be true no matter how small we choose ε.

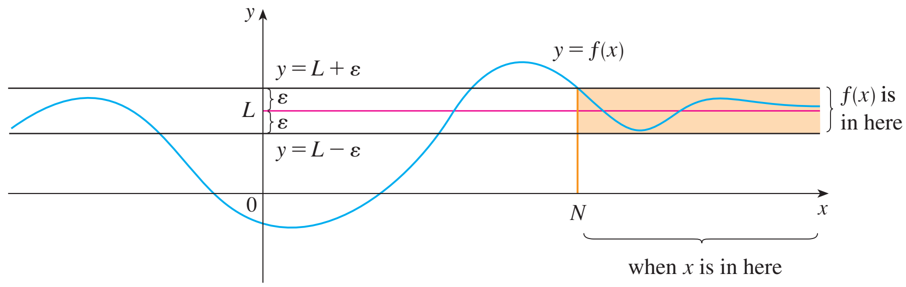


FIGURE 12
$\lim_{x \to \infty} f(x) = L$

Figure 13 shows that if a smaller value of ε is chosen, then a larger value of N may be required.

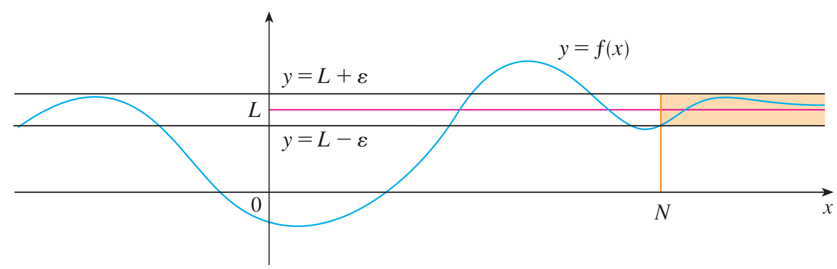


FIGURE 13
$\lim_{x \to \infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 6, which is illustrated in Figure 14.

6 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

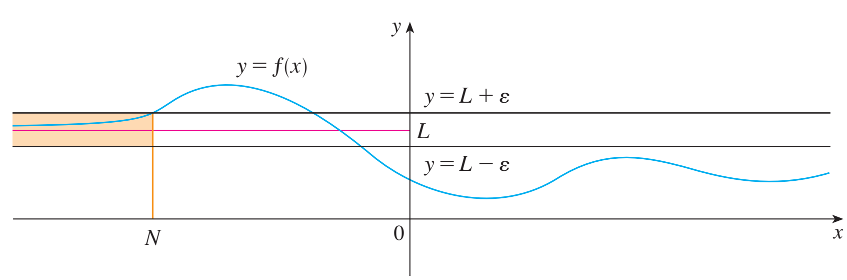


FIGURE 14
$\lim_{x \to -\infty} f(x) = L$

In Example 3 we calculated that

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 5 with $L = \frac{3}{5} = 0.6$ and $\varepsilon = 0.1$.

TEC In Module 1.7/3.4 you can explore the precise definition of a limit both graphically and numerically.

EXAMPLE 12 Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

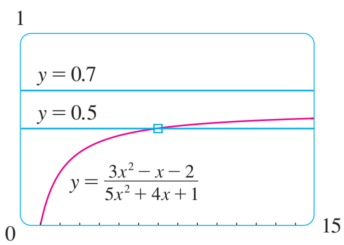


FIGURE 15

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 15. Then we use the cursor to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number it seems that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding up to be safe, we can say that

$$\text{if} \quad x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 5. ■

EXAMPLE 13 Use Definition 5 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

SOLUTION Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if} \quad x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 5,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 16 illustrates the proof by showing some values of ε and the corresponding values of N.

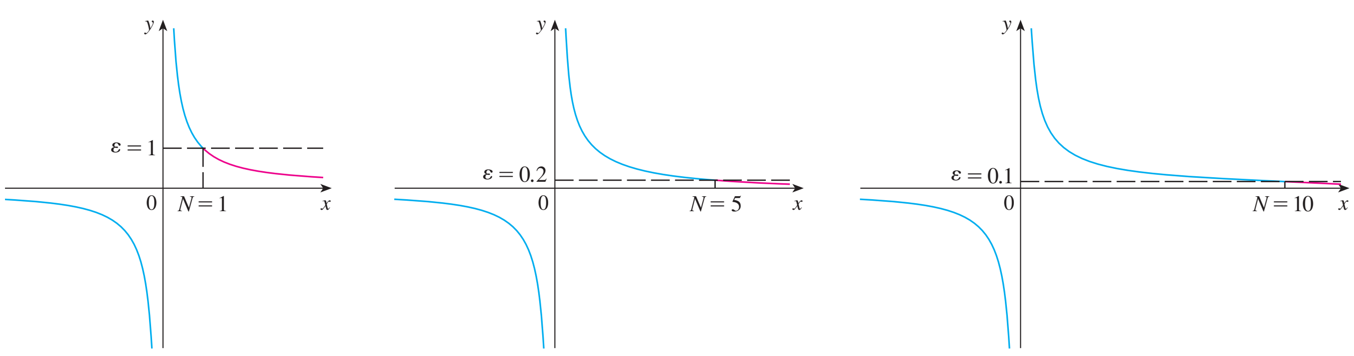


FIGURE 16 ■

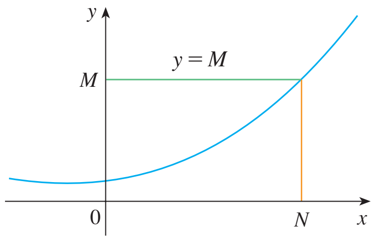


FIGURE 17
$\lim_{x \to \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.

7 Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

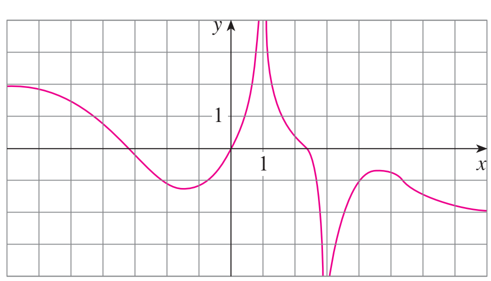
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 74.)

3.4 EXERCISES

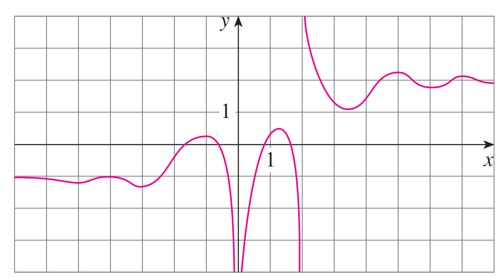
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$
(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$
(e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$
(e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes



5. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

6. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

7–8 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

7. $\lim_{x \to \infty} \dfrac{2x^2 - 7}{5x^2 + x - 3}$

8. $\lim_{x \to \infty} \sqrt{\dfrac{9x^3 + 8x - 4}{3 - 5x + x^3}}$

9–32 Find the limit or show that it does not exist.

9. $\displaystyle\lim_{x\to\infty} \frac{3x - 2}{2x + 1}$

10. $\displaystyle\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x + 1}$

11. $\displaystyle\lim_{x\to-\infty} \frac{x - 2}{x^2 + 1}$

12. $\displaystyle\lim_{x\to-\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

13. $\displaystyle\lim_{t\to\infty} \frac{\sqrt{t} + t^2}{2t - t^2}$

14. $\displaystyle\lim_{t\to\infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

15. $\displaystyle\lim_{x\to\infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

16. $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4 + 1}}$

17. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

18. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

19. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x + 3x^2}}{4x - 1}$

20. $\displaystyle\lim_{x\to\infty} \frac{x + 3x^2}{4x - 1}$

21. $\displaystyle\lim_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right)$

22. $\displaystyle\lim_{x\to-\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

23. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

24. $\displaystyle\lim_{x\to\infty} \cos x$

25. $\displaystyle\lim_{x\to\infty} \frac{x^4 - 3x^2 + x}{x^3 - x + 2}$

26. $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + 1}$

27. $\displaystyle\lim_{x\to-\infty} (x^2 + 2x^7)$

28. $\displaystyle\lim_{x\to-\infty} \frac{1 + x^6}{x^4 + 1}$

29. $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x}\right)$

30. $\displaystyle\lim_{x\to\infty} (x^2 - x^4)$

31. $\displaystyle\lim_{x\to\infty} x \sin \frac{1}{x}$

32. $\displaystyle\lim_{x\to\infty} \sqrt{x} \sin \frac{1}{x}$

---

33. (a) Estimate the value of

$$\lim_{x\to-\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

35–40 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

35. $y = \dfrac{5 + 4x}{x + 3}$

36. $y = \dfrac{2x^2 + 1}{3x^2 + 2x - 1}$

37. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

38. $y = \dfrac{1 + x^4}{x^2 - x^4}$

39. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$

40. $y = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

---

41. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

42. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x\to-\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

43. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

44. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$
(ii) $n > 0$, n odd
(iii) $n > 0$, n even
(iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\displaystyle\lim_{x\to0^+} x^n$
(b) $\displaystyle\lim_{x\to0^-} x^n$
(c) $\displaystyle\lim_{x\to\infty} x^n$
(d) $\displaystyle\lim_{x\to-\infty} x^n$

45. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x\to\pm\infty} f(x) = 0, \quad \lim_{x\to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x\to 3^-} f(x) = \infty, \quad \lim_{x\to 3^+} f(x) = -\infty$$

46. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

47. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x\to -1} f(x) = 2$. Evaluate

(a) $f(0)$ (b) $\lim_{x\to\infty} f(x)$

48–51 Find the horizontal asymptotes of the curve and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

48. $y = \dfrac{1 + 2x^2}{1 + x^2}$

49. $y = \dfrac{1 - x}{1 + x}$

50. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

51. $y = \dfrac{x}{x^2 + 1}$

52–56 Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 11.

52. $y = 2x^3 - x^4$

53. $y = x^4 - x^6$

54. $y = x^3(x + 2)^2(x - 1)$

55. $y = (3 - x)(1 + x)^2(1 - x)^4$

56. $y = x^2(x^2 - 1)^2(x + 2)$

57–60 Sketch the graph of a function that satisfies all of the given conditions.

57. $f'(2) = 0, \quad f(2) = -1, \quad f(0) = 0,$
$f'(x) < 0$ if $0 < x < 2, \quad f'(x) > 0$ if $x > 2,$
$f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4,$
$f''(x) > 0$ if $1 < x < 4, \quad \lim_{x\to\infty} f(x) = 1,$
$f(-x) = f(x)$ for all x

58. $f'(2) = 0, \quad f'(0) = 1, \quad f'(x) > 0$ if $0 < x < 2,$
$f'(x) < 0$ if $x > 2, \quad f''(x) < 0$ if $0 < x < 4,$
$f''(x) > 0$ if $x > 4, \quad \lim_{x\to\infty} f(x) = 0,$
$f(-x) = -f(x)$ for all x

59. $f(1) = f'(1) = 0, \quad \lim_{x\to 2^+} f(x) = \infty, \quad \lim_{x\to 2^-} f(x) = -\infty,$
$\lim_{x\to 0} f(x) = -\infty, \quad \lim_{x\to -\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0,$
$f''(x) > 0$ for $x > 2, \quad f''(x) < 0$ for $x < 0$ and for $0 < x < 2$

60. $g(0) = 0, \quad g''(x) < 0$ for $x \neq 0, \quad \lim_{x\to -\infty} g(x) = \infty,$
$\lim_{x\to\infty} g(x) = -\infty, \quad \lim_{x\to 0^-} g'(x) = -\infty,$
$\lim_{x\to 0^+} g'(x) = \infty$

61. (a) Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

62. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

63. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

64. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

65. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

66. For the limit

$$\lim_{x\to\infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = -3$$

illustrate Definition 5 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

67. For the limit

$$\lim_{x\to -\infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = 3$$

illustrate Definition 6 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

68. For the limit

$$\lim_{x \to \infty} \frac{3x}{\sqrt{x - 3}} = \infty$$

illustrate Definition 7 by finding a value of N that corresponds to $M = 100$.

69. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 4, we have the statement

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 5.

70. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?

(b) Taking $r = \frac{1}{2}$ in Theorem 4, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 5.

71. Use Definition 6 to prove that $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0$.

72. Prove, using Definition 7, that $\displaystyle\lim_{x \to \infty} x^3 = \infty$.

73. (a) Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

if these limits exist.

(b) Use part (a) and Exercise 61 to find

$$\lim_{x \to 0^+} x \sin \frac{1}{x}$$

74. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

3.5 Summary of Curve Sketching

So far we have been concerned with some particular aspects of curve sketching: domain, range, symmetry, limits, continuity, and vertical asymptotes in Chapter 1; derivatives and tangents in Chapter 2; and extreme values, intervals of increase and decrease, concavity, points of inflection, and horizontal asymptotes in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You might ask: Why don't we just use a graphing calculator or computer to graph a curve? Why do we need to use calculus?

It's true that current technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. It is easy to arrive at a misleading graph, or to miss important details of a curve, when relying solely on technology. (See "Graphing Calculators and Computers" at *www.stewartcalculus.com*, especially Examples 1, 3, 4, and 5. See also Section 3.6.) The use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points *exactly* instead of approximately.

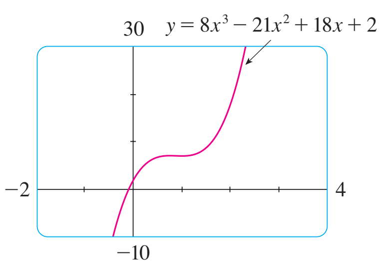


FIGURE 1

For instance, Figure 1 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: it has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 2. Without calculus, we could easily have overlooked it.

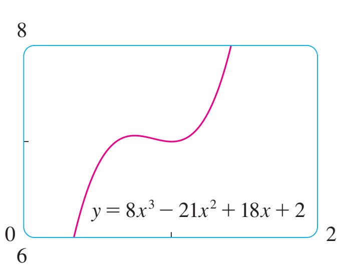


FIGURE 2

In the next section we will graph functions by using the interaction between calculus and graphing devices. In this section we draw graphs by first considering the following information. We don't assume that you have a graphing device, but if you do have one you should use it as a check on your work.

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an

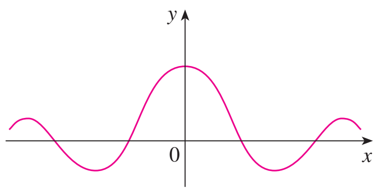
asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

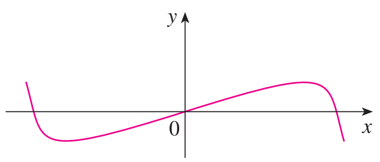
B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 3

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).

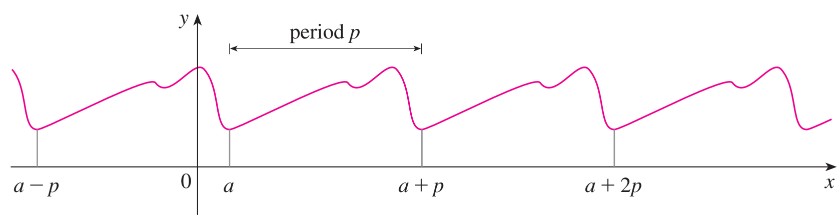


FIGURE 4
Periodic function: translational symmetry

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 3.4 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 1.5 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

1
$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$
$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

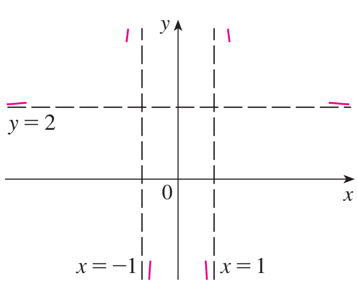


FIGURE 5
Preliminary sketch

We have shown the curve approaching its horizontal asymptote from above in Figure 5. This is confirmed by the intervals of increase and decrease.

E.
$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G. $$f''(x) = \frac{(x^2 - 1)^2(-4) + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure 6. ■

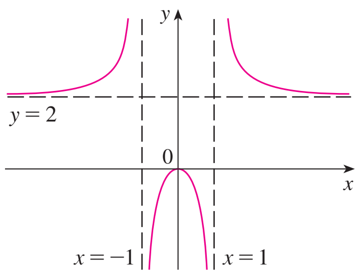


FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x + 1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x + 1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x + 1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E. $$f'(x) = \frac{\sqrt{x + 1}\,(2x) - x^2 \cdot 1/\left(2\sqrt{x + 1}\right)}{x + 1} = \frac{3x^2 + 4x}{2(x + 1)^{3/2}} = \frac{x(3x + 4)}{2(x + 1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. $$f''(x) = \frac{2(x + 1)^{3/2}(6x + 4) - (3x^2 + 4x)3(x + 1)^{1/2}}{4(x + 1)^3} = \frac{3x^2 + 8x + 8}{4(x + 1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

H. The curve is sketched in Figure 7. ■

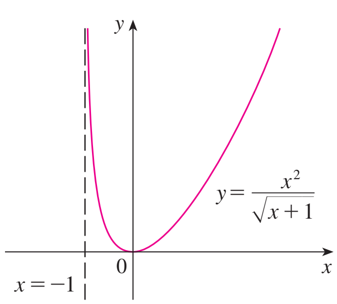


FIGURE 7

EXAMPLE 3 Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (\pi/2) + n\pi$, where n is an integer.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.

D. Asymptotes: None

E.
$$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

The denominator is always positive, so $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.

F. From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.

G. If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.

H. The graph of the function restricted to $0 \leqslant x \leqslant 2\pi$ is shown in Figure 8. Then we extend it, using periodicity, to the complete graph in Figure 9.

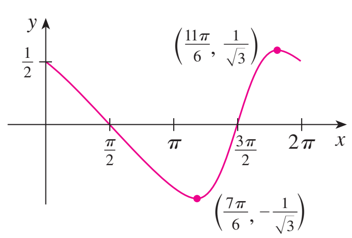


FIGURE 8

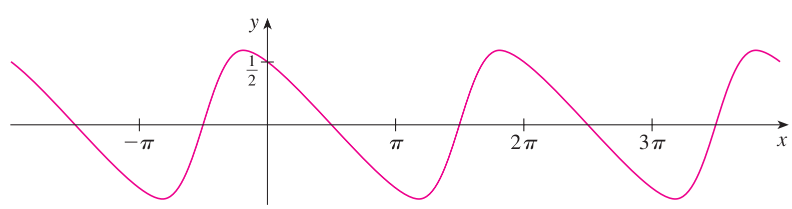


FIGURE 9

Slant Asymptotes

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

where $m \neq 0$, then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 10. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

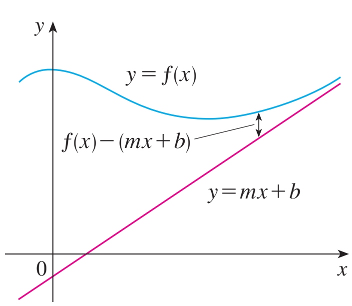


FIGURE 10

EXAMPLE 4 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.
B. The x- and y-intercepts are both 0.
C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.
D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

This equation suggests that $y = x$ is a candidate for a slant asymptote. In fact,

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E. $$f'(x) = \frac{(x^2 + 1)(3x^2) - x^3 \cdot 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.
F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.
G. $$f''(x) = \frac{(x^2 + 1)^2(4x^3 + 6x) - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{2x(3 - x^2)}{(x^2 + 1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

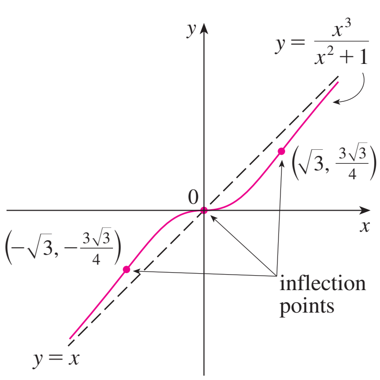


FIGURE 11

The points of inflection are $\left(-\sqrt{3}, -\frac{3}{4}\sqrt{3}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right)$.
H. The graph of f is sketched in Figure 11. ■

3.5 EXERCISES

1–40 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + 3x^2$

2. $y = 2 + 3x^2 - x^3$

3. $y = x^4 - 4x$

4. $y = x^4 - 8x^2 + 8$

5. $y = x(x - 4)^3$

6. $y = x^5 - 5x$

7. $y = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x$

8. $y = (4 - x^2)^5$

9. $y = \dfrac{x}{x - 1}$

10. $y = \dfrac{x^2 + 5x}{25 - x^2}$

11. $y = \dfrac{x - x^2}{2 - 3x + x^2}$

12. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

13. $y = \dfrac{x}{x^2 - 4}$

14. $y = \dfrac{1}{x^2 - 4}$

15. $y = \dfrac{x^2}{x^2 + 3}$

16. $y = \dfrac{(x - 1)^2}{x^2 + 1}$

17. $y = \dfrac{x - 1}{x^2}$

18. $y = \dfrac{x}{x^3 - 1}$

19. $y = \dfrac{x^3}{x^3 + 1}$

20. $y = \dfrac{x^3}{x - 2}$

21. $y = (x - 3)\sqrt{x}$

22. $y = (x - 4)\sqrt[3]{x}$

23. $y = \sqrt{x^2 + x - 2}$

24. $y = \sqrt{x^2 + x} - x$

25. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

26. $y = x\sqrt{2 - x^2}$

27. $y = \dfrac{\sqrt{1 - x^2}}{x}$

28. $y = \dfrac{x}{\sqrt{x^2 - 1}}$

29. $y = x - 3x^{1/3}$

30. $y = x^{5/3} - 5x^{2/3}$

31. $y = \sqrt[3]{x^2 - 1}$

32. $y = \sqrt[3]{x^3 + 1}$

33. $y = \sin^3 x$

34. $y = x + \cos x$

35. $y = x \tan x, \quad -\pi/2 < x < \pi/2$

36. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$

37. $y = \sin x + \sqrt{3} \cos x, \quad -2\pi \leqslant x \leqslant 2\pi$

38. $y = \csc x - 2 \sin x, \quad 0 < x < \pi$

39. $y = \dfrac{\sin x}{1 + \cos x}$

40. $y = \dfrac{\sin x}{2 + \cos x}$

---

41. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

42. In the theory of relativity, the energy of a particle is

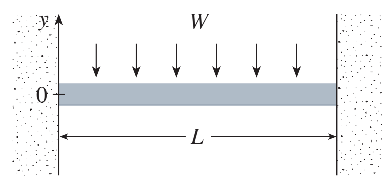
$$E = \sqrt{m_0^2 c^4 + h^2 c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

43. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

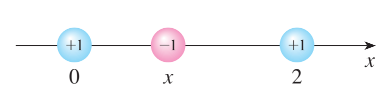
where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



44. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x - 2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?



45–48 Find an equation of the slant asymptote. Do not sketch the curve.

45. $y = \dfrac{x^2 + 1}{x + 1}$

46. $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$

47. $y = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}$ **48.** $y = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x}$

49–54 Use the guidelines of this section to sketch the curve. In guideline D find an equation of the slant asymptote.

49. $y = \dfrac{x^2}{x - 1}$ **50.** $y = \dfrac{1 + 5x - 2x^2}{x - 2}$

51. $y = \dfrac{x^3 + 4}{x^2}$ **52.** $y = \dfrac{x^3}{(x + 1)^2}$

53. $y = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$ **54.** $y = \dfrac{(x + 1)^3}{(x - 1)^2}$

55. Show that the curve $y = \sqrt{4x^2 + 9}$ has two slant asymptotes: $y = 2x$ and $y = -2x$. Use this fact to help sketch the curve.

56. Show that the curve $y = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$. Use this fact to help sketch the curve.

57. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

58. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

59. Discuss the asymptotic behavior of

$$f(x) = \frac{x^4 + 1}{x}$$

in the same manner as in Exercise 58. Then use your results to help sketch the graph of f.

60. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

3.6 Graphing with Calculus *and* Calculators

You may want to read "Graphing Calculators and Computers" at *www.stewartcalculus.com* if you haven't already. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of much of our study of differential calculus. The graph was the final object that we produced. In this section our point of view is completely different. Here we *start* with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve. And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the *interaction* between calculus and calculators.

EXAMPLE 1 Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

SOLUTION If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that $-5 \leqslant x \leqslant 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail. So we change to the viewing rectangle $[-3, 2]$ by $[-50, 100]$ shown in Figure 2.

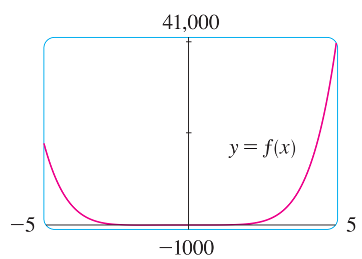


FIGURE 1

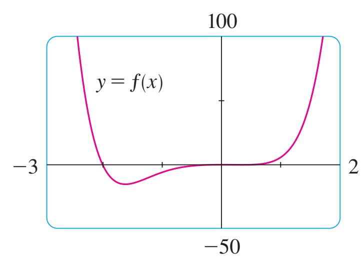


FIGURE 2

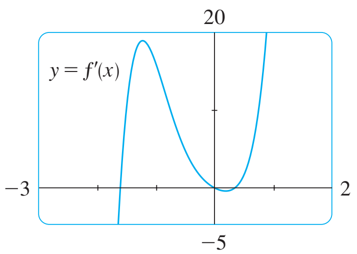


FIGURE 3

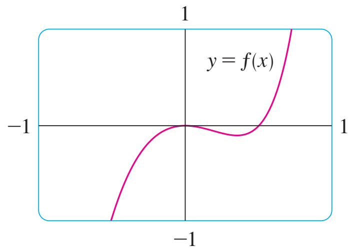


FIGURE 4

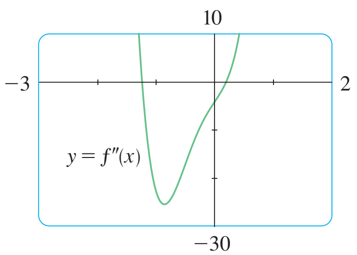


FIGURE 5

From Figure 2 it appears that there is an absolute minimum value of about -15.33 when $x \approx -1.62$ (by using the cursor) and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$. Also there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when x is somewhere between -2 and -1.

Now let's try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

When we graph f' in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$. This means that f has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about -0.1 when $x \approx 0.35$.

What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when x is a little to the left of -1 and when x is a little to the right of 0. But it's difficult to determine inflection points from the graph of f, so we graph the second derivative f'' in Figure 5. We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture. ■

EXAMPLE 2 Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

SOLUTION Figure 6, produced by a computer with automatic scaling, is a disaster. Some graphing calculators use $[-10, 10]$ by $[-10, 10]$ as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7; it's a major improvement.

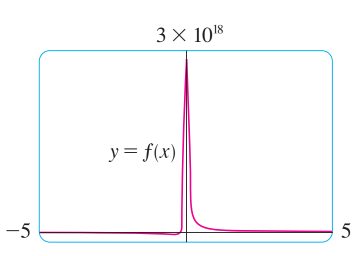


FIGURE 6

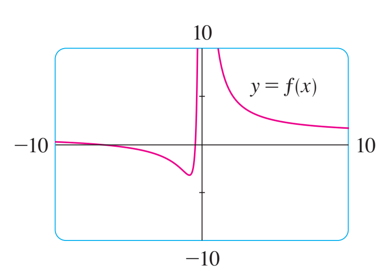


FIGURE 7

The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x\to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

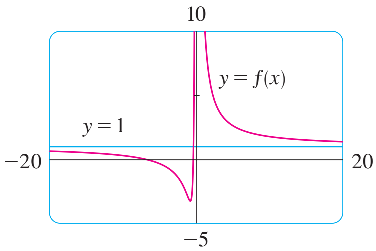


FIGURE 8

Figure 7 also allows us to estimate the x-intercepts: about -0.5 and -6.5. The exact values are obtained by using the quadratic formula to solve the equation $x^2 + 7x + 3 = 0$; we get $x = \left(-7 \pm \sqrt{37}\right)/2$.

To get a better look at horizontal asymptotes, we change to the viewing rectangle $[-20, 20]$ by $[-5, 10]$ in Figure 8. It appears that $y = 1$ is the horizontal asymptote and this is easily confirmed:

$$\lim_{x\to\pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x\to\pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1$$

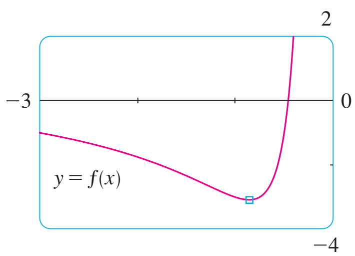


FIGURE 9

To estimate the minimum value we zoom in to the viewing rectangle $[-3, 0]$ by $[-4, 2]$ in Figure 9. The cursor indicates that the absolute minimum value is about -3.1 when $x \approx -0.9$, and we see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$ and increases on $(-0.9, 0)$. The exact values are obtained by differentiating:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3}$$

This shows that $f'(x) > 0$ when $-\frac{6}{7} < x < 0$ and $f'(x) < 0$ when $x < -\frac{6}{7}$ and when $x > 0$. The exact minimum value is $f\left(-\frac{6}{7}\right) = -\frac{37}{12} \approx -3.08$.

Figure 9 also shows that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate it much more accurately using the graph of the second derivative, but in this case it's just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = \frac{2(7x + 9)}{x^4}$$

we see that $f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$. So f is concave upward on $\left(-\frac{9}{7}, 0\right)$ and $(0, \infty)$ and concave downward on $\left(-\infty, -\frac{9}{7}\right)$. The inflection point is $\left(-\frac{9}{7}, -\frac{71}{27}\right)$.

The analysis using the first two derivatives shows that Figure 8 displays all the major aspects of the curve. ■

EXAMPLE 3 Graph the function $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

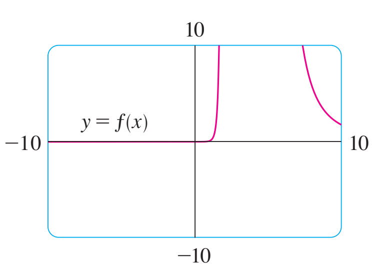


FIGURE 10

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x - 2)^2$ and $(x - 4)^4$ in the denominator, we expect $x = 2$ and $x = 4$ to be the vertical asymptotes. Indeed

$$\lim_{x\to 2} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty \qquad \text{and} \qquad \lim_{x\to 4} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty$$

To find the horizontal asymptotes, we divide numerator and denominator by x^6:

$$\frac{x^2(x+1)^3}{(x-2)^2(x-4)^4} = \frac{\dfrac{x^2}{x^3}\cdot\dfrac{(x+1)^3}{x^3}}{\dfrac{(x-2)^2}{x^2}\cdot\dfrac{(x-4)^4}{x^4}} = \frac{\dfrac{1}{x}\left(1+\dfrac{1}{x}\right)^3}{\left(1-\dfrac{2}{x}\right)^2\left(1-\dfrac{4}{x}\right)^4}$$

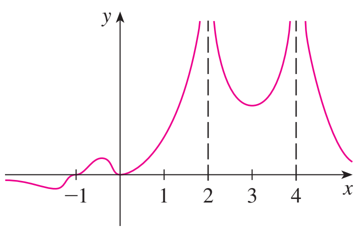


FIGURE 11

This shows that $f(x)\to 0$ as $x\to\pm\infty$, so the x-axis is a horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x-intercepts using an analysis like that in Example 3.4.11. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the x-axis at 0. But, because of the factor $(x+1)^3$, the graph does cross the x-axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

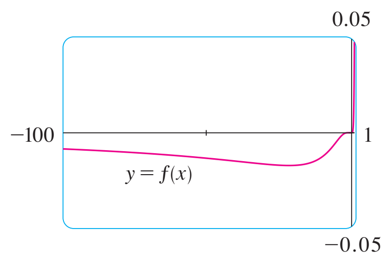


FIGURE 12

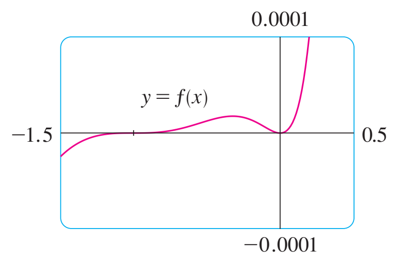


FIGURE 13

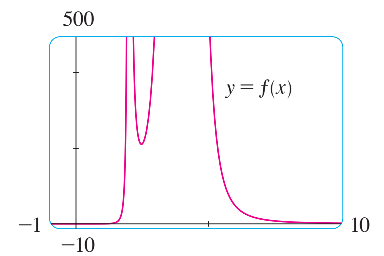


FIGURE 14

We can read from these graphs that the absolute minimum is about -0.02 and occurs when $x \approx -20$. There is also a local maximum ≈ 0.00002 when $x \approx -0.3$ and a local minimum ≈ 211 when $x \approx 2.5$. These graphs also show three inflection points near -35, -5, and -1 and two between -1 and 0. To estimate the inflection points closely we would need to graph f'', but to compute f'' by hand is an unreasonable chore. If you have a computer algebra system, then it's easy to do (see Exercise 13).

We have seen that, for this particular function, *three* graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function. ■

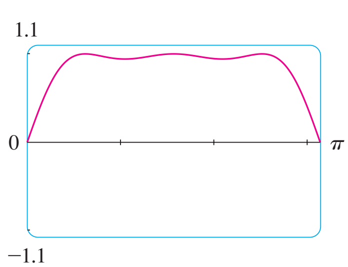


FIGURE 15

EXAMPLE 4 Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leqslant x \leqslant \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points.

SOLUTION We first note that f is periodic with period 2π. Also, f is odd and $|f(x)| \leqslant 1$ for all x. So the choice of a viewing rectangle is not a problem for this function: We start with $[0, \pi]$ by $[-1.1, 1.1]$. (See Figure 15.) It appears that there are three local maximum values and two local minimum values in that window. To confirm

The family of functions

$$f(x) = \sin(x + \sin cx)$$

where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency ($\sin cx$). The case where $c = 2$ is studied in Example 4. Exercise 19 explores another special case.

this and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

and graph both f and f' in Figure 16.

Using zoom-in and the First Derivative Test, we find the following approximate values:

Intervals of increase:	$(0, 0.6),\ (1.0, 1.6),\ (2.1, 2.5)$
Intervals of decrease:	$(0.6, 1.0),\ (1.6, 2.1),\ (2.5, \pi)$
Local maximum values:	$f(0.6) \approx 1,\ f(1.6) \approx 1,\ f(2.5) \approx 1$
Local minimum values:	$f(1.0) \approx 0.94,\ f(2.1) \approx 0.94$

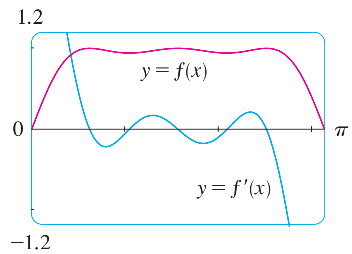


FIGURE 16

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4 \sin 2x \cos(x + \sin 2x)$$

Graphing both f and f'' in Figure 17, we obtain the following approximate values:

Concave upward on:	$(0.8, 1.3),\ (1.8, 2.3)$
Concave downward on:	$(0, 0.8),\ (1.3, 1.8),\ (2.3, \pi)$
Inflection points:	$(0, 0),\ (0.8, 0.97),\ (1.3, 0.97),\ (1.8, 0.97),\ (2.3, 0.97)$

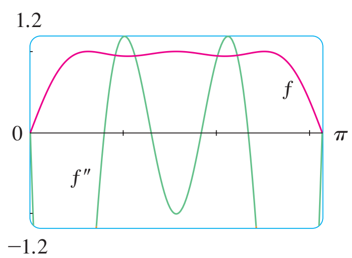


FIGURE 17

Having checked that Figure 15 does indeed represent f accurately for $0 \leqslant x \leqslant \pi$, we can state that the extended graph in Figure 18 represents f accurately for $-2\pi \leqslant x \leqslant 2\pi$. ■

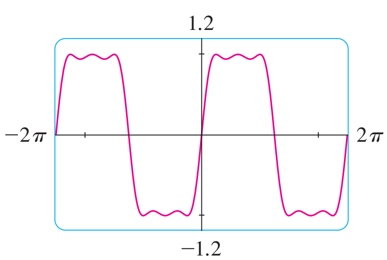


FIGURE 18

Our final example is concerned with *families* of functions. This means that the functions in the family are related to each other by a formula that contains one or more arbitrary constants. Each value of the constant gives rise to a member of the family and the idea is to see how the graph of the function changes as the constant changes.

EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies?

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) show two very different-looking curves.

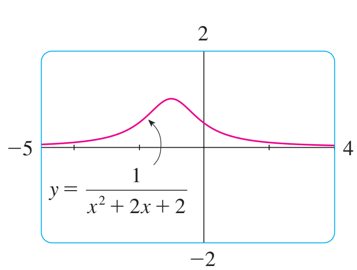


FIGURE 19
$c = 2$

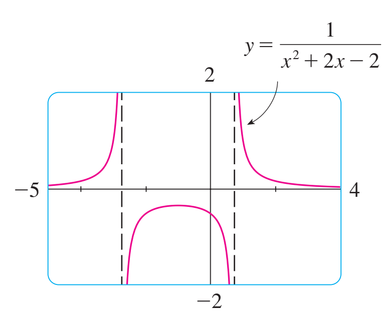


FIGURE 20
$c = -2$

Before drawing any more graphs, let's see what members of this family have in common. Since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

for any value of c, they all have the x-axis as a horizontal asymptote. A vertical asymptote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19). When $c = 1$, the graph has a single vertical asymptote $x = -1$ because

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

When $c < 1$, there are two vertical asymptotes: $x = -1 \pm \sqrt{1 - c}$ (as in Figure 20).

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geqslant 1$, this means that f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and the intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed in the viewing rectangle $[-5, 4]$ by $[-2, 2]$. As predicted, a transition takes place from two vertical asymptotes to one at $c = 1$, and then to none for $c > 1$. As c increases from 1, we see that the maximum point becomes lower; this is explained by the fact that $1/(c - 1) \to 0$ as $c \to \infty$. As c decreases from 1, the vertical asymptotes become more widely separated because the distance between them is $2\sqrt{1 - c}$, which becomes large as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c - 1) \to 0$ as $c \to -\infty$.

TEC See an animation of Figure 21 in Visual 3.6.

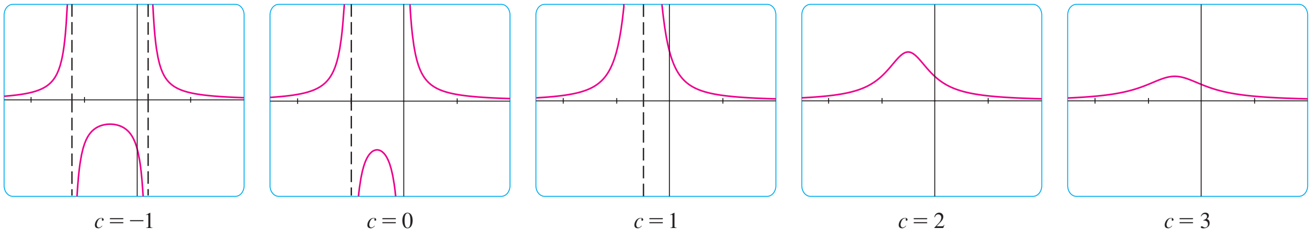


FIGURE 21
The family of functions $f(x) = 1/(x^2 + 2x + c)$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c - 1)}/3$. So the inflection points become more spread out as c increases and this seems plausible from the last two parts of Figure 21. ■

3.6 EXERCISES

1–8 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = x^5 - 5x^4 - x^3 + 28x^2 - 2x$

2. $f(x) = -2x^6 + 5x^5 + 140x^3 - 110x^2$

3. $f(x) = x^6 - 5x^5 + 25x^3 - 6x^2 - 48x$

4. $f(x) = \dfrac{x^4 - x^3 - 8}{x^2 - x - 6}$

5. $f(x) = \dfrac{x}{x^3 + x^2 + 1}$

6. $f(x) = 6 \sin x - x^2, \quad -5 \leqslant x \leqslant 3$

7. $f(x) = 6 \sin x + \cot x, \quad -\pi \leqslant x \leqslant \pi$

8. $f(x) = \dfrac{\sin x}{x}, \quad -2\pi \leqslant x \leqslant 2\pi$

9–10 Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease and intervals of concavity, and use calculus to find these intervals exactly.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$

10. $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

11–12 Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

11. $f(x) = \dfrac{(x + 4)(x - 3)^2}{x^4(x - 1)}$

12. $f(x) = \dfrac{(2x + 3)^2(x - 2)^5}{x^3(x - 5)^2}$

CAS **13.** If f is the function considered in Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f'' and use it to estimate the intervals of concavity and inflection points.

CAS **14.** If f is the function of Exercise 12, find f' and f'' and use their graphs to estimate the intervals of increase and decrease and concavity of f.

CAS **15–18** Use a computer algebra system to graph f and to find f' and f''. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of f.

15. $f(x) = \dfrac{x^3 + 5x^2 + 1}{x^4 + x^3 - x^2 + 2}$

16. $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

17. $f(x) = \sqrt{x + 5 \sin x}, \quad x \leqslant 20$

18. $f(x) = \dfrac{2x - 1}{\sqrt[4]{x^4 + x + 1}}$

19. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing f in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graph of f'' at the same time. Find all the maximum and minimum values and inflection points. Then graph f in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

20–25 Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

20. $f(x) = x^3 + cx$

21. $f(x) = x^2 + 6x + c/x \quad$ (Trident of Newton)

22. $f(x) = x\sqrt{c^2 - x^2}$

23. $f(x) = \dfrac{cx}{1 + c^2x^2}$

24. $f(x) = \dfrac{\sin x}{c + \cos x}$

25. $f(x) = cx + \sin x$

26. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

27. (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of c does the curve have minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

28. (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?

(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

3.7 Optimization Problems

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 98 and adapt them to this situation:

PS

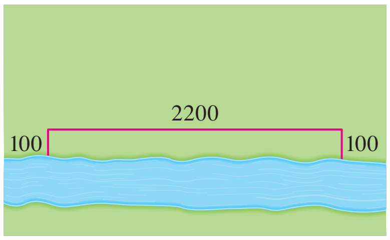
Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 3.1 and 3.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 3.1 can be used.

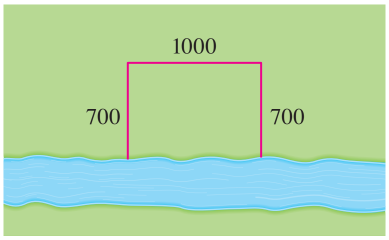
EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

PS Understand the problem
PS Analogy: Try special cases
PS Draw diagrams

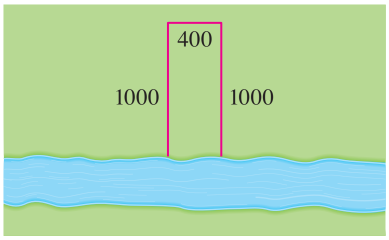
SOLUTION In order to get a feeling for what is happening in this problem, let's experiment with some specific cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing.



Area = 100 · 2200 = 220,000 ft²



Area = 700 · 1000 = 700,000 ft²



Area = 1000 · 400 = 400,000 ft²

FIGURE 1

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

PS Introduce notation

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y:

$$A = xy$$

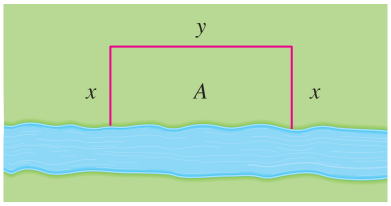


FIGURE 2

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = xy = x(2400 - 2x) = 2400x - 2x^2$$

Note that the largest x can be is 1200 (this uses all the fence for the depth and none for the width) and x can't be negative, so the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leqslant x \leqslant 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

The corresponding y-value is $y = 2400 - 2(600) = 1200$, so the rectangular field should be 600 ft deep and 1200 ft wide. ■

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

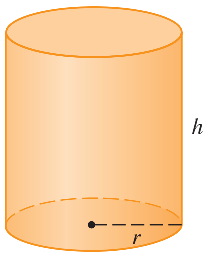


FIGURE 3

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

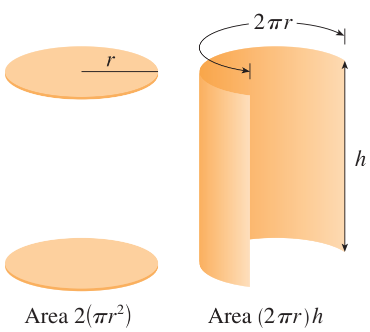


FIGURE 4

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know r must be positive, and there are no limitations on how large r can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

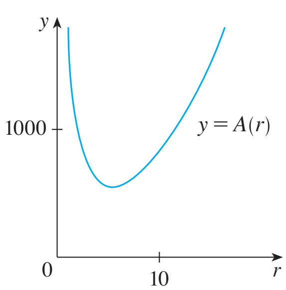


FIGURE 5

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for *all* r to the left of the critical number and increasing for *all* r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

In the Applied Project on page 270 we investigate the most economical shape for a can by taking into account other manufacturing costs.

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

NOTE 1 The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to *local* maximum or minimum values) and is stated here for future reference.

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

TEC Module 3.7 takes you through six additional optimization problems, including animations of the physical situations.

NOTE 2 An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 1000$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r:

$$A' = 4\pi r + 2\pi rh' + 2\pi h \qquad \pi r^2 h' + 2\pi rh = 0$$

The minimum occurs at a critical number, so we set $A' = 0$, simplify, and arrive at the equations

$$2r + rh' + h = 0 \qquad rh' + 2h = 0$$

Subtraction of these equations gives $2r - h = 0$, or $h = 2r$.

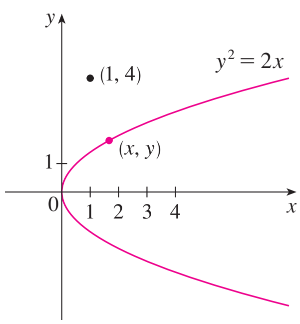


FIGURE 6

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$

(See Figure 6.) But if (x, y) lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.) Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Note that there are no restrictions on y, so the domain is all real numbers. Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y - 4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. [The distance between the points is $d = \sqrt{f(2)} = \sqrt{5}$.] ■

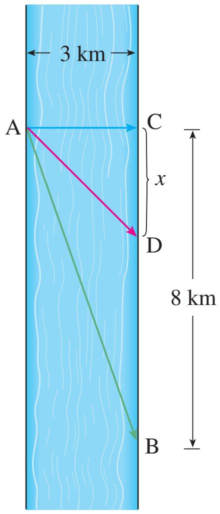


FIGURE 7

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

SOLUTION If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9}/6$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$, he rows to C and if $x = 8$, he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geqslant 0$, we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9}$$

$$\iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81$$

$$\iff x = \frac{9}{\sqrt{7}}$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we follow the Closed Interval Method by evaluating T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

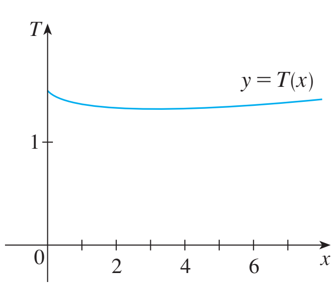


FIGURE 8

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Figure 8 illustrates this calculation by showing the graph of T.

Thus the man should land the boat at a point $9/\sqrt{7}$ km ($\approx$3.4 km) downstream from his starting point. ■

EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

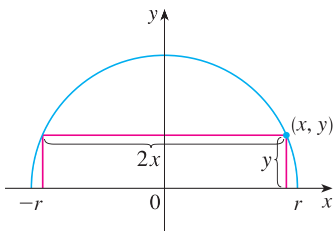


FIGURE 9

SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\,\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

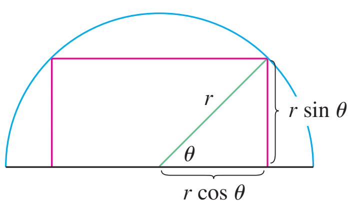


FIGURE 10

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all. ■

Applications to Business and Economics

In Section 2.7 we introduced the idea of marginal cost. Recall that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x. In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x. (More units sold corresponds to a lower price.) If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of

the profit function. In Exercises 59–63 you are asked to use the marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

EXAMPLE 6 A store has been selling 200 flat-screen TVs a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of TVs sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

SOLUTION If x is the number of TVs sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of \$125. ■

3.7 EXERCISES

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
(a) Make a table of values, like the one below, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.
(b) Use calculus to solve the problem and compare with your answer to part (a).

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. What is the maximum vertical distance between the line $y = x + 2$ and the parabola $y = x^2$ for $-1 \leqslant x \leqslant 2$?

6. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

7. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

8. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

9. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

10. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

11. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide

it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

12. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the volume as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

13. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

14. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

15. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

16. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

17. Do Exercise 16 assuming the container has a lid that is made from the same material as the sides.

18. A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs \$20 per linear foot to install and the farmer is not willing to spend more than \$5000, find the dimensions for the plot that would enclose the most area.

19. If the farmer in Exercise 18 wants to enclose 8000 square feet of land, what dimensions will minimize the cost of the fence?

20. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

21. Find the point on the line $y = 2x + 3$ that is closest to the origin.

22. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

23. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

24. Find, correct to two decimal places, the coordinates of the point on the curve $y = \sin x$ that is closest to the point $(4, 2)$.

25. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

26. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

27. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

28. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

29. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

30. If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

31. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

32. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

33. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

34. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 1.1.62.) If the perimeter of the window is 30 ft, find the dimensions

of the window so that the greatest possible amount of light is admitted.

35. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

36. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

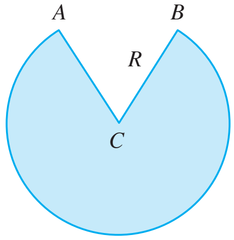
37. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

38. Solve Exercise 37 if one piece is bent into a square and the other into a circle.

39. If you are offered one slice from a round pizza (in other words, a sector of a circle) and the slice must have a perimeter of 32 inches, what diameter pizza will reward you with the largest slice?

40. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

41. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



42. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

43. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

44. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

45. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

46. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.

(a) Determine the value of v that minimizes E.

(b) Sketch the graph of E.

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

47. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the figure. It is believed that bees form their cells in such a way as to minimize the surface area for a given side length and height, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot \theta + \left(3s^2\sqrt{3}/2\right) \csc \theta$$

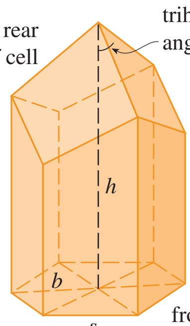
where s, the length of the sides of the hexagon, and h, the height, are constants.

(a) Calculate $dS/d\theta$.

(b) What angle should the bees prefer?

(c) Determine the minimum surface area of the cell (in terms of s and h).

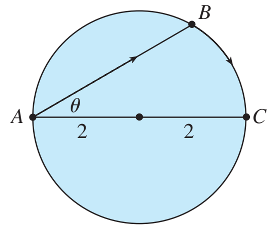
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



48. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

49. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

50. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?



51. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

52. Suppose the refinery in Exercise 51 is located 1 km north of the river. Where should P be located?

53. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

54. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

55. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

56. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

57. What is the shortest possible length of the line segment that is cut off by the first quadrant and is tangent to the curve $y = 3/x$ at some point?

58. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at some point?

59. (a) If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.
(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

60. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.
(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

61. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.
(a) Find the demand function, assuming that it is linear.
(b) How should ticket prices be set to maximize revenue?

62. During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that the average decreased by two sales per day.
(a) Find the demand function, assuming that it is linear.
(b) If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

63. A retailer has been selling 1200 tablet computers a week at \$350 each. The marketing department estimates that an additional 80 tablets will sell each week for every \$10 that the price is lowered.
(a) Find the demand function.
(b) What should the price be set at in order to maximize revenue?
(c) If the retailer's weekly cost function is

$$C(x) = 35{,}000 + 120x$$

what price should it choose in order to maximize its profit?

64. A company operates 16 oil wells in a designated area. Each pump, on average, extracts 240 barrels of oil daily. The company can add more wells but every added well reduces the average daily ouput of each of the wells by 8 barrels. How many wells should the company add in order to maximize daily production?

65. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

66. Consider the situation in Exercise 51 if the cost of laying pipe under the river is considerably higher than the cost of laying pipe over land ($400,000/km). You may suspect that in some instances, the minimum distance possible under the river should be used, and P should be located 6 km from the refinery, directly across from the storage tanks. Show that this is *never* the case, no matter what the "under river" cost is.

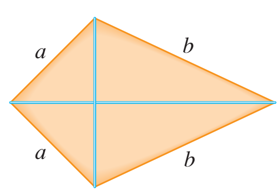
67. Consider the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at a point (p, q) in the first quadrant.

(a) Show that the tangent line has x-intercept a^2/p and y-intercept b^2/q.

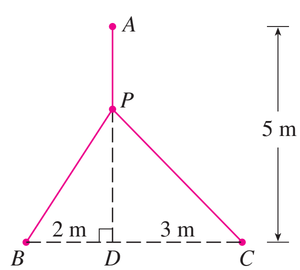
(b) Show that the portion of the tangent line cut off by the coordinate axes has minimum length $a + b$.

(c) Show that the triangle formed by the tangent line and the coordinate axes has minimum area ab.

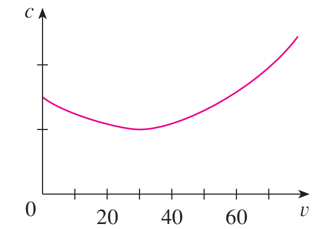
CAS **68.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



69. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B. and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value of L.



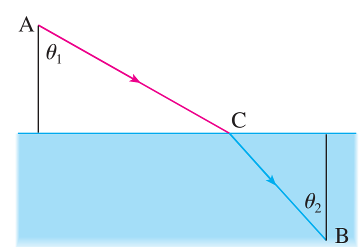
70. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



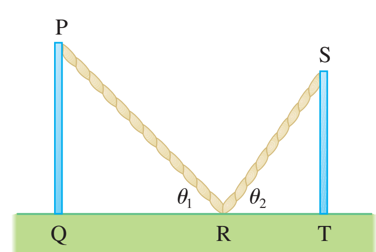
71. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

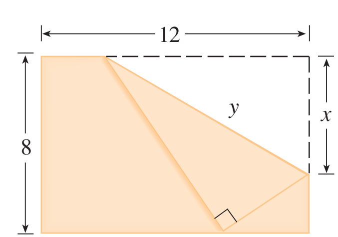
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



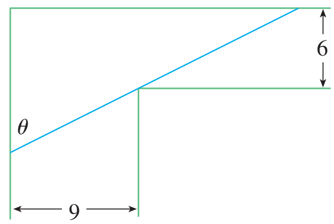
72. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



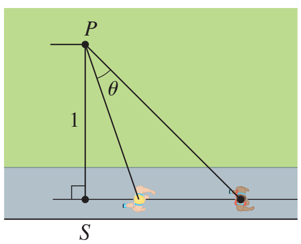
73. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



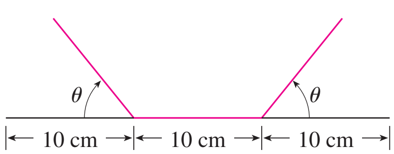
74. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



75. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners. [*Hint:* Maximize $\tan\theta$.]



76. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?

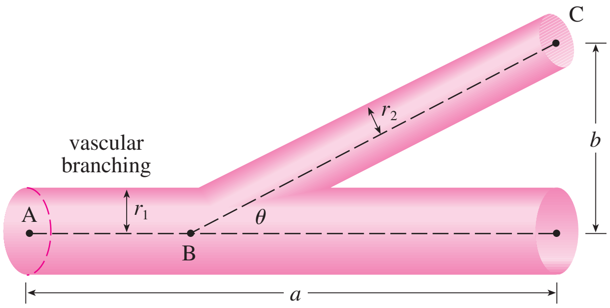


77. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W. [*Hint:* Express the area as a function of an angle θ.]

78. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 8.4.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

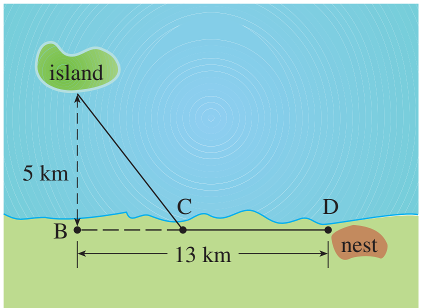
79. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as it does over land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than over land?



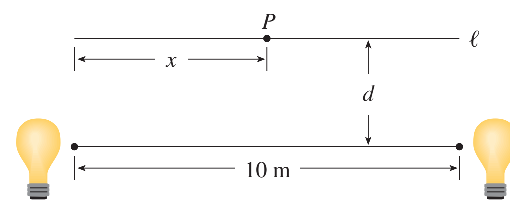
80. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ, parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

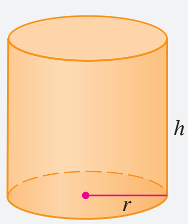
(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

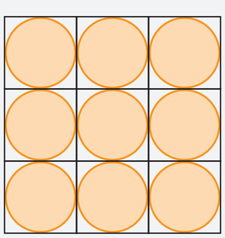
(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.

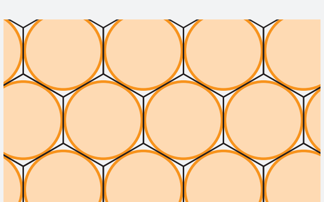


APPLIED PROJECT THE SHAPE OF A CAN





Discs cut from squares



Discs cut from hexagons

In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 3.7.2 and we found that $h = 2r$; that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this phenomenon.

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side $2r$ (as in the second figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$

2. A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the last figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$

3. The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate

the manufacturing of the can into the cost. Let's assume that most of the expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}\,r^2 + 2\pi rh + k(4\pi r + h)$$

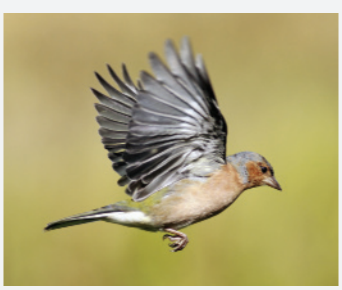
where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$$

4. Plot $\sqrt[3]{V}/k$ as a function of $x = h/r$ and use your graph to argue that when a can is large or joining is cheap, we should make h/r approximately 2.21 (as in Problem 2). But when the can is small or joining is costly, h/r should be substantially larger.

5. Our analysis shows that large cans should be almost square but small cans should be tall and thin. Take a look at the relative shapes of the cans in a supermarket. Is our conclusion usually true in practice? Are there exceptions? Can you suggest reasons why small cans are not always tall and thin?

APPLIED PROJECT — PLANES AND BIRDS: MINIMIZING ENERGY



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Small birds like finches alternate between flapping their wings and keeping them folded while gliding (see Figure 1). In this project we analyze this phenomenon and try to determine how frequently a bird should flap its wings. Some of the principles are the same as for fixed-wing aircraft and so we begin by considering how required power and energy depend on the speed of airplanes.[1]

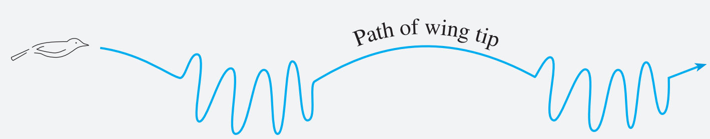


FIGURE 1

1. The power needed to propel an airplane forward at velocity v is

$$P = Av^3 + \frac{BL^2}{v}$$

where A and B are positive constants specific to the particular aircraft and L is the lift, the upward force supporting the weight of the plane. Find the speed that minimizes the required power.

2. The speed found in Problem 1 minimizes power but a faster speed might use less fuel. The energy needed to propel the airplane a unit distance is $E = P/v$. At what speed is energy minimized?

1. Adapted from R. McNeill Alexander, *Optima for Animals* (Princeton, NJ: Princeton University Press, 1996.)

3. Hows much faster is the speed for minimum energy than the speed for minimum power?

4. In applying the equation of Problem 1 to bird flight we split the term Av^3 into two parts: $A_b v^3$ for the bird's body and $A_w v^3$ for its wings. Let x be the fraction of flying time spent in flapping mode. If m is the bird's mass and all the lift occurs during flapping, then the lift is mg/x and so the power needed during flapping is

$$P_{\text{flap}} = (A_b + A_w)v^3 + \frac{B(mg/x)^2}{v}$$

The power while wings are folded is $P_{\text{fold}} = A_b v^3$. Show that the average power over an entire flight cycle is

$$\overline{P} = xP_{\text{flap}} + (1 - x)P_{\text{fold}} = A_b v^3 + xA_w v^3 + \frac{Bm^2g^2}{xv}$$

5. For what value of x is the average power a minimum? What can you conclude if the bird flies slowly? What can you conclude if the bird flies faster and faster?

6. The average energy over a cycle is $\overline{E} = \overline{P}/v$. What value of x minimizes $\overline{E}$?

3.8 Newton's Method

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

$$\boxed{1} \qquad 48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

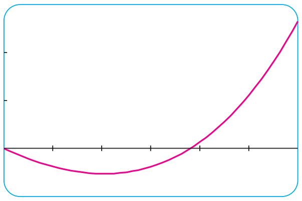


FIGURE 1

Try to solve Equation 1 numerically using your calculator or computer. Some machines are not able to solve it. Others are successful but require you to specify a starting point for the search.

(The details are explained in Exercise 39.) How would you solve such an equation?

For a quadratic equation $ax^2 + bx + c = 0$ there is a well-known formula for the solutions. For third- and fourth-degree equations there are also formulas for the solutions, but they are extremely complicated. If f is a polynomial of degree 5 or higher, there is no such formula (see the note on page 164). Likewise, there is no formula that will enable us to find the exact roots of a transcendental equation such as $\cos x = x$.

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation. Using a graphing device, and after experimenting with viewing rectangles, we produce the graph in Figure 1.

We see that in addition to the solution $x = 0$, which doesn't interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the root is approximately 0.0076. If we need more accuracy we could zoom in repeatedly, but that becomes tiresome. A faster alternative is to use a calculator or computer algebra system to solve the equation numerically. If we do so, we find that the root, correct to nine decimal places, is 0.007628603.

How do these devices solve equations? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**. We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

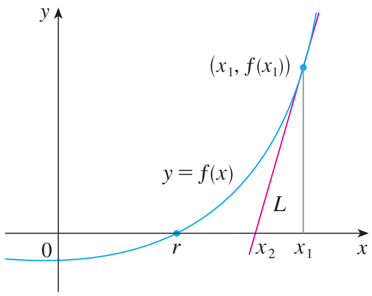


FIGURE 2

The geometry behind Newton's method is shown in Figure 2. We wish to solve an equation of the form $f(x) = 0$, so the roots of the equation correspond to the x-intercepts of the graph of f. The root that we are trying to find is labeled r in the figure. We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of f, or from a computer-generated graph of f. Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2. The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x-intercept.

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, we know that the point $(x_2, 0)$ is on the line, and so

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by the second approximation x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

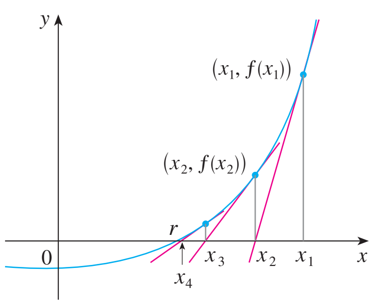


FIGURE 3

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in Figure 3. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

2
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Sequences were briefly introduced in *A Preview of Calculus* on page 5. A more thorough discussion starts in Section 11.1.

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

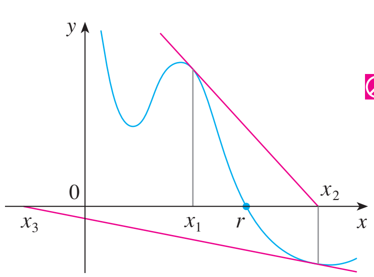


FIGURE 4

Although the sequence of successive approximations converges to the desired root for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge. For example, consider the situation shown in Figure 4. You can see that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation x_1 should be chosen. See Exercises 29–32 for specific examples in which Newton's method works very slowly or does not work at all.

EXAMPLE 1 Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

TEC In Module 3.8 you can investigate how Newton's method works for several functions and what happens when you change x_1.

SOLUTION We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Equation 2 becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places. ■

Figure 5 shows the geometry behind the first step in Newton's method in Example 1. Since $f'(2) = 10$, the tangent line to $y = x^3 - 2x - 5$ at $(2, -1)$ has equation $y = 10x - 21$ so its x-intercept is $x_2 = 2.1$.

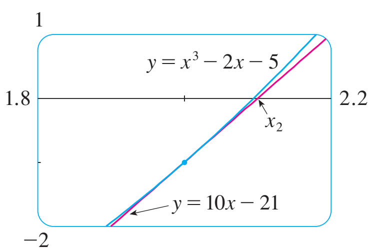


FIGURE 5

Suppose that we want to achieve a given accuracy, say to eight decimal places, using Newton's method. How do we know when to stop? The rule of thumb that is generally used is that we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places. (A precise statement concerning accuracy in Newton's method will be given in Exercise 11.11.39.)

Notice that the procedure in going from n to $n + 1$ is the same for all values of n. (It is called an *iterative* process.) This means that Newton's method is particularly convenient for use with a programmable calculator or a computer.

EXAMPLE 2 Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

SOLUTION First we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation

$$x^6 - 2 = 0$$

so we take $f(x) = x^6 - 2$. Then $f'(x) = 6x^5$ and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$x_2 \approx 1.16666667$$
$$x_3 \approx 1.12644368$$
$$x_4 \approx 1.12249707$$
$$x_5 \approx 1.12246205$$
$$x_6 \approx 1.12246205$$

Since x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places. ■

EXAMPLE 3 Find, correct to six decimal places, the root of the equation $\cos x = x$.

SOLUTION We first rewrite the equation in standard form:

$$\cos x - x = 0$$

Therefore we let $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$, so Formula 2 becomes

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

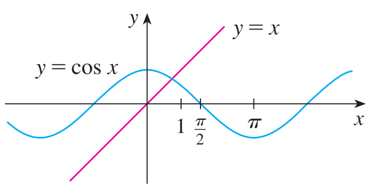


FIGURE 6

In order to guess a suitable value for x_1 we sketch the graphs of $y = \cos x$ and $y = x$ in Figure 6. It appears that they intersect at a point whose x-coordinate is somewhat less than 1, so let's take $x_1 = 1$ as a convenient first approximation. Then, remembering to put our calculator in radian mode, we get

$$x_2 \approx 0.75036387$$
$$x_3 \approx 0.73911289$$
$$x_4 \approx 0.73908513$$
$$x_5 \approx 0.73908513$$

Since x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085. ■

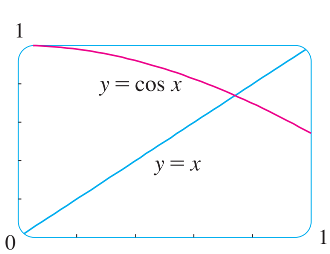


FIGURE 7

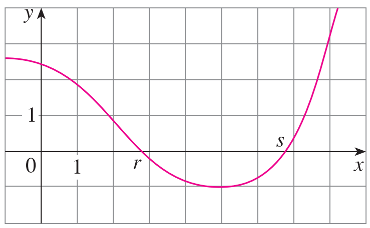
Instead of using the rough sketch in Figure 6 to get a starting approximation for Newton's method in Example 3, we could have used the more accurate graph that a calculator or computer provides. Figure 7 suggests that we use $x_1 = 0.75$ as the initial approximation. Then Newton's method gives

$$x_2 \approx 0.73911114$$
$$x_3 \approx 0.73908513$$
$$x_4 \approx 0.73908513$$

and so we obtain the same answer as before, but with one fewer step.

3.8 EXERCISES

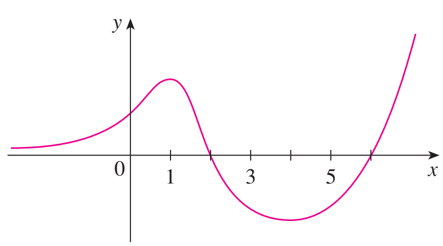
1. The figure shows the graph of a function f. Suppose that Newton's method is used to approximate the root s of the equation $f(x) = 0$ with initial approximation $x_1 = 6$.
(a) Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.
(b) Would $x_1 = 8$ be a better first approximation? Explain.



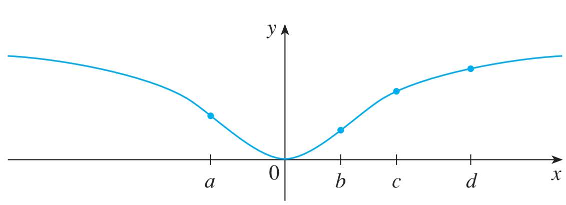
2. Follow the instructions for Exercise 1(a) but use $x_1 = 1$ as the starting approximation for finding the root r.

3. Suppose the tangent line to the curve $y = f(x)$ at the point $(2, 5)$ has the equation $y = 9 - 2x$. If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 2$, find the second approximation x_2.

4. For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.
(a) $x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$
(d) $x_1 = 4$ (e) $x_1 = 5$



5. For which of the initial approximations $x_1 = a, b, c$, and d do you think Newton's method will work and lead to the root of the equation $f(x) = 0$?



6–8 Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places.)

6. $2x^3 - 3x^2 + 2 = 0, \quad x_1 = -1$

7. $\dfrac{2}{x} - x^2 + 1 = 0, \quad x_1 = 2$

8. $x^7 + 4 = 0, \quad x_1 = -1$

9. Use Newton's method with initial approximation $x_1 = -1$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Explain how the method works by first graphing the function and its tangent line at $(-1, 1)$.

10. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 - x - 1 = 0$. Explain how the method works by first graphing the function and its tangent line at $(1, -1)$.

11–12 Use Newton's method to approximate the given number correct to eight decimal places.

11. $\sqrt[4]{75}$

12. $\sqrt[8]{500}$

13–14 (a) Explain how we know that the given equation must have a root in the given interval. (b) Use Newton's method to approximate the root correct to six decimal places.

13. $3x^4 - 8x^3 + 2 = 0, \quad [2, 3]$

14. $-2x^5 + 9x^4 - 7x^3 - 11x = 0, \quad [3, 4]$

15–16 Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

15. The positive root of $\sin x = x^2$

16. The positive root of $3 \sin x = x$

17–22 Use Newton's method to find all solutions of the equation correct to six decimal places.

17. $3 \cos x = x + 1$

18. $\sqrt{x + 1} = x^2 - x$

19. $\dfrac{1}{x} = \sqrt[3]{x} - 1$

20. $(x - 1)^2 = \sqrt{x}$

21. $x^3 = \cos x$

22. $\sin x = x^2 - 2$

23–26 Use Newton's method to find all the solutions of the equation correct to eight decimal places. Start by drawing a graph to find initial approximations.

23. $-2x^7 - 5x^4 + 9x^3 + 5 = 0$

24. $x^5 - 3x^4 + x^3 - x^2 - x + 6 = 0$

25. $\dfrac{x}{x^2 + 1} = \sqrt{1 - x}$

26. $\cos(x^2 - x) = x^4$

27. (a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm (used by the ancient Babylonians to compute $\sqrt{a}$):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Use part (a) to compute $\sqrt{1000}$ correct to six decimal places.

28. (a) Apply Newton's method to the equation $1/x - a = 0$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2$$

(This algorithm enables a computer to find reciprocals without actually dividing.)

(b) Use part (a) to compute $1/1.6984$ correct to six decimal places.

29. Explain why Newton's method doesn't work for finding the root of the equation

$$x^3 - 3x + 6 = 0$$

if the initial approximation is chosen to be $x_1 = 1$.

30. (a) Use Newton's method with $x_1 = 1$ to find the root of the equation $x^3 - x = 1$ correct to six decimal places.

(b) Solve the equation in part (a) using $x_1 = 0.6$ as the initial approximation.

(c) Solve the equation in part (a) using $x_1 = 0.57$. (You definitely need a programmable calculator for this part.)

(d) Graph $f(x) = x^3 - x - 1$ and its tangent lines at $x_1 = 1$, 0.6, and 0.57 to explain why Newton's method is so sensitive to the value of the initial approximation.

31. Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

32. If

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

then the root of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails to find the root no matter which initial approximation $x_1 \neq 0$ is used. Illustrate your explanation with a sketch.

33. (a) Use Newton's method to find the critical numbers of the function

$$f(x) = x^6 - x^4 + 3x^3 - 2x$$

correct to six decimal places.

(b) Find the absolute minimum value of f correct to four decimal places.

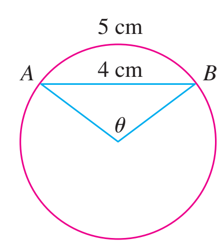
34. Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$, $0 \leqslant x \leqslant \pi$, correct to six decimal places.

35. Use Newton's method to find the coordinates of the inflection point of the curve $y = x^2 \sin x$, $0 \leqslant x \leqslant \pi$, correct to six decimal places.

36. Of the infinitely many lines that are tangent to the curve $y = -\sin x$ and pass through the origin, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.

37. Use Newton's method to find the coordinates, correct to six decimal places, of the point on the parabola $y = (x - 1)^2$ that is closest to the origin.

38. In the figure, the length of the chord AB is 4 cm and the length of the arc AB is 5 cm. Find the central angle θ, in radians, correct to four decimal places. Then give the answer to the nearest degree.



39. A car dealer sells a new car for \$18,000. He also offers to sell the same car for payments of \$375 per month for five years. What monthly interest rate is this dealer charging?

To solve this problem you will need to use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

$$A = \frac{R}{i}[1 - (1 + i)^{-n}]$$

Replacing i by x, show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

40. The figure shows the sun located at the origin and the earth at the point (1, 0). (The unit here is the distance between the centers of the earth and the sun, called an *astronomical unit:* 1 AU $\approx 1.496 \times 10^8$ km.) There are five locations L_1, L_2, L_3, L_4, and L_5 in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite (including the gravitational attractions of the earth and the sun) balance each other. These locations are called *libration points*. (A solar research satellite has been placed

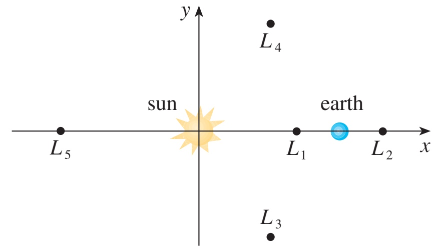
at one of these libration points.) If m_1 is the mass of the sun, m_2 is the mass of the earth, and $r = m_2/(m_1 + m_2)$, it turns out that the x-coordinate of L_1 is the unique root of the fifth-degree equation

$$p(x) = x^5 - (2 + r)x^4 + (1 + 2r)x^3 - (1 - r)x^2 + 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the root of the equation

$$p(x) - 2rx^2 = 0$$

Using the value $r \approx 3.04042 \times 10^{-6}$, find the locations of the libration points (a) L_1 and (b) L_2.



3.9 Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. The question arises: are there any others?

To answer this question, recall that in Section 3.2 we used the Mean Value Theorem to prove that if two functions have identical derivatives on an interval, then they must differ by a constant (Corollary 3.2.7). Thus if F and G are any two antiderivatives of f, then

$$F'(x) = f(x) = G'(x)$$

so $G(x) - F(x) = C$, where C is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

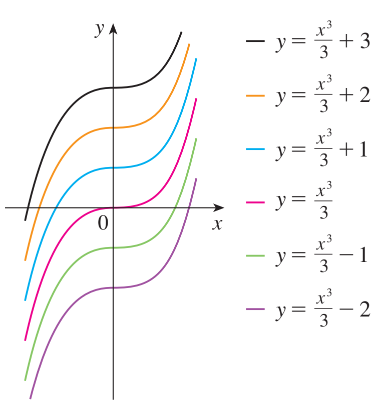


FIGURE 1
Members of the family of antiderivatives of $f(x) = x^2$

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3}x^3 + C$. By assigning specific values to the constant C, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1). This makes sense because each curve must have the same slope at any given value of x.

EXAMPLE 1 Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = x^n$, $n \geqslant 0$ (c) $f(x) = x^{-3}$

SOLUTION
(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

(b) We use the Power Rule to discover an antiderivative of x^n:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Therefore the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$ because then $f(x) = x^n$ is defined on an interval.

(c) If we put $n = -3$ in the antiderivative formula from part (b), we get the particular antiderivative $F(x) = x^{-2}/(-2)$. But notice that $f(x) = x^{-3}$ is not defined at $x = 0$. Thus Theorem 1 tells us only that the general antiderivative of f is $x^{-2}/(-2) + C$ on any interval that does not contain 0. So the general antiderivative of $f(x) = 1/x^3$ is

$$F(x) = \begin{cases} -\dfrac{1}{2x^2} + C_1 & \text{if } x > 0 \\[2ex] -\dfrac{1}{2x^2} + C_2 & \text{if } x < 0 \end{cases}$$ ■

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4 \sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4 \sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4 \sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

We often use a capital letter F to represent an antiderivative of a function f. If we begin with derivative notation, f', an antiderivative is f, of course.

$$g(x) = 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4 \cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$ ■

In applications of calculus it is very common to have a situation as in Example 2, where it is required to find a function, given knowledge about its derivatives. An equation that involves the derivatives of a function is called a **differential equation**. These will be studied in some detail in Chapter 9, but for the present we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants) as in Example 2. However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

EXAMPLE 3 Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

SOLUTION The general antiderivative of

$$f'(x) = x\sqrt{x} = x^{3/2}$$

is

$$f(x) = \frac{x^{5/2}}{\frac{5}{2}} + C = \tfrac{2}{5}x^{5/2} + C$$

To determine C we use the fact that $f(1) = 2$:

$$f(1) = \tfrac{2}{5} + C = 2$$

Solving for C, we get $C = 2 - \frac{2}{5} = \frac{8}{5}$, so the particular solution is

$$f(x) = \frac{2x^{5/2} + 8}{5}$$ ■

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$ ■

If we are given the graph of a function f, it seems reasonable that we should be able to sketch the graph of an antiderivative F. Suppose, for instance, that we are given that $F(0) = 1$. Then we have a place to start, the point $(0, 1)$, and the direction in which we move our pencil is given at each stage by the derivative $F'(x) = f(x)$. In the next example we use the principles of this chapter to show how to graph F even when we don't have a formula for f. This would be the case, for instance, when $f(x)$ is determined by experimental data.

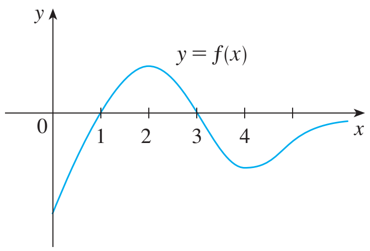


FIGURE 2

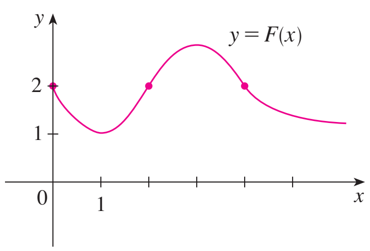


FIGURE 3

EXAMPLE 5 The graph of a function f is given in Figure 2. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.

SOLUTION We are guided by the fact that the slope of $y = F(x)$ is $f(x)$. We start at the point $(0, 2)$ and draw F as an initially decreasing function since $f(x)$ is negative when $0 < x < 1$. Notice that $f(1) = f(3) = 0$, so F has horizontal tangents when $x = 1$ and $x = 3$. For $1 < x < 3$, $f(x)$ is positive and so F is increasing. We see that F has a local minimum when $x = 1$ and a local maximum when $x = 3$. For $x > 3$, $f(x)$ is negative and so F is decreasing on $(3, \infty)$. Since $f(x) \to 0$ as $x \to \infty$, the graph of F becomes flatter as $x \to \infty$. Also notice that $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$, so F has inflection points when $x = 2$ and $x = 4$. We use this information to sketch the graph of the antiderivative in Figure 3. ■

■ Rectilinear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

EXAMPLE 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$ ■

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by g. For motion close to the ground we may assume that g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2).

EXAMPLE 7 A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

SOLUTION The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine C we use the given information that $v(0) = 48$. This gives $48 = 0 + C$, so

$$v(t) = -32t + 48$$

The maximum height is reached when $v(t) = 0$, that is, after 1.5 seconds. Since $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -16t^2 + 48t + D$$

Using the fact that $s(0) = 432$, we have $432 = 0 + D$ and so

$$s(t) = -16t^2 + 48t + 432$$

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-16t^2 + 48t + 432 = 0$$

or, equivalently,

$$t^2 - 3t - 27 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{3 \pm 3\sqrt{13}}{2}$$

We reject the solution with the minus sign since it gives a negative value for t. Therefore the ball hits the ground after $3(1 + \sqrt{13})/2 \approx 6.9$ seconds. ■

Figure 4 shows the position function of the ball in Example 7. The graph corroborates the conclusions we reached: The ball reaches its maximum height after 1.5 seconds and hits the ground after about 6.9 seconds.

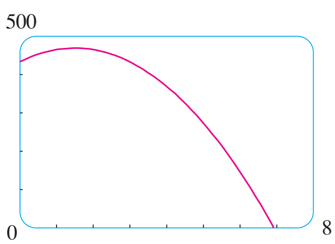


FIGURE 4

3.9 EXERCISES

1–20 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = 4x + 7$
2. $f(x) = x^2 - 3x + 2$
3. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x$
4. $f(x) = 6x^5 - 8x^4 - 9x^2$
5. $f(x) = x(12x + 8)$
6. $f(x) = (x - 5)^2$
7. $f(x) = 7x^{2/5} + 8x^{-4/5}$
8. $f(x) = x^{3.4} - 2x^{\sqrt{2}-1}$
9. $f(x) = \sqrt{2}$
10. $f(x) = \pi^2$
11. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$
12. $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$
13. $f(x) = \dfrac{10}{x^9}$
14. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$
15. $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}}$
16. $f(t) = 3\cos t - 4\sin t$
17. $h(\theta) = 2\sin\theta - \sec^2\theta$
18. $g(v) = 5 + 3\sec^2 v$
19. $f(t) = 8\sqrt{t} - \sec t \tan t$
20. $f(x) = 1 + 2\sin x + 3/\sqrt{x}$

21–22 Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

21. $f(x) = 5x^4 - 2x^5, \quad F(0) = 4$

22. $f(x) = x + 2\sin x, \quad F(0) = -6$

23–42 Find f.

23. $f''(x) = 20x^3 - 12x^2 + 6x$

24. $f''(x) = x^6 - 4x^4 + x + 1$

25. $f''(x) = 4 - \sqrt[3]{x}$

26. $f''(x) = x^{2/3} + x^{-2/3}$

27. $f'''(t) = 12 + \sin t$

28. $f'''(t) = \sqrt{t} - 2\cos t$

29. $f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$

30. $f'(x) = 5x^4 - 3x^2 + 4, \quad f(-1) = 2$

31. $f'(x) = \sqrt{x}\,(6 + 5x), \quad f(1) = 10$

32. $f'(t) = t + 1/t^3, \quad t > 0, \quad f(1) = 6$

33. $f'(t) = \sec t\,(\sec t + \tan t), \quad -\pi/2 < t < \pi/2,$ $f(\pi/4) = -1$

34. $f'(x) = (x + 1)/\sqrt{x}, \quad f(1) = 5$

35. $f''(x) = -2 + 12x - 12x^2, \quad f(0) = 4, \quad f'(0) = 12$

36. $f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$

37. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

38. $f''(t) = 4 - 6/t^4, \quad f(1) = 6, \quad f'(2) = 9, \quad t > 0$

39. $f''(x) = 4 + 6x + 24x^2, \quad f(0) = 3, \quad f(1) = 10$

40. $f''(x) = 20x^3 + 12x^2 + 4, \quad f(0) = 8, \quad f(1) = 5$

41. $f''(t) = \sqrt[3]{t} - \cos t, \quad f(0) = 2, \quad f(1) = 2$

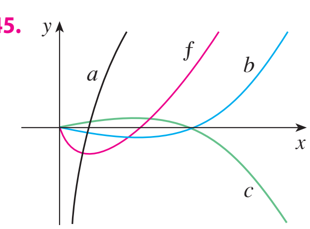
42. $f'''(x) = \cos x, \quad f(0) = 1, \quad f'(0) = 2, \quad f''(0) = 3$

43. Given that the graph of f passes through the point $(2, 5)$ and that the slope of its tangent line at $(x, f(x))$ is $3 - 4x$, find $f(1)$.

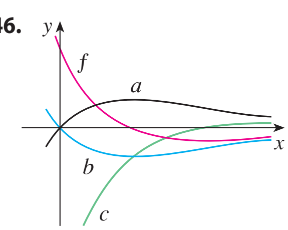
44. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

45–46 The graph of a function f is shown. Which graph is an antiderivative of f and why?

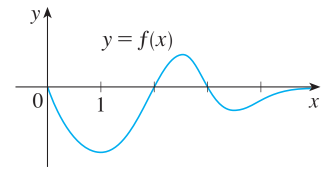
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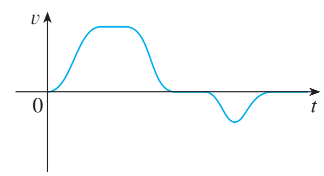
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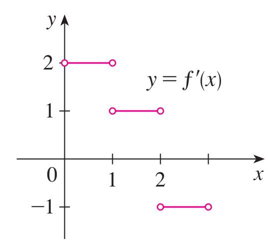
47. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.



48. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of a position function.



49. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous on $[0, 3]$ and $f(0) = -1$.



50. (a) Use a graphing device to graph $f(x) = 2x - 3\sqrt{x}$.
(b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies $F(0) = 1$.
(c) Use the rules of this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare with your sketch in part (b).

51–52 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

51. $f(x) = \dfrac{\sin x}{1 + x^2}, \quad -2\pi \leqslant x \leqslant 2\pi$

52. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \quad -3 \leqslant x \leqslant 3$

53–58 A particle is moving with the given data. Find the position of the particle.

53. $v(t) = \sin t - \cos t, \quad s(0) = 0$

54. $v(t) = t^2 - 3\sqrt{t}, \quad s(4) = 8$

55. $a(t) = 2t + 1, \quad s(0) = 3, \quad v(0) = -2$

56. $a(t) = 3\cos t - 2\sin t, \quad s(0) = 0, \quad v(0) = 4$

57. $a(t) = 10 \sin t + 3 \cos t, \quad s(0) = 0, \quad s(2\pi) = 12$

58. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

---

59. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

60. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0$$

61. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

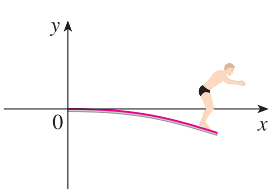
62. Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 48 ft/s and the other is thrown a second later with a speed of 24 ft/s. Do the balls ever pass each other?

63. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

64. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of a curve $y = f(x)$, where

$$EIy'' = mg(L - x) + \tfrac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and g (< 0) is the acceleration due to gravity.
(a) Find an expression for the shape of the curve.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.



65. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

66. The linear density of a rod of length 1 m is given by $\rho(x) = 1/\sqrt{x}$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.

67. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \leqslant t \leqslant 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

68. A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

69. What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 seconds?

70. A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

71. A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a pileup?

72. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 60t$, at which time the fuel is exhausted and it becomes a freely "falling" body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -18 ft/s in 5 seconds. The rocket then "floats" to the ground at that rate.
(a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
(b) At what time does the rocket reach its maximum height, and what is that height?
(c) At what time does the rocket land?

73. A high-speed bullet train accelerates and decelerates at the rate of 4 ft/s^2. Its maximum cruising speed is 90 mi/h.
(a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
(b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
(c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 miles apart.
(d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

3 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. What does the Extreme Value Theorem say?

3. (a) State Fermat's Theorem.
(b) Define a critical number of f.

4. Explain how the Closed Interval Method works.

5. (a) State Rolle's Theorem.
(b) State the Mean Value Theorem and give a geometric interpretation.

6. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave upward on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

7. (a) State the First Derivative Test.
(b) State the Second Derivative Test.
(c) What are the relative advantages and disadvantages of these tests?

8. Explain the meaning of each of the following statements.
(a) $\lim_{x \to \infty} f(x) = L$ (b) $\lim_{x \to -\infty} f(x) = L$ (c) $\lim_{x \to \infty} f(x) = \infty$
(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

9. If you have a graphing calculator or computer, why do you need calculus to graph a function?

10. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.
(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.
(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

11. (a) What is an antiderivative of a function f?
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.

2. If f has an absolute minimum value at c, then $f'(c) = 0$.

3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.

5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.

6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.

7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.

12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.

13. If f and g are increasing on an interval I, then fg is increasing on I.

14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.

15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.

16. If f is even, then f' is even.

17. If f is periodic, then f' is periodic.

18. The most general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = -\frac{1}{x} + C$$

19. If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.

EXERCISES

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 9x^2 + 24x - 2, \quad [0, 5]$

2. $f(x) = x\sqrt{1 - x}, \quad [-1, 1]$

3. $f(x) = \dfrac{3x - 4}{x^2 + 1}, \quad [-2, 2]$

4. $f(x) = \sqrt{x^2 + x + 1}, \quad [-2, 1]$

5. $f(x) = x + 2\cos x, \quad [-\pi, \pi]$

6. $f(x) = \sin x + \cos^2 x, \quad [0, \pi]$

7–12 Find the limit.

7. $\displaystyle \lim_{x\to\infty} \frac{3x^4 + x - 5}{6x^4 - 2x^2 + 1}$

8. $\displaystyle \lim_{t\to\infty} \frac{t^3 - t + 2}{(2t - 1)(t^2 + t + 1)}$

9. $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1}$

10. $\displaystyle \lim_{x\to-\infty} (x^2 + x^3)$

11. $\displaystyle \lim_{x\to\infty} \left(\sqrt{4x^2 + 3x} - 2x\right)$

12. $\displaystyle \lim_{x\to\infty} \frac{\sin^4 x}{\sqrt{x}}$

13–15 Sketch the graph of a function that satisfies the given conditions.

13. $f(0) = 0, \quad f'(-2) = f'(1) = f'(9) = 0,$

$\displaystyle \lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to 6} f(x) = -\infty,$

$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,

$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$

14. $f(0) = 0$, f is continuous and even,

$f'(x) = 2x$ if $0 < x < 1$, $\quad f'(x) = -1$ if $1 < x < 3$,

$f'(x) = 1$ if $x > 3$

15. f is odd, $\quad f'(x) < 0$ for $0 < x < 2$,

$f'(x) > 0$ for $x > 2$, $\quad f''(x) > 0$ for $0 < x < 3$,

$f''(x) < 0$ for $x > 3$, $\quad \displaystyle \lim_{x\to\infty} f(x) = -2$

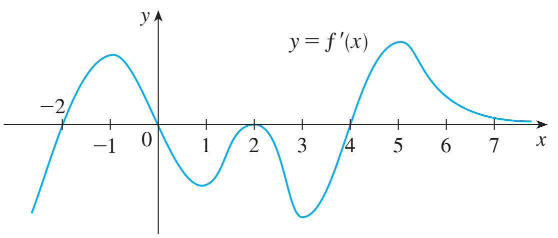
16. The figure shows the graph of the *derivative* f' of a function f.

(a) On what intervals is f increasing or decreasing?

(b) For what values of x does f have a local maximum or minimum?

(c) Sketch the graph of f''.

(d) Sketch a possible graph of f.



17–28 Use the guidelines of Section 3.5 to sketch the curve.

17. $y = 2 - 2x - x^3$

18. $y = -2x^3 - 3x^2 + 12x + 5$

19. $y = 3x^4 - 4x^3 + 2$

20. $y = \dfrac{x}{1 - x^2}$

21. $y = \dfrac{1}{x(x - 3)^2}$

22. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

23. $y = \dfrac{(x - 1)^3}{x^2}$

24. $y = \sqrt{1 - x} + \sqrt{1 + x}$

25. $y = x\sqrt{2 + x}$

26. $y = x^{2/3}(x - 3)^2$

27. $y = \sin^2 x - 2\cos x$

28. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$

29–32 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 29 use calculus to find these quantities exactly.

29. $f(x) = \dfrac{x^2 - 1}{x^3}$

30. $f(x) = \dfrac{x^3 + 1}{x^6 + 1}$

31. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

32. $f(x) = x^2 + 6.5\sin x, \quad -5 \leqslant x \leqslant 5$

33. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

34. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leqslant f'(x) \leqslant 5$ for all x in $(0, 4)$. Show that $9 \leqslant f(4) \leqslant 21$.

35. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval [32, 33], show that

$$2 < \sqrt[5]{33} < 2.0125$$

36. For what values of the constants a and b is (1, 3) a point of inflection of the curve $y = ax^3 + bx^2$?

37. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
(a) At what numbers does g have an extreme value?
(b) Discuss the concavity of g.

38. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

39. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

40. Find the point on the hyperbola $xy = 8$ that is closest to the point (3, 0).

41. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

42. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

43. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

44. Solve Exercise 43 when $|CD| = 2$ cm.

45. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

46. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

47. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

48. A manufacturer determines that the cost of making x units of a commodity is

$$C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$$

and the demand function is $p(x) = 48.2 - 0.03x$.
(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
(b) Use calculus to find the production level for maximum profit.
(c) Estimate the production level that minimizes the average cost.

49. Use Newton's method to find the root of the equation

$$x^5 - x^4 + 3x^2 - 3x - 2 = 0$$

in the interval [1, 2] correct to six decimal places.

50. Use Newton's method to find all solutions of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

51. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

52. Use the guidelines in Section 3.5 to sketch the curve $y = x \sin x$, $0 \leqslant x \leqslant 2\pi$. Use Newton's method when necessary.

53–54 Find the most general antiderivative of the function.

53. $f(x) = 4\sqrt{x} - 6x^2 + 3$

54. $g(x) = \cos x + 2\sec^2 x$

55–58 Find f.

55. $f'(t) = 2t - 3\sin t, \quad f(0) = 5$

56. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}, \quad f(1) = 3$

57. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

58. $f''(x) = 5x^3 + 6x^2 + 2, \quad f(0) = 3, \quad f(1) = -2$

59–60 A particle is moving according to the given data. Find the position of the particle.

59. $v(t) = 2t - \sin t, \quad s(0) = 3$

60. $a(t) = \sin t + 3\cos t, \quad s(0) = 0, \quad v(0) = 2$

61. Use a graphing device to draw a graph of the function $f(x) = x^2\sin(x^2)$, $0 \leqslant x \leqslant \pi$, and use that graph to sketch the antiderivative F of f that satisfies the initial condition $F(0) = 0$.

62. Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

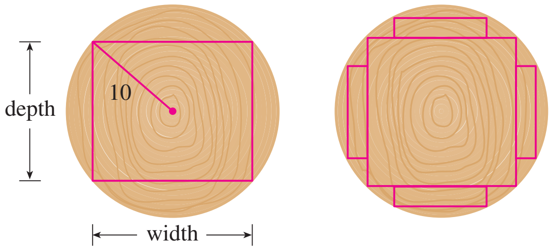
63. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

64. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

65. A rectangular beam will be cut from a cylindrical log of radius 10 inches.

(a) Show that the beam of maximal cross-sectional area is a square.

(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.



(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.

66. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

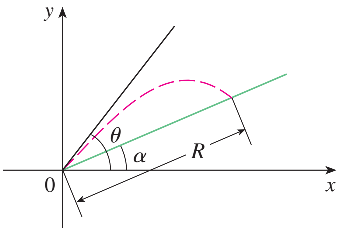
$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

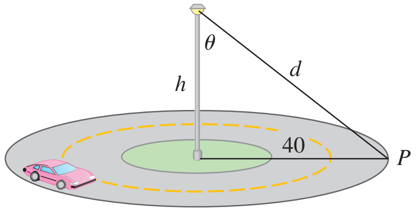
(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.



67. A light is to be placed atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ (see the figure) and inversely proportional to the square of the distance d from the source.

(a) How tall should the light pole be to maximize I?

(b) Suppose that the light pole is h feet tall and that a woman is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on her back 4 ft above the ground decreasing when she reaches the outer edge of the traffic circle?



CAS **68.** If $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}$, $-\pi \leqslant x \leqslant \pi$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

Problems Plus

One of the most important principles of problem solving is *analogy* (see page 98). If you are having trouble getting started on a problem, it is sometimes helpful to start by solving a similar, but simpler, problem. The following example illustrates the principle. Cover up the solution and try solving it yourself first.

EXAMPLE If x, y, and z are positive numbers, prove that

$$\frac{(x^2 + 1)(y^2 + 1)(z^2 + 1)}{xyz} \geqslant 8$$

SOLUTION It may be difficult to get started on this problem. (Some students have tackled it by multiplying out the numerator, but that just creates a mess.) Let's try to think of a similar, simpler problem. When several variables are involved, it's often helpful to think of an analogous problem with fewer variables. In the present case we can reduce the number of variables from three to one and prove the analogous inequality

$$\boxed{1} \qquad \frac{x^2 + 1}{x} \geqslant 2 \qquad \text{for } x > 0$$

In fact, if we are able to prove (1), then the desired inequality follows because

$$\frac{(x^2 + 1)(y^2 + 1)(z^2 + 1)}{xyz} = \left(\frac{x^2 + 1}{x}\right)\left(\frac{y^2 + 1}{y}\right)\left(\frac{z^2 + 1}{z}\right) \geqslant 2 \cdot 2 \cdot 2 = 8$$

The key to proving (1) is to recognize that it is a disguised version of a minimum problem. If we let

$$f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x} \qquad x > 0$$

then $f'(x) = 1 - (1/x^2)$, so $f'(x) = 0$ when $x = 1$. Also, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Therefore the absolute minimum value of f is $f(1) = 2$. This means that

$$\frac{x^2 + 1}{x} \geqslant 2 \qquad \text{for all positive values of } x$$

and, as previously mentioned, the given inequality follows by multiplication.

The inequality in (1) could also be proved without calculus. In fact, if $x > 0$, we have

$$\frac{x^2 + 1}{x} \geqslant 2 \iff x^2 + 1 \geqslant 2x \iff x^2 - 2x + 1 \geqslant 0$$

$$\iff (x - 1)^2 \geqslant 0$$

Because the last inequality is obviously true, the first one is true too. ■

PS Look Back

What have we learned from the solution to this example?

- To solve a problem involving several variables, it might help to solve a similar problem with just one variable.
- When trying to prove an inequality, it might help to think of it as a maximum or minimum problem.

PROBLEMS

1. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

2. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant 2$.

3. Show that the inflection points of the curve $y = (\sin x)/x$ lie on the curve $y^2(x^4 + 4) = 4$.

4. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

5. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

6. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.
(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
(b) Is $V''(t)$ positive, negative, or zero? Explain.
(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

7. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

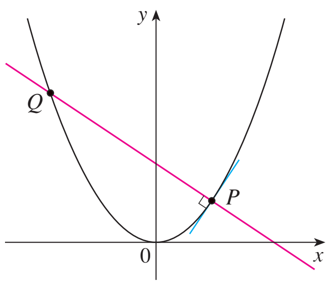
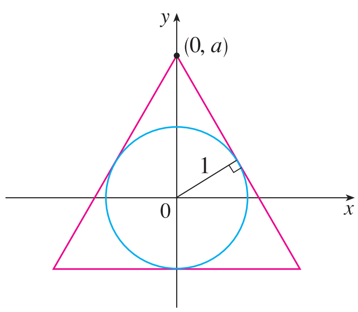


FIGURE FOR PROBLEM 9

8. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

9. If $P(a, a^2)$ is any point on the parabola $y = x^2$, except for the origin, let Q be the point where the normal line at P intersects the parabola again (see the figure).
(a) Show that the y-coordinate of Q is smallest when $a = 1/\sqrt{2}$.
(b) Show that the line segment PQ has the shortest possible length when $a = 1/\sqrt{2}$.

10. An isosceles triangle is circumscribed about the unit circle so that the equal sides meet at the point $(0, a)$ on the y-axis (see the figure). Find the value of a that minimizes the lengths of the equal sides. (You may be surprised that the result does not give an equilateral triangle.).



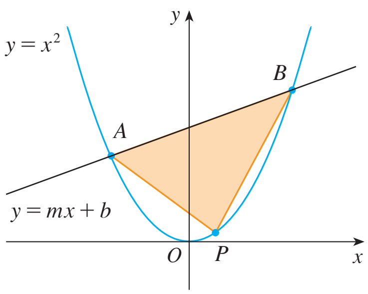


FIGURE FOR PROBLEM 11

11. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B. (See the figure.) Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

12. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty} [f(x) + x] = 0$.

13. Determine the values of the number a for which the function f has no critical number:

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

14. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

15. Let ABC be a triangle with $\angle BAC = 120°$ and $|AB| \cdot |AC| = 1$.
(a) Express the length of the angle bisector AD in terms of $x = |AB|$.
(b) Find the largest possible value of $|AD|$.

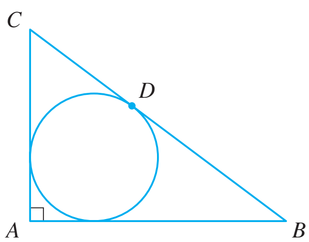


FIGURE FOR PROBLEM 16

16. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

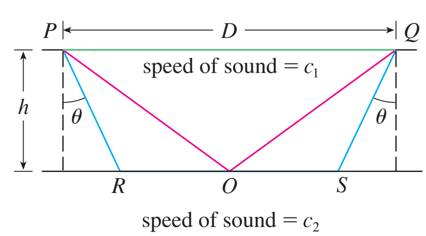
$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
(c) If a is fixed and θ varies, find the maximum value of r.

17. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
(a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
(b) Express da/dt in terms of the quantities in part (a).

18. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have.

19. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q. (See the figure.)



(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.
(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.
(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.

Note: Geophysicists use this technique when studying the structure of the earth's crust, whether searching for oil or examining fault lines.

20. For what values of c is there a straight line that intersects the curve

$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

in four distinct points?

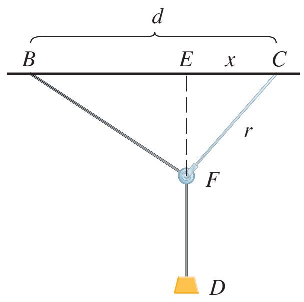


FIGURE FOR PROBLEM 21

21. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. (See the figure.) As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

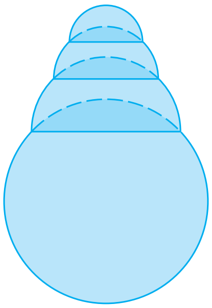
$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

22. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon? (A regular n-gon is a polygon with n equal sides and angles.) (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

23. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

24. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.



4 Integrals



The photo shows Lake Lanier, which is a reservoir in Georgia, USA. In Exercise 63 in Section 4.4 you will estimate the amount of water that flowed into Lake Lanier during a certain time period.

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IN CHAPTER 2 WE USED the tangent and velocity problems to introduce the derivative, which is the central idea in differential calculus. In much the same way, this chapter starts with the area and distance problems and uses them to formulate the idea of a definite integral, which is the basic concept of integral calculus. We will see in Chapters 5 and 8 how to use the integral to solve problems concerning volumes, lengths of curves, population predictions, cardiac output, forces on a dam, work, consumer surplus, and baseball, among many others.

There is a connection between integral calculus and differential calculus. The Fundamental Theorem of Calculus relates the integral to the derivative, and we will see in this chapter that it greatly simplifies the solution of many problems.

4.1 Areas and Distances

Now is a good time to read (or reread) *A Preview of Calculus* (see page 1). It discusses the unifying ideas of calculus and helps put in perspective where we have been and where we are going.

In this section we discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

The Area Problem

We begin by attempting to solve the *area problem:* find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

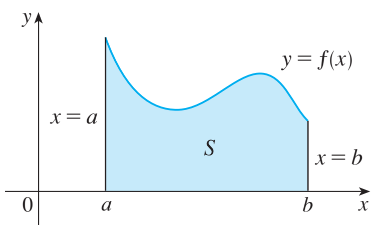


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

In trying to solve the area problem we have to ask ourselves: what is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

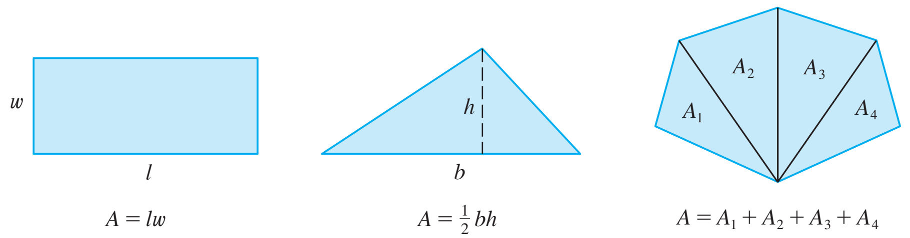


FIGURE 2

However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

EXAMPLE 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

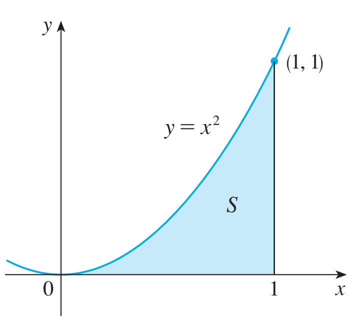


FIGURE 3

SOLUTION We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

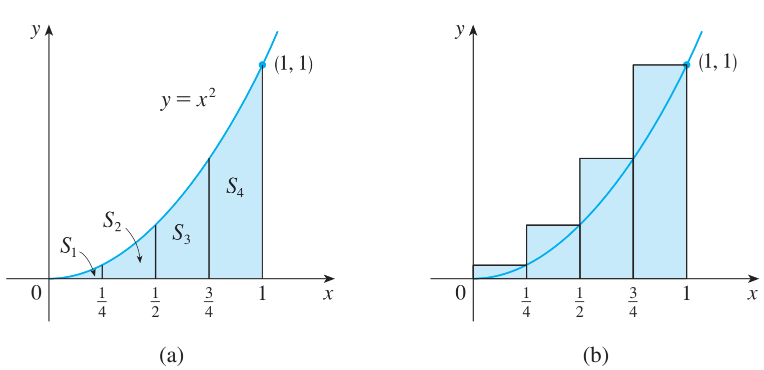


FIGURE 4

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)]. In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4} \cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

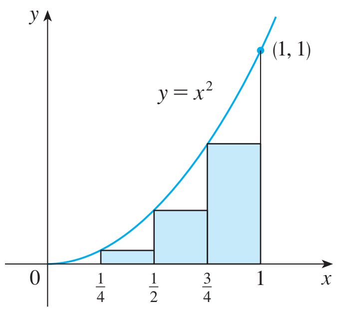


FIGURE 5

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of f at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width.

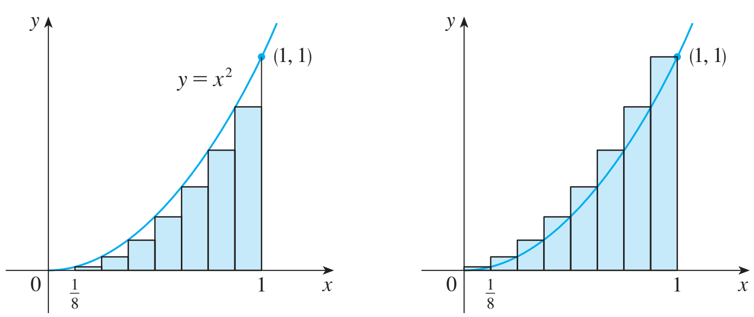


FIGURE 6
Approximating S with eight rectangles

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$. ■

From the values in the table in Example 1, it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

EXAMPLE 2 For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}$$

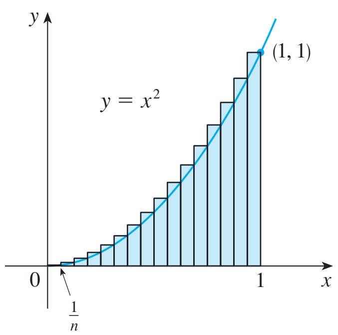


FIGURE 7

SOLUTION R_n is the sum of the areas of the n rectangles in Figure 7. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

$$\boxed{1} \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Perhaps you have seen this formula before. It is proved in Example 5 in Appendix E.

Putting Formula 1 into our expression for R_n, we get

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Here we are computing the limit of the sequence $\{R_n\}$. Sequences and their limits were discussed in *A Preview of Calculus* and will be studied in detail in Section 11.1. The idea is very similar to a limit at infinity (Section 3.4) except that in writing $\lim_{n\to\infty}$ we restrict n to be a positive integer. In particular, we know that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

When we write $\lim_{n\to\infty} R_n = \frac{1}{3}$ we mean that we can make R_n as close to $\frac{1}{3}$ as we like by taking n sufficiently large.

Thus we have

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\ &= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3} \end{aligned}$$

■

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

TEC In Visual 4.1 you can create pictures like those in Figures 8 and 9 for other values of n.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

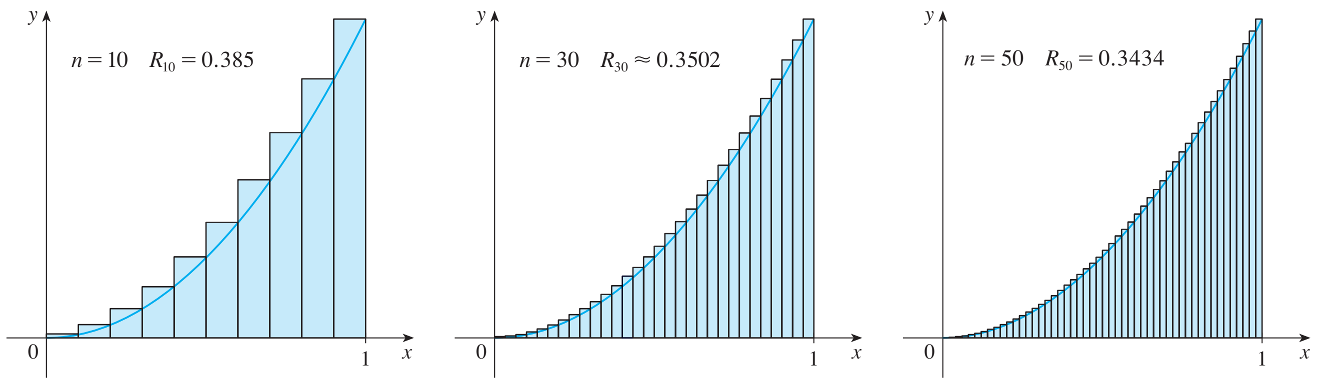


FIGURE 8 Right endpoints produce upper sums because $f(x) = x^2$ is increasing.

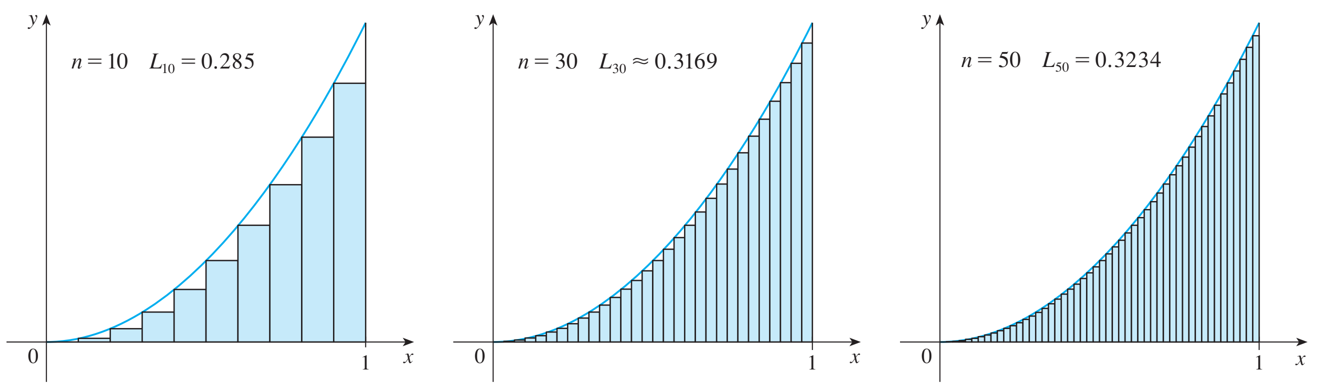


FIGURE 9 Left endpoints produce lower sums because $f(x) = x^2$ is increasing.

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

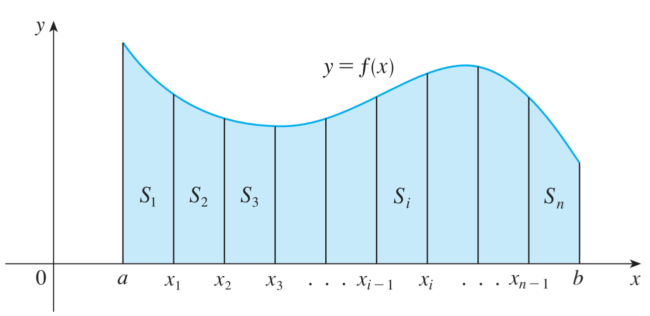


FIGURE 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$
$$x_2 = a + 2\,\Delta x,$$
$$x_3 = a + 3\,\Delta x,$$
$$\vdots$$

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the ith rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

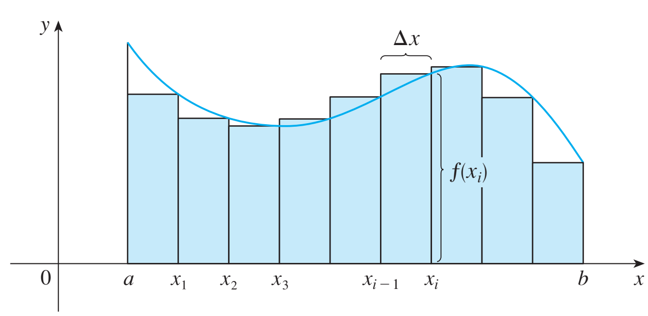
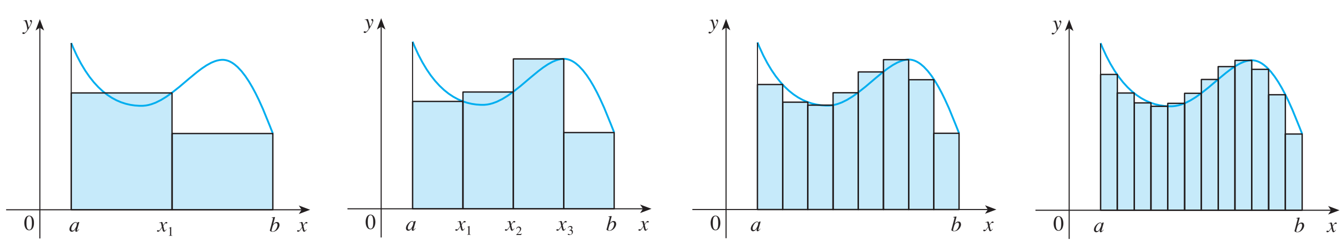


FIGURE 11

Figure 12 shows this approximation for $n = 2$, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore we define the area A of the region S in the following way.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

FIGURE 12

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

3
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

4
$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

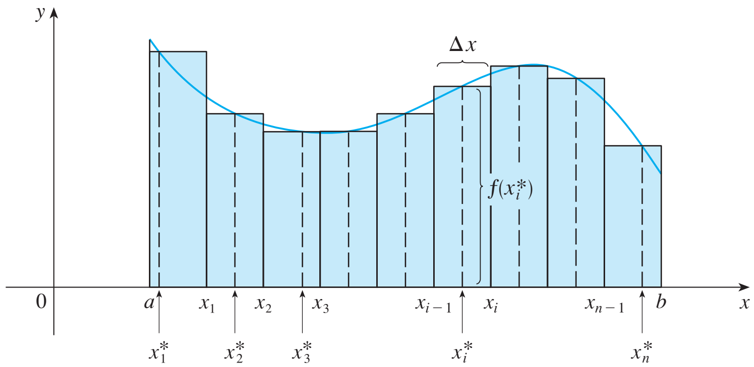


FIGURE 13

NOTE It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.* We saw in Examples 1 and 2, for instance, that the area $\left(A = \frac{1}{3}\right)$ is trapped between all the left approximating sums L_n and all the right approximating sums R_n. The function in those examples, $f(x) = x^2$, happens to be increasing on $[0, 1]$ and so the lower sums arise from left endpoints and the upper sums from right endpoints. (See Figures 8 and 9.) In general, we form **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the ith subinterval. (See Figure 14 and Exercises 7–8.)

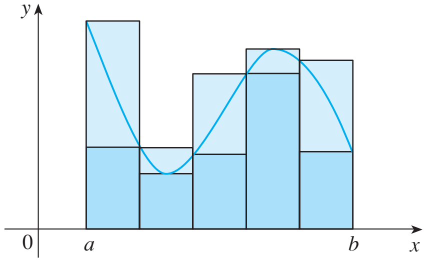


FIGURE 14
Lower sums (short rectangles) and upper sums (tall rectangles)

This tells us to end with $i = n$.

This tells us to add. $\sum_{i=m}^{n} f(x_i)\,\Delta x$

This tells us to start with $i = m$.

If you need practice with sigma notation, look at the examples and try some of the exercises in Appendix E.

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

We can also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = \cos x$ between $x = 0$ and $x = b$, where $0 \leqslant b \leqslant \pi/2$.
(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.
(b) Estimate the area for the case $b = \pi/2$ by taking the sample points to be midpoints and using four subintervals.

SOLUTION
(a) Since $a = 0$, the width of a subinterval is

$$\Delta x = \frac{b-0}{n} = \frac{b}{n}$$

So $x_1 = b/n$, $x_2 = 2b/n$, $x_3 = 3b/n$, $x_i = ib/n$, and $x_n = nb/n$. The sum of the areas of the approximating rectangles is

$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= (\cos x_1)\,\Delta x + (\cos x_2)\,\Delta x + \cdots + (\cos x_n)\,\Delta x \\ &= \left(\cos\frac{b}{n}\right)\frac{b}{n} + \left(\cos\frac{2b}{n}\right)\frac{b}{n} + \cdots + \left(\cos\frac{nb}{n}\right)\frac{b}{n} \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{b}{n}\left(\cos\frac{b}{n} + \cos\frac{2b}{n} + \cos\frac{3b}{n} + \cdots + \cos\frac{nb}{n}\right)$$

Using sigma notation we could write

$$A = \lim_{n\to\infty} \frac{b}{n} \sum_{i=1}^{n} \cos\frac{ib}{n}$$

It is very difficult to evaluate this limit directly by hand, but with the aid of a computer algebra system it isn't hard (see Exercise 31). In Section 4.3 we will be able to find A more easily using a different method.

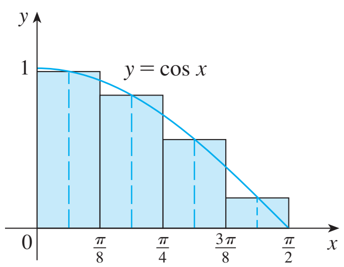


FIGURE 15

(b) With $n = 4$ and $b = \pi/2$ we have $\Delta x = (\pi/2)/4 = \pi/8$, so the subintervals are $[0, \pi/8]$, $[\pi/8, \pi/4]$, $[\pi/4, 3\pi/8]$, and $[3\pi/8, \pi/2]$. The midpoints of these subintervals are

$$x_1^* = \frac{\pi}{16} \qquad x_2^* = \frac{3\pi}{16} \qquad x_3^* = \frac{5\pi}{16} \qquad x_4^* = \frac{7\pi}{16}$$

and the sum of the areas of the four approximating rectangles (see Figure 15) is

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(x_i^*)\,\Delta x \\ &= f(\pi/16)\,\Delta x + f(3\pi/16)\,\Delta x + f(5\pi/16)\,\Delta x + f(7\pi/16)\,\Delta x \\ &= \left(\cos\frac{\pi}{16}\right)\frac{\pi}{8} + \left(\cos\frac{3\pi}{16}\right)\frac{\pi}{8} + \left(\cos\frac{5\pi}{16}\right)\frac{\pi}{8} + \left(\cos\frac{7\pi}{16}\right)\frac{\pi}{8} \\ &= \frac{\pi}{8}\left(\cos\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos\frac{5\pi}{16} + \cos\frac{7\pi}{16}\right) \approx 1.006 \end{aligned}$$

So an estimate for the area is

$$A \approx 1.006$$

■

The Distance Problem

Now let's consider the *distance problem:* find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. (In a sense this is the inverse problem of the velocity problem that we discussed in Section 1.4.) If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, it's not so easy to find the distance traveled. We investigate the problem in the following example.

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we

take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s. So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) = 1130 \text{ ft}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (45 \times 5) + (41 \times 5) = 1210 \text{ ft}$$

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second. ■

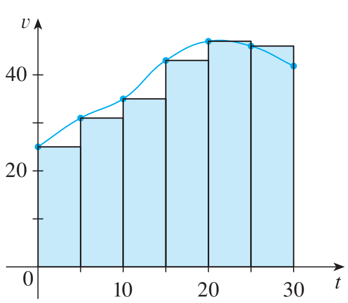


FIGURE 16

Perhaps the calculations in Example 4 remind you of the sums we used earlier to estimate areas. The similarity is explained when we sketch a graph of the velocity function of the car in Figure 16 and draw rectangles whose heights are the initial velocities for each time interval. The area of the first rectangle is $25 \times 5 = 125$, which is also our estimate for the distance traveled in the first five seconds. In fact, the area of each rectangle can be interpreted as a distance because the height represents velocity and the width represents time. The sum of the areas of the rectangles in Figure 16 is $L_6 = 1130$, which is our initial estimate for the total distance traveled.

In general, suppose an object moves with velocity $v = f(t)$, where $a \leqslant t \leqslant b$ and $f(t) \geqslant 0$ (so the object always moves in the positive direction). We take velocity readings at times $t_0 (= a), t_1, t_2, \ldots, t_n (= b)$ so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become,

so it seems plausible that the *exact* distance d traveled is the *limit* of such expressions:

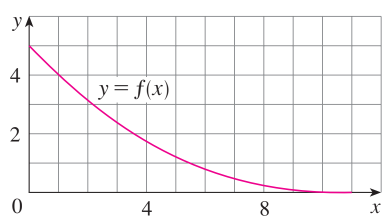
5
$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

We will see in Section 4.4 that this is indeed true.

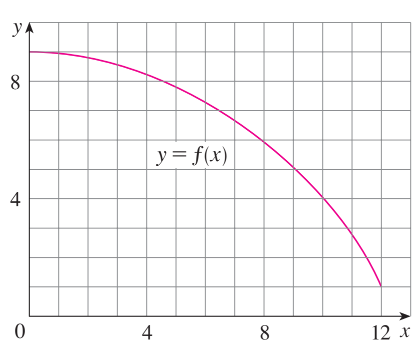
Because Equation 5 has the same form as our expressions for area in Equations 2 and 3, it follows that the distance traveled is equal to the area under the graph of the velocity function. In Chapter 5 we will see that other quantities of interest in the natural and social sciences—such as the work done by a variable force or the cardiac output of the heart—can also be interpreted as the area under a curve. So when we compute areas in this chapter, bear in mind that they can be interpreted in a variety of practical ways.

4.1 EXERCISES

1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.
(b) Find new estimates using ten rectangles in each case.



2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
(i) L_6 (sample points are left endpoints)
(ii) R_6 (sample points are right endpoints)
(iii) M_6 (sample points are midpoints)
(b) Is L_6 an underestimate or overestimate of the true area?
(c) Is R_6 an underestimate or overestimate of the true area?
(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
(b) Repeat part (a) using left endpoints.
(c) Repeat part (a) using midpoints.
(d) From your sketches in parts (a)–(c), which appears to be the best estimate?

6. (a) Graph the function

$$f(x) = 1/(1 + x^2) \qquad -2 \leqslant x \leqslant 2$$

(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be (i) right endpoints and (ii) midpoints. In each case sketch the curve and the rectangles.
(c) Improve your estimates in part (b) by using eight rectangles.

7. Evaluate the upper and lower sums for $f(x) = 2 + \sin x$, $0 \leqslant x \leqslant \pi$, with $n = 2$, 4, and 8. Illustrate with diagrams like Figure 14.

8. Evaluate the upper and lower sums for $f(x) = 1 + x^2$, $-1 \leqslant x \leqslant 1$, with $n = 3$ and 4. Illustrate with diagrams like Figure 14.

9–10 With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Then guess the value of the exact area.

9. The region under $y = x^4$ from 0 to 1

10. The region under $y = \cos x$ from 0 to $\pi/2$

CAS **11.** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)
(a) If $f(x) = 1/(x^2 + 1)$, $0 \leqslant x \leqslant 1$, find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 0.780 and 0.791.

CAS **12.** (a) If $f(x) = x/(x + 2)$, $1 \leqslant x \leqslant 4$, use the commands discussed in Exercise 11 to find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 1.603 and 1.624.

13. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

14. The table shows speedometer readings at 10-second intervals during a 1-minute period for a car racing at the Daytona International Speedway in Florida.
(a) Estimate the distance the race car traveled during this time period using the velocities at the beginning of the time intervals.
(b) Give another estimate using the velocities at the end of the time periods.
(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

Time (s)	Velocity (mi/h)
0	182.9
10	168.0
20	106.6
30	99.8
40	124.5
50	176.1
60	175.6

15. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

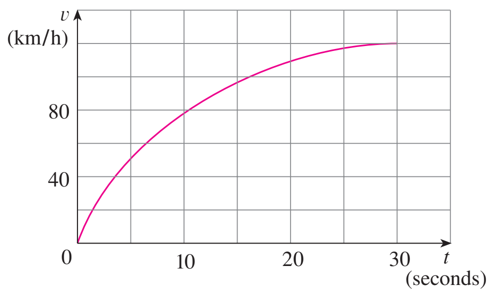
16. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour*, 62 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

17. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



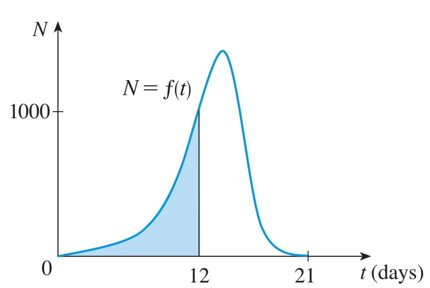
18. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



19. In someone infected with measles, the virus level N (measured in number of infected cells per mL of blood plasma) reaches a peak density at about $t = 12$ days (when a rash appears) and then decreases fairly rapidly as a result of immune response. The area under the graph of $N(t)$ from $t = 0$ to $t = 12$ (as shown in the figure) is equal to the total amount of infection needed to develop symptoms (measured in density of infected cells $\times$ time). The function N has been modeled by the function

$$f(t) = -t(t - 21)(t + 1)$$

Use this model with six subintervals and their midpoints to estimate the total amount of infection needed to develop symptoms of measles.



Source: J. M. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2006): 134–47.

20. The table shows the number of people per day who died from SARS in Singapore at two-week intervals beginning on March 1, 2003.

Date	Deaths per day	Date	Deaths per day
March 1	0.0079	April 26	0.5620
March 15	0.0638	May 10	0.4630
March 29	0.1944	May 24	0.2897
April 12	0.4435		

(a) By using an argument similar to that in Example 4, estimate the number of people who died of SARS in Singapore between March 1 and May 24, 2003, using both left endpoints and right endpoints.

(b) How would you interpret the number of SARS deaths as an area under a curve?

Source: A. Gumel et al., "Modelling Strategies for Controlling SARS Outbreaks," *Proceedings of the Royal Society of London: Series B* 271 (2004): 2223–32.

21–23 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

21. $f(x) = \dfrac{2x}{x^2 + 1}, \quad 1 \leqslant x \leqslant 3$

22. $f(x) = x^2 + \sqrt{1 + 2x}, \quad 4 \leqslant x \leqslant 7$

23. $f(x) = \sqrt{\sin x}, \quad 0 \leqslant x \leqslant \pi$

24–25 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

24. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

25. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

26. (a) Use Definition 2 to find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.

(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix E. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n + 1)}{2}\right]^2$$

27. Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n, and R_n related?

(b) Show that

$$R_n - L_n = \frac{b - a}{n}[f(b) - f(a)]$$

Then draw a diagram to illustrate this equation by showing that the n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle whose area is the right side of the equation.

(c) Deduce that

$$R_n - A < \frac{b - a}{n}[f(b) - f(a)]$$

28. If A is the area under the curve $y = \sin x$ from 0 to $\pi/2$, use Exercise 27 to find a value of n such that $R_n - A < 0.0001$.

CAS **29.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.
(b) Use a computer algebra system to find the sum in your expression from part (a).
(c) Evaluate the limit in part (a).

CAS **30.** (a) Express the area under the curve $y = x^4 + 5x^2 + x$ from 2 to 7 as a limit.
(b) Use a computer algebra system to evaluate the sum in part (a).
(c) Use a computer algebra system to find the exact area by evaluating the limit of the expression in part (b).

CAS **31.** Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leqslant b \leqslant \pi/2$. (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

32. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \tfrac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 2.4.2 on page 145.]

4.2 The Definite Integral

We saw in Section 4.1 that a limit of the form

$$\boxed{1} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 5 and 8 we will see that limits of the form (1) also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

NOTE 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation

$\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

NOTE 2 The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

NOTE 3 The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 4.1, we see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

Riemann

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

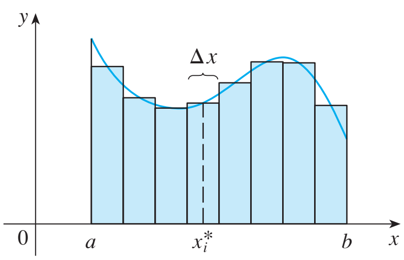


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

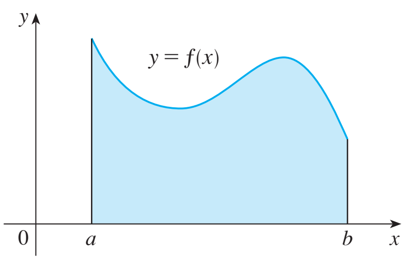


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

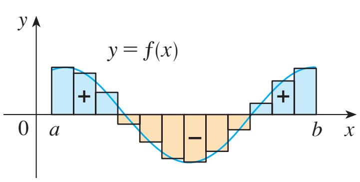


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

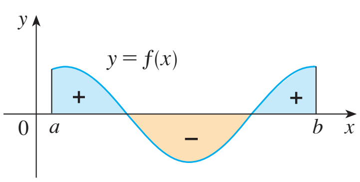


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles). When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

NOTE 4 Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a, b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. For instance, in Exercise 4.1.16, NASA provided velocity data at times that were not equally spaced, but we were still able to estimate the distance traveled. And there are methods for numerical integration that take advantage of unequal subintervals.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

NOTE 5 We have defined the definite integral for an integrable function, but not all functions are integrable (see Exercises 71–72). The following theorem shows that the most commonly occurring functions are in fact integrable. The theorem is proved in more advanced courses.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\,\Delta x$$

EXAMPLE 1 Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

SOLUTION Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi$. Therefore, by Theorem 4, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^\pi (x^3 + x \sin x)\,dx$$ ■

Later, when we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals, as we did in Example 1. When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process. In

general, when we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

we replace $\lim \Sigma$ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Integrals

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following three equations give formulas for sums of powers of positive integers. Equation 5 may be familiar to you from a course in algebra. Equations 6 and 7 were discussed in Section 4.1 and are proved in Appendix E.

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

Formulas 8–11 are proved by writing out each side in expanded form. The left side of Equation 9 is

$$ca_1 + ca_2 + \cdots + ca_n$$

The right side is

$$c(a_1 + a_2 + \cdots + a_n)$$

These are equal by the distributive property. The other formulas are discussed in Appendix E.

8 $$\sum_{i=1}^{n} c = nc$$

9 $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

EXAMPLE 2

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION

(a) With $n = 6$ the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, and

$x_6 = 3.0$. So the Riemann sum is

$$\begin{aligned}R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x \\ &= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375\end{aligned}$$

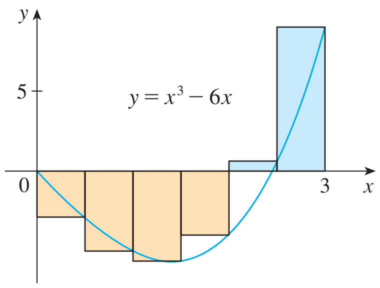


FIGURE 5

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the gold rectangles (below the x-axis) in Figure 5.

(b) With n subintervals we have

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

So $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$, and, in general, $x_i = 3i/n$. Since we are using right endpoints, we can use Theorem 4:

In the sum, n is a constant (unlike i), so we can move $3/n$ in front of the $\sum$ sign.

$$\begin{aligned}\int_0^3 (x^3 - 6x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \\ &= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \qquad \text{(Equation 9 with } c = 3/n\text{)} \\ &= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right] \\ &= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] \qquad \text{(Equations 11 and 9)} \\ &= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\} \qquad \text{(Equations 7 and 5)} \\ &= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right] \\ &= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75\end{aligned}$$

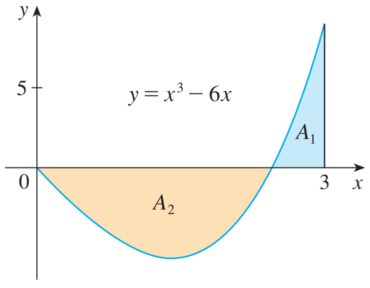


FIGURE 6
$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 6.

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -6.75, as $n \to \infty$.

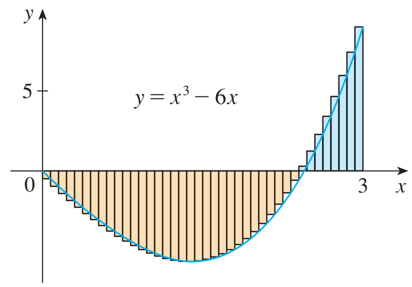


FIGURE 7
$R_{40} \approx -6.3998$

n	R_n
40	-6.3998
100	-6.6130
500	-6.7229
1000	-6.7365
5000	-6.7473

■

A much simpler method for evaluating the integral in Example 2 will be given in Example 4.4.3.

Because $f(x) = x^4$ is positive, the integral in Example 3 represents the area shown in Figure 8.

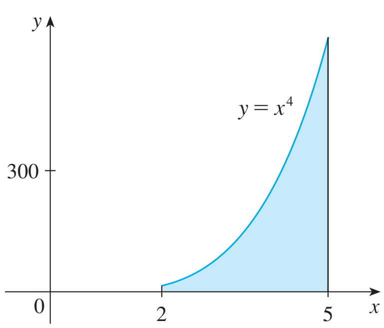


FIGURE 8

EXAMPLE 3

(a) Set up an expression for $\int_2^5 x^4\,dx$ as a limit of sums.
(b) Use a computer algebra system to evaluate the expression.

SOLUTION

(a) Here we have $f(x) = x^4$, $a = 2$, $b = 5$, and

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}$$

$x_0 = 2$, $x_1 = 2 + 3/n$, $x_2 = 2 + 6/n$, $x_3 = 2 + 9/n$, and

$$x_i = 2 + \frac{3i}{n}$$

From Theorem 4, we get

$$\int_2^5 x^4\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^4$$

(b) If we ask a computer algebra system to evaluate the sum and simplify, we obtain

$$\sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^4 = \frac{2062n^4 + 3045n^3 + 1170n^2 - 27}{10n^3}$$

Now we ask the computer algebra system to evaluate the limit:

$$\int_2^5 x^4\,dx = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^4 = \lim_{n\to\infty} \frac{3(2062n^4 + 3045n^3 + 1170n^2 - 27)}{10n^4}$$

$$= \frac{3(2062)}{10} = \frac{3093}{5} = 618.6$$

We will learn a much easier method for the evaluation of integrals in the next section. ■

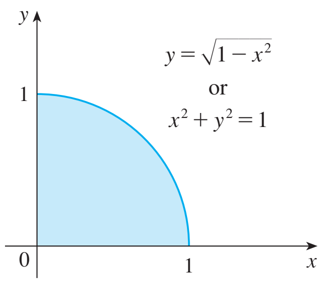


FIGURE 9

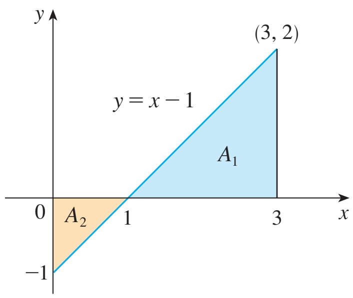


FIGURE 10

EXAMPLE 4 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^3 (x-1)\,dx$

SOLUTION

(a) Since $f(x)=\sqrt{1-x^2}\geqslant 0$, we can interpret this integral as the area under the curve $y=\sqrt{1-x^2}$ from 0 to 1. But, since $y^2=1-x^2$, we get $x^2+y^2=1$, which shows that the graph of f is the quarter-circle with radius 1 in Figure 9. Therefore

$$\int_0^1 \sqrt{1-x^2}\,dx = \tfrac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

(In Section 7.3 we will be able to *prove* that the area of a circle of radius r is πr^2.)

(b) The graph of $y=x-1$ is the line with slope 1 shown in Figure 10. We compute the integral as the difference of the areas of the two triangles:

$$\int_0^3 (x-1)\,dx = A_1 - A_2 = \tfrac{1}{2}(2\cdot 2) - \tfrac{1}{2}(1\cdot 1) = 1.5$$ ■

■ The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

TEC Module 4.2/7.7 shows how the Midpoint Rule estimates improve as n increases.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\,[f(\bar{x}_1)+\cdots+f(\bar{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\bar{x}_i = \tfrac{1}{2}(x_{i-1}+x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

EXAMPLE 5 Use the Midpoint Rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

SOLUTION The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. The width of the subintervals is $\Delta x = (2-1)/5 = \frac{1}{5}$, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right)$$

$$\approx 0.691908$$

Since $f(x)=1/x>0$ for $1\leqslant x\leqslant 2$, the integral represents an area, and the approxi-

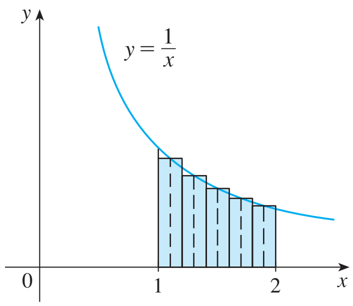


FIGURE 11

mation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 11. ■

At the moment we don't know how accurate the approximation in Example 5 is, but in Section 7.7 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will discuss other methods for approximating definite integrals.

If we apply the Midpoint Rule to the integral in Example 2, we get the picture in Figure 12. The approximation $M_{40} \approx -6.7563$ is much closer to the true value -6.75 than the right endpoint approximation, $R_{40} \approx -6.3998$, shown in Figure 7.

TEC In Visual 4.2 you can compare left, right, and midpoint approximations to the integral in Example 2 for different values of n.

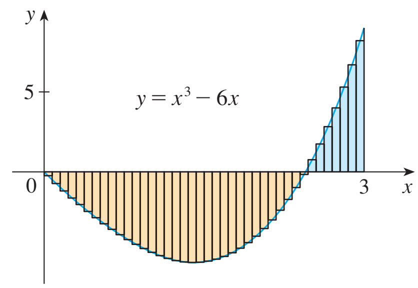


FIGURE 12
$M_{40} \approx -6.7563$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Property 1 says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval. If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 13.

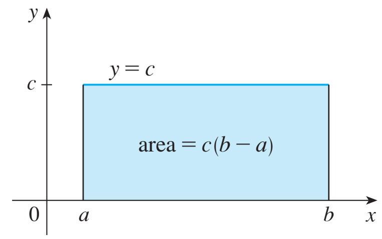


FIGURE 13
$\int_a^b c\,dx = c(b - a)$

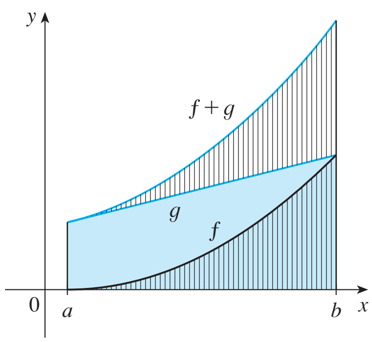


FIGURE 14
$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Property 2 says that the integral of a sum is the sum of the integrals. For positive functions it says that the area under $f + g$ is the area under f plus the area under g. Figure 14 helps us understand why this is true: in view of how graphical addition works, the corresponding vertical line segments have equal height.

In general, Property 2 follows from Theorem 4 and the fact that the limit of a sum is the sum of the limits:

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\,\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

Property 3 seems intuitively reasonable because we know that multiplying a function by a positive number c stretches or shrinks its graph vertically by a factor of c. So it stretches or shrinks each approximating rectangle by a factor c and therefore it has the effect of multiplying the area by c.

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (but *only* a constant) can be taken in front of an integral sign. Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

EXAMPLE 6 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0) = 4$$

and we found in Example 4.1.2 that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\begin{aligned}\int_0^1 (4 + 3x^2)\,dx &= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx \\ &= 4 + 3 \cdot \tfrac{1}{3} = 5\end{aligned}$$

■

The next property tells us how to combine integrals of the same function over adjacent intervals.

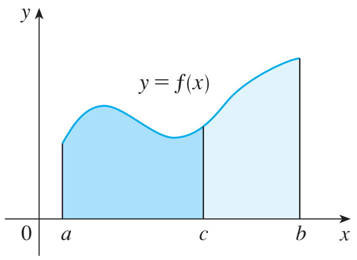


FIGURE 15

$$\textbf{5.} \qquad \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This is not easy to prove in general, but for the case where $f(x) \geqslant 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: the area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

EXAMPLE 7 If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

SOLUTION By Property 5, we have

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

so $$\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5$$ ■

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leqslant b$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

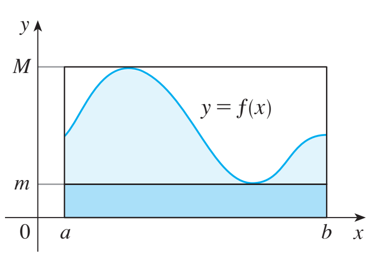


FIGURE 16

If $f(x) \geqslant 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.) Property 7 says that a bigger function has a bigger integral. It follows from Properties 6 and 4 because $f - g \geqslant 0$.

Property 8 is illustrated by Figure 16 for the case where $f(x) \geqslant 0$. If f is continuous, we could take m and M to be the absolute minimum and maximum values of f on the interval $[a, b]$. In this case Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

PROOF OF PROPERTY 8 Since $m \leqslant f(x) \leqslant M$, Property 7 gives

$$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx$$

Using Property 1 to evaluate the integrals on the left and right sides, we obtain

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$ ■

Property 8 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.

EXAMPLE 8 Use Property 8 to estimate $\int_1^4 \sqrt{x}\, dx$.

SOLUTION Since $f(x) = \sqrt{x}$ is an increasing function, its absolute minimum on $[1, 4]$ is $m = f(1) = 1$ and its absolute maximum on $[1, 4]$ is $M = f(4) = \sqrt{4} = 2$. Thus Property 8 gives

$$1(4 - 1) \leqslant \int_1^4 \sqrt{x}\, dx \leqslant 2(4 - 1)$$

or

$$3 \leqslant \int_1^4 \sqrt{x}\, dx \leqslant 6$$ ■

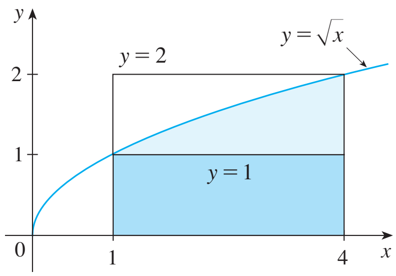


FIGURE 17

The result of Example 8 is illustrated in Figure 17. The area under $y = \sqrt{x}$ from 1 to 4 is greater than the area of the lower rectangle and less than the area of the large rectangle.

4.2 EXERCISES

1. Evaluate the Riemann sum for $f(x) = x - 1$, $-6 \leqslant x \leqslant 4$, with five subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

2. If

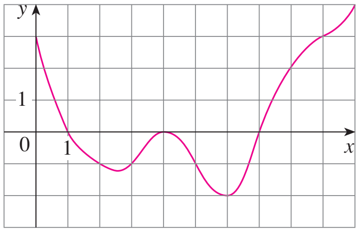
$$f(x) = \cos x \qquad 0 \leqslant x \leqslant 3\pi/4$$

evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints. (Give your answer correct to six decimal places.) What does the Riemann sum represent? Illustrate with a diagram.

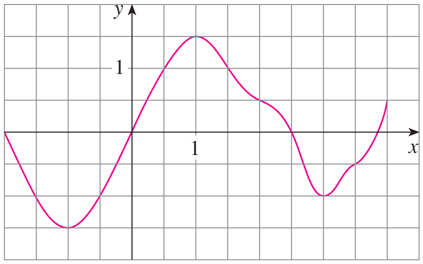
3. If $f(x) = x^2 - 4$, $0 \leqslant x \leqslant 3$, find the Riemann sum with $n = 6$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

4. (a) Find the Riemann sum for $f(x) = 1/x$, $1 \leqslant x \leqslant 2$, with four terms, taking the sample points to be right endpoints. (Give your answer correct to six decimal places.) Explain what the Riemann sum represents with the aid of a sketch.
(b) Repeat part (a) with midpoints as the sample points.

5. The graph of a function f is given. Estimate $\int_0^{10} f(x)\, dx$ using five subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



6. The graph of g is shown. Estimate $\int_{-2}^{4} g(x)\, dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.



7. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_{10}^{30} f(x)\, dx$.

x	10	14	18	22	26	30
$f(x)$	-12	-6	-2	1	3	8

8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_3^9 f(x)\, dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

9–12 Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

9. $\int_0^8 \sin\sqrt{x}\,dx, \quad n = 4$

10. $\int_0^1 \sqrt{x^3 + 1}\,dx, \quad n = 5$

11. $\int_0^2 \frac{x}{x + 1}\,dx, \quad n = 5$

12. $\int_0^\pi x \sin^2 x\,dx, \quad n = 4$

CAS **13.** If you have a CAS that evaluates midpoint approximations and graphs the corresponding rectangles (use `RiemannSum` or `middlesum` and `middlebox` commands in Maple), check the answer to Exercise 11 and illustrate with a graph. Then repeat with $n = 10$ and $n = 20$.

14. With a programmable calculator or computer (see the instructions for Exercise 4.1.9), compute the left and right Riemann sums for the function $f(x) = x/(x + 1)$ on the interval $[0, 2]$ with $n = 100$. Explain why these estimates show that

$$0.8946 < \int_0^2 \frac{x}{x + 1}\,dx < 0.9081$$

15. Use a calculator or computer to make a table of values of right Riemann sums R_n for the integral $\int_0^\pi \sin x\,dx$ with $n = 5, 10, 50,$ and 100. What value do these numbers appear to be approaching?

16. Use a calculator or computer to make a table of values of left and right Riemann sums L_n and R_n for the integral $\int_0^2 \sqrt{1 + x^4}\,dx$ with $n = 5, 10, 50,$ and 100. Between what two numbers must the value of the integral lie? Can you make a similar statement for the integral $\int_{-1}^2 \sqrt{1 + x^4}\,dx$? Explain.

17–20 Express the limit as a definite integral on the given interval.

17. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1 + x_i}\,\Delta x, \quad [0, \pi]$

18. $\lim_{n\to\infty} \sum_{i=1}^{n} x_i\sqrt{1 + x_i^3}\,\Delta x, \quad [2, 5]$

19. $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x, \quad [2, 7]$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2 + 4}\,\Delta x, \quad [1, 3]$

21–25 Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

21. $\int_2^5 (4 - 2x)\,dx$

22. $\int_1^4 (x^2 - 4x + 2)\,dx$

23. $\int_{-2}^0 (x^2 + x)\,dx$

24. $\int_0^2 (2x - x^3)\,dx$

25. $\int_0^1 (x^3 - 3x^2)\,dx$

26. (a) Find an approximation to the integral $\int_0^4 (x^2 - 3x)\,dx$ using a Riemann sum with right endpoints and $n = 8$.
(b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).
(c) Use Theorem 4 to evaluate $\int_0^4 (x^2 - 3x)\,dx$.
(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.

27. Prove that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

28. Prove that $\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

29–30 Express the integral as a limit of Riemann sums. Do not evaluate the limit.

29. $\int_1^3 \sqrt{4 + x^2}\,dx$

30. $\int_2^5 \left(x^2 + \frac{1}{x}\right) dx$

CAS **31–32** Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.

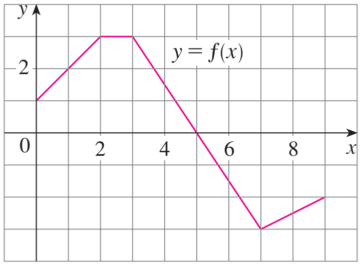
31. $\int_0^\pi \sin 5x\,dx$

32. $\int_2^{10} x^6\,dx$

33. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

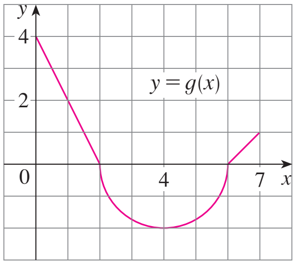
(a) $\int_0^2 f(x)\,dx$ (b) $\int_0^5 f(x)\,dx$

(c) $\int_5^7 f(x)\,dx$ (d) $\int_0^9 f(x)\,dx$



34. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ (c) $\int_0^7 g(x)\,dx$



35–40 Evaluate the integral by interpreting it in terms of areas.

35. $\int_{-1}^{2} (1 - x)\, dx$

36. $\int_{0}^{9} \left(\frac{1}{3}x - 2\right) dx$

37. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$

38. $\int_{-5}^{5} \left(x - \sqrt{25 - x^2}\right) dx$

39. $\int_{-4}^{3} \left|\frac{1}{2}x\right| dx$

40. $\int_{0}^{1} |2x - 1|\, dx$

41. Evaluate $\int_{1}^{1} \sqrt{1 + x^4}\, dx$.

42. Given that $\int_{0}^{\pi} \sin^4 x\, dx = \frac{3}{8}\pi$, what is $\int_{\pi}^{0} \sin^4 \theta\, d\theta$?

43. In Example 4.1.2 we showed that $\int_0^1 x^2\, dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 6x^2)\, dx$.

44. Use the properties of integrals and the result of Example 3 to evaluate $\int_2^5 (1 + 3x^4)\, dx$.

45. Use the results of Exercises 27 and 28 and the properties of integrals to evaluate $\int_1^4 (2x^2 - 3x + 1)\, dx$.

46. Use the result of Exercise 27 and the fact that $\int_0^{\pi/2} \cos x\, dx = 1$ (from Exercise 4.1.31), together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2 \cos x - 5x)\, dx$.

47. Write as a single integral in the form $\int_a^b f(x)\, dx$:

$$\int_{-2}^{2} f(x)\, dx + \int_{2}^{5} f(x)\, dx - \int_{-2}^{-1} f(x)\, dx$$

48. If $\int_2^8 f(x)\, dx = 7.3$ and $\int_2^4 f(x)\, dx = 5.9$, find $\int_4^8 f(x)\, dx$.

49. If $\int_0^9 f(x)\, dx = 37$ and $\int_0^9 g(x)\, dx = 16$, find

$$\int_0^9 [2f(x) + 3g(x)]\, dx$$

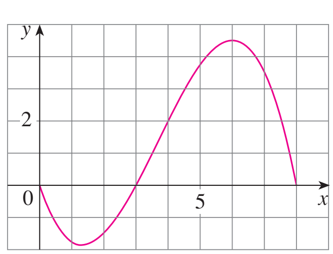
50. Find $\int_0^5 f(x)\, dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

51. For the function f whose graph is shown, list the following quantities in increasing order, from smallest to largest, and explain your reasoning.

(A) $\int_0^8 f(x)\, dx$ (B) $\int_0^3 f(x)\, dx$ (C) $\int_3^8 f(x)\, dx$

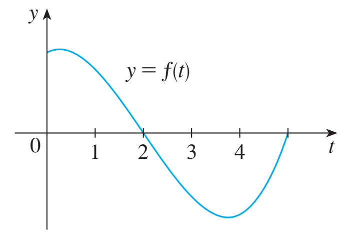
(D) $\int_4^8 f(x)\, dx$ (E) $f'(1)$



52. If $F(x) = \int_2^x f(t)\, dt$, where f is the function whose graph is given, which of the following values is largest?

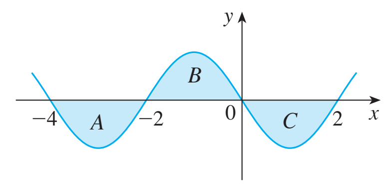
(A) $F(0)$ (B) $F(1)$ (C) $F(2)$

(D) $F(3)$ (E) $F(4)$



53. Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} [f(x) + 2x + 5]\, dx$$



54. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\, dx$ lie? Which property of integrals allows you to make your conclusion?

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

55. $\int_0^4 (x^2 - 4x + 4)\, dx \geqslant 0$

56. $\int_0^1 \sqrt{1 + x^2}\, dx \leqslant \int_0^1 \sqrt{1 + x}\, dx$

57. $2 \leqslant \int_{-1}^{1} \sqrt{1 + x^2}\, dx \leqslant 2\sqrt{2}$

58. $\dfrac{\pi}{12} \leqslant \int_{\pi/6}^{\pi/3} \sin x\, dx \leqslant \dfrac{\sqrt{3}\,\pi}{12}$

59–64 Use Property 8 of integrals to estimate the value of the integral.

59. $\int_0^1 x^3\, dx$

60. $\int_0^3 \dfrac{1}{x + 4}\, dx$

61. $\int_{\pi/4}^{\pi/3} \tan x\, dx$

62. $\int_0^2 (x^3 - 3x + 3)\, dx$

63. $\int_{-1}^{1} \sqrt{1 + x^4}\, dx$

64. $\int_{\pi}^{2\pi} (x - 2 \sin x)\, dx$

65–66 Use properties of integrals, together with Exercises 27 and 28, to prove the inequality.

65. $\int_1^3 \sqrt{x^4 + 1}\, dx \geqslant \dfrac{26}{3}$

66. $\int_0^{\pi/2} x \sin x\, dx \leqslant \dfrac{\pi^2}{8}$

67. Which of the integrals $\int_1^2 \sqrt{x}\,dx$, $\int_1^2 \sqrt{1/x}\,dx$, and $\int_1^2 \sqrt{\sqrt{x}}\,dx$ has the largest value? Why?

68. Which of the integrals $\int_0^{0.5} \cos(x^2)\,dx$, $\int_0^{0.5} \cos\sqrt{x}\,dx$ is larger? Why?

69. Prove Property 3 of integrals.

70. (a) If f is continuous on $[a, b]$, show that

$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$

[*Hint:* $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.]

(b) Use the result of part (a) to show that

$$\left| \int_0^{2\pi} f(x) \sin 2x\,dx \right| \leqslant \int_0^{2\pi} |f(x)|\,dx$$

71. Let $f(x) = 0$ if x is any rational number and $f(x) = 1$ if x is any irrational number. Show that f is not integrable on $[0, 1]$.

72. Let $f(0) = 0$ and $f(x) = 1/x$ if $0 < x \leqslant 1$. Show that f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_1^*)\,\Delta x$, can be made arbitrarily large.]

73–74 Express the limit as a definite integral.

73. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

74. $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}$

75. Find $\int_1^2 x^{-2}\,dx$. *Hint:* Choose x_i^* to be the geometric mean of x_{i-1} and x_i (that is, $x_i^* = \sqrt{x_{i-1}x_i}$) and use the identity

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

DISCOVERY PROJECT AREA FUNCTIONS

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Differentiate the area function $A(x)$. What do you notice?

2. (a) If $x \geqslant -1$, let

$$A(x) = \int_{-1}^{x} (1 + t^2)\,dt$$

$A(x)$ represents the area of a region. Sketch that region.

(b) Use the result of Exercise 4.2.28 to find an expression for $A(x)$.

(c) Find $A'(x)$. What do you notice?

(d) If $x \geqslant -1$ and h is a small positive number, then $A(x + h) - A(x)$ represents the area of a region. Describe and sketch the region.

(e) Draw a rectangle that approximates the region in part (d). By comparing the areas of these two regions, show that

$$\frac{A(x+h) - A(x)}{h} \approx 1 + x^2$$

(f) Use part (e) to give an intuitive explanation for the result of part (c).

3. (a) Draw the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2]$ by $[-1.25, 1.25]$.

(b) If we define a new function g by

$$g(x) = \int_0^x \cos(t^2)\,dt$$

then $g(x)$ is the area under the graph of f from 0 to x [until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas]. Use part (a) to determine the value

of x at which $g(x)$ starts to decrease. [Unlike the integral in Problem 2, it is impossible to evaluate the integral defining g to obtain an explicit expression for $g(x)$.]

(c) Use the integration command on your calculator or computer to estimate $g(0.2)$, $g(0.4)$, $g(0.6)$, . . . , $g(1.8)$, $g(2)$. Then use these values to sketch a graph of g.

(d) Use your graph of g from part (c) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?

4. Suppose f is a continuous function on the interval $[a, b]$ and we define a new function g by the equation

$$g(x) = \int_a^x f(t)\,dt$$

Based on your results in Problems 1–3, conjecture an expression for $g'(x)$.

4.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's mentor at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums as we did in Sections 4.1 and 4.2.

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

1 $$g(x) = \int_a^x f(t)\,dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of x denoted by $g(x)$.

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.)

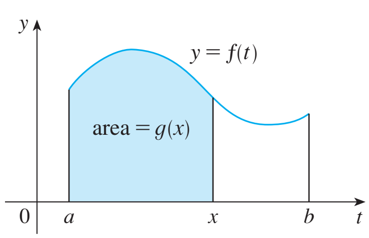


FIGURE 1

EXAMPLE 1 If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

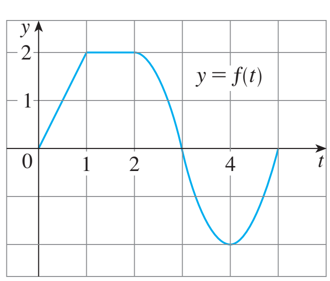


FIGURE 2

SOLUTION First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$. From Figure 3 we see that $g(1)$ is the area of a triangle:

$$g(1) = \int_0^1 f(t)\,dt = \tfrac{1}{2}(1 \cdot 2) = 1$$

To find $g(2)$ we add to $g(1)$ the area of a rectangle:

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \cdot 2) = 3$$

We estimate that the area under f from 2 to 3 is about 1.3, so

$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

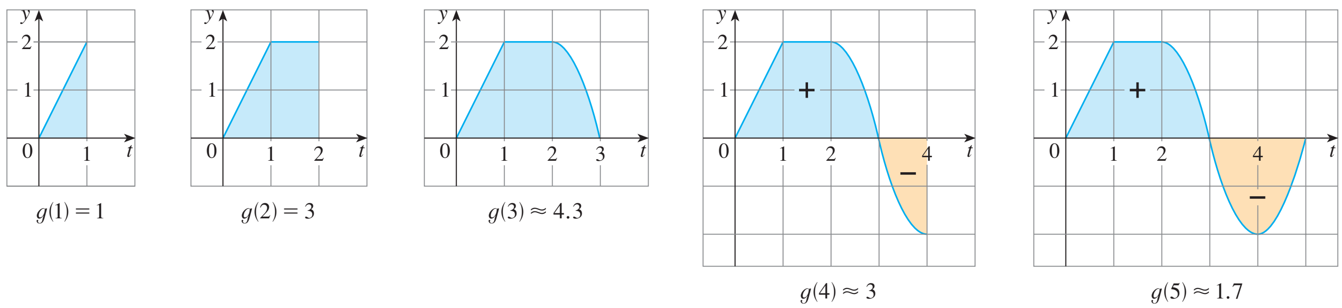


FIGURE 3

For $t > 3$, $f(t)$ is negative and so we start subtracting areas:

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$

We use these values to sketch the graph of g in Figure 4. Notice that, because $f(t)$ is positive for $t < 3$, we keep adding area for $t < 3$ and so g is increasing up to $x = 3$, where it attains a maximum value. For $x > 3$, g decreases because $f(t)$ is negative. ■

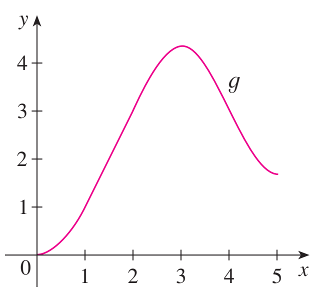


FIGURE 4

$g(x) = \int_0^x f(t)\,dt$

If we take $f(t) = t$ and $a = 0$, then, using Exercise 4.2.27, we have

$$g(x) = \int_0^x t\,dt = \frac{x^2}{2}$$

Notice that $g'(x) = x$, that is, $g' = f$. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case. And if we sketch the derivative of the function g shown in Figure 4 by estimating slopes of tangents, we get a graph like that of f in Figure 2. So we suspect that $g' = f$ in Example 1 too.

To see why this might be generally true we consider any continuous function f with $f(x) \geqslant 0$. Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x + h$ (the blue area in Figure 5). For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x + h) - g(x) \approx hf(x)$$

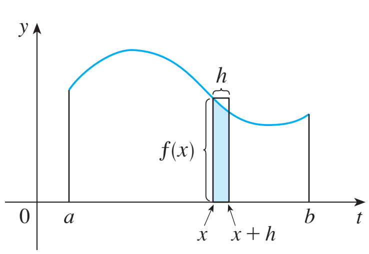


FIGURE 5

so

$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

We abbreviate the name of this theorem as FTC1. In words, it says that the derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

PROOF If x and $x + h$ are in (a, b), then

$$\begin{aligned} g(x + h) - g(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \right) - \int_a^x f(t)\,dt \qquad \text{(by Property 5)} \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

and so, for $h \neq 0$,

2
$$\frac{g(x + h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

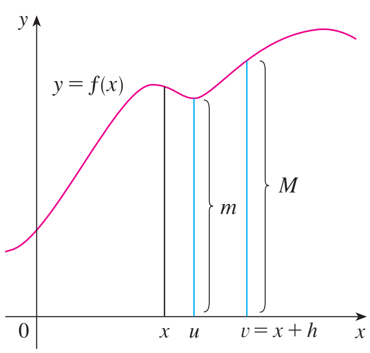


FIGURE 6

For now let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. (See Figure 6.)

By Property 8 of integrals, we have

$$mh \leqslant \int_x^{x+h} f(t)\,dt \leqslant Mh$$

that is,
$$f(u)h \leqslant \int_x^{x+h} f(t)\,dt \leqslant f(v)h$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leqslant \frac{1}{h} \int_x^{x+h} f(t)\,dt \leqslant f(v)$$

Now we use Equation 2 to replace the middle part of this inequality:

3
$$f(u) \leqslant \frac{g(x + h) - g(x)}{h} \leqslant f(v)$$

TEC Module 4.3 provides visual evidence for FTC1.

Inequality 3 can be proved in a similar manner for the case where $h < 0$. (See Exercise 69.)

Now we let $h \to 0$. Then $u \to x$ and $v \to x$, since u and v lie between x and $x + h$. Therefore

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Squeeze Theorem, that

4
$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

If $x = a$ or b, then Equation 4 can be interpreted as a one-sided limit. Then Theorem 2.2.4 (modified for one-sided limits) shows that g is continuous on $[a, b]$. ■

Using Leibniz notation for derivatives, we can write FTC1 as

5
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

when f is continuous. Roughly speaking, Equation 5 says that if we first integrate f and then differentiate the result, we get back to the original function f.

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

SOLUTION Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$ ■

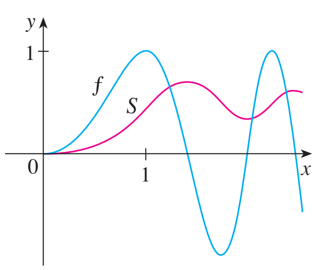


FIGURE 7
$f(x) = \sin(\pi x^2/2)$
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

EXAMPLE 3 Although a formula of the form $g(x) = \int_a^x f(t)\, dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics. This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin(\pi x^2/2)$$

This means that we can apply all the methods of differential calculus to analyze S (see Exercise 63).

Figure 7 shows the graphs of $f(x) = \sin(\pi x^2/2)$ and the Fresnel function $S(x) = \int_0^x f(t)\, dt$. A computer was used to graph S by computing the value of this integral for many values of x. It does indeed look as if $S(x)$ is the area under the graph of f from 0 to x [until $x \approx 1.4$ when $S(x)$ becomes a difference of areas]. Figure 8 shows a larger part of the graph of S.

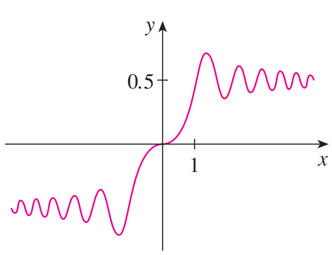


FIGURE 8
The Fresnel function
$S(x) = \int_0^x \sin(\pi t^2/2)\, dt$

If we now start with the graph of S in Figure 7 and think about what its derivative should look like, it seems reasonable that $S'(x) = f(x)$. [For instance, S is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.] So this gives a visual confirmation of Part 1 of the Fundamental Theorem of Calculus. ■

EXAMPLE 4 Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\, dt$.

SOLUTION Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t\, dt &= \frac{d}{dx}\int_1^{u} \sec t\, dt\\ &= \frac{d}{du}\left[\int_1^{u} \sec t\, dt\right]\frac{du}{dx} && \text{(by the Chain Rule)}\\ &= \sec u\,\frac{du}{dx} && \text{(by FTC1)}\\ &= \sec(x^4)\cdot 4x^3\end{aligned}$$

In Section 4.2 we computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

We abbreviate this theorem as FTC2.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

PROOF Let $g(x) = \int_a^x f(t)\, dt$. We know from Part 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then we know from Corollary 3.2.7 that F and g differ by a constant:

6 $$F(x) = g(x) + C$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of Equation 6 (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$. So $F(x) = g(x) + C$ for all x in $[a, b]$.

If we put $x = a$ in the formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)\, dt = 0$$

So, using Equation 6 with $x = b$ and $x = a$, we have

$$\begin{aligned}F(b) - F(a) &= [g(b) + C] - [g(a) + C]\\ &= g(b) - g(a) = g(b) = \int_a^b f(t)\, dt\end{aligned}$$

Part 2 of the Fundamental Theorem states that if we know an antiderivative F of f, then we can evaluate $\int_a^b f(x)\, dx$ simply by subtracting the values of F at the endpoints of the interval $[a, b]$. It's very surprising that $\int_a^b f(x)\, dx$, which was defined by a complicated procedure involving all of the values of $f(x)$ for $a \leqslant x \leqslant b$, can be found by knowing the values of $F(x)$ at only two points, a and b.

Although the theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms. If $v(t)$ is the velocity of an object and $s(t)$ is its position at time t, then $v(t) = s'(t)$, so s is an antiderivative of v. In Section 4.1 we considered an object that always moves in the positive direction and made the guess that the area under the velocity curve is equal to the distance traveled. In symbols:

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

That is exactly what FTC2 says in this context.

EXAMPLE 5 Evaluate the integral $\int_{-2}^{1} x^3\,dx$.

SOLUTION The function $f(x) = x^3$ is continuous on $[-2, 1]$ and we know from Section 3.9 that an antiderivative is $F(x) = \frac{1}{4}x^4$, so Part 2 of the Fundamental Theorem gives

$$\int_{-2}^{1} x^3\,dx = F(1) - F(-2) = \tfrac{1}{4}(1)^4 - \tfrac{1}{4}(-2)^4 = -\tfrac{15}{4}$$

Notice that FTC2 says we can use *any* antiderivative F of f. So we may as well use the simplest one, namely $F(x) = \frac{1}{4}x^4$, instead of $\frac{1}{4}x^4 + 7$ or $\frac{1}{4}x^4 + C$. ■

We often use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

So the equation of FTC2 can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f$$

Other common notations are $F(x)\big|_a^b$ and $[F(x)]_a^b$.

EXAMPLE 6 Find the area under the parabola $y = x^2$ from 0 to 1.

SOLUTION An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

In applying the Fundamental Theorem we use a particular antiderivative F of f. It is not necessary to use the most general antiderivative.

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$ ■

If you compare the calculation in Example 6 with the one in Example 4.1.2, you will see that the Fundamental Theorem gives a *much* shorter method.

EXAMPLE 7 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$. (See Figure 9.) ■

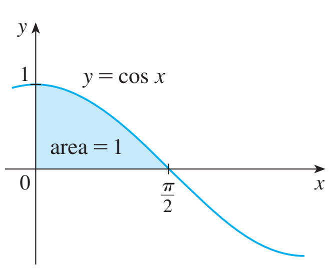


FIGURE 9

When the French mathematician Gilles de Roberval first found the area under the sine and cosine curves in 1635, this was a very challenging problem that required a great deal of ingenuity. If we didn't have the benefit of the Fundamental Theorem, we would

have to compute a difficult limit of sums using obscure trigonometric identities (or a computer algebra system as in Exercise 4.1.31). It was even more difficult for Roberval because the apparatus of limits had not been invented in 1635. But in the 1660s and 1670s, when the Fundamental Theorem was discovered by Barrow and exploited by Newton and Leibniz, such problems became very easy, as you can see from Example 7.

EXAMPLE 8 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

SOLUTION To start, we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \geqslant 0$ and Property 6 of integrals says that $\int_a^b f(x)\,dx \geqslant 0$ when $f \geqslant 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has an infinite discontinuity at $x = 0$, so

$$\int_{-1}^{3} \frac{1}{x^2}\,dx \qquad \text{does not exist.}$$

Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f. Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

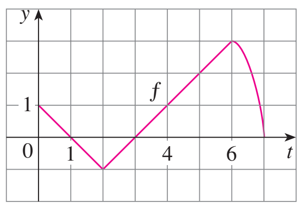
This version says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function F, but in the form $F(b) - F(a)$. Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

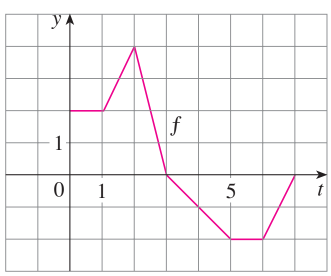
4.3 EXERCISES

1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

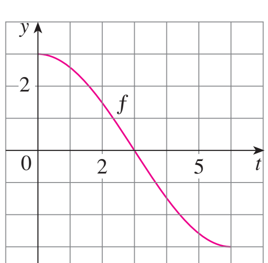
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.
(b) Estimate $g(7)$.
(c) Where does g have a maximum value? Where does it have a minimum value?
(d) Sketch a rough graph of g.



3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
(b) On what interval is g increasing?
(c) Where does g have a maximum value?
(d) Sketch a rough graph of g.



4. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$ and $g(6)$.
(b) Estimate $g(x)$ for $x = 1, 2, 3, 4$, and 5.
(c) On what interval is g increasing?
(d) Where does g have a maximum value?
(e) Sketch a rough graph of g.
(f) Use the graph in part (e) to sketch the graph of $g'(x)$. Compare with the graph of f.



5–6 Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

5. $g(x) = \int_1^x t^2\,dt$

6. $g(x) = \int_0^x (2 + \sin t)\,dt$

7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \int_0^x \sqrt{t + t^3}\,dt$

8. $g(x) = \int_1^x \cos(t^2)\,dt$

9. $g(s) = \int_5^s (t - t^2)^8\,dt$

10. $h(u) = \int_0^u \dfrac{\sqrt{t}}{t + 1}\,dt$

11. $F(x) = \int_x^0 \sqrt{1 + \sec t}\,dt$

$\left[\textit{Hint: } \int_x^0 \sqrt{1 + \sec t}\,dt = -\int_0^x \sqrt{1 + \sec t}\,dt\right]$

12. $R(y) = \int_y^2 t^3 \sin t\,dt$

13. $h(x) = \int_2^{1/x} \sin^4 t\,dt$

14. $h(x) = \int_1^{\sqrt{x}} \dfrac{z^2}{z^4 + 1}\,dz$

15. $y = \int_1^{3x+2} \dfrac{t}{1 + t^3}\,dt$

16. $y = \int_0^{x^4} \cos^2\theta\,d\theta$

17. $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan\theta\,d\theta$

18. $y = \int_{\sin x}^1 \sqrt{1 + t^2}\,dt$

19–38 Evaluate the integral.

19. $\int_1^3 (x^2 + 2x - 4)\,dx$

20. $\int_{-1}^1 x^{100}\,dx$

21. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right)dt$

22. $\int_0^1 \left(1 - 8v^3 + 16v^7\right)dv$

23. $\int_1^9 \sqrt{x}\,dx$

24. $\int_1^8 x^{-2/3}\,dx$

25. $\int_{\pi/6}^{\pi} \sin\theta\,d\theta$

26. $\int_{-5}^5 \pi\,dx$

27. $\int_0^1 (u + 2)(u - 3)\,du$

28. $\int_0^4 (4 - t)\sqrt{t}\,dt$

29. $\int_1^4 \dfrac{2 + x^2}{\sqrt{x}}\,dx$

30. $\int_{-1}^2 (3u - 2)(u + 1)\,du$

31. $\int_{\pi/6}^{\pi/2} \csc t \cot t\,dt$

32. $\int_{\pi/4}^{\pi/3} \csc^2\theta\,d\theta$

33. $\displaystyle\int_0^1 (1+r)^3\,dr$

34. $\displaystyle\int_1^2 \frac{s^4+1}{s^2}\,ds$

35. $\displaystyle\int_1^2 \frac{v^5+3v^6}{v^4}\,dv$

36. $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz$

37. $\displaystyle\int_0^{\pi} f(x)\,dx$ where $f(x)=\begin{cases}\sin x & \text{if } 0 \le x < \pi/2\\ \cos x & \text{if } \pi/2 \le x \le \pi\end{cases}$

38. $\displaystyle\int_{-2}^{2} f(x)\,dx$ where $f(x)=\begin{cases}2 & \text{if } -2 \le x \le 0\\ 4-x^2 & \text{if } 0 < x \le 2\end{cases}$

39–42 Sketch the region enclosed by the given curves and calculate its area.

39. $y=\sqrt{x},\quad y=0,\quad x=4$

40. $y=x^3,\quad y=0,\quad x=1$

41. $y=4-x^2,\quad y=0$

42. $y=2x-x^2,\quad y=0$

43–46 Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

43. $y=\sqrt[3]{x},\ 0 \le x \le 27$

44. $y=x^{-4},\ 1 \le x \le 6$

45. $y=\sin x,\ 0 \le x \le \pi$

46. $y=\sec^2 x,\ 0 \le x \le \pi/3$

47–48 Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

47. $\displaystyle\int_{-1}^{2} x^3\,dx$

48. $\displaystyle\int_{\pi/6}^{2\pi} \cos x\,dx$

49–52 What is wrong with the equation?

49. $\displaystyle\int_{-2}^{1} x^{-4}\,dx = \frac{x^{-3}}{-3}\bigg]_{-2}^{1} = -\frac{3}{8}$

50. $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\,dx = -\frac{2}{x^2}\bigg]_{-1}^{2} = \frac{3}{2}$

51. $\displaystyle\int_{\pi/3}^{\pi} \sec\theta\tan\theta\,d\theta = \sec\theta\Big]_{\pi/3}^{\pi} = -3$

52. $\displaystyle\int_0^{\pi} \sec^2 x\,dx = \tan x\Big]_0^{\pi} = 0$

53–56 Find the derivative of the function.

53. $\displaystyle g(x)=\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$

$$\left[\textit{Hint: } \int_{2x}^{3x} f(u)\,du = \int_{2x}^{0} f(u)\,du + \int_0^{3x} f(u)\,du\right]$$

54. $\displaystyle g(x)=\int_{1-2x}^{1+2x} t\sin t\,dt$

55. $\displaystyle h(x)=\int_{\sqrt{x}}^{x^3} \cos(t^2)\,dt$

56. $\displaystyle g(x)=\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$

57. Let $\displaystyle F(x)=\int_{\pi}^{x} \frac{\cos t}{t}\,dt$. Find an equation of the tangent line to the curve $y=F(x)$ at the point with x-coordinate π.

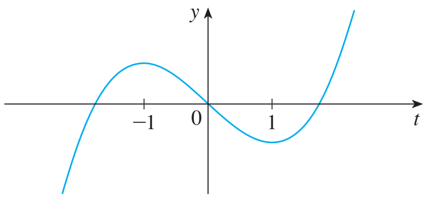
58. If $f(x)=\int_0^x (1-t^2)\cos^2 t\,dt$, on what interval is f increasing?

59. On what interval is the curve

$$y=\int_0^x \frac{t^2}{t^2+t+2}\,dt$$

concave downward?

60. Let $F(x)=\int_1^x f(t)\,dt$, where f is the function whose graph is shown. Where is F concave downward?



61. If $f(1)=12$, f' is continuous, and $\int_1^4 f'(x)\,dx=17$, what is the value of $f(4)$?

62. If $f(x)=\int_0^{\sin x} \sqrt{1+t^2}\,dt$ and $g(y)=\int_3^y f(x)\,dx$, find $g''(\pi/6)$.

63. The Fresnel function S was defined in Example 3 and graphed in Figures 7 and 8.

(a) At what values of x does this function have local maximum values?

(b) On what intervals is the function concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \sin(\pi t^2/2)\,dt = 0.2$$

CAS **64.** The **sine integral function**

$$\text{Si}(x)=\int_0^x \frac{\sin t}{t}\,dt$$

is important in electrical engineering. [The integrand $f(t)=(\sin t)/t$ is not defined when $t=0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0)=1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.

(b) At what values of x does this function have local maximum values?

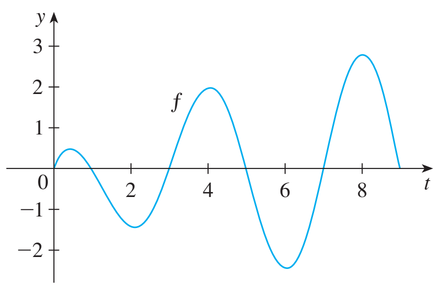
(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?
(e) Solve the following equation correct to one decimal place:

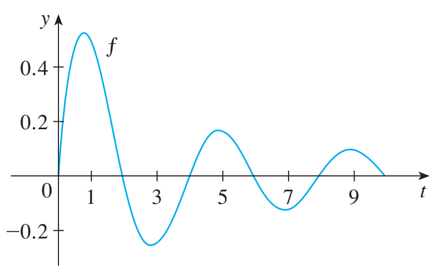
$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

65–66 Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) At what values of x do the local maximum and minimum values of g occur?
(b) Where does g attain its absolute maximum value?
(c) On what intervals is g concave downward?
(d) Sketch the graph of g.

65.



66.



67–68 Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on [0, 1].

67. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^4}{n^5} + \frac{i}{n^2}\right)$

68. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

69. Justify (3) for the case $h < 0$.

70. If f is continuous and g and h are differentiable functions, find a formula for

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$

71. (a) Show that $1 \leqslant \sqrt{1 + x^3} \leqslant 1 + x^3$ for $x \geqslant 0$.
(b) Show that $1 \leqslant \int_0^1 \sqrt{1 + x^3}\,dx \leqslant 1.25$.

72. (a) Show that $\cos(x^2) \geqslant \cos x$ for $0 \leqslant x \leqslant 1$.
(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \geqslant \frac{1}{2}$.

73. Show that

$$0 \leqslant \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx \leqslant 0.1$$

by comparing the integrand to a simpler function.

74. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and $$g(x) = \int_0^x f(t)\,dt$$

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f differentiable? Where is g differentiable?

75. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

76. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

77. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.
(a) Explain why $\int_0^t f(s)\,ds$ represents the loss in value of the machine over the period of time t since the last overhaul.
(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

What does C represent and why would the company want to minimize C?
(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

78. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measured in months. The company wants to determine the optimal time to replace the system.
(a) Let

$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$$

Show that the critical numbers of C occur at the numbers t where $C(t) = f(t) + g(t)$.

(b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450}t & \text{if } 0 < t \leqslant 30 \\ 0 & \text{if } t > 30 \end{cases}$$

and $$g(t) = \frac{Vt^2}{12{,}900} \qquad t > 0$$

Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)\,ds$ to equal the initial value V.

(c) Determine the absolute minimum of C on $(0, T]$.

(d) Sketch the graphs of C and $f + g$ in the same coordinate system, and verify the result in part (a) in this case.

The following exercises are intended only for those who have already covered Chapter 6.

79–84 Evaluate the integral.

79. $\displaystyle\int_1^9 \frac{1}{2x}\,dx$

80. $\displaystyle\int_0^1 10^x\,dx$

81. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\,dx$

82. $\displaystyle\int_0^1 \frac{4}{t^2 + 1}\,dt$

83. $\displaystyle\int_{-1}^1 e^{u+1}\,du$

84. $\displaystyle\int_1^3 \frac{y^3 - 2y^2 - y}{y^2}\,dy$

4.4 Indefinite Integrals and the Net Change Theorem

We saw in Section 4.3 that the second part of the Fundamental Theorem of Calculus provides a very powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals. We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where F is an antiderivative of f.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant C).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions). The connection between them is given by Part 2 of the Fundamental Theorem: if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions. We therefore restate the Table of Antidifferentiation Formulas from Section 3.9, together with a few others, in the notation of indefinite integrals. Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance,

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Recall from Theorem 3.9.1 that the most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative. **We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.** Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\[2ex] -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

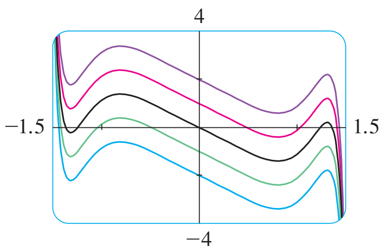


FIGURE 1

The indefinite integral in Example 1 is graphed in Figure 1 for several values of C. Here the value of C is the y-intercept.

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\, dx$$

SOLUTION Using our convention and Table 1, we have

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\, dx &= 10\int x^4\, dx - 2\int \sec^2 x\, dx \\ &= 10\frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C\end{aligned}$$

You should check this answer by differentiating it. ■

EXAMPLE 2 Evaluate $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$.

SOLUTION This indefinite integral isn't immediately apparent from Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta &= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta \\ &= \int \csc\theta\,\cot\theta\, d\theta = -\csc\theta + C\end{aligned}$$ ■

EXAMPLE 3 Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\, dx$.

SOLUTION Using FTC2 and Table 1, we have

$$\begin{aligned}\int_0^3 (x^3 - 6x)\, dx &= \frac{x^4}{4} - 6\frac{x^2}{2}\Big]_0^3 \\ &= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right) \\ &= \tfrac{81}{4} - 27 - 0 + 0 = -6.75\end{aligned}$$

Compare this calculation with Example 4.2.2(b). ■

EXAMPLE 4 Find $\displaystyle\int_0^{12} (x - 12\sin x)\, dx$.

SOLUTION The Fundamental Theorem gives

$$\begin{aligned}\int_0^{12} (x - 12\sin x)\, dx &= \frac{x^2}{2} - 12(-\cos x)\Big]_0^{12} \\ &= \tfrac{1}{2}(12)^2 + 12(\cos 12 - \cos 0) \\ &= 72 + 12\cos 12 - 12 \\ &= 60 + 12\cos 12\end{aligned}$$

This is the exact value of the integral. If a decimal approximation is desired, we can use

Figure 2 shows the graph of the integrand in Example 4. We know from Section 4.2 that the value of the integral can be interpreted as a net area: the sum of the areas labeled with a plus sign minus the areas labeled with a minus sign.

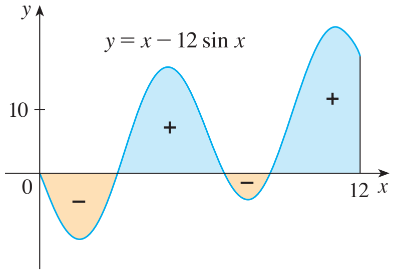


FIGURE 2

a calculator to approximate cos 12. Doing so, we get

$$\int_0^{12} (x - 12\sin x)\,dx \approx 70.1262$$ ■

EXAMPLE 5 Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\bigg]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\bigg]_1^9$$

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$ ■

Applications

Part 2 of the Fundamental Theorem says that if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b. [Note that y could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.] So we can reformulate FTC2 in words as follows.

> **Net Change Theorem** The integral of a rate of change is the net change:
>
> $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences that we discussed in Section 2.7. Here are a few instances of this idea:

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If $[C](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$. So

$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

2 $$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2. In Section 4.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the particle moves to the right) and also the intervals when $v(t) \leqslant 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

3 $$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

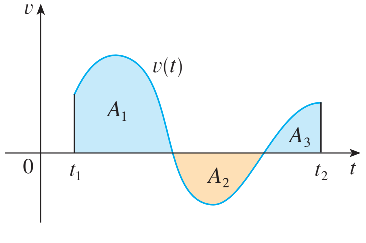


FIGURE 3

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

EXAMPLE 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$.
(b) Find the distance traveled during this time period.

SOLUTION
(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.
(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

To integrate the absolute value of $v(t)$, we use Property 5 of integrals from Section 4.2 to split the integral into two parts, one where $v(t) \leqslant 0$ and one where $v(t) \geqslant 0$.

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6} \approx 10.17 \text{ m}$$

■

EXAMPLE 7 Figure 4 shows the power consumption in the city of San Francisco for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.

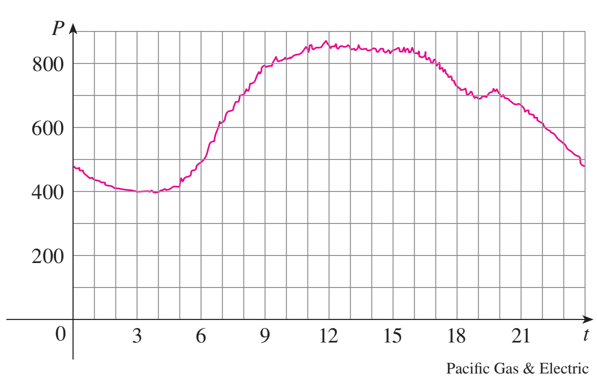


FIGURE 4

SOLUTION Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used on that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$:

$$\begin{aligned} \int_0^{24} P(t)\,dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\,\Delta t \\ &\approx (440 + 400 + 420 + 620 + 790 + 840 + 850 \\ &\qquad + 840 + 810 + 690 + 670 + 550)(2) \\ &= 15{,}840 \end{aligned}$$

The energy used was approximately 15,840 megawatt-hours. ■

A note on units

How did we know what units to use for energy in Example 7? The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$. Now $P(t_i^*)$ is measured in megawatts and Δt is measured in hours, so their product is measured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x.

4.4 EXERCISES

1–4 Verify by differentiation that the formula is correct.

1. $\displaystyle\int \frac{1}{x^2\sqrt{1 + x^2}}\,dx = -\frac{\sqrt{1 + x^2}}{x} + C$

2. $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C$

3. $\displaystyle\int \tan^2 x\,dx = \tan x - x + C$

4. $\displaystyle\int x\sqrt{a + bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a + bx)^{3/2} + C$

5–16 Find the general indefinite integral.

5. $\displaystyle\int (x^{1.3} + 7x^{2.5})\,dx$

6. $\displaystyle\int \sqrt[4]{x^5}\,dx$

7. $\displaystyle\int \left(5 + \tfrac{2}{3}x^2 + \tfrac{3}{4}x^3\right) dx$

8. $\displaystyle\int \left(u^6 - 2u^5 - u^3 + \tfrac{2}{7}\right) du$

9. $\displaystyle\int (u + 4)(2u + 1)\,du$

10. $\displaystyle\int \sqrt{t}\,(t^2 + 3t + 2)\,dt$

11. $\displaystyle\int \frac{1 + \sqrt{x} + x}{\sqrt{x}}\,dx$

12. $\displaystyle\int \left(u^2 + 1 + \frac{1}{u^2}\right) du$

13. $\displaystyle\int (2 + \tan^2\theta)\,d\theta$

14. $\displaystyle\int \sec t\,(\sec t + \tan t)\,dt$

15. $\displaystyle\int \frac{1 - \sin^3 t}{\sin^2 t}\,dt$

16. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx$

17–18 Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

17. $\displaystyle\int \left(\cos x + \tfrac{1}{2}x\right) dx$

18. $\displaystyle\int (1 - x^2)^2\,dx$

19–42 Evaluate the integral.

19. $\displaystyle\int_{-2}^{3} (x^2 - 3)\,dx$

20. $\displaystyle\int_{1}^{2} (4x^3 - 3x^2 + 2x)\,dx$

21. $\displaystyle\int_{-2}^{0} \left(\tfrac{1}{2}t^4 + \tfrac{1}{4}t^3 - t\right) dt$

22. $\displaystyle\int_{0}^{3} (1 + 6w^2 - 10w^4)\,dw$

23. $\int_0^2 (2x - 3)(4x^2 + 1)\,dx$

24. $\int_{-1}^1 t(1 - t)^2\,dt$

25. $\int_0^\pi (4\sin\theta - 3\cos\theta)\,d\theta$

26. $\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx$

27. $\int_1^4 \left(\frac{4 + 6u}{\sqrt{u}}\right) du$

28. $\int_1^2 \left(2 - \frac{1}{p^2}\right)^2 dp$

29. $\int_1^4 \sqrt{\frac{5}{x}}\,dx$

30. $\int_1^8 \left(\frac{2}{\sqrt[3]{w}} - \sqrt[3]{w}\right) dw$

31. $\int_1^4 \sqrt{t}\,(1 + t)\,dt$

32. $\int_0^{\pi/4} \sec\theta\,\tan\theta\,d\theta$

33. $\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\,d\theta$

34. $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta$

35. $\int_1^8 \frac{2 + t}{\sqrt[3]{t^2}}\,dt$

36. $\int_0^{64} \sqrt{u}\,(u - \sqrt[3]{u})\,du$

37. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

38. $\int_0^1 (1 + x^2)^3\,dx$

39. $\int_2^5 |x - 3|\,dx$

40. $\int_0^2 |2x - 1|\,dx$

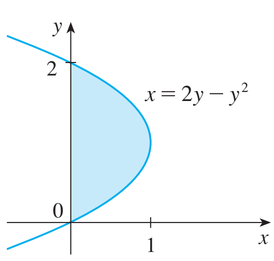
41. $\int_{-1}^2 \left(x - 2|x|\right) dx$

42. $\int_0^{3\pi/2} |\sin x|\,dx$

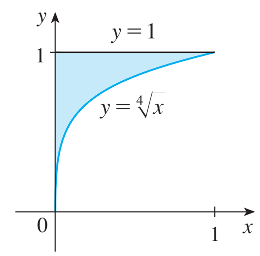
43. Use a graph to estimate the x-intercepts of the curve $y = 1 - 2x - 5x^4$. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.

44. Repeat Exercise 43 for the curve $y = 2x + 3x^4 - 2x^6$.

45. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\,dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.



46. The boundaries of the shaded region in the figure are the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y (as in Exercise 45).



47. If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

48. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 2.7.3.) What does $\int_a^b I(t)\,dt$ represent?

49. If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\,dt$ represent?

50. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

51. In Section 3.7 we defined the marginal revenue function $R'(x)$ as the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)\,dx$ represent?

52. If $f(x)$ is the slope of a trail at a distance of x miles from the start of the trail, what does $\int_3^5 f(x)\,dx$ represent?

53. If x is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\,dx$?

54. If the units for x are feet and the units for $a(x)$ are pounds per foot, what are the units for da/dx? What units does $\int_2^8 a(x)\,dx$ have?

55–56 The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

55. $v(t) = 3t - 5, \quad 0 \leqslant t \leqslant 3$

56. $v(t) = t^2 - 2t - 3, \quad 2 \leqslant t \leqslant 4$

57–58 The acceleration function (in m/s^2) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

57. $a(t) = t + 4, \quad v(0) = 5, \quad 0 \leqslant t \leqslant 10$

58. $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \leqslant t \leqslant 3$

59. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

60. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leqslant t \leqslant 50$. Find the amount of water that flows from the tank during the first 10 minutes.

61. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

t (s)	v (mi/h)	t (s)	v (mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

62. Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time t is measured in seconds and the units for $r(t)$ are tonnes (metric tons) per second.

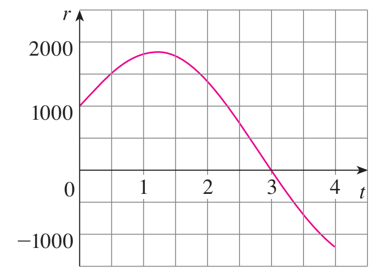
t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after six seconds.
(b) Use the Midpoint Rule to estimate $Q(6)$.

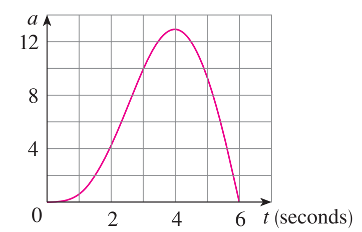
63. Lake Lanier in Georgia, USA, is a reservoir created by Buford Dam on the Chattahoochee River. The table shows the rate of inflow of water, in cubic feet per second, as measured every morning at 7:30 AM by the US Army Corps of Engineers. Use the Midpoint Rule to estimate the amount of water that flowed into Lake Lanier from July 18th, 2013, at 7:30 AM to July 26th at 7:30 AM.

Day	Inflow rate (ft^3/s)
July 18	5275
July 19	6401
July 20	2554
July 21	4249
July 22	3016
July 23	3821
July 24	2462
July 25	2628
July 26	3003

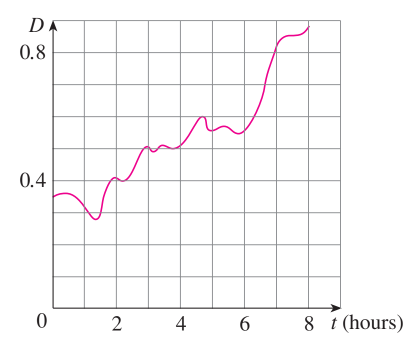
64. Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown. If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule to estimate the amount of water in the tank four days later.



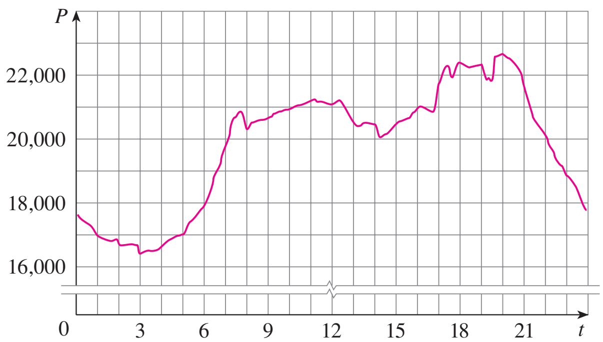
65. The graph of the acceleration $a(t)$ of a car measured in ft/s^2 is shown. Use the Midpoint Rule to estimate the increase in the velocity of the car during the six-second time interval.



66. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use the Midpoint Rule to estimate the total amount of data transmitted during that time period.



67. The following graph shows the power consumption in the province of Ontario, Canada, for December 9, 2004 (P is measured in megawatts; t is measured in hours starting at midnight). Using the fact that power is the rate of change of energy, estimate the energy used on that day.



Independent Electricity Market Operator

68. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

(a) Use a graphing calculator or computer to model these data by a third-degree polynomial.

(b) Use the model in part (a) to estimate the height reached by the *Endeavour,* 125 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

The following exercises are intended only for those who have already covered Chapter 6.

69–73 Evaluate the integral.

69. $\displaystyle\int (\sin x + \sinh x)\, dx$

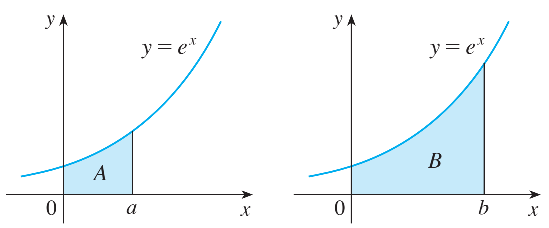
70. $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\, dx$

71. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx$

72. $\displaystyle\int_1^2 \frac{(x-1)^3}{x^2}\, dx$

73. $\displaystyle\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1}\, dt$

74. The area labeled B is three times the area labeled A. Express b in terms of a.



WRITING PROJECT NEWTON, LEIBNIZ, AND THE INVENTION OF CALCULUS

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow—who taught at Cambridge and was a major influence on Newton—was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that Newton and Leibniz are credited with the invention of calculus.

Read about the contributions of these men in one or more of the given references and write a report on one of the following three topics. You can include biographical details, but the main

focus of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the sourcebooks, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

References

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1987), Chapter 19.
2. Carl Boyer, *The History of the Calculus and Its Conceptual Development* (New York: Dover, 1959), Chapter V.
3. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), Chapters 8 and 9.
4. Howard Eves, *An Introduction to the History of Mathematics,* 6th ed. (New York: Saunders, 1990), Chapter 11.
5. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Leibniz by Joseph Hofmann in Volume VIII and the article on Newton by I. B. Cohen in Volume X.
6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), Chapter 17.

Sourcebooks

1. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., *A Sourcebook in Mathematics* (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., *A Sourcebook in Mathematics,* 1200–1800 (Princeton, NJ: Princeton University Press, 1969), Chapter V.

4.5 The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\, dx$$

PS To find this integral we use the problem-solving strategy of *introducing something extra.* Here the "something extra" is a new variable; we change from the variable x to a new variable u. Suppose that we let u be the quantity under the root sign in (1), $u = 1 + x^2$. Then the differential of u is $du = 2x\, dx$. Notice that if the dx in the notation for an inte-

Differentials were defined in Section 2.9. If $u = f(x)$, then
$$du = f'(x)\, dx$$

gral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in (1) and so, formally, without justifying our calculation, we could write

2
$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$
$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1+x^2)^{3/2} + C$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(1+x^2)^{3/2} + C\right] = \tfrac{2}{3}\cdot\tfrac{3}{2}(1+x^2)^{1/2}\cdot 2x = 2x\sqrt{1+x^2}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\,dx$. Observe that if $F' = f$, then

3
$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\,g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du$$

or, writing $F' = f$, we get

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Thus we have proved the following rule.

> **4 The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then
>
> $$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in (4) as differentials.

Thus the Substitution Rule says: **it is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

EXAMPLE 1 Find $\int x^3\cos(x^4 + 2)\,dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using

$x^3\,dx = \frac{1}{4}\,du$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$
$$= \tfrac{1}{4}\sin u + C$$
$$= \tfrac{1}{4}\sin(x^4 + 2) + C$$

Check the answer by differentiating it.

Notice that at the final stage we had to return to the original variable x. ■

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral. This is accomplished by changing from the original variable x to a new variable u that is a function of x. Thus in Example 1 we replaced the integral $\int x^3 \cos(x^4 + 2)\,dx$ by the simpler integral $\frac{1}{4}\int \cos u\,du$.

The main challenge in using the Substitution Rule is to think of an appropriate substitution. You should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). This was the case in Example 1. If that is not possible, try choosing u to be some complicated part of the integrand (perhaps the inner function in a composite function). Finding the right substitution is a bit of an art. It's not unusual to guess wrong; if your first guess doesn't work, try another substitution.

EXAMPLE 2 Evaluate $\int \sqrt{2x + 1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\int \sqrt{2x + 1}\,dx = \int \sqrt{u}\,\cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$
$$= \tfrac{1}{3}(2x + 1)^{3/2} + C$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x + 1}$. Then

$$du = \frac{dx}{\sqrt{2x + 1}} \qquad \text{so} \qquad dx = \sqrt{2x + 1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\int \sqrt{2x + 1}\,dx = \int u \cdot u\,du = \int u^2\,du$$
$$= \frac{u^3}{3} + C = \tfrac{1}{3}(2x + 1)^{3/2} + C$$ ■

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1 - 4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1 - 4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$
$$= -\tfrac{1}{8}\left(2\sqrt{u}\right) + C = -\tfrac{1}{4}\sqrt{1 - 4x^2} + C$$ ■

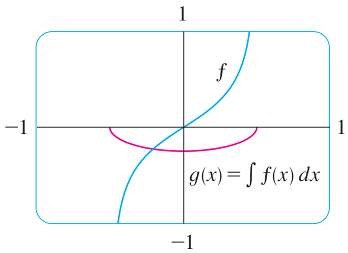


FIGURE 1

$f(x) = \dfrac{x}{\sqrt{1 - 4x^2}}$

$g(x) = \int f(x)\,dx = -\frac{1}{4}\sqrt{1 - 4x^2}$

The answer to Example 3 could be checked by differentiation, but instead let's check it with a graph. In Figure 1 we have used a computer to graph both the integrand $f(x) = x/\sqrt{1 - 4x^2}$ and its indefinite integral $g(x) = -\frac{1}{4}\sqrt{1 - 4x^2}$ (we take the case $C = 0$). Notice that $g(x)$ decreases when $f(x)$ is negative, increases when $f(x)$ is positive, and has its minimum value when $f(x) = 0$. So it seems reasonable, from the graphical evidence, that g is an antiderivative of f.

EXAMPLE 4 Evaluate $\int \cos 5x\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int \cos 5x\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$ ■

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3\cos(x^4 + 2)\,dx = \int \cos(x^4 + 2)\cdot x^3\,dx = \frac{1}{4}\int \cos(x^4 + 2)\cdot(4x^3)\,dx$$

$$= \frac{1}{4}\int \cos(x^4 + 2)\cdot\frac{d}{dx}(x^4 + 2)\,dx = \frac{1}{4}\sin(x^4 + 2) + C$$

Similarly, the solution to Example 4 could be written like this:

$$\int \cos 5x\,dx = \frac{1}{5}\int 5\cos 5x\,dx = \frac{1}{5}\int \frac{d}{dx}(\sin 5x)\,dx = \frac{1}{5}\sin 5x + C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1 + x^2}\,x^5\,dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\int \sqrt{1 + x^2}\,x^5\,dx = \int \sqrt{1 + x^2}\,x^4\cdot x\,dx$$

$$= \int \sqrt{u}\,(u - 1)^2\cdot\frac{1}{2}\,du = \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du$$

$$= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{7}(1 + x^2)^{7/2} - \frac{2}{5}(1 + x^2)^{5/2} + \frac{1}{3}(1 + x^2)^{3/2} + C$$ ■

Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For instance, using the result of Example 2, we have

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4$$

$$= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2}$$

$$= \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Let F be an antiderivative of f. Then, by (3), $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

EXAMPLE 6 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using (5).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3}u^{3/2}\Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

Observe that when using (5) we do *not* return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u. ■

The integral given in Example 7 is an abbreviation for

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx$$

EXAMPLE 7 Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7}\frac{du}{u^2}$$

$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\bigg]_{-2}^{-7}$$

$$= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}$$

Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (5) to simplify the calculation of integrals of functions that possess symmetry properties.

6 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF We split the integral in two:

7 $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)\,(-du) = \int_0^a f(-u)\,du$$

and so Equation 7 becomes

8 $$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

(a) If f is even, then $f(-u) = f(u)$ so Equation 8 gives

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 8 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$$

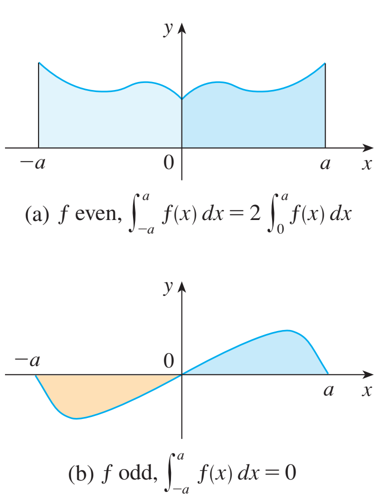


FIGURE 2

Theorem 6 is illustrated by Figure 2. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. Thus part (b) says the integral is 0 because the areas cancel.

EXAMPLE 8 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_0^2 (x^6 + 1)\,dx$$

$$= 2\Big[\tfrac{1}{7}x^7 + x\Big]_0^2 = 2\big(\tfrac{128}{7} + 2\big) = \tfrac{284}{7}$$ ■

EXAMPLE 9 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$ ■

4.5 EXERCISES

1–6 Evaluate the integral by making the given substitution.

1. $\int \cos 2x\,dx, \quad u = 2x$
2. $\int x(2x^2 + 3)^4\,dx, \quad u = 2x^2 + 3$
3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$
4. $\int \sin^2\theta\cos\theta\,d\theta, \quad u = \sin\theta$
5. $\int \frac{x^3}{(x^4 - 5)^2}\,dx, \quad u = x^4 - 5$
6. $\int \sqrt{2t + 1}\,dt, \quad u = 2t + 1$

7–30 Evaluate the indefinite integral.

7. $\int x\sqrt{1 - x^2}\,dx$
8. $\int x^2\sin(x^3)\,dx$
9. $\int (1 - 2x)^9\,dx$
10. $\int \sin t\sqrt{1 + \cos t}\;dt$
11. $\int \sin(2\theta/3)\,d\theta$
12. $\int \sec^2 2\theta\,d\theta$
13. $\int \sec 3t\tan 3t\,dt$
14. $\int y^2(4 - y^3)^{2/3}\,dy$
15. $\int \cos(1 + 5t)\,dt$
16. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
17. $\int \sec^2\theta\tan^3\theta\,d\theta$
18. $\int \sin x\sin(\cos x)\,dx$
19. $\int (x^2 + 1)(x^3 + 3x)^4\,dx$
20. $\int x\sqrt{x + 2}\,dx$
21. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$
22. $\int \frac{\cos(\pi/x)}{x^2}\,dx$
23. $\int \frac{z^2}{\sqrt[3]{1 + z^3}}\,dz$
24. $\int \frac{dt}{\cos^2 t\sqrt{1 + \tan t}}$
25. $\int \sqrt{\cot x}\csc^2 x\,dx$
26. $\int \frac{\sec^2 x}{\tan^2 x}\,dx$
27. $\int \sec^3 x\tan x\,dx$
28. $\int x^2\sqrt{2 + x}\,dx$
29. $\int x(2x + 5)^8\,dx$
30. $\int x^3\sqrt{x^2 + 1}\,dx$

31–34 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

31. $\int x(x^2 - 1)^3\,dx$
32. $\int \tan^2\theta\sec^2\theta\,d\theta$
33. $\int \sin^3 x\cos x\,dx$
34. $\int \sin x\cos^4 x\,dx$

35–51 Evaluate the definite integral.

35. $\int_0^1 \cos(\pi t/2)\,dt$
36. $\int_0^1 (3t - 1)^{50}\,dt$
37. $\int_0^1 \sqrt[3]{1 + 7x}\,dx$
38. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$
39. $\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt$
40. $\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right)dt$
41. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x)\,dx$
42. $\int_0^{\pi/2} \cos x\sin(\sin x)\,dx$
43. $\int_0^{13} \frac{dx}{\sqrt[3]{(1 + 2x)^2}}$
44. $\int_0^a x\sqrt{a^2 - x^2}\,dx$
45. $\int_0^a x\sqrt{x^2 + a^2}\,dx \quad (a > 0)$
46. $\int_{-\pi/3}^{\pi/3} x^4\sin x\,dx$

47. $\int_1^2 x\sqrt{x-1}\,dx$

48. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

49. $\int_{1/2}^1 \frac{\cos(x^{-2})}{x^3}\,dx$

50. $\int_0^{T/2} \sin(2\pi t/T - \alpha)\,dt$

51. $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

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52. Verify that $f(x) = \sin \sqrt[3]{x}$ is an odd function and use that fact to show that

$$0 \leqslant \int_{-2}^3 \sin \sqrt[3]{x}\,dx \leqslant 1$$

53–54 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

53. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

54. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

---

55. Evaluate $\int_{-2}^2 (x+3)\sqrt{4-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

56. Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

57. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

58. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is this man's total basal metabolism, $\int_0^{24} R(t)\,dt$, over a 24-hour time period?

59. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

60. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

61. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

62. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

63. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

64. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

65. If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

66. Use Exercise 65 to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

The following exercises are intended only for those who have already covered Chapter 6.

67–84 Evaluate the integral.

67. $\int \frac{dx}{5-3x}$

68. $\int e^{-5r}\,dr$

69. $\int \frac{(\ln x)^2}{x}\,dx$

70. $\int \frac{dx}{ax+b} \quad (a \neq 0)$

71. $\int e^x\sqrt{1+e^x}\,dx$

72. $\int e^{\cos t}\sin t\,dt$

73. $\int \frac{(\arctan x)^2}{x^2+1}\,dx$

74. $\int \frac{x}{x^2+4}\,dx$

75. $\int \frac{1+x}{1+x^2}\,dx$

76. $\int \frac{\sin(\ln x)}{x}\,dx$

77. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

78. $\int \frac{\sin x}{1+\cos^2 x}\,dx$

79. $\int \cot x\,dx$

80. $\int \frac{x}{1+x^4}\,dx$

81. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

82. $\int_0^1 xe^{-x^2}\,dx$

83. $\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

84. $\int_0^2 (x-1)e^{(x-1)^2}\,dx$

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85. Use Exercise 64 to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

4 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) Write an expression for a Riemann sum of a function f on an interval $[a, b]$. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a continuous function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State the Midpoint Rule.

4. State both parts of the Fundamental Theorem of Calculus.

5. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

9. State the Substitution Rule. In practice, how do you use it?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then

$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then

$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then

$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. If f' is continuous on $[1, 3]$, then $\int_1^3 f'(v)\,dv = f(3) - f(1)$.

7. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

8. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

9. $\displaystyle\int_{-1}^{1}\left(x^5 - 6x^9 + \frac{\sin x}{(1 + x^4)^2}\right)dx = 0$

10. $\displaystyle\int_{-5}^{5}(ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

11. All continuous functions have derivatives.

12. All continuous functions have antiderivatives.

13. $\displaystyle\int_{\pi}^{2\pi}\frac{\sin x}{x}\,dx = \int_{\pi}^{3\pi}\frac{\sin x}{x}\,dx + \int_{3\pi}^{2\pi}\frac{\sin x}{x}\,dx$

14. If $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

15. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

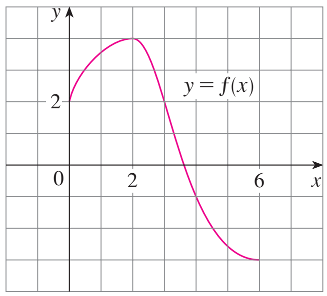
16. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

17. $\displaystyle\int_{-2}^{1}\frac{1}{x^4}\,dx = -\frac{3}{8}$

18. If f has a discontinuity at 0, then $\displaystyle\int_{-1}^{1} f(x)\,dx$ does not exist.

EXERCISES

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \leqslant x \leqslant 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_0^2 (x^2 - x)\, dx$$

(c) Use the Fundamental Theorem to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$

by interpreting it in terms of areas.

4. Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sin x_i \, \Delta x$$

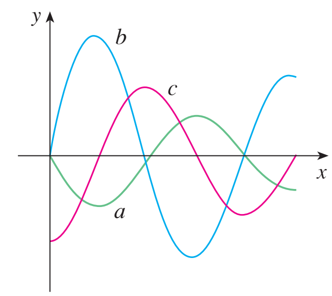
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\, dx = 10$ and $\int_0^4 f(x)\, dx = 7$, find $\int_4^6 f(x)\, dx$.

CAS **6.** (a) Write $\int_1^5 (x + 2x^5)\, dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem to check your answer to part (a).

7. The figure shows the graphs of f, f', and $\int_0^x f(t)\, dt$. Identify each graph, and explain your choices.



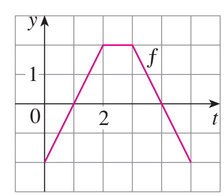
8. Evaluate:

(a) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right) dx$

(b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\, dx$

(c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\, dt$

9. The graph of f consists of the three line segments shown. If $g(x) = \int_0^x f(t)\, dt$, find $g(4)$ and $g'(4)$.



10. If f is the function in Exercise 9, find $g''(4)$.

11–30 Evaluate the integral, if it exists.

11. $\displaystyle\int_1^2 (8x^3 + 3x^2)\, dx$

12. $\displaystyle\int_0^T (x^4 - 8x + 7)\, dx$

13. $\displaystyle\int_0^1 (1 - x^9)\, dx$

14. $\displaystyle\int_0^1 (1 - x)^9\, dx$

15. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\, du$

16. $\displaystyle\int_0^1 \left(\sqrt[4]{u} + 1\right)^2 du$

17. $\displaystyle\int_0^1 y(y^2 + 1)^5\, dy$

18. $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\, dy$

19. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

20. $\displaystyle\int_0^1 \sin(3\pi t)\, dt$

21. $\displaystyle\int_0^1 v^2\cos(v^3)\, dv$

22. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\, dx$

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\, dt$

24. $\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\, dx$

25. $\int \sin \pi t \cos \pi t \, dt$

26. $\int \sin x \cos(\cos x) \, dx$

27. $\int_0^{\pi/8} \sec 2\theta \tan 2\theta \, d\theta$

28. $\int_0^{\pi/4} (1 + \tan t)^3 \sec^2 t \, dt$

29. $\int_0^3 |x^2 - 4| \, dx$

30. $\int_0^4 |\sqrt{x} - 1| \, dx$

31–32 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

31. $\int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx$

32. $\int \frac{x^3}{\sqrt{x^2 + 1}} \, dx$

33. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

34. Graph the function $f(x) = \cos^2 x \sin x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x) \, dx$. Then evaluate the integral to confirm your guess.

35–40 Find the derivative of the function.

35. $F(x) = \int_0^x \frac{t^2}{1 + t^3} \, dt$

36. $F(x) = \int_x^1 \sqrt{t + \sin t} \, dt$

37. $g(x) = \int_0^{x^4} \cos(t^2) \, dt$

38. $g(x) = \int_1^{\sin x} \frac{1 - t^2}{1 + t^4} \, dt$

39. $y = \int_{\sqrt{x}}^x \frac{\cos \theta}{\theta} \, d\theta$

40. $y = \int_{2x}^{3x+1} \sin(t^4) \, dt$

41–42 Use Property 8 of integrals to estimate the value of the integral.

41. $\int_1^3 \sqrt{x^2 + 3} \, dx$

42. $\int_3^5 \frac{1}{x + 1} \, dx$

43–44 Use the properties of integrals to verify the inequality.

43. $\int_0^1 x^2 \cos x \, dx \leqslant \frac{1}{3}$

44. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{x} \, dx \leqslant \frac{\sqrt{2}}{2}$

45. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3) \, dx$.

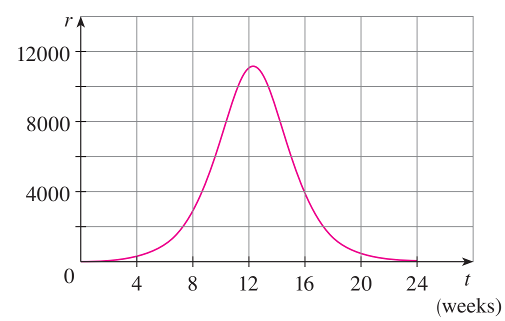
46. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

47. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^8 r(t) \, dt$ represent?

48. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

49. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



50. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leqslant x \leqslant 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$$

Evaluate $\int_{-3}^1 f(x) \, dx$ by interpreting the integral as a difference of areas.

51. If f is continuous and $\int_0^2 f(x) \, dx = 6$, evaluate $\int_0^{\pi/2} f(2 \sin \theta) \cos \theta \, d\theta$.

52. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 4.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

CAS (d) Plot the graphs of C and S on the same screen. How are these graphs related?

53. If f is a continuous function such that

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt$$

for all x, find an explicit formula for $f(x)$.

54. Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin x - 1$$

55. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x)f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

56. Find

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$$

57. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$$

58. Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

Problems Plus

Before you look at the solution of the following example, cover it up and first try to solve the problem yourself.

EXAMPLE Evaluate $\displaystyle\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right)$.

SOLUTION Let's start by having a preliminary look at the ingredients of the function. What happens to the first factor, $x/(x-3)$, when x approaches 3? The numerator approaches 3 and the denominator approaches 0, so we have

$$\frac{x}{x-3}\to\infty \quad \text{as} \quad x\to 3^+ \qquad \text{and} \qquad \frac{x}{x-3}\to-\infty \quad \text{as} \quad x\to 3^-$$

The second factor approaches $\int_3^3 (\sin t)/t\,dt$, which is 0. It's not clear what happens to the function as a whole. (One factor is becoming large while the other is becoming small.) So how do we proceed?

PS The principles of problem solving are discussed on page 98.

One of the principles of problem solving is *recognizing something familiar*. Is there a part of the function that reminds us of something we've seen before? Well, the integral

$$\int_3^x \frac{\sin t}{t}\,dt$$

has x as its upper limit of integration and that type of integral occurs in Part 1 of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This suggests that differentiation might be involved.

Once we start thinking about differentiation, the denominator $(x-3)$ reminds us of something else that should be familiar: One of the forms of the definition of the derivative in Chapter 2 is

$$F'(a)=\lim_{x\to a}\frac{F(x)-F(a)}{x-a}$$

and with $a=3$ this becomes

$$F'(3)=\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

So what is the function F in our situation? Notice that if we define

$$F(x)=\int_3^x \frac{\sin t}{t}\,dt$$

then $F(3)=0$. What about the factor x in the numerator? That's just a red herring, so let's factor it out and put together the calculation:

$$\begin{aligned}
\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right) &= \lim_{x\to 3} x\cdot\lim_{x\to 3}\frac{\displaystyle\int_3^x \frac{\sin t}{t}\,dt}{x-3}\\
&= 3\lim_{x\to 3}\frac{F(x)-F(3)}{x-3}\\
&= 3F'(3)=3\,\frac{\sin 3}{3} \qquad \text{(FTC1)}\\
&= \sin 3
\end{aligned}$$

■

Problems

1. If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$, where f is a continuous function, find $f(4)$.

2. Find the minimum value of the area of the region under the curve $y = 4x - x^3$ from $x = a$ to $x = a + 1$, for all $a > 0$.

3. If f is a differentiable function such that $f(x)$ is never 0 and $\int_0^x f(t)\,dt = [f(x)]^2$ for all x, find f.

4. (a) Graph several members of the family of functions $f(x) = (2cx - x^2)/c^3$ for $c > 0$ and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.

(b) Prove your conjecture in part (a).

(c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?

(d) Find an equation of the curve you sketched in part (c).

5. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\,dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\,dt$, find $f'(\pi/2)$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt$, find $f'(x)$.

7. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\,dx$ is a maximum.

8. Use an integral to estimate the sum $\sum_{i=1}^{10000} \sqrt{i}$.

9. (a) Evaluate $\int_0^n [\![x]\!]\,dx$, where n is a positive integer.

(b) Evaluate $\int_a^b [\![x]\!]\,dx$, where a and b are real numbers with $0 \leqslant a < b$.

10. Find $\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\,du \right) dt$.

11. Suppose the coefficients of the cubic polynomial $P(x) = a + bx + cx^2 + dx^3$ satisfy the equation

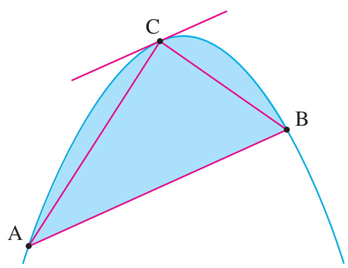
$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = 0$$

Show that the equation $P(x) = 0$ has a root between 0 and 1. Can you generalize this result for an nth-degree polynomial?

12. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height $r/\sqrt{1 + \pi^2}$ above the surface of the liquid.

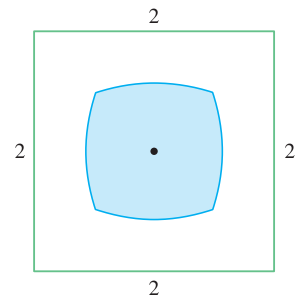
13. Prove that if f is continuous, then $\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$.

14. The figure shows a parabolic segment, that is, a portion of a parabola cut off by a chord AB. It also shows a point C on the parabola with the property that the tangent line at C is parallel to the chord AB. Archimedes proved that the area of the parabolic segment is $\frac{4}{3}$ times the area of the inscribed triangle ABC. Verify Archimedes' result for the parabola $y = 4 - x^2$ and the line $y = x + 2$.



15. Given the point (a, b) in the first quadrant, find the downward-opening parabola that passes through the point (a, b) and the origin such that the area under the parabola is a minimum.

16. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.



17. Evaluate $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$.

18. For any number c, we let $f_c(x)$ be the smaller of the two numbers $(x-c)^2$ and $(x-c-2)^2$. Then we define $g(c)=\int_0^1 f_c(x)\,dx$. Find the maximum and minimum values of $g(c)$ if $-2 \leqslant c \leqslant 2$.

5 Applications of Integration

When a bat strikes a baseball, the collision lasts only about a thousandth of a second. In the project on page 392, you will use calculus to find the average force on the bat when this happens. Several other applications of calculus to the game of baseball are explored as well.



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IN THIS CHAPTER WE EXPLORE some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, and the work done by a varying force. The common theme is the following general method, which is similar to the one we used to find areas under curves: we break up a quantity Q into a large number of small parts. We next approximate each small part by a quantity of the form $f(x_i^*)\,\Delta x$ and thus approximate Q by a Riemann sum. Then we take the limit and express Q as an integral. Finally we evaluate the integral using the Fundamental Theorem of Calculus or the Midpoint Rule.

5.1 Areas Between Curves

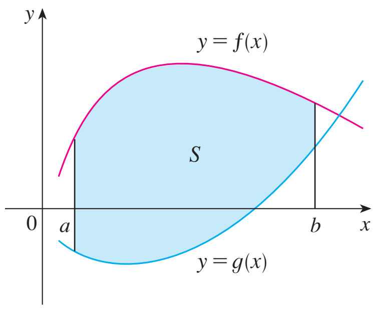


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)\}$

In Chapter 4 we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geqslant g(x)$ for all x in $[a, b]$. (See Figure 1.)

Just as we did for areas under curves in Section 4.1, we divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.) The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.

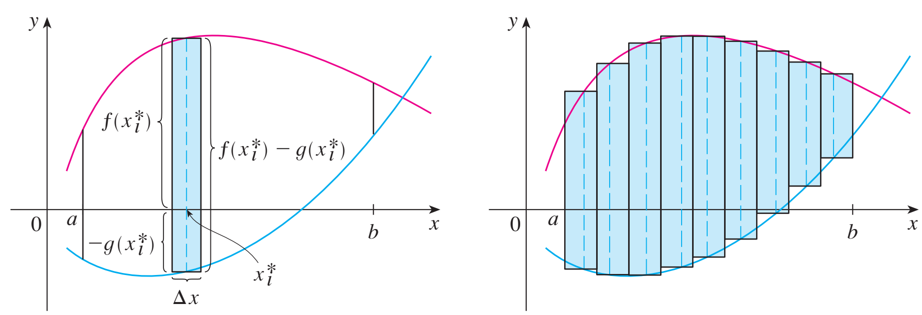


FIGURE 2 (a) Typical rectangle (b) Approximating rectangles

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

1
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

We recognize the limit in (1) as the definite integral of $f - g$. Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces to our previous definition (Definition 4.1.2).

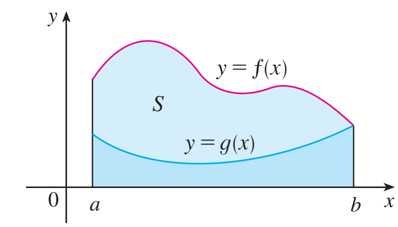


FIGURE 3
$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

In the case where both f and g are positive, you can see from Figure 3 why (2) is true:

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

EXAMPLE 1 Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

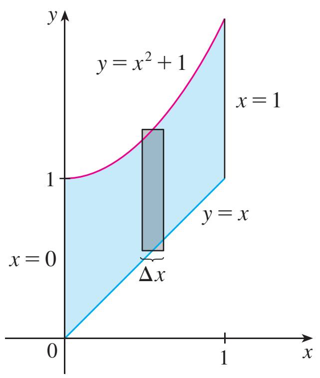


FIGURE 4

SOLUTION The region is shown in Figure 4. The upper boundary curve is $y = x^2 + 1$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = x^2 + 1$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 [(x^2 + 1) - x]\,dx = \int_0^1 (x^2 - x + 1)\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x\Big]_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$ ■

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in (1). In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle as in Figure 5. Then the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

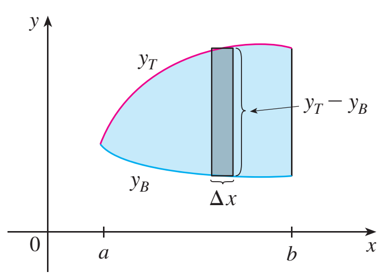


FIGURE 5

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point. In the next example both of the side boundaries reduce to a point, so the first step is to find a and b.

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

SOLUTION We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are (0, 0) and (1, 1).

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

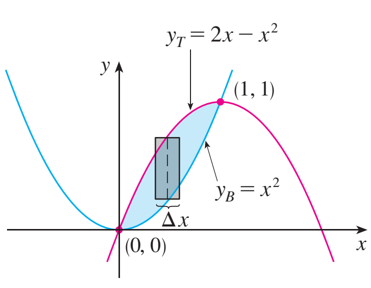


FIGURE 6

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$ ■

Sometimes it's difficult, or even impossible, to find the points of intersection of two curves exactly. As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

EXAMPLE 3 Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2 + 1}$ and $y = x^4 - x$.

SOLUTION If we were to try to find the exact intersection points, we would have to solve the equation

$$\frac{x}{\sqrt{x^2 + 1}} = x^4 - x$$

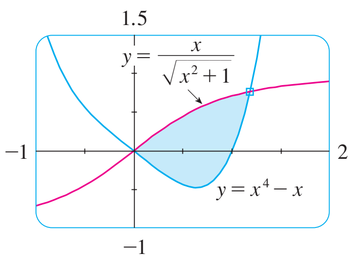


FIGURE 7

This looks like a very difficult equation to solve exactly (in fact, it's impossible), so instead we use a graphing device to draw the graphs of the two curves in Figure 7. One intersection point is the origin. We zoom in toward the other point of intersection and find that $x \approx 1.18$. (If greater accuracy is required, we could use Newton's method or solve numerically on our graphing device.) So an approximation to the area between the curves is

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2 + 1}} - (x^4 - x) \right] dx$$

To integrate the first term we use the substitution $u = x^2 + 1$. Then $du = 2x\,dx$, and when $x = 1.18$, we have $u \approx 2.39$; when $x = 0$, $u = 1$. So

$$A \approx \tfrac{1}{2} \int_1^{2.39} \frac{du}{\sqrt{u}} - \int_0^{1.18} (x^4 - x)\,dx$$

$$= \sqrt{u}\,\Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^{1.18}$$

$$= \sqrt{2.39} - 1 - \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2}$$

$$\approx 0.785$$

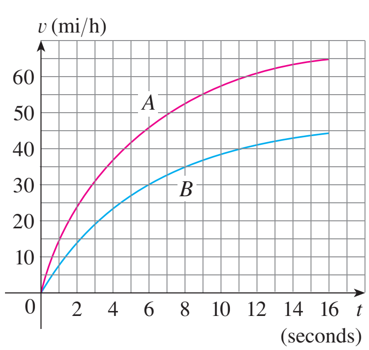


FIGURE 8

EXAMPLE 4 Figure 8 shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.

SOLUTION We know from Section 4.4 that the area under the velocity curve A represents the distance traveled by car A during the first 16 seconds. Similarly, the area under curve B is the distance traveled by car B during that time period. So the area between these curves, which is the difference of the areas under the curves, is the distance between the cars after 16 seconds. We read the velocities from the graph and convert them to feet per second ($1 \text{ mi/h} = \frac{5280}{3600} \text{ ft/s}$).

t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - v_B$	0	13	20	23	25	28	29	29	30

We use the Midpoint Rule with $n = 4$ intervals, so that $\Delta t = 4$. The midpoints of the intervals are $\bar{t}_1 = 2$, $\bar{t}_2 = 6$, $\bar{t}_3 = 10$, and $\bar{t}_4 = 14$. We estimate the distance between the cars after 16 seconds as follows:

$$\int_0^{16} (v_A - v_B)\, dt \approx \Delta t\,[13 + 23 + 28 + 29]$$

$$= 4(93) = 372 \text{ ft}$$

EXAMPLE 5 Figure 9 is an example of a *pathogenesis curve* for a measles infection. It shows how the disease develops in an individual with no immunity after the measles virus spreads to the bloodstream from the respiratory tract.

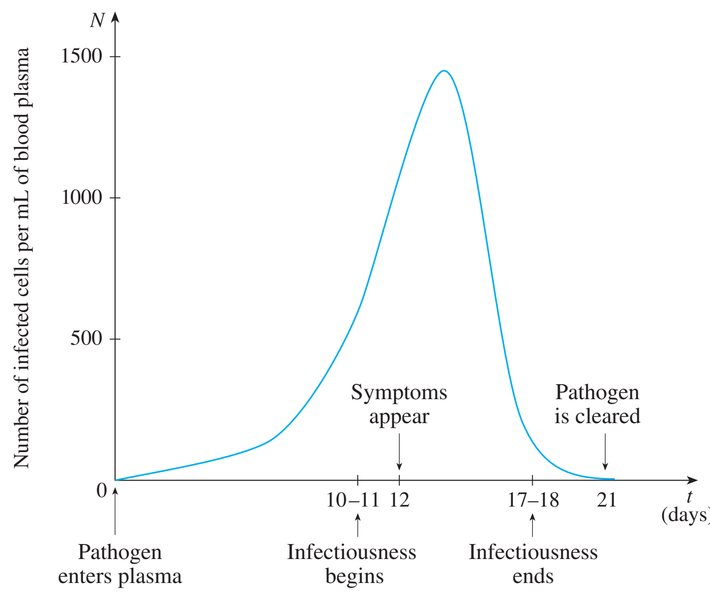


FIGURE 9

Measles pathogenesis curve

Source: J. Heffernan et al., "An In-Host Model of Acute Infection: Measles as a Case Study," *Theoretical Population Biology* 73 (2008): 134–47.

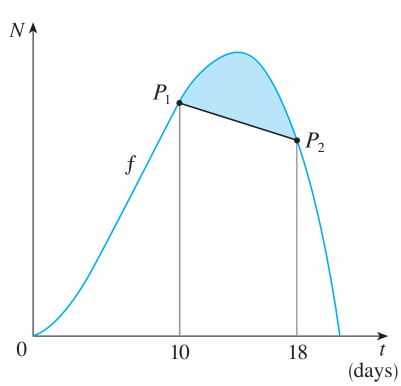


FIGURE 10

The patient becomes infectious to others once the concentration of infected cells becomes great enough, and he or she remains infectious until the immune system manages to prevent further transmission. However, symptoms don't develop until the "amount of infection" reaches a particular threshold. The amount of infection needed to develop symptoms depends on both the concentration of infected cells and time, and corresponds to the area under the pathogenesis curve until symptoms appear. (See Exercise 4.1.19.)

(a) The pathogenesis curve in Figure 9 has been modeled by $f(t) = -t(t - 21)(t + 1)$. If infectiousness begins on day $t_1 = 10$ and ends on day $t_2 = 18$, what are the corresponding concentration levels of infected cells?

(b) The *level of infectiousness* for an infected person is the area between $N = f(t)$ and the line through the points $P_1(t_1, f(t_1))$ and $P_2(t_2, f(t_2))$, measured in (cells/mL) · days. (See Figure 10.) Compute the level of infectiousness for this particular patient.

SOLUTION

(a) Infectiousness begins when the concentration reaches $f(10) = 1210$ cells/mL and ends when the concentration reduces to $f(18) = 1026$ cells/mL.

(b) The line through P_1 and P_2 has slope $\frac{1026 - 1210}{18 - 10} = -\frac{184}{8} = -23$ and equation $N - 1210 = -23(t - 10)$, or $N = -23t + 1440$. The area between f and this line is

$$\int_{10}^{18} [f(t) - (-23t + 1440)]\,dt = \int_{10}^{18} (-t^3 + 20t^2 + 21t + 23t - 1440)\,dt$$

$$= \int_{10}^{18} (-t^3 + 20t^2 + 44t - 1440)\,dt$$

$$= \left[-\frac{t^4}{4} + 20\,\frac{t^3}{3} + 44\,\frac{t^2}{2} - 1440t\right]_{10}^{18}$$

$$= -6156 - \left(-8033\tfrac{1}{3}\right) \approx 1877$$

Thus the level of infectiousness for this patient is about 1877 (cells/mL) · days. ■

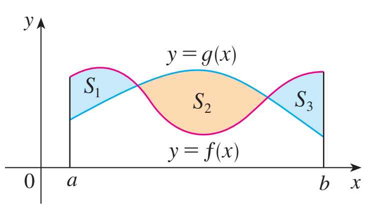


FIGURE 11

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geqslant g(x)$ for some values of x but $g(x) \geqslant f(x)$ for other values of x, then we split the given region S into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$ as shown in Figure 11. We then define the area of the region S to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$, that is, $A = A_1 + A_2 + \cdots$. Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geqslant g(x) \\ g(x) - f(x) & \text{when } g(x) \geqslant f(x) \end{cases}$$

we have the following expression for A.

> **3** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is
>
> $$A = \int_a^b |f(x) - g(x)|\,dx$$

When evaluating the integral in (3), however, we must still split it into integrals corresponding to $A_1, A_2, \ldots$.

EXAMPLE 6 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

SOLUTION The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 12.

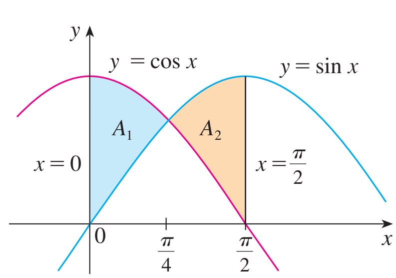


FIGURE 12

Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$\begin{aligned} A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\ &= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} - 2 \end{aligned}$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\, dx$$ ■

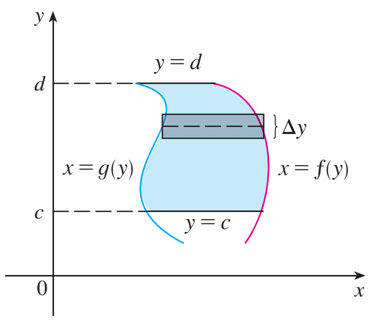


FIGURE 13

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$ (see Figure 13), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

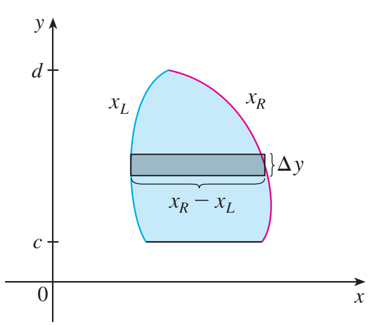


FIGURE 14

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 14 illustrates, we have

$$A = \int_c^d (x_R - x_L)\, dy$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and Δy.

EXAMPLE 7 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and notice from Figure 15 that the left and right boundary curves are

$$x_L = \tfrac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1$$

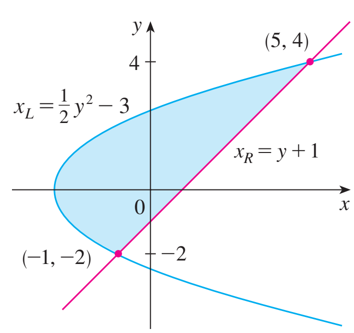


FIGURE 15

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^{4} (x_R - x_L)\,dy = \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\Bigg]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18$$ ■

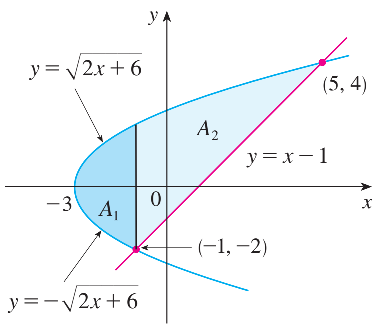


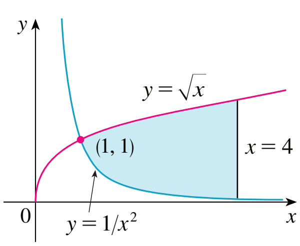
FIGURE 16

NOTE We could have found the area in Example 7 by integrating with respect to x instead of y, but the calculation is much more involved. Because the bottom boundary consists of two different curves, it would have meant splitting the region in two and computing the areas labeled A_1 and A_2 in Figure 16. The method we used in Example 7 is *much* easier.

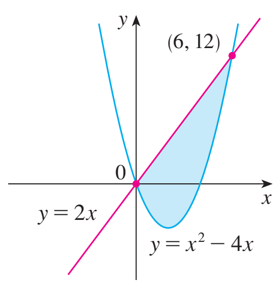
5.1 EXERCISES

1–4 Find the area of the shaded region.

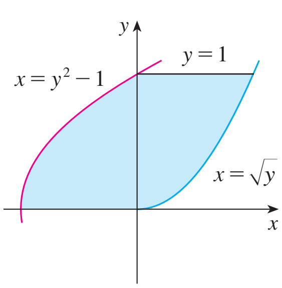
1.



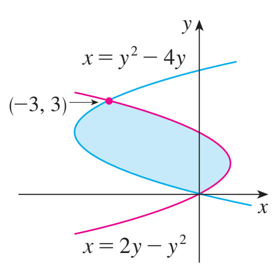
2.



3.



4.



5–12 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

5. $y = x + 1,\quad y = 9 - x^2,\quad x = -1,\quad x = 2$

6. $y = \sin x,\ \ y = x,\quad x = \pi/2,\quad x = \pi$

7. $y = (x - 2)^2,\ \ y = x$

8. $y = x^2 - 4x,\ \ y = 2x$

9. $y = \sqrt{x + 3},\quad y = (x + 3)/2$

10. $y = \sin x,\quad y = 2x/\pi,\quad x \geqslant 0$

11. $x = 1 - y^2,\quad x = y^2 - 1$

12. $4x + y^2 = 12,\quad x = y$

13–28 Sketch the region enclosed by the given curves and find its area.

13. $y = 12 - x^2,\quad y = x^2 - 6$

14. $y = x^2,\quad y = 4x - x^2$

15. $y = \sec^2 x,\quad y = 8\cos x,\quad -\pi/3 \leqslant x \leqslant \pi/3$

16. $y = \cos x,\quad y = 2 - \cos x,\quad 0 \leqslant x \leqslant 2\pi$

17. $x = 2y^2,\quad x = 4 + y^2$

18. $y = \sqrt{x - 1},\quad x - y = 1$

19. $y = \cos \pi x,\quad y = 4x^2 - 1$

20. $x = y^4,\quad y = \sqrt{2 - x},\quad y = 0$

21. $y = \cos x,\quad y = 1 - 2x/\pi$

22. $y = x^3,\quad y = x$

23. $y = \sqrt[3]{2x},\quad y = \tfrac{1}{8}x^2,\quad 0 \leqslant x \leqslant 6$

24. $y = \cos x,\quad y = 1 - \cos x,\quad 0 \leqslant x \leqslant \pi$

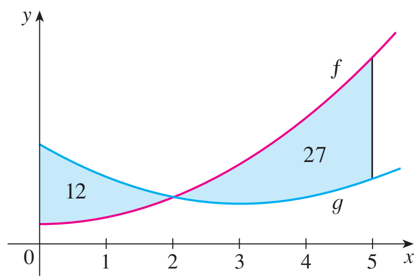
25. $y = x^4,\quad y = 2 - |x|$

26. $y = 3x - x^2,\quad y = x,\quad x = 3$

27. $y = 1/x^2,\quad y = x,\quad y = \tfrac{1}{8}x$

28. $y = \tfrac{1}{4}x^2,\quad y = 2x^2,\quad x + y = 3,\quad x \geqslant 0$

29. The graphs of two functions are shown with the areas of the regions between the curves indicated.
(a) What is the total area between the curves for $0 \leqslant x \leqslant 5$?
(b) What is the value of $\int_0^5 [f(x) - g(x)]\,dx$?



30–32 Sketch the region enclosed by the given curves and find its area.

30. $y = \dfrac{x}{\sqrt{1 + x^2}}, \quad y = \dfrac{x}{\sqrt{9 - x^2}}, \quad x \geqslant 0$

31. $y = \cos^2 x \sin x, \quad y = \sin x, \quad 0 \leqslant x \leqslant \pi$

32. $y = x\sqrt{x^2 + 1}, \quad y = x^2\sqrt{x^3 + 1}$

33–34 Use calculus to find the area of the triangle with the given vertices.

33. $(0, 0), \quad (3, 1), \quad (1, 2)$

34. $(2, 0), \quad (0, 2), \quad (-1, 1)$

35–36 Evaluate the integral and interpret it as the area of a region. Sketch the region.

35. $\displaystyle\int_0^{\pi/2} |\sin x - \cos 2x|\,dx$ **36.** $\displaystyle\int_0^4 |\sqrt{x + 2} - x|\,dx$

37–40 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

37. $y = x\sin(x^2), \quad y = x^4, \quad x \geqslant 0$

38. $y = \dfrac{x}{(x^2 + 1)^2}, \quad y = x^5 - x, \quad x \geqslant 0$

39. $y = 3x^2 - 2x, \quad y = x^3 - 3x + 4$

40. $y = x - \cos x, \quad y = 2 - x^2$

41–44 Graph the region between the curves and use your calculator to compute the area correct to five decimal places.

41. $y = \dfrac{2}{1 + x^4}, \quad y = x^2$ **42.** $y = x^6, \quad y = \sqrt{2 - x^4}$

43. $y = \tan^2 x, \quad y = \sqrt{x}$ **44.** $y = \cos x, \quad y = x + 2\sin^4 x$

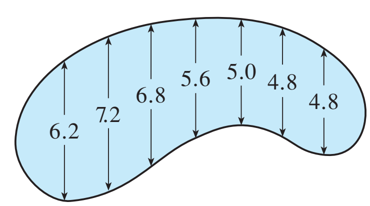
CAS **45.** Use a computer algebra system to find the exact area enclosed by the curves $y = x^5 - 6x^3 + 4x$ and $y = x$.

46. Sketch the region in the xy-plane defined by the inequalities $x - 2y^2 \geqslant 0$, $1 - x - |y| \geqslant 0$ and find its area.

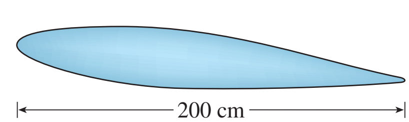
47. Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in miles per hour) during the first ten seconds of the race. Use the Midpoint Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.

t	v_C	v_K	t	v_C	v_K
0	0	0	6	69	80
1	20	22	7	75	86
2	32	37	8	81	93
3	46	52	9	86	98
4	54	61	10	90	102
5	62	71			

48. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use the Midpoint Rule to estimate the area of the pool.



49. A cross-section of an airplane wing is shown. Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing's cross-section.



50. If the birth rate of a population is

$$b(t) = 2200 + 52.3t + 0.74t^2 \text{ people per year}$$

and the death rate is

$$d(t) = 1460 + 28.8t \text{ people per year}$$

find the area between these curves for $0 \leqslant t \leqslant 10$. What does this area represent?

51. In Example 5, we modeled a measles pathogenesis curve by a function f. A patient infected with the measles virus

who has some immunity to the virus has a pathogenesis curve that can be modeled by, for instance, $g(t) = 0.9f(t)$.

(a) If the same threshold concentration of the virus is required for infectiousness to begin as in Example 5, on what day does this occur?

(b) Let P_3 be the point on the graph of g where infectiousness begins. It has been shown that infectiousness ends at a point P_4 on the graph of g where the line through P_3, P_4 has the same slope as the line through P_1, P_2 in Example 5(b). On what day does infectiousness end?

(c) Compute the level of infectiousness for this patient.

52. The rates at which rain fell, in inches per hour, in two different locations t hours after the start of a storm are given by $f(t) = 0.73t^3 - 2t^2 + t + 0.6$ and $g(t) = 0.17t^2 - 0.5t + 1.1$. Compute the area between the graphs for $0 \leqslant t \leqslant 2$ and interpret your result in this context.

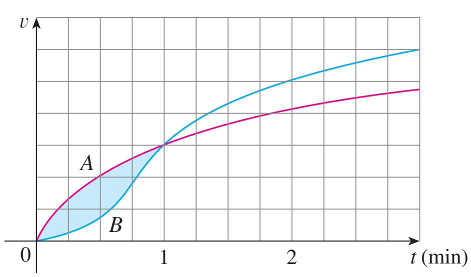
53. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

(a) Which car is ahead after one minute? Explain.

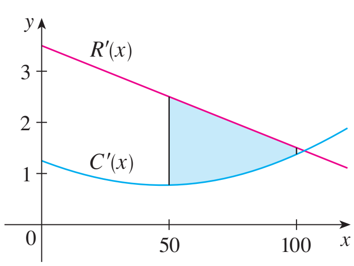
(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.

(d) Estimate the time at which the cars are again side by side.



54. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 3.7 that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.



55. The curve with equation $y^2 = x^2(x + 3)$ is called **Tschirnhausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

56. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

57. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

58. (a) Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \leqslant x \leqslant 4$.

(b) Find the number b such that the line $y = b$ bisects the area in part (a).

59. Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

60. Suppose that $0 < c < \pi/2$. For what value of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$, and $y = 0$?

The following exercises are intended only for those who have already covered Chapter 6.

61–63 Sketch the region bounded by the given curves and find the area of the region.

61. $y = 1/x, \quad y = 1/x^2, \quad x = 2$

62. $y = \sin x, \quad y = e^x, \quad x = 0, \quad x = \pi/2$

63. $y = \tan x, \quad y = 2\sin x, \quad -\pi/3 \leqslant x \leqslant \pi/3$

64. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

APPLIED PROJECT THE GINI INDEX

How is it possible to measure the distribution of income among the inhabitants of a given country? One such measure is the *Gini index*, named after the Italian economist Corrado Gini, who first devised it in 1912.

We first rank all households in a country by income and then we compute the percentage of households whose income is at most a given percentage of the country's total income. We

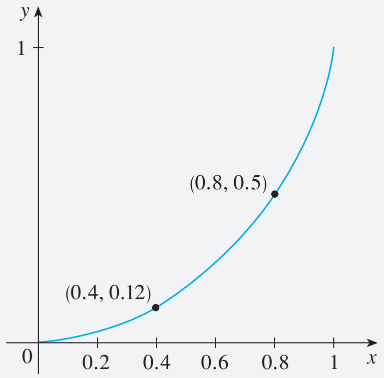


FIGURE 1
Lorenz curve for the US in 2010

define a **Lorenz curve** $y = L(x)$ on the interval $[0, 1]$ by plotting the point $(a/100, b/100)$ on the curve if the bottom $a\%$ of households receive at most $b\%$ of the total income. For instance, in Figure 1 the point (0.4, 0.12) is on the Lorenz curve for the United States in 2010 because the poorest 40% of the population received just 12% of the total income. Likewise, the bottom 80% of the population received 50% of the total income, so the point (0.8, 0.5) lies on the Lorenz curve. (The Lorenz curve is named after the American economist Max Lorenz.)

Figure 2 shows some typical Lorenz curves. They all pass through the points (0, 0) and (1, 1) and are concave upward. In the extreme case $L(x) = x$, society is perfectly egalitarian: the poorest $a\%$ of the population receives $a\%$ of the total income and so everybody receives the same income. The area between a Lorenz curve $y = L(x)$ and the line $y = x$ measures how much the income distribution differs from absolute equality. The **Gini index** (sometimes called the **Gini coefficient** or the **coefficient of inequality**) is the area between the Lorenz curve and the line $y = x$ (shaded in Figure 3) divided by the area under $y = x$.

1. (a) Show that the Gini index G is twice the area between the Lorenz curve and the line $y = x$, that is,

$$G = 2\int_0^1 [x - L(x)]\,dx$$

(b) What is the value of G for a perfectly egalitarian society (everybody has the same income)? What is the value of G for a perfectly totalitarian society (a single person receives all the income?)

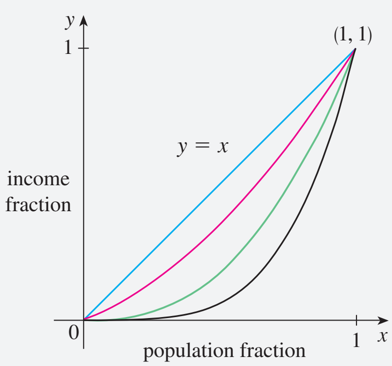


FIGURE 2

2. The following table (derived from data supplied by the US Census Bureau) shows values of the Lorenz function for income distribution in the United States for the year 2010.

x	0.0	0.2	0.4	0.6	0.8	1.0
$L(x)$	0.000	0.034	0.120	0.266	0.498	1.000

(a) What percentage of the total US income was received by the richest 20% of the population in 2010?
(b) Use a calculator or computer to fit a quadratic function to the data in the table. Graph the data points and the quadratic function. Is the quadratic model a reasonable fit?
(c) Use the quadratic model for the Lorenz function to estimate the Gini index for the United States in 2010.

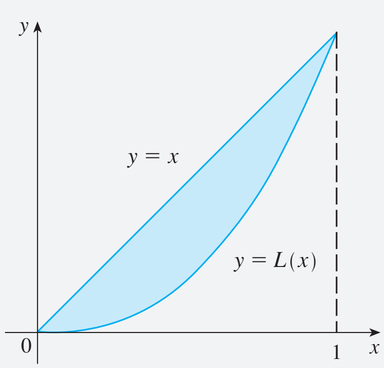


FIGURE 3

3. The following table gives values for the Lorenz function in the years 1970, 1980, 1990, and 2000. Use the method of Problem 2 to estimate the Gini index for the United States for those years and compare with your answer to Problem 2(c). Do you notice a trend?

x	0.0	0.2	0.4	0.6	0.8	1.0
1970	0.000	0.041	0.149	0.323	0.568	1.000
1980	0.000	0.042	0.144	0.312	0.559	1.000
1990	0.000	0.038	0.134	0.293	0.530	1.000
2000	0.000	0.036	0.125	0.273	0.503	1.000

CAS **4.** A power model often provides a more accurate fit than a quadratic model for a Lorenz function. If you have a computer with Maple or Mathematica, fit a power function $(y = ax^k)$ to the data in Problem 2 and use it to estimate the Gini index for the United States in 2010. Compare with your answer to parts (b) and (c) of Problem 2.

5.2 Volumes

In trying to find the volume of a solid we face the same type of problem as in finding areas. We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

We start with a simple type of solid called a **cylinder** (or, more precisely, a *right cylinder*). As illustrated in Figure 1(a), a cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume V of the cylinder is defined as

$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 1(c)].

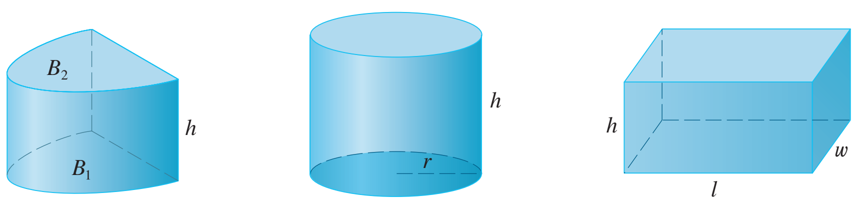


FIGURE 1 (a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

For a solid S that isn't a cylinder we first "cut" S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.

We start by intersecting S with a plane and obtaining a plane region that is called a **cross-section** of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leqslant x \leqslant b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.) The cross-sectional area $A(x)$ will vary as x increases from a to b.

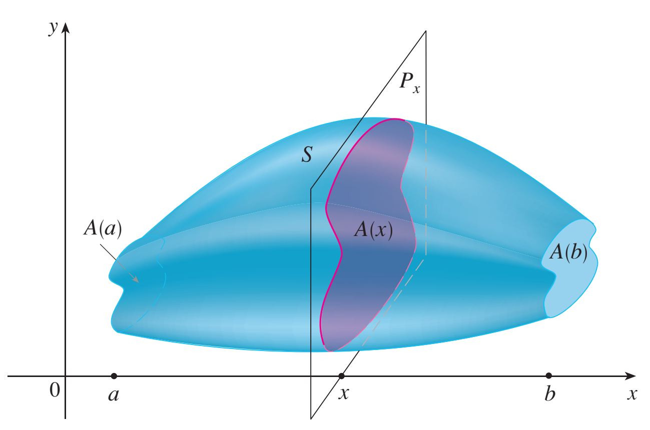


FIGURE 2

Let's divide S into n "slabs" of equal width Δx by using the planes $P_{x_1}, P_{x_2}, \ldots$ to slice the solid. (Think of slicing a loaf of bread.) If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx. (See Figure 3.)

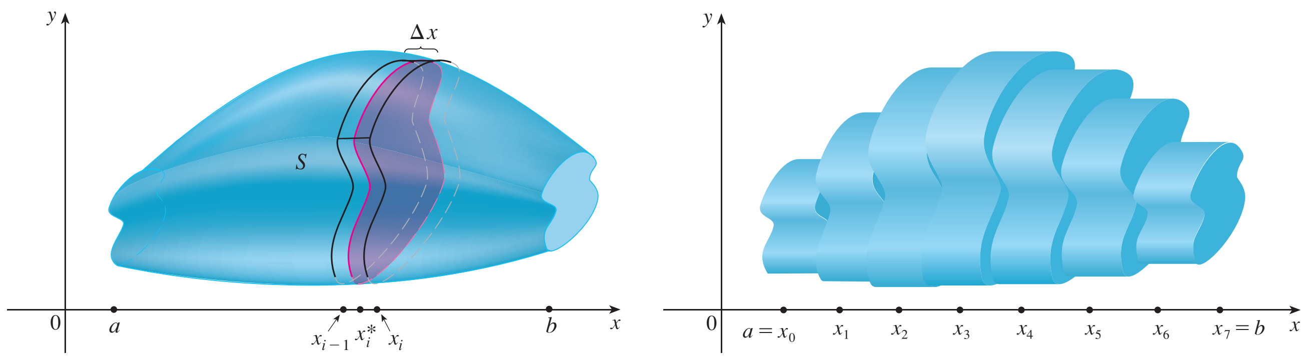


FIGURE 3

The volume of this cylinder is $A(x_i^*)\,\Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\,\Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

This approximation appears to become better and better as $n \to \infty$. (Think of the slices as becoming thinner and thinner.) Therefore we *define* the volume as the limit of these sums as $n \to \infty$. But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

It can be proved that this definition is independent of how S is situated with respect to the x-axis. In other words, no matter how we slice S with parallel planes, we always get the same answer for V.

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

When we use the volume formula $V = \int_a^b A(x)\,dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A\,dx = A(b - a)$; this agrees with the formula $V = Ah$.

EXAMPLE 1 Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

SOLUTION If we place the sphere so that its center is at the origin, then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is

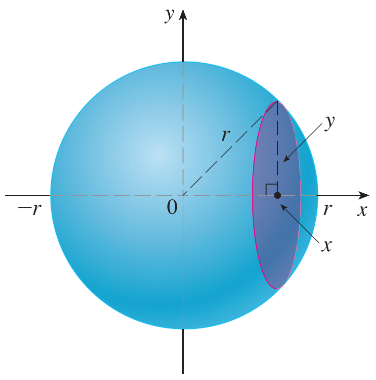


FIGURE 4

$y = \sqrt{r^2 - x^2}$. (See Figure 4.) So the cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Using the definition of volume with $a = -r$ and $b = r$, we have

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx \qquad \text{(The integrand is even.)}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right)$$

$$= \tfrac{4}{3}\pi r^3$$

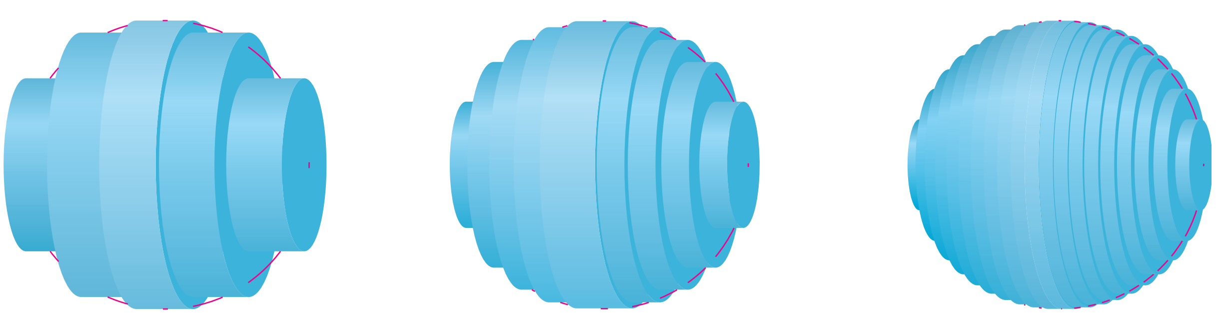
■

Figure 5 illustrates the definition of volume when the solid is a sphere with radius $r = 1$. From the result of Example 1, we know that the volume of the sphere is $\frac{4}{3}\pi$, which is approximately 4.18879. Here the slabs are circular cylinders, or *disks*, and the three parts of Figure 5 show the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\bar{x}_i)\,\Delta x = \sum_{i=1}^{n} \pi(1^2 - \bar{x}_i^2)\,\Delta x$$

when $n = 5$, 10, and 20 if we choose the sample points x_i^* to be the midpoints $\bar{x}_i$. Notice that as we increase the number of approximating cylinders, the corresponding Riemann sums become closer to the true volume.

TEC Visual 5.2A shows an animation of Figure 5.



(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

FIGURE 5
Approximating the volume of a sphere with radius 1

EXAMPLE 2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

SOLUTION The region is shown in Figure 6(a). If we rotate about the x-axis, we get the solid shown in Figure 6(b). When we slice through the point x, we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$$

and the volume of the approximating cylinder (a disk with thickness Δx) is

$$A(x)\,\Delta x = \pi x\,\Delta x$$

The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\Bigg]_0^1 = \frac{\pi}{2}$$

Did we get a reasonable answer in Example 2? As a check on our work, let's replace the given region by a square with base [0, 1] and height 1. If we rotate this square, we get a cylinder with radius 1, height 1, and volume $\pi \cdot 1^2 \cdot 1 = \pi$. We computed that the given solid has half this volume. That seems about right.

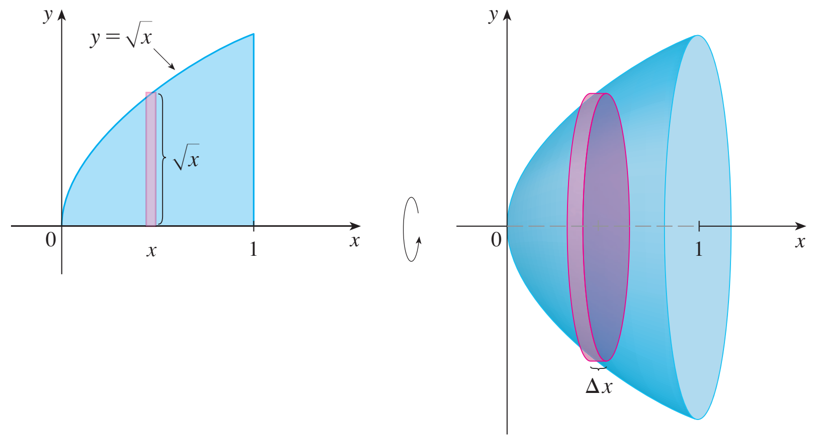


FIGURE 6 (a) (b)

EXAMPLE 3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

SOLUTION The region is shown in Figure 7(a) and the resulting solid is shown in Figure 7(b). Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis (obtaining circular cross-sections) and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

and the volume of the approximating cylinder pictured in Figure 7(b) is

$$A(y)\,\Delta y = \pi y^{2/3}\,\Delta y$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

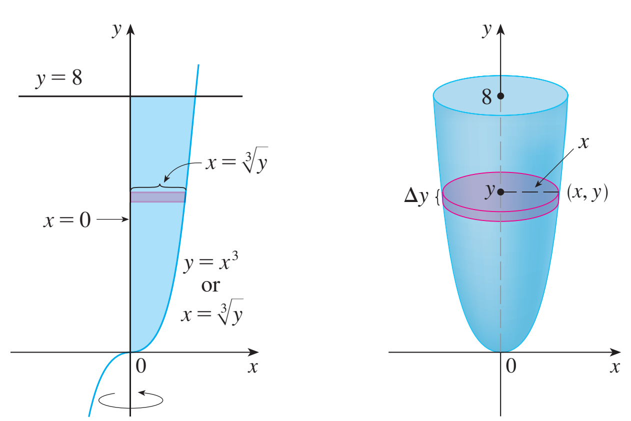


FIGURE 7 (a) (b)

EXAMPLE 4 The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

SOLUTION The curves $y = x$ and $y = x^2$ intersect at the points (0, 0) and (1, 1). The region between them, the solid of rotation, and a cross-section perpendicular to the x-axis are shown in Figure 8. A cross-section in the plane P_x has the shape of a *washer* (an annular ring) with inner radius x^2 and outer radius x, so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

TEC Visual 5.2B shows how solids of revolution are formed.

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

Therefore we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

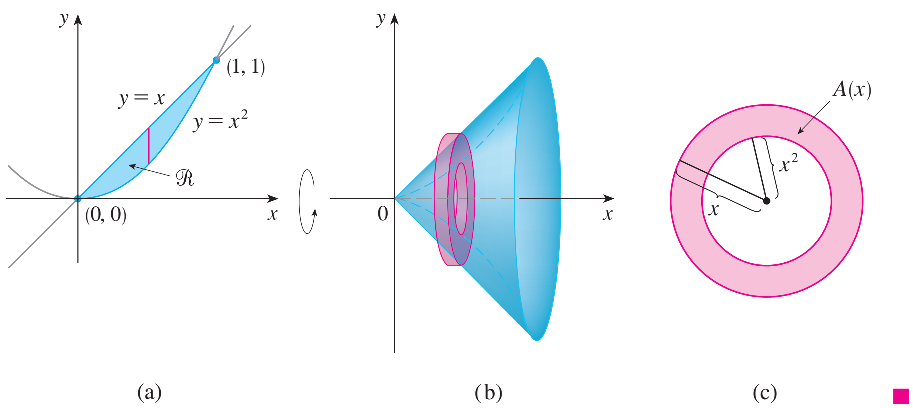


FIGURE 8 ∎

EXAMPLE 5 Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.

SOLUTION The solid and a cross-section are shown in Figure 9. Again the cross-section is a washer, but this time the inner radius is $2 - x$ and the outer radius is $2 - x^2$.

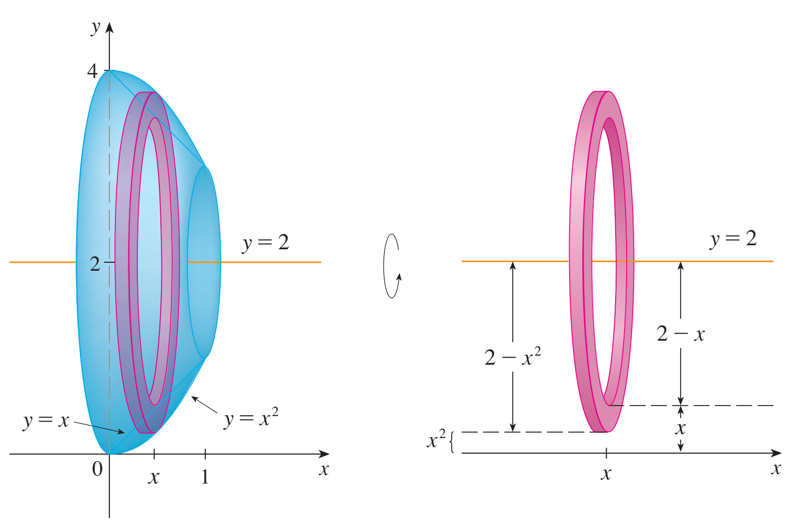


FIGURE 9

The cross-sectional area is

$$A(x) = \pi(2 - x^2)^2 - \pi(2 - x)^2$$

and so the volume of S is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \pi \int_0^1 [(2 - x^2)^2 - (2 - x)^2]\,dx \\ &= \pi \int_0^1 (x^4 - 5x^2 + 4x)\,dx \\ &= \pi \left[\frac{x^5}{5} - 5\frac{x^3}{3} + 4\frac{x^2}{2} \right]_0^1 \\ &= \frac{8\pi}{15} \end{aligned}$$

■

The solids in Examples 1–5 are all called **solids of revolution** because they are obtained by revolving a region about a line. In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)\,dx \qquad \text{or} \qquad V = \int_c^d A(y)\,dy$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

- If the cross-section is a disk (as in Examples 1–3), we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

- If the cross-section is a washer (as in Examples 4 and 5), we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figures 8, 9, and 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

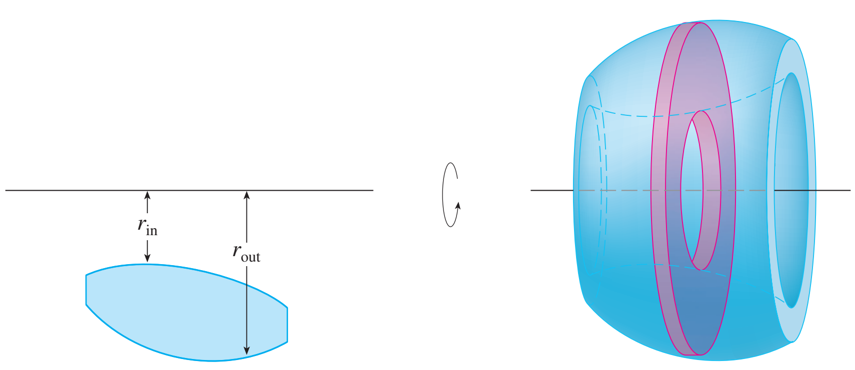


FIGURE 10

The next example gives a further illustration of the procedure.

EXAMPLE 6 Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.

SOLUTION Figure 11 shows a horizontal cross-section. It is a washer with inner radius $1 + y$ and outer radius $1 + \sqrt{y}$, so the cross-sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi(1 + \sqrt{y})^2 - \pi(1 + y)^2$$

The volume is

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 \left[(1 + \sqrt{y})^2 - (1 + y)^2\right] dy$$
$$= \pi \int_0^1 \left(2\sqrt{y} - y - y^2\right) dy = \pi \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{2}$$

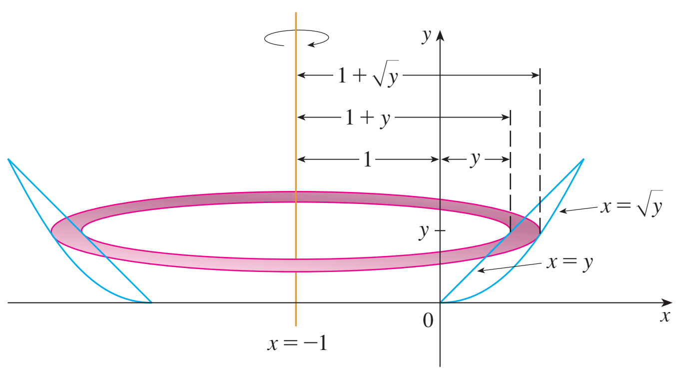


FIGURE 11

■

We now find the volumes of three solids that are *not* solids of revolution.

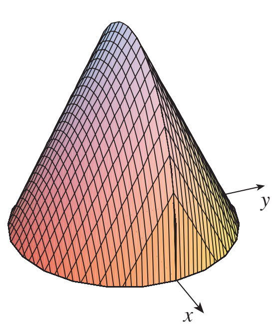


FIGURE 12
Computer-generated picture of the solid in Example 7

TEC Visual 5.2C shows how the solid in Figure 12 is generated.

EXAMPLE 7 Figure 12 shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

SOLUTION Let's take the circle to be $x^2 + y^2 = 1$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 13.

Since B lies on the circle, we have $y = \sqrt{1 - x^2}$ and so the base of the triangle ABC is $|AB| = 2y = 2\sqrt{1 - x^2}$. Since the triangle is equilateral, we see from Figure 13(c)

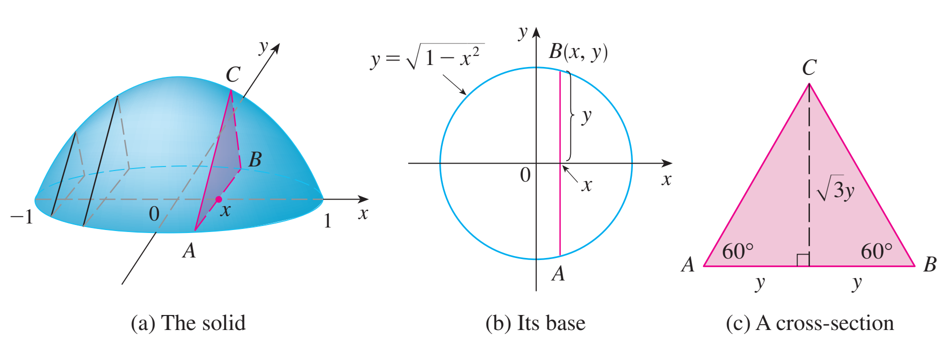


FIGURE 13

that its height is $\sqrt{3}\,y = \sqrt{3}\sqrt{1 - x^2}$. The cross-sectional area is therefore

$$A(x) = \tfrac{1}{2} \cdot 2\sqrt{1 - x^2} \cdot \sqrt{3}\sqrt{1 - x^2} = \sqrt{3}\,(1 - x^2)$$

and the volume of the solid is

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\,dx$$

$$= 2\int_{0}^{1} \sqrt{3}\,(1 - x^2)\,dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}$$ ■

EXAMPLE 8 Find the volume of a pyramid whose base is a square with side L and whose height is h.

SOLUTION We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 14. Any plane P_x that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 15 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$. [Another method is to observe that the line OP has slope $L/(2h)$ and so its equation is $y = Lx/(2h)$.] Therefore the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2$$

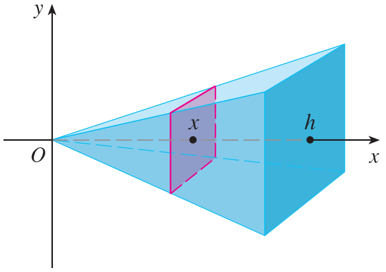


FIGURE 14

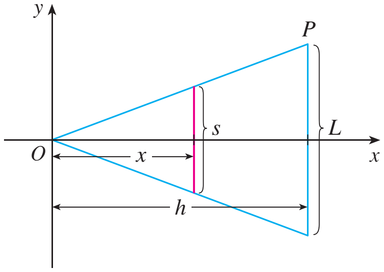


FIGURE 15

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx = \frac{L^2}{h^2}\frac{x^3}{3}\Big]_0^h = \frac{L^2h}{3}$$ ■

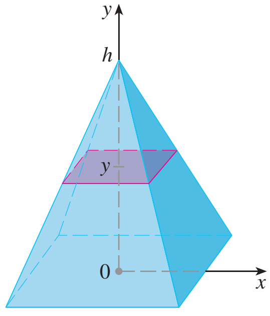


FIGURE 16

NOTE We didn't need to place the vertex of the pyramid at the origin in Example 8. We did so merely to make the equations simple. If, instead, we had placed the center of the base at the origin and the vertex on the positive y-axis, as in Figure 16, you can verify that we would have obtained the integral

$$V = \int_0^h \frac{L^2}{h^2}(h - y)^2\,dy = \frac{L^2h}{3}$$

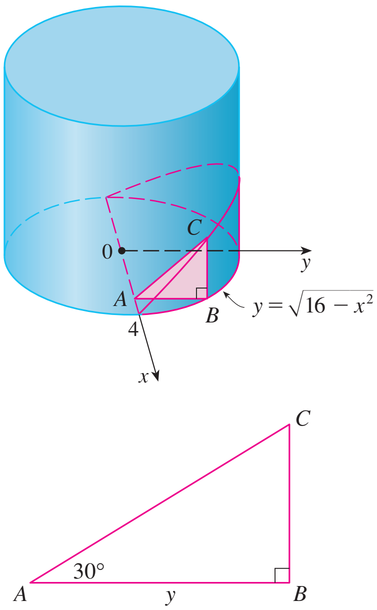


FIGURE 17

EXAMPLE 9 A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

SOLUTION If we place the x-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation $y = \sqrt{16 - x^2}$, $-4 \leqslant x \leqslant 4$. A cross-section perpendicular to the x-axis at a distance x from the origin is a triangle ABC, as shown in Figure 17, whose base is $y = \sqrt{16 - x^2}$ and whose height is $|BC| = y \tan 30° = \sqrt{16 - x^2}/\sqrt{3}$. So the cross-sectional area is

$$A(x) = \tfrac{1}{2}\sqrt{16 - x^2} \cdot \frac{1}{\sqrt{3}}\sqrt{16 - x^2} = \frac{16 - x^2}{2\sqrt{3}}$$

and the volume is

$$\begin{aligned} V &= \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx \\ &= \frac{1}{\sqrt{3}}\int_0^4 (16 - x^2)\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4 \\ &= \frac{128}{3\sqrt{3}} \end{aligned}$$

For another method see Exercise 64. ■

5.2 EXERCISES

1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = x + 1,\ y = 0,\ x = 0,\ x = 2;$ about the x-axis

2. $y = 1/x,\ y = 0,\ x = 1,\ x = 4;$ about the x-axis

3. $y = \sqrt{x - 1},\ y = 0,\ x = 5;$ about the x-axis

4. $y = \sqrt{25 - x^2},\ y = 0,\ x = 2,\ x = 4;$ about the x-axis

5. $x = 2\sqrt{y},\ x = 0,\ y = 9;$ about the y-axis

6. $2x = y^2,\ x = 0,\ y = 4;$ about the y-axis

7. $y = x^3,\ y = x,\ x \geqslant 0;$ about the x-axis

8. $y = 6 - x^2,\ y = 2;$ about the x-axis

9. $y^2 = x,\ x = 2y;$ about the y-axis

10. $x = 2 - y^2,\ x = y^4;$ about the y-axis

11. $y = x^2,\ x = y^2;$ about $y = 1$

12. $y = x^3,\ y = 1,\ x = 2;$ about $y = -3$

13. $y = 1 + \sec x,\ y = 3;$ about $y = 1$

14. $y = \sin x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/4;$ about $y = -1$

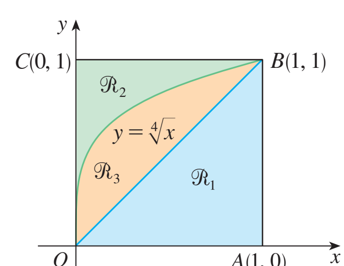
15. $y = x^3,\ y = 0,\ x = 1;$ about $x = 2$

16. $y = x^2,\ x = y^2;$ about $x = -1$

17. $x = y^2,\ x = 1 - y^2;$ about $x = 3$

18. $y = x,\ y = 0,\ x = 2,\ x = 4;$ about $x = 1$

19–30 Refer to the figure and find the volume generated by rotating the given region about the specified line.



19. $\mathscr{R}_1$ about OA

20. $\mathscr{R}_1$ about OC

21. $\mathscr{R}_1$ about AB

22. $\mathscr{R}_1$ about BC

23. $\mathcal{R}_2$ about OA

24. $\mathcal{R}_2$ about OC

25. $\mathcal{R}_2$ about AB

26. $\mathcal{R}_2$ about BC

27. $\mathcal{R}_3$ about OA

28. $\mathcal{R}_3$ about OC

29. $\mathcal{R}_3$ about AB

30. $\mathcal{R}_3$ about BC

31–34 Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use your calculator to evaluate the integral correct to five decimal places.

31. $y = \tan x,\ y = 0,\ x = \pi/4$
(a) About the x-axis (b) About $y = -1$

32. $y = 0,\ y = \cos^2 x,\ -\pi/2 \leqslant x \leqslant \pi/2$
(a) About the x-axis (b) About $y = 1$

33. $x^2 + 4y^2 = 4$
(a) About $y = 2$ (b) About $x = 2$

34. $y = x^2,\ x^2 + y^2 = 1,\ y \geqslant 0$
(a) About the x-axis (b) About the y-axis

35–36 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then use your calculator to find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

35. $y = 1 + x^4, \quad y = \sqrt{3 - x^3}$

36. $y = \sqrt[3]{2x - x^2}, \quad y = x^2/(x^2 + 1)$

CAS **37–38** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

37. $y = \sin^2 x,\ y = 0,\ 0 \leqslant x \leqslant \pi;\quad$ about $y = -1$

38. $y = x^2 - 2x,\ y = x\cos(\pi x/4);\quad$ about $y = 2$

39–42 Each integral represents the volume of a solid. Describe the solid.

39. $\pi \int_0^{\pi} \sin x\, dx$

40. $\pi \int_{-1}^{1} (1 - y^2)^2\, dy$

41. $\pi \int_0^{1} (y^4 - y^8)\, dy$

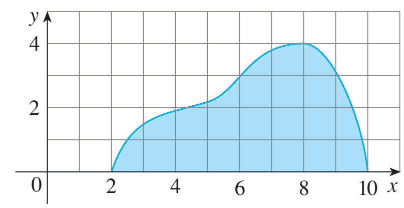
42. $\pi \int_1^{4} \left[3^2 - \left(3 - \sqrt{x}\,\right)^2\right] dx$

43. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use the Midpoint Rule to estimate the volume of the liver.

44. A log 10 m long is cut at 1-meter intervals and its cross-sectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

x (m)	A (m^2)	x (m)	A (m^2)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

45. (a) If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 4$ to estimate the volume of the solid.



(b) Estimate the volume if the region is rotated about the y-axis. Again use the Midpoint Rule with $n = 4$.

CAS **46.** (a) A model for the shape of a bird's egg is obtained by rotating about the x-axis the region under the graph of

$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$

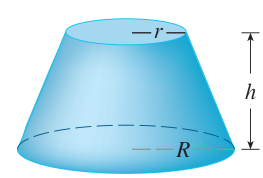
Use a CAS to find the volume of such an egg.

(b) For a red-throated loon, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Graph f and find the volume of an egg of this species.

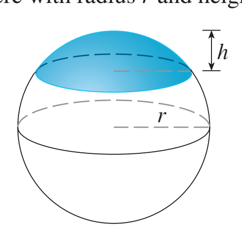
47–61 Find the volume of the described solid S.

47. A right circular cone with height h and base radius r

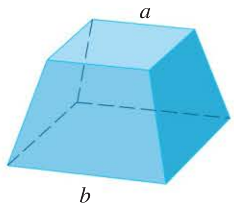
48. A frustum of a right circular cone with height h, lower base radius R, and top radius r



49. A cap of a sphere with radius r and height h



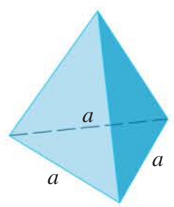
50. A frustum of a pyramid with square base of side b, square top of side a, and height h



What happens if $a = b$? What happens if $a = 0$?

51. A pyramid with height h and rectangular base with dimensions b and $2b$

52. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



53. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm

54. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.



55. The base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

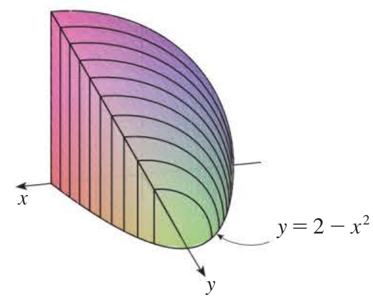
56. The base of S is the triangular region with vertices (0, 0), (1, 0), and (0, 1). Cross-sections perpendicular to the y-axis are equilateral triangles.

57. The base of S is the same base as in Exercise 56, but cross-sections perpendicular to the x-axis are squares.

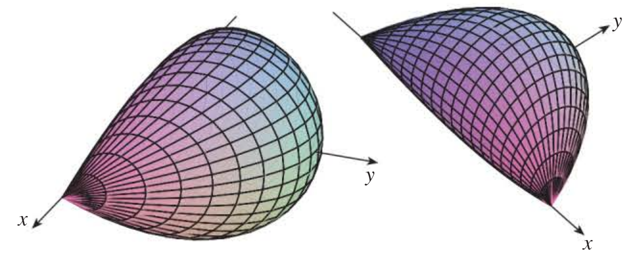
58. The base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are squares.

59. The base of S is the same base as in Exercise 58, but cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base.

60. The base of S is the region enclosed by $y = 2 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are quarter-circles.



61. The solid S is bounded by circles that are perpendicular to the x-axis, intersect the x-axis, and have centers on the parabola $y = \frac{1}{2}(1 - x^2)$, $-1 \leqslant x \leqslant 1$.



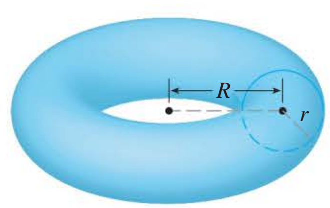
62. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.

(a) Set up an integral for the volume of S.

(b) By interpreting the integral as an area, find the volume of S.

63. (a) Set up an integral for the volume of a solid *torus* (the donut-shaped solid shown in the figure) with radii r and R.

(b) By interpreting the integral as an area, find the volume of the torus.

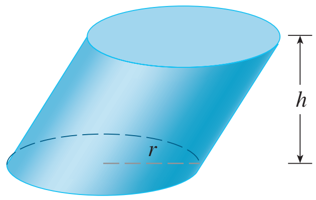


64. Solve Example 9 taking cross-sections to be parallel to the line of intersection of the two planes.

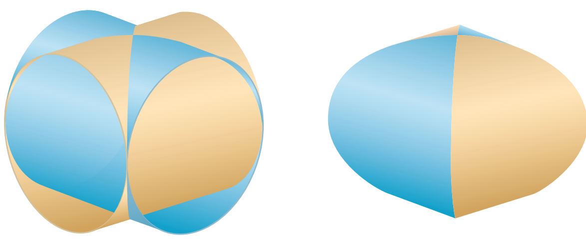
65. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids

S_1 and S_2, then the volumes of S_1 and S_2 are equal. Prove this principle.

(b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



66. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.



67. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

68. A bowl is shaped like a hemisphere with diameter 30 cm. A heavy ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

69. A hole of radius r is bored through the middle of a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.

70. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

71. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact Kepler published a book *Stereometria doliorum* in 1615 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.

(b) Show that the volume enclosed by the barrel is

$$V = \tfrac{1}{3}\pi h\left(2R^2 + r^2 - \tfrac{2}{5}d^2\right)$$

72. Suppose that a region $\mathcal{R}$ has area A and lies above the x-axis. When $\mathcal{R}$ is rotated about the x-axis, it sweeps out a solid with volume V_1. When $\mathcal{R}$ is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.

5.3 Volumes by Cylindrical Shells

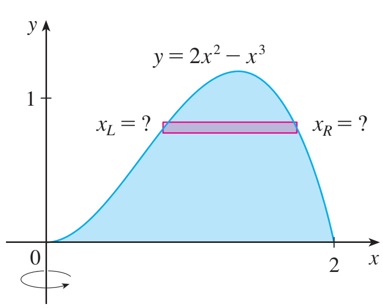


FIGURE 1

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.) If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius

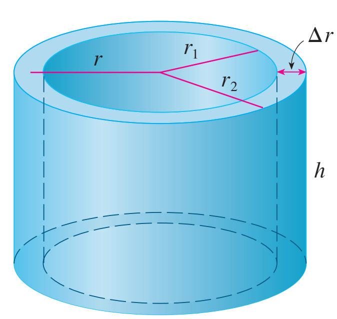


FIGURE 2

r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$\begin{aligned} V &= V_2 - V_1 \\ &= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \end{aligned}$$

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

1 $$V = 2\pi rh\,\Delta r$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geqslant 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geqslant 0$. (See Figure 3.)

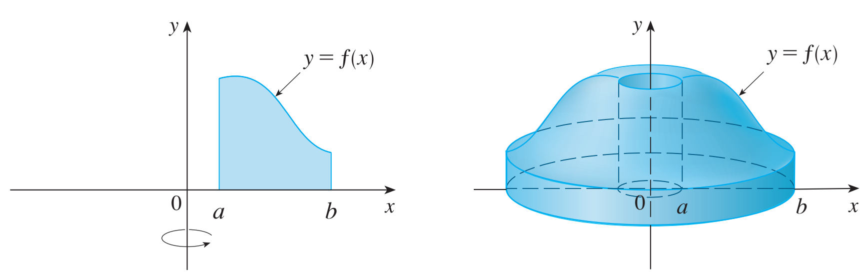


FIGURE 3

We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx. (See Figure 4.) So by Formula 1 its volume is

$$V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

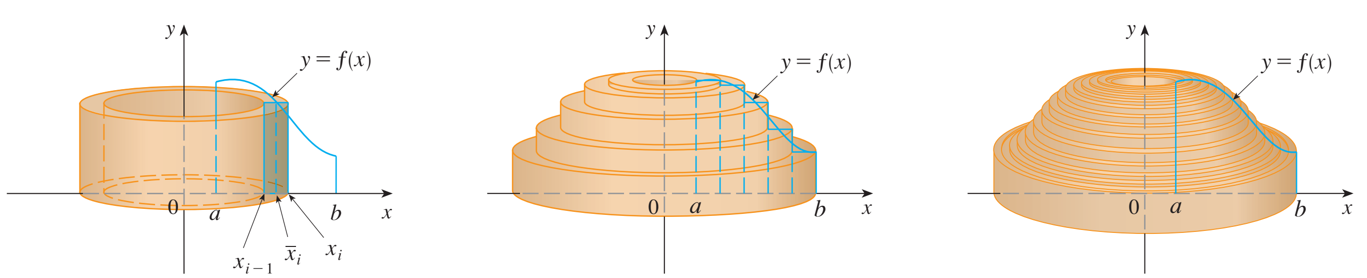


FIGURE 4

Therefore an approximation to the volume V of S is given by the sum of the volumes of

these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\, \Delta x$$

This approximation appears to become better as $n \to \infty$. But, from the definition of an integral, we know that

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\, \Delta x = \int_a^b 2\pi x f(x)\, dx$$

Thus the following appears plausible:

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{where } 0 \leqslant a < b$$

The argument using cylindrical shells makes Formula 2 seem reasonable, but later we will be able to prove it (see Exercise 7.1.73).

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \ \underbrace{[f(x)]}_{\text{height}} \ \underbrace{dx}_{\text{thickness}}$$

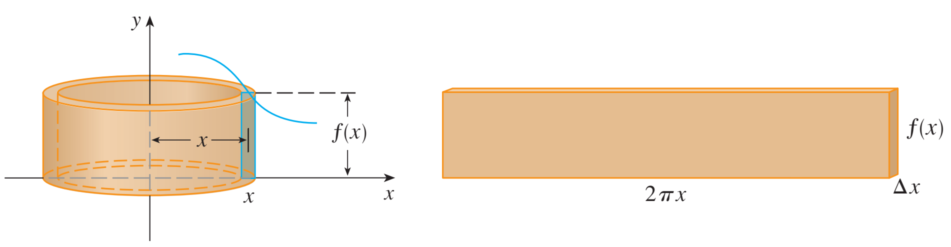


FIGURE 5

This type of reasoning will be helpful in other situations, such as when we rotate about lines other than the y-axis.

EXAMPLE 1 Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

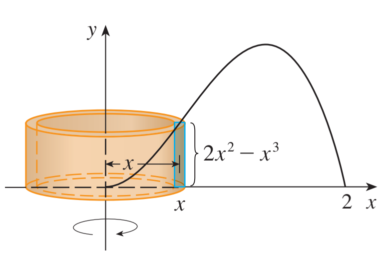


FIGURE 6

SOLUTION From the sketch in Figure 6 we see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$. So, by the shell method, the volume is

$$V = \int_0^2 \underbrace{(2\pi x)}_{\text{circumference}} \ \underbrace{(2x^2 - x^3)}_{\text{height}} \ \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx = 2\pi \left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right) = \tfrac{16}{5}\pi$$

It can be verified that the shell method gives the same answer as slicing. ■

TEC Visual 5.3 shows how the solid and shells in Example 1 are formed.

Figure 7 shows a computer-generated picture of the solid whose volume we computed in Example 1.

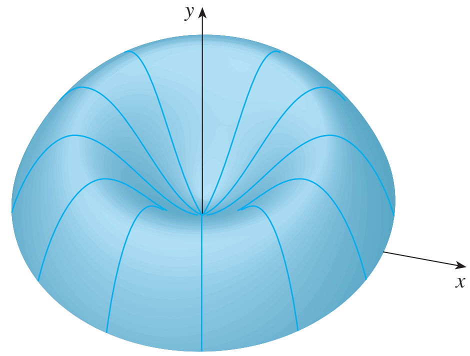


FIGURE 7

NOTE Comparing the solution of Example 1 with the remarks at the beginning of this section, we see that the method of cylindrical shells is much easier than the washer method for this problem. We did not have to find the coordinates of the local maximum and we did not have to solve the equation of the curve for x in terms of y. However, in other examples the methods of the preceding section may be easier.

EXAMPLE 2 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

SOLUTION The region and a typical shell are shown in Figure 8. We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

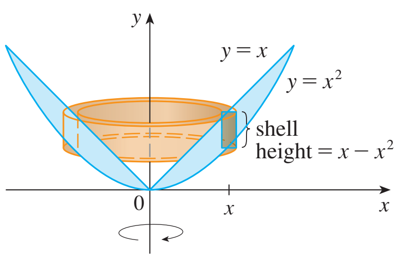


FIGURE 8

As the following example shows, the shell method works just as well if we rotate about the x-axis. We simply have to draw a diagram to identify the radius and height of a shell.

EXAMPLE 3 Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

SOLUTION This problem was solved using disks in Example 5.2.2. To use shells we relabel the curve $y = \sqrt{x}$ (in the figure in that example) as $x = y^2$ in Figure 9. For rotation about the x-axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\,dy = 2\pi \int_0^1 (y - y^3)\,dy$$

$$= 2\pi\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

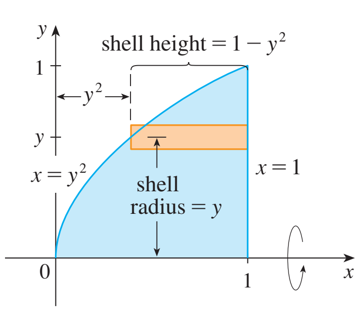


FIGURE 9

In this problem the disk method was simpler.

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

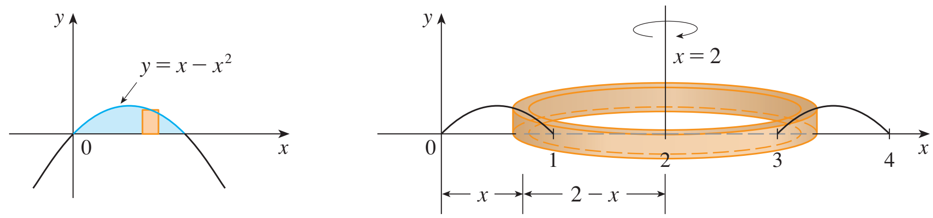


FIGURE 10

The volume of the given solid is

$$\begin{aligned} V &= \int_0^1 2\pi(2 - x)(x - x^2)\,dx \\ &= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx \\ &= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2} \end{aligned}$$

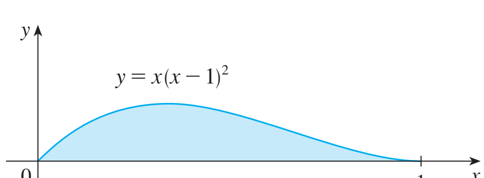
Disks and Washers versus Cylindrical Shells

When computing the volume of a solid of revolution, how do we know whether to use disks (or washers) or cylindrical shells? There are several considerations to take into account: Is the region more easily described by top and bottom boundary curves of the form $y = f(x)$, or by left and right boundaries $x = g(y)$? Which choice is easier to work with? Are the limits of integration easier to find for one variable versus the other? Does the region require two separate integrals when using x as the variable but only one integral in y? Are we able to evaluate the integral we set up with our choice of variable?

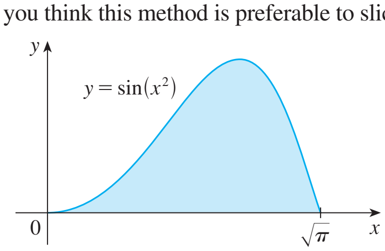
If we decide that one variable is easier to work with than the other, then this dictates which method to use. Draw a sample rectangle in the region, corresponding to a cross-section of the solid. The thickness of the rectangle, either Δx or Δy, corresponds to the integration variable. If you imagine the rectangle revolving, it becomes either a disk (washer) or a shell.

5.3 EXERCISES

1. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use slicing to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use shells to find V.



2. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Sketch a typical cylindrical shell and find its circumference and height. Use shells to find the volume of S. Do you think this method is preferable to slicing? Explain.



3–7 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

3. $y = \sqrt[3]{x}, \quad y = 0, \quad x = 1$

4. $y = x^3, \quad y = 0, \quad x = 1, \quad x = 2$

5. $y = x^2, \quad 0 \leqslant x \leqslant 2, \quad y = 4, \quad x = 0$

6. $y = 4x - x^2, \quad y = x$

7. $y = x^2, \quad y = 6x - 2x^2$

8. Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sqrt{x}$ and $y = x^2$. Find V both by slicing and by cylindrical shells. In both cases draw a diagram to explain your method.

9–14 Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

9. $xy = 1, \quad x = 0, \quad y = 1, \quad y = 3$

10. $y = \sqrt{x}, \quad x = 0, \quad y = 2$

11. $y = x^{3/2}, \quad y = 8, \quad x = 0$

12. $x = -3y^2 + 12y - 9, \quad x = 0$

13. $x = 1 + (y - 2)^2, \quad x = 2$

14. $x + y = 4, \quad x = y^2 - 4y + 4$

15–20 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

15. $y = x^3, \ y = 8, \ x = 0; \quad$ about $x = 3$

16. $y = 4 - 2x, \ y = 0, \ x = 0; \quad$ about $x = -1$

17. $y = 4x - x^2, \ y = 3; \quad$ about $x = 1$

18. $y = \sqrt{x}, \ x = 2y; \quad$ about $x = 5$

19. $x = 2y^2, \ y \geqslant 0, \ x = 2; \quad$ about $y = 2$

20. $x = 2y^2, \ x = y^2 + 1; \quad$ about $y = -2$

21–26
(a) Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curve about the specified axis.
(b) Use your calculator to evaluate the integral correct to five decimal places.

21. $y = \sin x, \ y = 0, \ x = 2\pi, \ x = 3\pi; \quad$ about the y-axis

22. $y = \tan x, \ y = 0, \ x = \pi/4; \quad$ about $x = \pi/2$

23. $y = \cos^4 x, \ y = -\cos^4 x, \ -\pi/2 \leqslant x \leqslant \pi/2; \quad$ about $x = \pi$

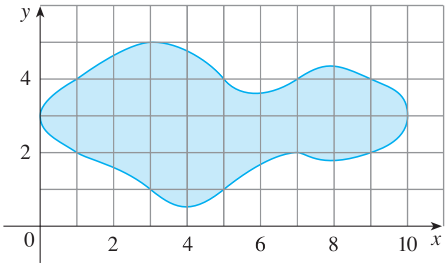
24. $y = x, \ y = 2x/(1 + x^3); \quad$ about $x = -1$

25. $x = \sqrt{\sin y}, \ 0 \leqslant y \leqslant \pi, \ x = 0; \quad$ about $y = 4$

26. $x^2 - y^2 = 7, \ x = 4; \quad$ about $y = 5$

27. Use the Midpoint Rule with $n = 5$ to estimate the volume obtained by rotating about the y-axis the region under the curve $y = \sqrt{1 + x^3}, 0 \leqslant x \leqslant 1$.

28. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 5$ to estimate the volume of the solid.



29–32 Each integral represents the volume of a solid. Describe the solid.

29. $\int_0^3 2\pi x^5 \, dx$

30. $\int_1^5 2\pi y \sqrt{y - 1} \, dy$

31. $2\pi \int_1^4 \frac{y + 2}{y^2} \, dy$

32. $\int_0^{\pi/2} 2\pi (x + 1)(2x - \sin x) \, dx$

33–34 Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information and your calculator to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

33. $y = x^2 - 2x, \quad y = \dfrac{x}{x^2 + 1}$

34. $y = 3 \sin x, \quad y = x^2 - 4x + 5$

CAS **35–36** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

35. $y = \sin^2 x, \ y = \sin^4 x, \ 0 \leqslant x \leqslant \pi; \quad$ about $x = \pi/2$

36. $y = x^3 \sin x, \ y = 0, \ 0 \leqslant x \leqslant \pi; \quad$ about $x = -1$

37–43 The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

37. $y = -x^2 + 6x - 8,\ y = 0;$ about the y-axis

38. $y = -x^2 + 6x - 8,\ y = 0;$ about the x-axis

39. $y^2 - x^2 = 1,\ y = 2;$ about the x-axis

40. $y^2 - x^2 = 1,\ y = 2;$ about the y-axis

41. $x^2 + (y - 1)^2 = 1;$ about the y-axis

42. $x = (y - 3)^2,\ x = 4;$ about $y = 1$

43. $x = (y - 1)^2,\ x - y = 1;$ about $x = -1$

44. Let T be the triangular region with vertices (0, 0), (1, 0), and (1, 2), and let V be the volume of the solid generated when T is rotated about the line $x = a$, where $a > 1$. Express a in terms of V.

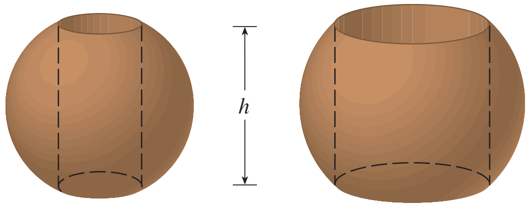
45–47 Use cylindrical shells to find the volume of the solid.

45. A sphere of radius r

46. The solid torus of Exercise 5.2.63

47. A right circular cone with height h and base radius r

48. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.
(a) Guess which ring has more wood in it.
(b) Check your guess: use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.



5.4 Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task. In physics it has a technical meaning that depends on the idea of a *force*. Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is given by Newton's Second Law of Motion as the product of its mass m and its acceleration a:

1
$$F = ma = m\frac{d^2s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($\text{N} = \text{kg}\cdot\text{m/s}^2$). Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2. In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves:

2
$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J). If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

EXAMPLE 1
(a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.
(b) How much work is done in lifting a 20-lb weight 6 ft off the ground?

SOLUTION
(a) The force exerted is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76 \text{ N})(0.7 \text{ m}) \approx 8.2 \text{ J}$$

(b) Here the force is given as $F = 20$ lb, so the work done is

$$W = Fd = (20 \text{ lb})(6 \text{ ft}) = 120 \text{ ft-lb}$$

Notice that in part (b), unlike part (a), we did not have to multiply by g because we were given the *weight* (which is a force) and not the mass of the object. ■

Equation 2 defines work as long as the force is constant, but what happens if the force is variable? Let's suppose that the object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function. We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$. If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$. In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:

$$W_i \approx f(x_i^*)\,\Delta x$$

Thus we can approximate the total work by

3
$$W \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

It seems that this approximation becomes better as we make n larger. Therefore we define the **work done in moving the object from *a* to *b*** as the limit of this quantity as $n \to \infty$. Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

4
$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

EXAMPLE 2 When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

SOLUTION
$$W = \int_1^3 (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 \Big]_1^3 = \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb. ■

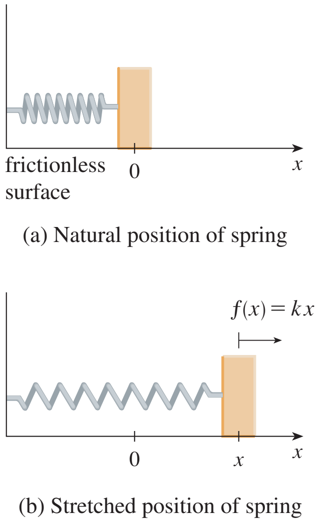


FIGURE 1
Hooke's Law

In the next example we use a law from physics. **Hooke's Law** states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant called the **spring constant** (see Figure 1). Hooke's Law holds provided that x is not too large.

EXAMPLE 3 A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

SOLUTION According to Hooke's Law, the force required to hold the spring stretched x meters beyond its natural length is $f(x) = kx$. When the spring is stretched from 10 cm to 15 cm, the amount stretched is 5 cm = 0.05 m. This means that $f(0.05) = 40$, so

$$0.05k = 40 \qquad k = \frac{40}{0.05} = 800$$

Thus $f(x) = 800x$ and the work done in stretching the spring from 15 cm to 18 cm is

$$W = \int_{0.05}^{0.08} 800x\,dx = 800\,\frac{x^2}{2}\bigg]_{0.05}^{0.08}$$

$$= 400[(0.08)^2 - (0.05)^2] = 1.56 \text{ J}$$ ■

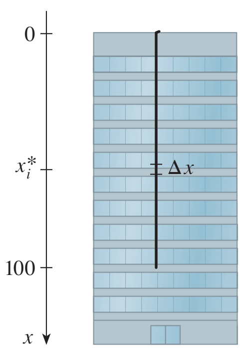


FIGURE 2

EXAMPLE 4 A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

SOLUTION Here we don't have a formula for the force function, but we can use an argument similar to the one that led to Definition 4.

Let's place the origin at the top of the building and the x-axis pointing downward as in Figure 2. We divide the cable into small parts with length Δx. If x_i^* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely, x_i^*. The cable weighs 2 pounds per foot, so the weight of the ith part is $(2 \text{ lb/ft})(\Delta x \text{ ft}) = 2\Delta x$ lb. Thus the work done on the ith part, in foot-pounds, is

$$\underbrace{(2\Delta x)}_{\text{force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 2x_i^*\,\Delta x$$

If we had placed the origin at the bottom of the cable and the x-axis upward, we would have gotten

$$W = \int_0^{100} 2(100 - x)\,dx$$

which gives the same answer.

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{100} 2x\,dx$$

$$= x^2\Big]_0^{100} = 10{,}000 \text{ ft-lb}$$ ■

EXAMPLE 5 A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

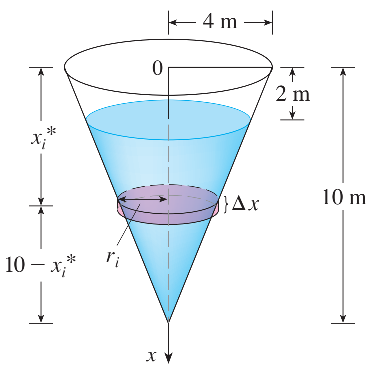


FIGURE 3

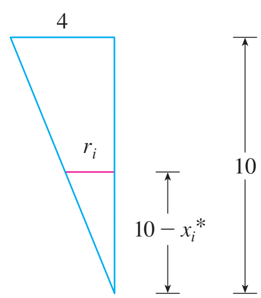


FIGURE 4

SOLUTION Let's measure depths from the top of the tank by introducing a vertical coordinate line as in Figure 3. The water extends from a depth of 2 m to a depth of 10 m and so we divide the interval $[2, 10]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval. This divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx. We can compute r_i from similar triangles, using Figure 4, as follows:

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \qquad r_i = \tfrac{2}{5}(10 - x_i^*)$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \, \Delta x = \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x$$

and so its mass is

$$\begin{aligned} m_i &= \text{density} \times \text{volume} \\ &\approx 1000 \cdot \frac{4\pi}{25}(10 - x_i^*)^2 \, \Delta x = 160\pi(10 - x_i^*)^2 \, \Delta x \end{aligned}$$

The force required to raise this layer must overcome the force of gravity and so

$$\begin{aligned} F_i = m_i g &\approx (9.8)160\pi(10 - x_i^*)^2 \, \Delta x \\ &= 1568\pi(10 - x_i^*)^2 \, \Delta x \end{aligned}$$

Each particle in the layer must travel a distance upward of approximately x_i^*. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 1568\pi x_i^*(10 - x_i^*)^2 \, \Delta x$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 1568\pi x_i^*(10 - x_i^*)^2 \, \Delta x = \int_2^{10} 1568\pi x(10 - x)^2 \, dx \\ &= 1568\pi \int_2^{10} (100x - 20x^2 + x^3) \, dx = 1568\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10} \\ &= 1568\pi \left(\tfrac{2048}{3}\right) \approx 3.4 \times 10^6 \text{ J} \end{aligned}$$

5.4 EXERCISES

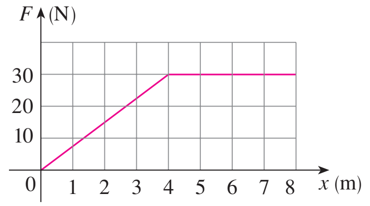
1. A 360-lb gorilla climbs a tree to a height of 20 ft. Find the work done if the gorilla reaches that height in
(a) 10 seconds (b) 5 seconds

2. How much work is done when a hoist lifts a 200-kg rock to a height of 3 m?

3. A variable force of $5x^{-2}$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work done in moving the object from $x = 1$ ft to $x = 10$ ft.

4. When a particle is located a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

5. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done by the force in moving an object a distance of 8 m?



6. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use the Midpoint Rule to estimate the work done by the force in moving an object from $x = 4$ to $x = 20$.

x	4	6	8	10	12	14	16	18	20
$f(x)$	5	5.8	7.0	8.8	9.6	8.2	6.7	5.2	4.1

7. A force of 10 lb is required to hold a spring stretched 4 in. beyond its natural length. How much work is done in stretching it from its natural length to 6 in. beyond its natural length?

8. A spring has a natural length of 40 cm. If a 60-N force is required to keep the spring compressed 10 cm, how much work is done during this compression? How much work is required to compress the spring to a length of 25 cm?

9. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
(a) How much work is needed to stretch the spring from 35 cm to 40 cm?
(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

10. If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in. beyond its natural length?

11. A spring has natural length 20 cm. Compare the work W_1 done in stretching the spring from 20 cm to 30 cm with the work W_2 done in stretching it from 30 cm to 40 cm. How are W_2 and W_1 related?

12. If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

13–22 Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

13. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.
(a) How much work is done in pulling the rope to the top of the building?
(b) How much work is done in pulling half the rope to the top of the building?

14. A thick cable, 60 ft long and weighing 180 lb, hangs from a winch on a crane. Compute in two different ways the work done if the winch winds up 25 ft of the cable.
(a) Follow the method of Example 4.
(b) Write a function for the weight of the remaining cable after x feet has been wound up by the winch. Estimate the amount of work done when the winch pulls up Δx ft of cable.

15. A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

16. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

17. A leaky 10-kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12-m level. How much work is done?

18. A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.

19. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it is level with the upper end.

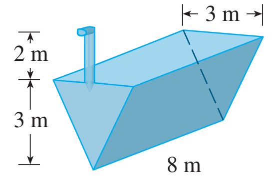
20. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft^3.)

21. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3.)

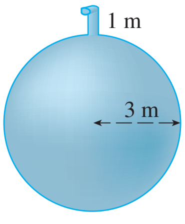
22. A spherical water tank, 24 ft in diameter, sits atop a 60 ft tower. The tank is filled by a hose attached to the bottom of the sphere. If a 1.5 horsepower pump is used to deliver water up to the tank, how long will it take to fill the tank? (One horsepower = 550 ft-lb of work per second.)

23–26 A tank is full of water. Find the work required to pump the water out of the spout. In Exercises 25 and 26 use the fact that water weighs 62.5 lb/ft^3.

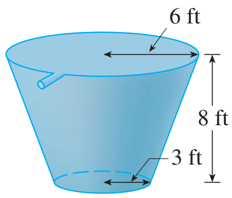
23.



24.

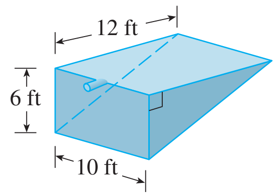


25.



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26.

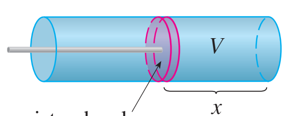


27. Suppose that for the tank in Exercise 23 the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank?

28. Solve Exercise 24 if the tank is half full of oil that has a density of 900 kg/m^3.

29. When gas expands in a cylinder with radius r, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area: $F = \pi r^2 P$. Show that the work done by the gas when the volume expands from volume V_1 to volume V_2 is

$$W = \int_{V_1}^{V_2} P\,dV$$



30. In a steam engine the pressure P and volume V of steam satisfy the equation $PV^{1.4} = k$, where k is a constant. (This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings.) Use Exercise 29 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 160 lb/in^2 and a volume of 100 in^3 and expands to a volume of 800 in^3.

31. The kinetic energy KE of an object of mass m moving with velocity v is defined as $\text{KE} = \frac{1}{2}mv^2$. If a force $f(x)$ acts on the object, moving it along the x-axis from x_1 to x_2, the *Work-Energy Theorem* states that the net work done is equal to the change in kinetic energy: $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$, where v_1 is the velocity at x_1 and v_2 is the velocity at x_2.

(a) Let $x = s(t)$ be the position function of the object at time t and $v(t)$, $a(t)$ the velocity and acceleration functions. Prove the Work-Energy Theorem by first using the Substitution Rule for Definite Integrals (4.5.5) to show that

$$W = \int_{x_1}^{x_2} f(x)\,dx = \int_{t_1}^{t_2} f(s(t))\,v(t)\,dt$$

Then use Newton's Second Law of Motion (force = mass × acceleration) and the substitution $u = v(t)$ to evaluate the integral.

(b) How much work (in ft-lb) is required to hurl a 12-lb bowling ball at 20 mi/h? (*Note:* Divide the weight in pounds by 32 ft/s^2, the acceleration due to gravity, to find the mass, measured in slugs.)

32. Suppose that when launching an 800-kg roller coaster car an electromagnetic propulsion system exerts a force of $5.7x^2 + 1.5x$ newtons on the car at a distance x meters along the track. Use Exercise 31(a) to find the speed of the car when it has traveled 60 meters.

33. (a) Newton's Law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.

(b) Compute the work required to launch a 1000-kg satellite vertically to a height of 1000 km. You may assume that the earth's mass is 5.98×10^{24} kg and is concentrated at its center. Take the radius of the earth to be 6.37×10^6 m and $G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

34. The Great Pyramid of King Khufu was built of limestone in Egypt over a 20-year time period from 2580 BC to 2560 BC. Its base is a square with side length 756 ft and its height when built was 481 ft. (It was the tallest man-made structure in the world for more than 3800 years.) The density of the limestone is about 150 lb/ft^3.

(a) Estimate the total work done in building the pyramid.

(b) If each laborer worked 10 hours a day for 20 years, for 340 days a year, and did 200 ft-lb/h of work in lifting the limestone blocks into place, about how many laborers were needed to construct the pyramid?



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5.5 Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

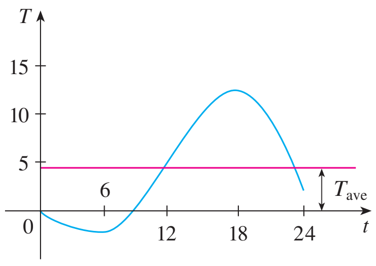


FIGURE 1

But how do we compute the average temperature during a day if infinitely many temperature readings are possible? Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

In general, let's try to compute the average value of a function $y = f(x)$, $a \leqslant x \leqslant b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.) Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*) + \cdots + f(x_n^*)\right]\Delta x$$

$$= \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values. (For example, we would be averaging temperature readings taken every minute or even every second.) The limiting value is

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of f** on the interval $[a, b]$ as

For a positive function, we can think of this definition as saying

$$\frac{\text{area}}{\text{width}} = \text{average height}$$

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EXAMPLE 1 Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

SOLUTION With $a = -1$ and $b = 2$ we have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1 + x^2)\,dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2 \quad ■$$

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature. For the temperature function

graphed in Figure 1, we see that there are two such times—just before noon and just before midnight. In general, is there a number c at which the value of a function f is exactly equal to the average value of the function, that is, $f(c) = f_{ave}$? The following theorem says that this is true for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

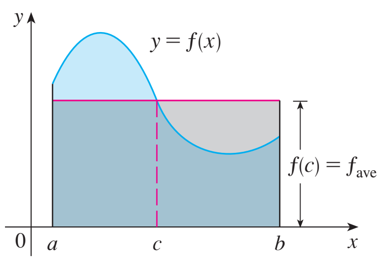


FIGURE 2

You can always chop off the top of a (two-dimensional) mountain at a certain height and use it to fill in the valleys so that the mountain becomes completely flat.

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus. The proof is outlined in Exercise 23.

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. (See Figure 2 and the more picturesque interpretation in the margin note.)

EXAMPLE 2 Since $f(x) = 1 + x^2$ is continuous on the interval $[-1, 2]$, the Mean Value Theorem for Integrals says there is a number c in $[-1, 2]$ such that

$$\int_{-1}^{2} (1 + x^2)\,dx = f(c)[2 - (-1)]$$

In this particular case we can find c explicitly. From Example 1 we know that $f_{ave} = 2$, so the value of c satisfies

$$f(c) = f_{ave} = 2$$

Therefore $\qquad 1 + c^2 = 2 \qquad \text{so} \qquad c^2 = 1$

So in this case there happen to be two numbers $c = \pm 1$ in the interval $[-1, 2]$ that work in the Mean Value Theorem for Integrals. ■

Examples 1 and 2 are illustrated by Figure 3.

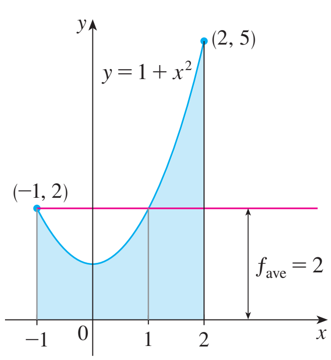


FIGURE 3

EXAMPLE 3 Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

SOLUTION If $s(t)$ is the displacement of the car at time t, then, by definition, the average velocity of the car over the interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

On the other hand, the average value of the velocity function on the interval is

$$\begin{aligned} v_{ave} &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt \\ &= \frac{1}{t_2 - t_1}[s(t_2) - s(t_1)] \qquad \text{(by the Net Change Theorem)} \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity} \end{aligned}$$ ■

5.5 EXERCISES

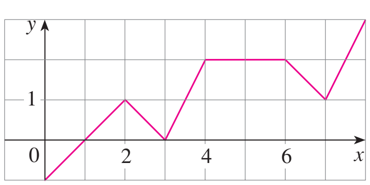
1–8 Find the average value of the function on the given interval.

1. $f(x) = 3x^2 + 8x, \quad [-1, 2]$
2. $f(x) = \sqrt{x}, \quad [0, 4]$
3. $g(x) = 3 \cos x, \quad [-\pi/2, \pi/2]$
4. $g(t) = \dfrac{t}{\sqrt{3 + t^2}}, \quad [1, 3]$
5. $f(t) = t^2(1 + t^3)^4, \quad [0, 2]$
6. $f(x) = x^2/(x^3 + 3)^2, \quad [-1, 1]$
7. $h(x) = \cos^4 x \sin x, \quad [0, \pi]$
8. $h(t) = (1 + \sin t)^2 \cos t, \quad [\pi/2, 3\pi/2]$

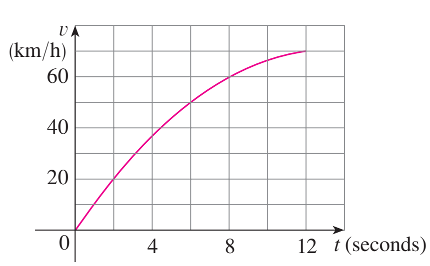
9–12
(a) Find the average value of f on the given interval.
(b) Find c such that $f_{ave} = f(c)$.
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

9. $f(x) = (x - 3)^2, \quad [2, 5]$
10. $f(x) = \sqrt[3]{x}, \quad [0, 8]$
11. $f(x) = 2 \sin x - \sin 2x, \quad [0, \pi]$
12. $f(x) = x\sqrt{4 - x^2}, \quad [0, 2]$

13. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.
14. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.
15. Find the average value of f on $[0, 8]$.



16. The velocity graph of an accelerating car is shown.



(a) Use the Midpoint Rule to estimate the average velocity of the car during the first 12 seconds.
(b) At what time was the instantaneous velocity equal to the average velocity?

17. In a certain city the temperature (in °F) t hours after 9 AM was modeled by the function

$$T(t) = 50 + 14 \sin \frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

18. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Example 2.7.7). Find the average velocity (with respect to r) over the interval $0 \leqslant r \leqslant R$. Compare the average velocity with the maximum velocity.

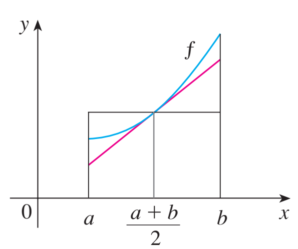
19. The linear density in a rod 8 m long is $12/\sqrt{x + 1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

20. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{ave} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{ave} = \frac{2}{3}v_T$.

21. Use the result of Exercise 4.5.57 to compute the average volume of inhaled air in the lungs in one respiratory cycle.

22. Use the diagram to show that if f is concave upward on $[a, b]$, then

$$f_{ave} > f\left(\frac{a + b}{2}\right)$$



23. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 3.2) to the function

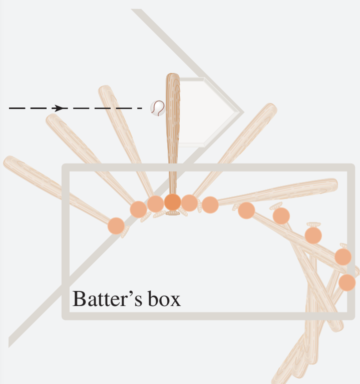
$$F(x) = \int_a^x f(t)\, dt$$

24. If $f_{ave}[a, b]$ denotes the average value of f on the interval $[a, b]$ and $a < c < b$, show that

$$f_{ave}[a, b] = \left(\frac{c-a}{b-a}\right) f_{ave}[a, c] + \left(\frac{b-c}{b-a}\right) f_{ave}[c, b]$$

APPLIED PROJECT CALCULUS AND BASEBALL

In this project we explore two of the many applications of calculus to baseball. The physical interactions of the game, especially the collision of ball and bat, are quite complex and their models are discussed in detail in a book by Robert Adair, *The Physics of Baseball,* 3d ed. (New York, 2002).



An overhead view of the position of a baseball bat, shown every fiftieth of a second during a typical swing. (Adapted from *The Physics of Baseball*)

1. It may surprise you to learn that the collision of baseball and bat lasts only about a thousandth of a second. Here we calculate the average force on the bat during this collision by first computing the change in the ball's momentum.

The *momentum* p of an object is the product of its mass m and its velocity v, that is, $p = mv$. Suppose an object, moving along a straight line, is acted on by a force $F = F(t)$ that is a continuous function of time.

(a) Show that the change in momentum over a time interval $[t_0, t_1]$ is equal to the integral of F from t_0 to t_1; that is, show that

$$p(t_1) - p(t_0) = \int_{t_0}^{t_1} F(t)\, dt$$

This integral is called the *impulse* of the force over the time interval.

(b) A pitcher throws a 90-mi/h fastball to a batter, who hits a line drive directly back to the pitcher. The ball is in contact with the bat for 0.001 s and leaves the bat with velocity 110 mi/h. A baseball weighs 5 oz and, in US Customary units, its mass is measured in slugs: $m = w/g$, where $g = 32 \text{ ft/s}^2$.

(i) Find the change in the ball's momentum.

(ii) Find the average force on the bat.

2. In this problem we calculate the work required for a pitcher to throw a 90-mi/h fastball by first considering kinetic energy.

The *kinetic energy* K of an object of mass m and velocity v is given by $K = \frac{1}{2}mv^2$. Suppose an object of mass m, moving in a straight line, is acted on by a force $F = F(s)$ that depends on its position s. According to Newton's Second Law

$$F(s) = ma = m\frac{dv}{dt}$$

where a and v denote the acceleration and velocity of the object.

(a) Show that the work done in moving the object from a position s_0 to a position s_1 is equal to the change in the object's kinetic energy; that is, show that

$$W = \int_{s_0}^{s_1} F(s)\, ds = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

where $v_0 = v(s_0)$ and $v_1 = v(s_1)$ are the velocities of the object at the positions s_0 and s_1. *Hint:* By the Chain Rule,

$$m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

(b) How many foot-pounds of work does it take to throw a baseball at a speed of 90 mi/h?

NOTE Another application of calculus to baseball can be found in Problem 16 on page 678.

5 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.
   (b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.
2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?
3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.
   (b) If S is a solid of revolution, how do you find the cross-sectional areas?
4. (a) What is the volume of a cylindrical shell?
   (b) Explain how to use cylindrical shells to find the volume of a solid of revolution.
   (c) Why might you want to use the shell method instead of slicing?
5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?
6. (a) What is the average value of a function f on an interval $[a, b]$?
   (b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

EXERCISES

1–6 Find the area of the region bounded by the given curves.

1. $y = x^2, \quad y = 4x - x^2$
2. $y = \sqrt{x}, \quad y = -\sqrt[3]{x}, \quad y = x - 2$
3. $y = 1 - 2x^2, \quad y = |x|$
4. $x + y = 0, \quad x = y^2 + 3y$
5. $y = \sin(\pi x/2), \quad y = x^2 - 2x$
6. $y = \sqrt{x}, \quad y = x^2, \quad x = 2$

7–11 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

7. $y = 2x,\ y = x^2;$ about the x-axis
8. $x = 1 + y^2,\ y = x - 3;$ about the y-axis
9. $x = 0,\ x = 9 - y^2;$ about $x = -1$
10. $y = x^2 + 1,\ y = 9 - x^2;$ about $y = -1$
11. $x^2 - y^2 = a^2,\ x = a + h$ (where $a > 0, h > 0$); about the y-axis

12–14 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

12. $y = \tan x,\ y = x,\ x = \pi/3;$ about the y-axis
13. $y = \cos^2 x,\ |x| \leqslant \pi/2,\ y = \frac{1}{4};$ about $x = \pi/2$
14. $y = \sqrt{x},\ y = x^2;$ about $y = 2$

15. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines.
   (a) The x-axis (b) The y-axis (c) $y = 2$
16. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities.
   (a) The area of $\mathcal{R}$
   (b) The volume obtained by rotating $\mathcal{R}$ about the x-axis
   (c) The volume obtained by rotating $\mathcal{R}$ about the y-axis
17. Let $\mathcal{R}$ be the region bounded by the curves $y = \tan(x^2)$, $x = 1$, and $y = 0$. Use the Midpoint Rule with $n = 4$ to estimate the following quantities.
   (a) The area of $\mathcal{R}$
   (b) The volume obtained by rotating $\mathcal{R}$ about the x-axis
18. Let $\mathcal{R}$ be the region bounded by the curves $y = 1 - x^2$ and $y = x^6 - x + 1$. Estimate the following quantities.
   (a) The x-coordinates of the points of intersection of the curves
   (b) The area of $\mathcal{R}$
   (c) The volume generated when $\mathcal{R}$ is rotated about the x-axis
   (d) The volume generated when $\mathcal{R}$ is rotated about the y-axis

19–22 Each integral represents the volume of a solid. Describe the solid.

19. $\int_0^{\pi/2} 2\pi x \cos x \, dx$

20. $\int_0^{\pi/2} 2\pi \cos^2 x \, dx$

21. $\int_0^{\pi} \pi(2 - \sin x)^2 \, dx$

22. $\int_0^{4} 2\pi(6 - y)(4y - y^2) \, dy$

---

23. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.

24. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base.

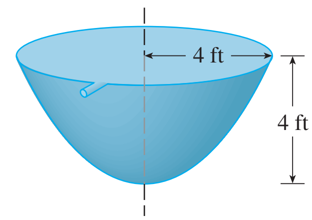
25. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

26. (a) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.
(b) Show that the volume of the solid of part (a) can be computed more simply by first cutting the solid and rearranging it to form a cone.

27. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

28. A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

29. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.
(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?



30. A steel tank has the shape of a circular cylinder oriented vertically with diameter 4 m and height 5 m. The tank is currently filled to a level of 3 m with cooking oil that has a density of 920 kg/m^3. Compute the work required to pump the oil out through a 1-meter spout at the top of the tank.

31. Find the average value of the function $f(t) = \sec^2 t$ on the interval $[0, \pi/4]$.

32. (a) Find the average value of the function $f(x) = 1/\sqrt{x}$ on the interval $[1, 4]$.
(b) Find the value c guaranteed by the Mean Value Theorem for Integrals such that $f_{\text{ave}} = f(c)$.
(c) Sketch the graph of f on $[1, 4]$ and a rectangle whose area is the same as the area under the graph of f.

33. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x + h]$?

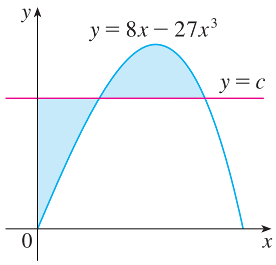
34. Let $\mathcal{R}_1$ be the region bounded by $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let $\mathcal{R}_2$ be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$.
(a) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same area?
(b) Is there a value of b such that $\mathcal{R}_1$ sweeps out the same volume when rotated about the x-axis and the y-axis?
(c) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the x-axis?
(d) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the y-axis?

Problems Plus

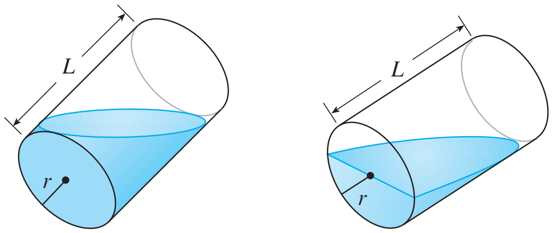
1. (a) Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.
   (b) A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geqslant 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

2. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

3. The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



4. A cylindrical glass of radius r and height L is filled with water and then tilted until the water remaining in the glass exactly covers its base.
   (a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.
   (b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.
   (c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).
   (d) Find the volume of the water in the glass from purely geometric considerations.
   (e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.



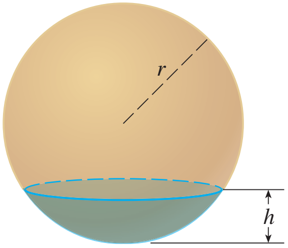


FIGURE FOR PROBLEM 5

5. (a) Show that the volume of a segment of height h of a sphere of radius r is

$$V = \tfrac{1}{3}\pi h^2(3r - h)$$

   (See the figure.)
   (b) Show that if a sphere of radius 1 is sliced by a plane at a distance x from the center in such a way that the volume of one segment is twice the volume of the other, then x is a solution of the equation

$$3x^3 - 9x + 2 = 0$$

   where $0 < x < 1$. Use Newton's method to find x accurate to four decimal places.
   (c) Using the formula for the volume of a segment of a sphere, it can be shown that the depth x to which a floating sphere of radius r sinks in water is a root of the equation

$$x^3 - 3rx^2 + 4r^3s = 0$$

where s is the specific gravity of the sphere. Suppose a wooden sphere of radius 0.5 m has specific gravity 0.75. Calculate, to four-decimal-place accuracy, the depth to which the sphere will sink.

(d) A hemispherical bowl has radius 5 inches and water is running into the bowl at the rate of 0.2 in^3/s.

(i) How fast is the water level in the bowl rising at the instant the water is 3 inches deep?

(ii) At a certain instant, the water is 4 inches deep. How long will it take to fill the bowl?

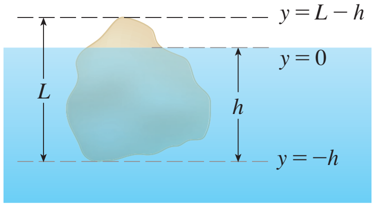


FIGURE FOR PROBLEM 6

6. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density ρ_0 floating partly submerged in a fluid of density ρ_f, the buoyant force is given by $F = \rho_f g \int_{-h}^{0} A(y)\,dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object (see the figure). The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$$

(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

(b) The density of ice is 917 kg/m^3 and the density of seawater is 1030 kg/m^3. What percentage of the volume of an iceberg is above water?

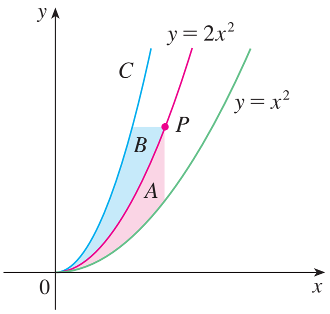
(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 kg/m^3.

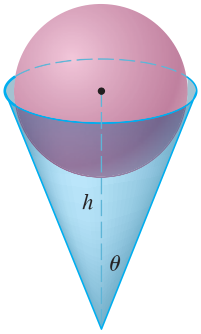
7. Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

8. A sphere of radius 1 overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r. (In other words, they intersect in a great circle of the small sphere.) Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

9. The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.



10. A paper drinking cup filled with water has the shape of a cone with height h and semi-vertical angle θ. (See the figure.) A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?



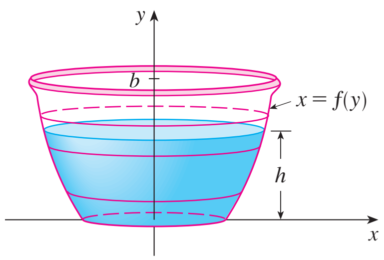


FIGURE FOR PROBLEM 11

11. A *clepsydra*, or water clock, is a glass container with a small hole in the bottom through which water can flow. The "clock" is calibrated for measuring time by placing markings on the container corresponding to water levels at equally spaced times. Let $x = f(y)$ be continuous on the interval $[0, b]$ and assume that the container is formed by rotating the graph of f about the y-axis. Let V denote the volume of water and h the height of the water level at time t.

(a) Determine V as a function of h.

(b) Show that

$$\frac{dV}{dt} = \pi[f(h)]^2 \frac{dh}{dt}$$

(c) Suppose that A is the area of the hole in the bottom of the container. It follows from Torricelli's Law that the rate of change of the volume of the water is given by

$$\frac{dV}{dt} = kA\sqrt{h}$$

where k is a negative constant. Determine a formula for the function f such that dh/dt is a constant C. What is the advantage in having $dh/dt = C$?

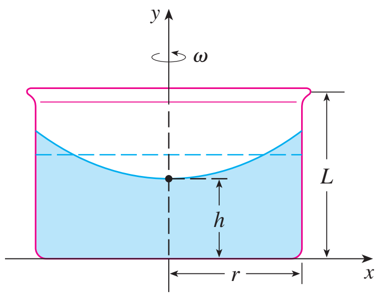


FIGURE FOR PROBLEM 12

12. A cylindrical container of radius r and height L is partially filled with a liquid whose volume is V. If the container is rotated about its axis of symmetry with constant angular speed ω, then the container will induce a rotational motion in the liquid around the same axis. Eventually, the liquid will be rotating at the same angular speed as the container. The surface of the liquid will be convex, as indicated in the figure, because the centrifugal force on the liquid particles increases with the distance from the axis of the container. It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the y-axis, where g is the acceleration due to gravity.

(a) Determine h as a function of ω.

(b) At what angular speed will the surface of the liquid touch the bottom? At what speed will it spill over the top?

(c) Suppose the radius of the container is 2 ft, the height is 7 ft, and the container and liquid are rotating at the same constant angular speed. The surface of the liquid is 5 ft below the top of the tank at the central axis and 4 ft below the top of the tank 1 ft out from the central axis.

 (i) Determine the angular speed of the container and the volume of the fluid.

 (ii) How far below the top of the tank is the liquid at the wall of the container?

13. Suppose the graph of a cubic polynomial intersects the parabola $y = x^2$ when $x = 0$, $x = a$, and $x = b$, where $0 < a < b$. If the two regions between the curves have the same area, how is b related to a?

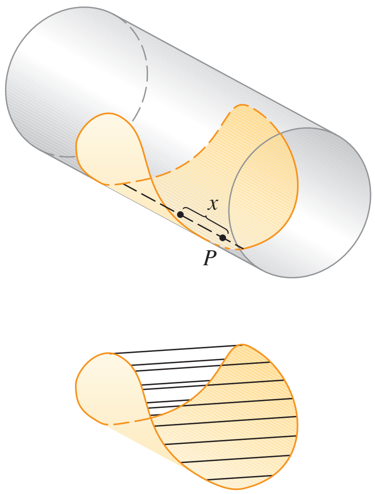


FIGURE FOR PROBLEM 14

CAS **14.** Suppose we are planning to make a taco from a round tortilla with diameter 8 inches by bending the tortilla so that it is shaped as if it is partially wrapped around a circular cylinder. We will fill the tortilla to the edge (but no more) with meat, cheese, and other ingredients. Our problem is to decide how to curve the tortilla in order to maximize the volume of food it can hold.

(a) We start by placing a circular cylinder of radius r along a diameter of the tortilla and folding the tortilla around the cylinder. Let x represent the distance from the center of the tortilla to a point P on the diameter (see the figure). Show that the cross-sectional area of the filled taco in the plane through P perpendicular to the axis of the cylinder is

$$A(x) = r\sqrt{16 - x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16 - x^2}\right)$$

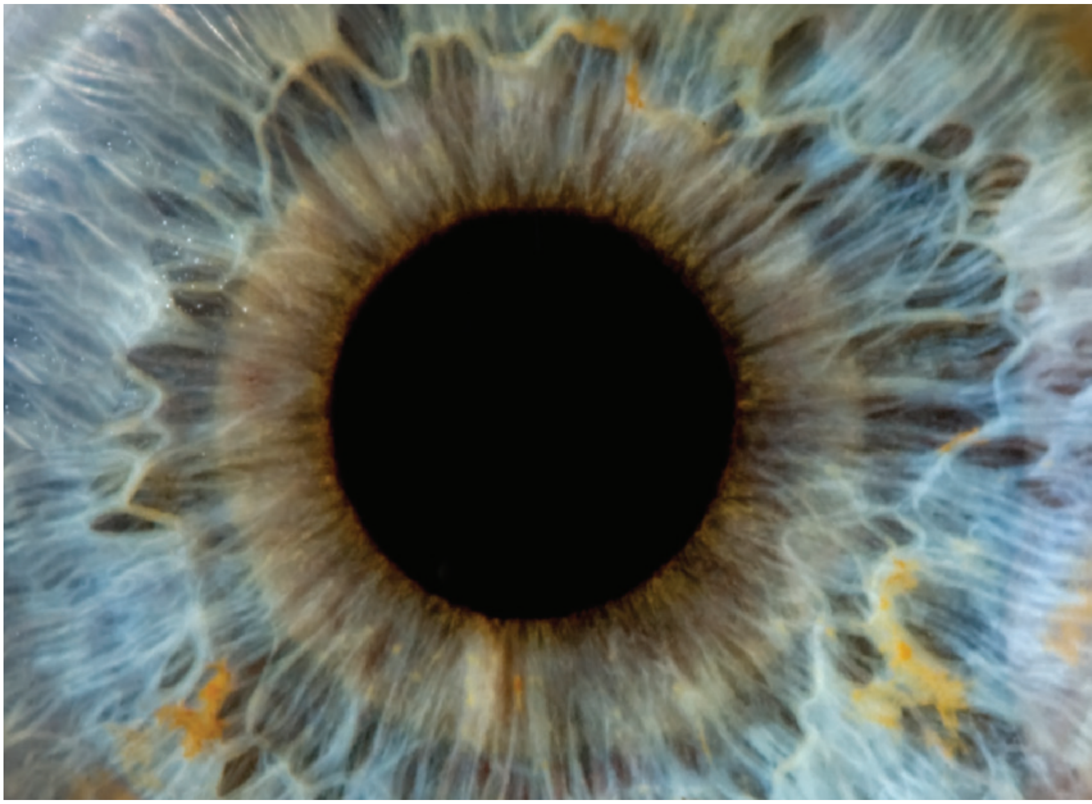
and write an expression for the volume of the filled taco.

(b) Determine (approximately) the value of r that maximizes the volume of the taco. (Use a graphical approach with your CAS.)

15. If the tangent at a point P on the curve $y = x^3$ intersects the curve again at Q, let A be the area of the region bounded by the curve and the line segment PQ. Let B be the area of the region defined in the same way starting with Q instead of P. What is the relationship between A and B?

6 Inverse Functions

Exponential, Logarithmic, and Inverse Trigonometric Functions



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When we view the world around us, the light entering the eye near the center of the pupil is perceived brighter than light entering closer to the edges of the pupil. This phenomenon, known as the *Stiles–Crawford effect*, is explored as the pupil changes in radius in Exercise 90 on page 501.

THE COMMON THEME THAT LINKS the functions of this chapter is that they occur as pairs of inverse functions. In particular, two of the most important functions that occur in mathematics and its applications are the exponential function $f(x) = b^x$ and its inverse function, the logarithmic function $g(x) = \log_b x$. In this chapter we investigate their properties, compute their derivatives, and use them to describe exponential growth and decay in biology, physics, chemistry, and other sciences. We also study the inverses of trigonometric and hyperbolic functions. Finally, we look at a method (l'Hospital's Rule) for computing difficult limits and apply it to sketching curves.

There are two possible ways of defining the exponential and logarithmic functions and developing their properties and derivatives. One is to start with the exponential function (defined as in algebra or precalculus courses) and then define the logarithm as its inverse. That is the approach taken in Sections 6.2, 6.3, and 6.4 and is probably the most intuitive method. The other way is to start by defining the logarithm as an integral and then define the exponential function as its inverse. This approach is followed in Sections 6.2*, 6.3*, and 6.4* and, although it is less intuitive, many instructors prefer it because it is more rigorous and the properties follow more easily. You need only read one of these two approaches (whichever your instructor recommends).

6.1 Inverse Functions

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Thus $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

Table 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

Table 2 t as a function of N

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

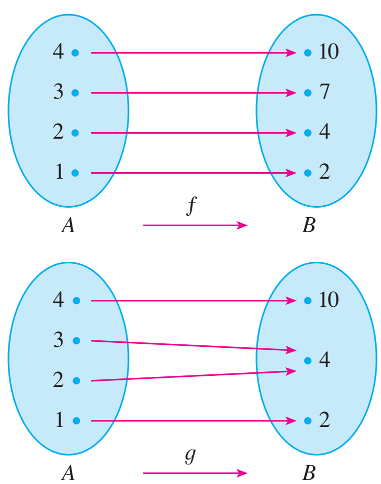


FIGURE 1
f is one-to-one; g is not.

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but
$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that share this property with f are called *one-to-one functions*.

In the language of inputs and outputs, this definition says that f is one-to-one if each output corresponds to only one input.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

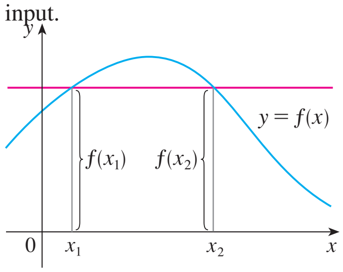


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

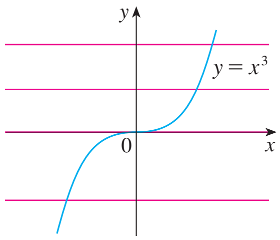


FIGURE 3
$f(x) = x^3$ is one-to-one.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. ■

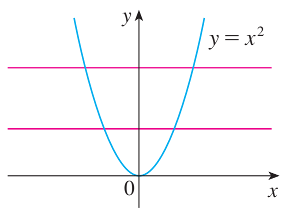


FIGURE 4
$g(x) = x^2$ is not one-to-one.

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. ■

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

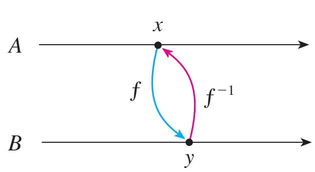


FIGURE 5

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } not \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

EXAMPLE 3 If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$

$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case. ■

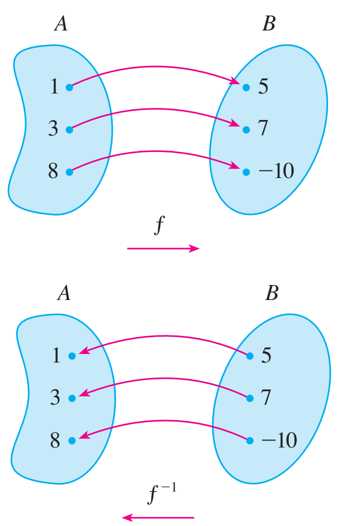


FIGURE 6
The inverse function reverses inputs and outputs.

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

3

$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

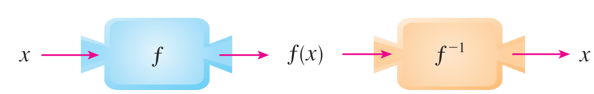
4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7). Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

FIGURE 7



For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$. ■

In Example 4, notice how f^{-1} reverses the effect of f. The function f is the rule "Cube, then add 2"; f^{-1} is the rule "Subtract 2, then take the cube root."

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

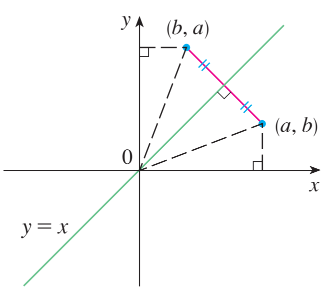


FIGURE 8

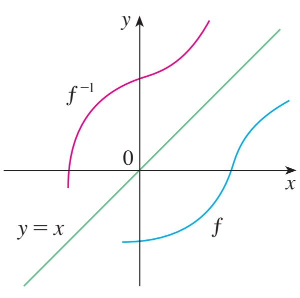


FIGURE 9

Therefore, as illustrated by Figure 9:

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

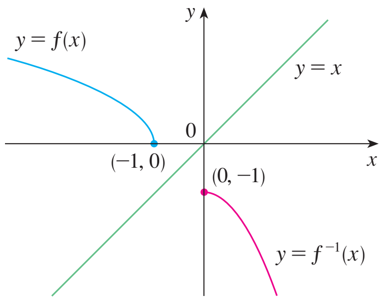


FIGURE 10

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10. ■

The Calculus of Inverse Functions

Now let's look at inverse functions from the point of view of calculus. Suppose that f is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.) Since the graph of f^{-1} is obtained from the graph of f by reflecting about the line $y = x$, the graph of f^{-1} has no break in it either (see Figure 9). Thus we might expect that f^{-1} is also a continuous function.

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible. A proof can be found in Appendix F.

> **6 Theorem** If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

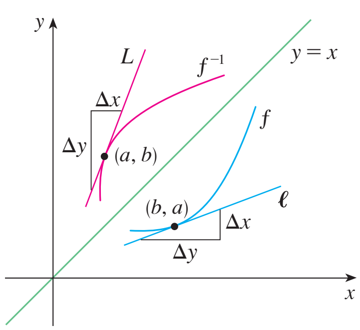


FIGURE 11

Now suppose that f is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it. We get the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or kink in it either. We therefore expect that f^{-1} is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of f^{-1} at a given point by a geometric argument. In Figure 11 the graphs of f and its inverse f^{-1} are shown. If $f(b) = a$, then $f^{-1}(a) = b$ and $(f^{-1})'(a)$ is the slope of the tangent line L to the graph of f^{-1} at (a, b), which is $\Delta y/\Delta x$. Reflecting in the line $y = x$ has the effect of interchanging the x- and y-coordinates. So the slope of the reflected line ℓ [the tangent to the graph of f at (b, a)] is $\Delta x/\Delta y$. Thus the slope of L is the reciprocal of the slope of ℓ, that is,

$$(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}$$

> **7 Theorem** If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and
>
> $$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

PROOF Write the definition of derivative as in Equation 2.1.5:

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

If $f(b) = a$, then $f^{-1}(a) = b$. And if we let $y = f^{-1}(x)$, then $f(y) = x$. Since f is differentiable, it is continuous, so f^{-1} is continuous by Theorem 6. Thus if $x \to a$,

then $f^{-1}(x) \to f^{-1}(a)$, that is, $y \to b$. Therefore

Note that $x \neq a \;\Rightarrow\; f(y) \neq f(b)$ because f is one-to-one.

$$\begin{aligned}(f^{-1})'(a) &= \lim_{x\to a}\frac{f^{-1}(x)-f^{-1}(a)}{x-a} = \lim_{y\to b}\frac{y-b}{f(y)-f(b)} \\ &= \lim_{y\to b}\frac{1}{\dfrac{f(y)-f(b)}{y-b}} = \frac{1}{\displaystyle\lim_{y\to b}\frac{f(y)-f(b)}{y-b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}\end{aligned}$$

■

NOTE 1 Replacing a by the general number x in the formula of Theorem 7, we get

8 $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

NOTE 2 If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, remembering that y is a function of x, and using the Chain Rule, we get

$$f'(y)\frac{dy}{dx} = 1$$

Therefore $$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$

EXAMPLE 6 Although the function $y = x^2$, $x \in \mathbb{R}$, is not one-to-one and therefore does not have an inverse function, we can turn it into a one-to-one function by restricting its domain. For instance, the function $f(x) = x^2$, $0 \leqslant x \leqslant 2$, is one-to-one (by the Horizontal Line Test) and has domain $[0, 2]$ and range $[0, 4]$. (See Figure 12.) Thus f has an inverse function f^{-1} with domain $[0, 4]$ and range $[0, 2]$.

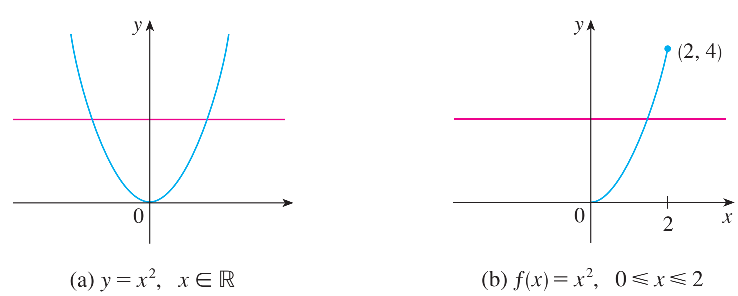


FIGURE 12

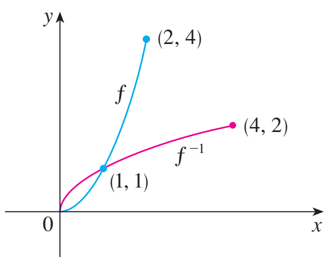


FIGURE 13

Without computing a formula for $(f^{-1})'$ we can still calculate $(f^{-1})'(1)$. Since $f(1) = 1$, we have $f^{-1}(1) = 1$. Also $f'(x) = 2x$. So by Theorem 7 we have

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{2}$$

In this case it is easy to find f^{-1} explicitly. In fact, $f^{-1}(x) = \sqrt{x}$, $0 \leqslant x \leqslant 4$. [In general, we could use the method given by (5).] Then $(f^{-1})'(x) = 1/(2\sqrt{x})$, so $(f^{-1})'(1) = \frac{1}{2}$, which agrees with the preceding computation. The functions f and f^{-1} are graphed in Figure 13. ■

EXAMPLE 7 If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

SOLUTION Notice that f is one-to-one because

$$f'(x) = 2 - \sin x > 0$$

and so f is increasing. To use Theorem 7 we need to know $f^{-1}(1)$ and we can find it by inspection:

$$f(0) = 1 \quad \Rightarrow \quad f^{-1}(1) = 0$$

Therefore $$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$ ■

6.1 EXERCISES

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

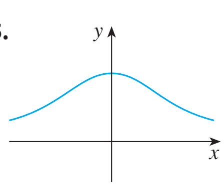
3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

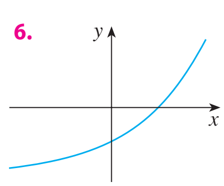
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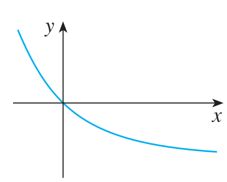
x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

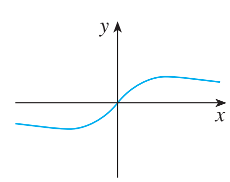
4.

x	1	2	3	4	5	6
$f(x)$	1.0	1.9	2.8	3.5	3.1	2.9

5.


6.


7.


8.


9. $f(x) = 2x - 3$

10. $f(x) = x^4 - 16$

11. $g(x) = 1 - \sin x$

12. $g(x) = \sqrt[3]{x}$

13. $h(x) = 1 + \cos x$

14. $h(x) = 1 + \cos x$, $\quad 0 \leqslant x \leqslant \pi$

15. $f(t)$ is the height of a football t seconds after kickoff.

16. $f(t)$ is your height at age t.

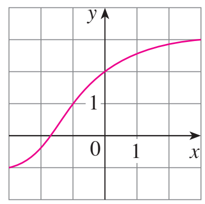
17. Assume that f is a one-to-one function.
(a) If $f(6) = 17$, what is $f^{-1}(17)$?
(b) If $f^{-1}(3) = 2$, what is $f(2)$?

18. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

19. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) What are the domain and range of f^{-1}?

(c) What is the value of $f^{-1}(2)$?
(d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. $f(x) = 5 - 4x$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = 1 + \sqrt{2 + 3x}$

26. $y = x^2 - x, \ x \geqslant \frac{1}{2}$

27. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

28. $f(x) = 2x^2 - 8x, \ x \geqslant 2$

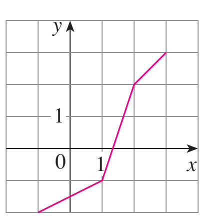
29–30 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

29. $f(x) = \sqrt{4x + 3}$

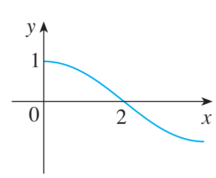
30. $f(x) = 2 - x^4, \ x \geqslant 0$

31–32 Use the given graph of f to sketch the graph of f^{-1}.

31.



32.



33. Let $f(x) = \sqrt{1 - x^2}, \ 0 \leqslant x \leqslant 1$.
(a) Find f^{-1}. How is it related to f?
(b) Identify the graph of f and explain your answer to part (a).

34. Let $g(x) = \sqrt[3]{1 - x^3}$.
(a) Find g^{-1}. How is it related to g?
(b) Graph g. How do you explain your answer to part (a)?

35–38
(a) Show that f is one-to-one.
(b) Use Theorem 7 to find $(f^{-1})'(a)$.
(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.
(d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).
(e) Sketch the graphs of f and f^{-1} on the same axes.

35. $f(x) = x^3, \quad a = 8$

36. $f(x) = \sqrt{x - 2}, \quad a = 2$

37. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

38. $f(x) = 1/(x - 1), \quad x > 1, \quad a = 2$

39–42 Find $(f^{-1})'(a)$.

39. $f(x) = 3x^3 + 4x^2 + 6x + 5, \quad a = 5$

40. $f(x) = x^3 + 3\sin x + 2\cos x, \quad a = 2$

41. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

42. $f(x) = \sqrt{x^3 + 4x + 4}, \quad a = 3$

43. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

44. If g is an increasing function such that $g(2) = 8$ and $g'(2) = 5$, calculate $(g^{-1})'(8)$.

45. If $f(x) = \int_3^x \sqrt{1 + t^3}\, dt$, find $(f^{-1})'(0)$.

46. Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

CAS **47.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

48. Show that $h(x) = \sin x$, $x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one. Compute the derivative of $f^{-1} = \sin^{-1}$ by the method of Note 2.

49. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

50. (a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

(b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

6.2 Exponential Functions and Their Derivatives

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

where b is a positive constant. Let's recall what this means.

If your instructor has assigned Sections 6.2*, 6.3*, and 6.4* (pp. 438–465), you don't need to read Sections 6.2–6.4 (pp. 408–438).

If $x = n$, a positive integer, then

$$b^n = \underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}$$

If $x = 0$, then $b^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$b^{-n} = \frac{1}{b^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$b^x = b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

But what is the meaning of b^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^π?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

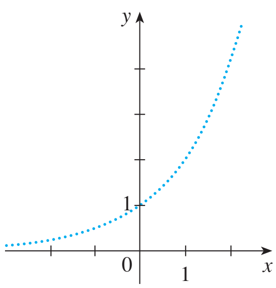


FIGURE 1
Representation of $y = 2^x$, x rational

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{array}{rcl}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \quad \vdots & & \vdots \quad \vdots
\end{array}$$

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1981). For an online version, see resolver.caltech.edu/CaltechBOOK:1981.001

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can

compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

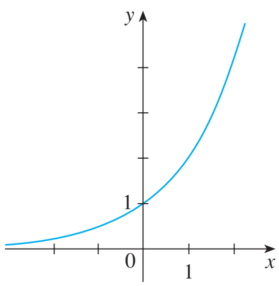


FIGURE 2
$y = 2^x$, x real

Similarly, we can define 2^x (or b^x, if $b > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

In general, if b is any positive number, we define

1
$$b^x = \lim_{r \to x} b^r \qquad r \text{ rational}$$

This definition makes sense because any irrational number can be approximated as closely as we like by a rational number. For instance, because $\sqrt{3}$ has the decimal representation $\sqrt{3} = 1.7320508\ldots$, Definition 1 says that $2^{\sqrt{3}}$ is the limit of the sequence of numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad 2^{1.732050}, \quad 2^{1.7320508}, \quad \ldots$$

Similarly, 5^{π} is the limit of the sequence of numbers

$$5^{3.1}, \quad 5^{3.14}, \quad 5^{3.141}, \quad 5^{3.1415}, \quad 5^{3.14159}, \quad 5^{3.141592}, \quad 5^{3.1415926}, \quad \ldots$$

It can be shown that Definition 1 uniquely specifies b^x and makes the function $f(x) = b^x$ continuous.

The graphs of members of the family of functions $y = b^x$ are shown in Figure 3 for various values of the base b. Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$. Notice also that as the base b gets larger, the exponential function grows more rapidly (for $x > 0$).

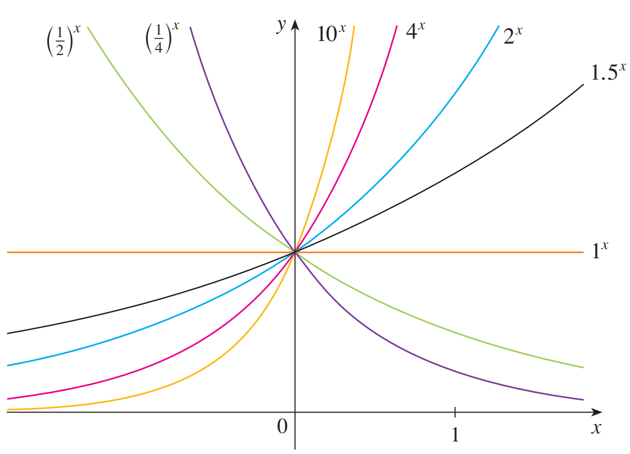


FIGURE 3
Members of the family of exponential functions

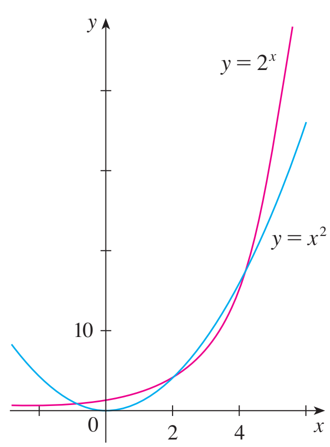


FIGURE 4

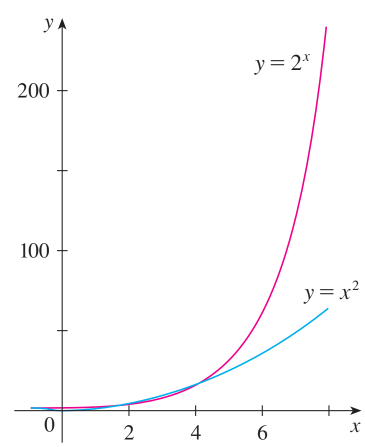


FIGURE 5

Figure 4 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

You can see from Figure 3 that there are basically three kinds of exponential functions $y = b^x$. If $0 < b < 1$, the exponential function decreases; if $b = 1$, it is a constant; and if $b > 1$, it increases. These three cases are illustrated in Figure 6. Because

$(1/b)^x = 1/b^x = b^{-x}$, the graph of $y = (1/b)^x$ is just the reflection of the graph of $y = b^x$ about the y-axis.

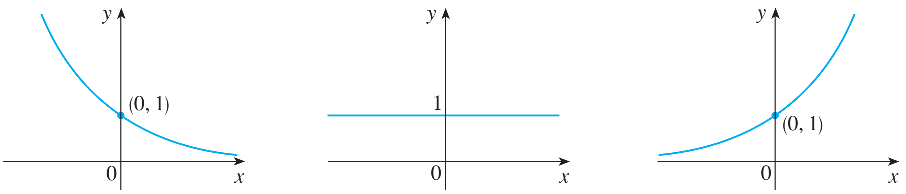


FIGURE 6 (a) $y = b^x$, $0 < b < 1$ (b) $y = 1^x$ (c) $y = b^x$, $b > 1$

The properties of the exponential function are summarized in the following theorem.

> **2 Theorem** If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $b^x > 0$ for all x. If $0 < b < 1$, $f(x) = b^x$ is a decreasing function; if $b > 1$, f is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then
>
> **1.** $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

www.stewartcalculus.com
For review and practice using the Laws of Exponents, click on *Review of Algebra*.

The reason for the importance of the exponential function lies in properties 1–4, which are called the **Laws of Exponents**. If x and y are rational numbers, then these laws are well known from elementary algebra. For arbitrary real numbers x and y these laws can be deduced from the special case where the exponents are rational by using Equation 1.

The following limits can be read from the graphs shown in Figure 6 or proved from the definition of a limit at infinity. (See Exercise 6.3.71.)

> **3** If $b > 1$, then $\lim_{x\to\infty} b^x = \infty$ and $\lim_{x\to-\infty} b^x = 0$
>
> If $0 < b < 1$, then $\lim_{x\to\infty} b^x = 0$ and $\lim_{x\to-\infty} b^x = \infty$

In particular, if $b \neq 1$, then the x-axis is a horizontal asymptote of the graph of the exponential function $y = b^x$.

EXAMPLE 1
(a) Find $\lim_{x\to\infty} (2^{-x} - 1)$.
(b) Sketch the graph of the function $y = 2^{-x} - 1$.

SOLUTION

(a)
$$\lim_{x\to\infty} (2^{-x} - 1) = \lim_{x\to\infty} \left[\left(\tfrac{1}{2}\right)^x - 1\right]$$
$$= 0 - 1 \qquad [\text{by (3) with } b = \tfrac{1}{2} < 1]$$
$$= -1$$

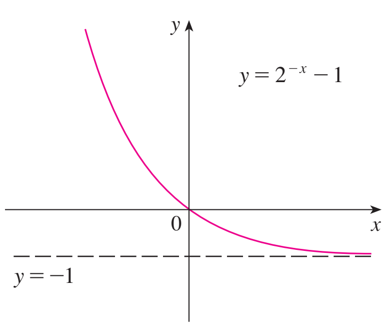


FIGURE 7

(b) We write $y = \left(\frac{1}{2}\right)^x - 1$ as in part (a). The graph of $y = \left(\frac{1}{2}\right)^x$ is shown in Figure 3, so we shift it down one unit to obtain the graph of $y = \left(\frac{1}{2}\right)^x - 1$ shown in Figure 7. (For a review of shifting graphs, see Section 1.3.) Part (a) shows that the line $y = -1$ is a horizontal asymptote. ■

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth. In Section 6.5 we will pursue these and other applications in greater detail.

In Section 2.7 we considered a bacteria population that doubles every hour and saw that if the initial population is n_0, then the population after t hours is given by the function $f(t) = n_0 2^t$. This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 5. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

Table 1

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

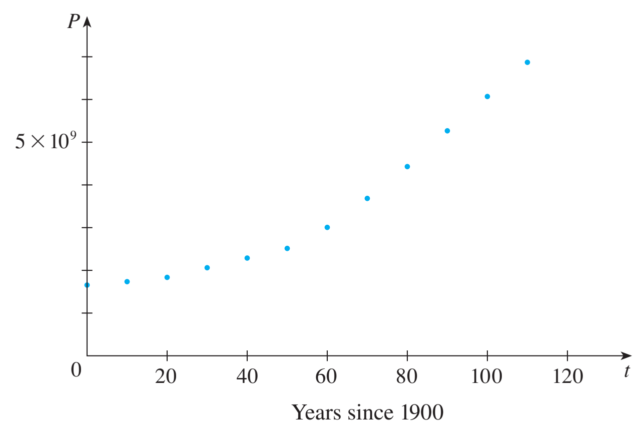


FIGURE 8
Scatter plot for world population growth

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to the year 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

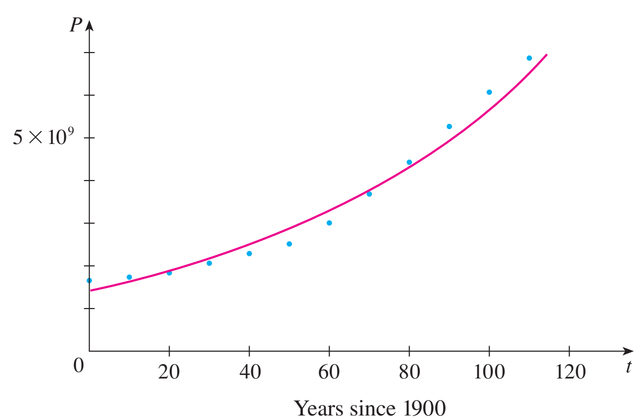


FIGURE 9
Exponential model for population growth

In 1995 a paper appeared detailing the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1.[1] Table 2 shows values of the plasma viral load $V(t)$ of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10.

Table 2

t (days)	$V(t)$
1	76.0
4	53.0
8	18.0
11	9.4
15	5.2
22	3.6

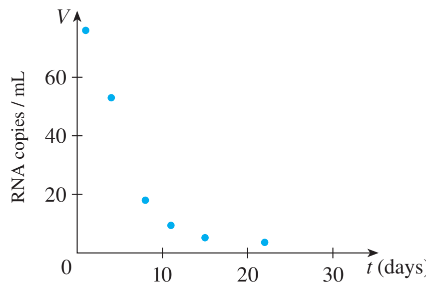


FIGURE 10 Plasma viral load in patient 303

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function $y = b^x$ in Figures 3 and 6(a) for the case where the base b is less than 1. So let's model the function $V(t)$ by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form $y = a \cdot b^t$, we obtain the model

$$V = 96.39785 \cdot (0.818656)^t$$

In Figure 11 we graph this exponential function with the data points and see that the model represents the viral load reasonably well for the first month of treatment.

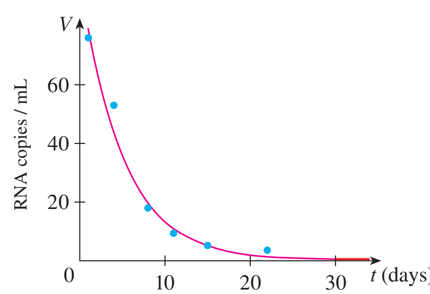


FIGURE 11
Exponential model for viral load

We could use the graph in Figure 11 to estimate the *half-life* of V, that is, the time required for the viral load to be reduced to half its initial value (see Exercise 63).

Derivatives of Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

1. D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

The factor b^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{b^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = b^x$ is differentiable at 0, then it is differentiable everywhere and

4 $$f'(x) = f'(0)b^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself*. (The slope is proportional to the height.)

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $b = 2$ and $b = 3$. (Values are stated correct to four decimal places.) It appears that the limits exist and

$$\text{for } b = 2, \qquad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } b = 3, \qquad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$$

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

5 $$\left.\frac{d}{dx}(2^x)\right|_{x=0} \approx 0.693147 \qquad \left.\frac{d}{dx}(3^x)\right|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

6 $$\frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Of all possible choices for the base b in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $b = 2$ and $b = 3$, it seems reasonable that there is a number b between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e. Thus we have the following definition.

7 Definition of the Number *e*

$$e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Geometrically, this means that of all the possible exponential functions $y = b^x$, the function $f(x) = e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)$ that is exactly 1. (See Figures 12 and 13.) We call the function $f(x) = e^x$ the *natural exponential function*.

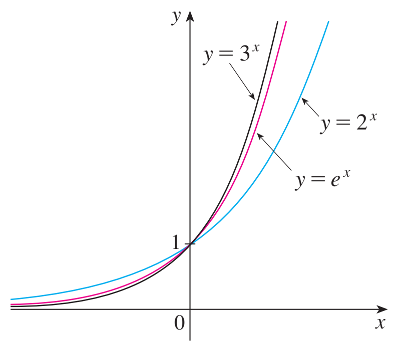


FIGURE 12

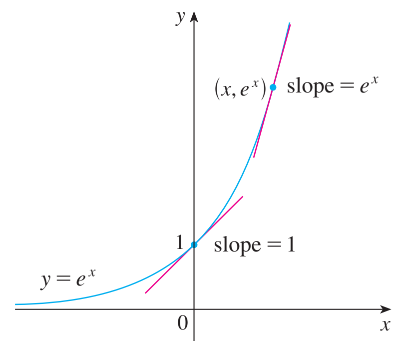


FIGURE 13

If we put $b = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

TEC Visual 6.2/6.3* uses the slope-a-scope to illustrate this formula.

8 Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point (see Figure 13).

EXAMPLE 2 Differentiate the function $y = e^{\tan x}$.

SOLUTION To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \frac{du}{dx} = e^{\tan x} \sec^2 x$$ ■

In general if we combine Formula 8 with the Chain Rule, as in Example 2, we get

9
$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

EXAMPLE 3 Find y' if $y = e^{-4x} \sin 5x$.

SOLUTION Using Formula 9 and the Product Rule, we have

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5 \cos 5x - 4 \sin 5x)$$ ■

We have seen that e is a number that lies somewhere between 2 and 3, but we can use Equation 4 to estimate the numerical value of e more accurately. Let $e = 2^c$. Then $e^x = 2^{cx}$. If $f(x) = 2^x$, then from Equation 4 we have $f'(x) = k2^x$, where the value of k

is $f'(0) \approx 0.693147$ (see Equations 5). Thus, by the Chain Rule,

$$e^x = \frac{d}{dx}(e^x) = \frac{d}{dx}(2^{cx}) = k2^{cx}\frac{d}{dx}(cx) = ck2^{cx}$$

Putting $x = 0$, we have $1 = ck$, so $c = 1/k$ and

$$e = 2^{1/k} \approx 2^{1/0.693147} \approx 2.71828$$

It can be shown that the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

EXAMPLE 4 In Example 2.7.6 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after t hours is

$$n = n_0 2^t$$

where n_0 is the initial population. Now we can use (4) and (5) to compute the growth rate:

The rate of growth is proportional to the size of the population.

$$\frac{dn}{dt} \approx n_0(0.693147)2^t$$

For instance, if the initial population is $n_0 = 1000$ cells, then the growth rate after two hours is

$$\left.\frac{dn}{dt}\right|_{t=2} \approx (1000)(0.693147)2^t\big|_{t=2}$$

$$= (4000)(0.693147) \approx 2773 \text{ cells/h}$$

■

EXAMPLE 5 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

SOLUTION We differentiate to find any critical numbers:

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, when $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37$$

■

Exponential Graphs

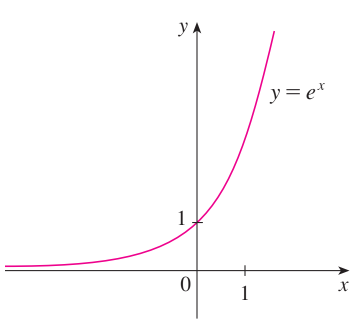


FIGURE 14
The natural exponential function

The exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 14) and properties. We summarize these properties as follows, using the fact that this function is just a special case of the exponential functions considered in Theorem 2 but with base $b = e > 1$.

10 Properties of the Natural Exponential Function The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

EXAMPLE 6 Find $\displaystyle\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1}$.

SOLUTION We divide numerator and denominator by e^{2x}:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \lim_{x \to \infty} e^{-2x}}$$

$$= \frac{1}{1 + 0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0$$

EXAMPLE 7 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote (both to the left and right).

Now let's compute the derivative. The Chain Rule gives

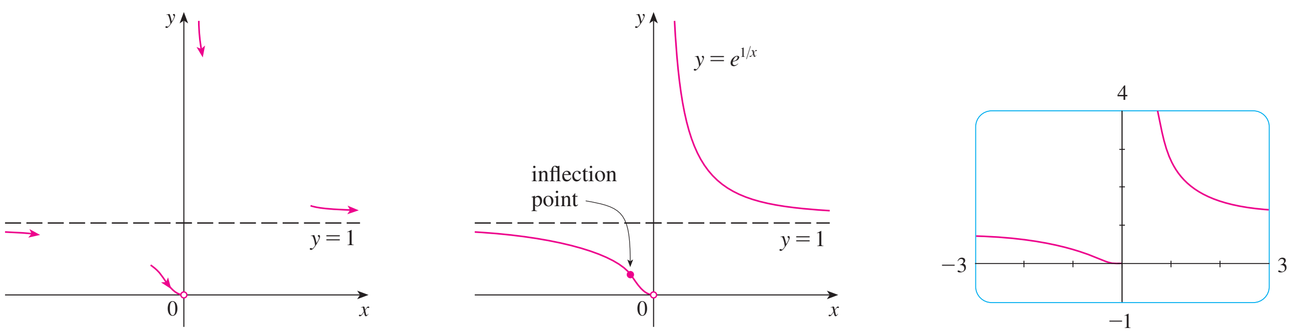
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no local maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 15(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 15(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 15(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

FIGURE 15 ■

■ Integration

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

11
$$\int e^x\,dx = e^x + C$$

EXAMPLE 8 Evaluate $\int x^2 e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$ ■

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big]_0^1 = \frac{1}{3}(1 - e^{-3})$$ ■

6.2 EXERCISES

1. (a) Write an equation that defines the exponential function with base $b > 0$.
(b) What is the domain of this function?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
 (i) $b > 1$
 (ii) $b = 1$
 (iii) $0 < b < 1$

2. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

4. $y = e^x, \quad y = e^{-x}, \quad y = 8^x, \quad y = 8^{-x}$

5. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x, \quad y = 0.6^x, \quad y = 0.3^x, \quad y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 14 and, if necessary, the transformations of Section 1.3.

7. $y = 4^x - 1$

8. $y = (0.5)^{x-1}$

9. $y = -2^{-x}$

10. $y = e^{|x|}$

11. $y = 1 - \frac{1}{2}e^{-x}$

12. $y = 2(1 - e^x)$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward.
(b) shifting 2 units to the right.
(c) reflecting about the x-axis.
(d) reflecting about the y-axis.
(e) reflecting about the x-axis and then about the y-axis.

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$.
(b) reflecting about the line $x = 2$.

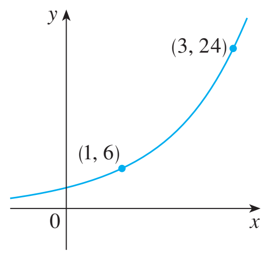
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

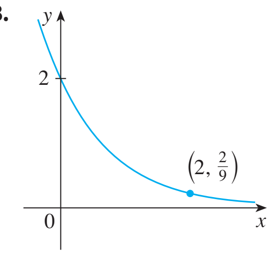
16. (a) $g(t) = \sqrt{10^t - 100}$ (b) $g(t) = \sin(e^t - 1)$

17–18 Find the exponential function $f(x) = Cb^x$ whose graph is given.

17.



18.



19. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

20. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

21. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

22. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

23–30 Find the limit.

23. $\lim_{x \to \infty} (1.001)^x$

24. $\lim_{x \to -\infty} (1.001)^x$

25. $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

26. $\lim_{x \to \infty} e^{-x^2}$

27. $\lim_{x \to 2^+} e^{3/(2-x)}$

28. $\lim_{x \to 2^-} e^{3/(2-x)}$

29. $\lim_{x \to \infty} (e^{-2x} \cos x)$

30. $\lim_{x \to (\pi/2)^+} e^{\tan x}$

31–50 Differentiate the function.

31. $f(x) = e^5$

32. $k(r) = e^r + r^e$

33. $f(x) = (3x^2 - 5x)e^x$

34. $y = \dfrac{e^x}{1 - e^x}$

35. $y = e^{ax^3}$

36. $g(x) = e^{x^2 - x}$

37. $y = e^{\tan \theta}$

38. $V(t) = \dfrac{4 + t}{te^t}$

39. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$

40. $y = x^2 e^{-1/x}$

41. $y = x^2 e^{-3x}$

42. $f(t) = \tan(1 + e^{2t})$

43. $f(t) = e^{at} \sin bt$

44. $f(z) = e^{z/(z-1)}$

45. $F(t) = e^{t \sin 2t}$

46. $y = e^{\sin 2x} + \sin(e^{2x})$

47. $g(u) = e^{\sqrt{\sec u^2}}$

48. $y = \sqrt{1 + xe^{-2x}}$

49. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

50. $f(t) = \sin^2(e^{\sin^2 t})$

51–52 Find an equation of the tangent line to the curve at the given point.

51. $y = e^{2x} \cos \pi x, \quad (0, 1)$

52. $y = \dfrac{e^x}{x}, \quad (1, e)$

53. Find y' if $e^{x/y} = x - y$.

54. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

55. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

56. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

57. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' + 6y' + 8y = 0$?

58. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

59. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

60. Find the thousandth derivative of $f(x) = xe^{-x}$.

61. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

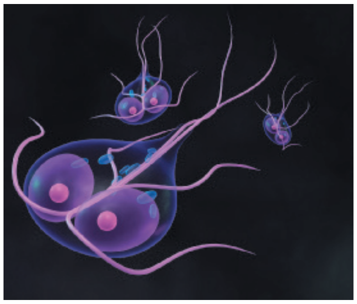
62. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

63. Use the graph of V in Figure 11 to estimate the half-life of the viral load of patient 303 during the first month of treatment.

64. A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia*. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.
(b) Use a graphing calculator to find an exponential curve $f(t) = a \cdot b^t$ that models the bacteria population t hours later.
(c) Graph the model from part (b) together with the scatter plot in part (a). Use the TRACE feature to determine how long it takes for the bacteria count to double.


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G. lamblia

65. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable model for $p(t)$.]
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

66. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

67. Find the absolute maximum value of the function $f(x) = x - e^x$.

68. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

69–70 Find the absolute maximum and absolute minimum values of f on the given interval.

69. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

70. $f(x) = xe^{x/2}, \quad [-3, 1]$

71–72 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

71. $f(x) = (1 - x)e^{-x}$

72. $f(x) = \dfrac{e^x}{x^2}$

73–75 Discuss the curve using the guidelines of Section 3.5.

73. $y = e^{-1/(x+1)}$

74. $y = e^{-x} \sin x, \quad 0 \leqslant x \leqslant 2\pi$

75. $y = 1/(1 + e^{-x})$

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

78. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/mL. What is the maximum concentration of the antibiotic during the first 12 hours?

79. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the average BAC, measured in mg/mL, of a group of eight male subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

80–81 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

80. $f(x) = e^{\cos x}$

81. $f(x) = e^{x^3 - x}$

82. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

83–94 Evaluate the integral.

83. $\displaystyle\int_0^1 (x^e + e^x)\, dx$

84. $\displaystyle\int_{-5}^5 e\, dx$

85. $\displaystyle\int_0^2 \frac{dx}{e^{\pi x}}$

86. $\displaystyle\int x^2 e^{x^3} dx$

87. $\displaystyle\int e^x\sqrt{1 + e^x}\, dx$

88. $\displaystyle\int \frac{(1 + e^x)^2}{e^x}\, dx$

89. $\displaystyle\int (e^x + e^{-x})^2\, dx$

90. $\displaystyle\int e^x(4 + e^x)^5\, dx$

91. $\displaystyle\int \frac{e^u}{(1 - e^u)^2}\, du$

92. $\displaystyle\int e^{\sin\theta}\cos\theta\, d\theta$

93. $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

94. $\displaystyle\int_0^1 \frac{\sqrt{1 + e^{-x}}}{e^x}\, dx$

95. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

96. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

97. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

98. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

99. The **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

is used in probability, statistics, and engineering. Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\text{erf}(b) - \text{erf}(a)]$.

100. Show that the function

$$y = e^{x^2}\text{erf}(x)$$

satisfies the differential equation

$$y' = 2xy + 2/\sqrt{\pi}$$

101. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

102. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

103. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V}C_0e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^{30} u(t)\,dt$ and interpret it.

104. The rate of growth of a fish population was modeled by the equation

$$G(t) = \frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}$$

where t is measured in years and G in kilograms per year. If the biomass (the total mass of the population) was 25,000 kg in the year 2000, what is the predicted biomass for the year 2020?

105. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

106. Evaluate $\displaystyle\lim_{x\to\pi}\frac{e^{\sin x} - 1}{x - \pi}$.

107. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

108. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

109. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

110. (a) Use the inequality of Exercise 109(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 109(b).

111. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x\to\infty}\frac{e^x}{x^k} = \infty$$

for any positive integer k.

6.3 Logarithmic Functions

If $b > 0$ and $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *b*** and is denoted by $\log_b$. If we use the formulation of an inverse function given by (6.1.3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

1

$$\log_b x = y \iff b^y = x$$

Thus, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x.

EXAMPLE 1 Evaluate (a) $\log_3 81$, (b) $\log_{25} 5$, and (c) $\log_{10} 0.001$.

SOLUTION

(a) $\log_3 81 = 4$ because $3^4 = 81$

(b) $\log_{25} 5 = \frac{1}{2}$ because $25^{1/2} = 5$

(c) $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$ ■

The cancellation equations (6.1.4), when applied to the functions $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, become

2
$$\log_b(b^x) = x \quad \text{for every } x \in \mathbb{R}$$
$$b^{\log_b x} = x \quad \text{for every } x > 0$$

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$ and is continuous since it is the inverse of a continuous function, namely, the exponential function. Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.

Figure 1 shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

Figure 2 shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

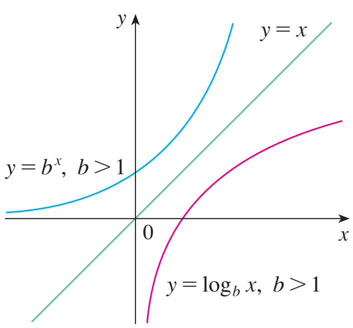


FIGURE 1

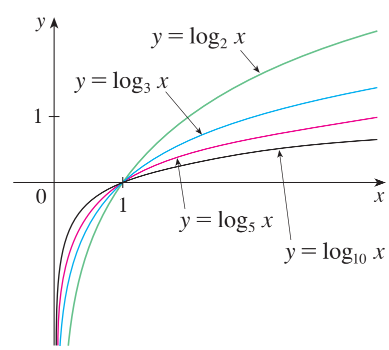


FIGURE 2

The following theorem summarizes the properties of logarithmic functions.

3 Theorem If $b > 1$, the function $f(x) = \log_b x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$ and r is any real number, then

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r \log_b x$

Properties 1, 2, and 3 follow from the corresponding properties of exponential functions given in Theorem 6.2.2.

EXAMPLE 2 Use the properties of logarithms in Theorem 3 to evaluate the following.
(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$

SOLUTION
(a) Using Property 1 in Theorem 3, we have

$$\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3$$

since $4^3 = 64$.
(b) Using Property 2 we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

since $2^4 = 16$. ■

The limits of exponential functions given in Section 6.2 are reflected in the following limits of logarithmic functions. (Compare with Figure 1.)

4 If $b > 1$, then

$$\lim_{x \to \infty} \log_b x = \infty \qquad \text{and} \qquad \lim_{x \to 0^+} \log_b x = -\infty$$

In particular, the y-axis is a vertical asymptote of the curve $y = \log_b x$.

EXAMPLE 3 Find $\lim_{x \to 0} \log_{10}(\tan^2 x)$.

SOLUTION As $x \to 0$, we know that $t = \tan^2 x \to \tan^2 0 = 0$ and the values of t are positive. So by (4) with $b = 10 > 1$, we have

$$\lim_{x \to 0} \log_{10}(\tan^2 x) = \lim_{t \to 0^+} \log_{10} t = -\infty$$ ■

Natural Logarithms

Of all possible bases b for logarithms, we will see in the next section that the most convenient choice of a base is the number e, which was defined in Section 6.2. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

Notation for Logarithms
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

If we put $b = e$ and replace $\log_e$ with "ln" in (1) and (2), then the defining properties of the natural logarithm function become

5 $$\ln x = y \iff e^y = x$$

6 $$\ln(e^x) = x \qquad x \in \mathbb{R}$$
$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

EXAMPLE 4 Find x if $\ln x = 5$.

SOLUTION 1 From (5) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (6) says that $e^{\ln x} = x$. Therefore $x = e^5$. ■

EXAMPLE 5 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (6):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \tfrac{1}{3}(5 - \ln 10) \end{aligned}$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

EXAMPLE 6 Express $\ln a + \frac{1}{2} \ln b$ as a single logarithm.

SOLUTION Using Properties 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \tfrac{1}{2} \ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln \sqrt{b} \\ &= \ln\left(a\sqrt{b}\right) \end{aligned}$$ ■

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

7 Change of Base Formula For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

PROOF Let $y = \log_b x$. Then, from (1), we have $b^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln b = \ln x$. Therefore

$$y = \frac{\ln x}{\ln b}$$ ■

Scientific calculators have a key for natural logarithms, so Formula 7 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 7 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 20–22).

EXAMPLE 7 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 7 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

■

■ Graph and Growth of the Natural Logarithm

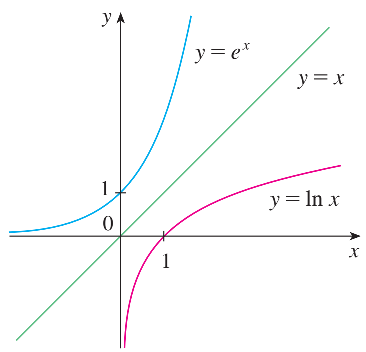


FIGURE 3
The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 3. Because the curve $y = e^x$ crosses the y-axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x-axis with a slope of 1.

In common with all other logarithmic functions with base greater than 1, the natural logarithm is a continuous, increasing function defined on $(0, \infty)$ and the y-axis is a vertical asymptote.

If we put $b = e$ in (4), then we have the following limits:

8
$$\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$$

EXAMPLE 8 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 3. Using the transformations of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 4.)

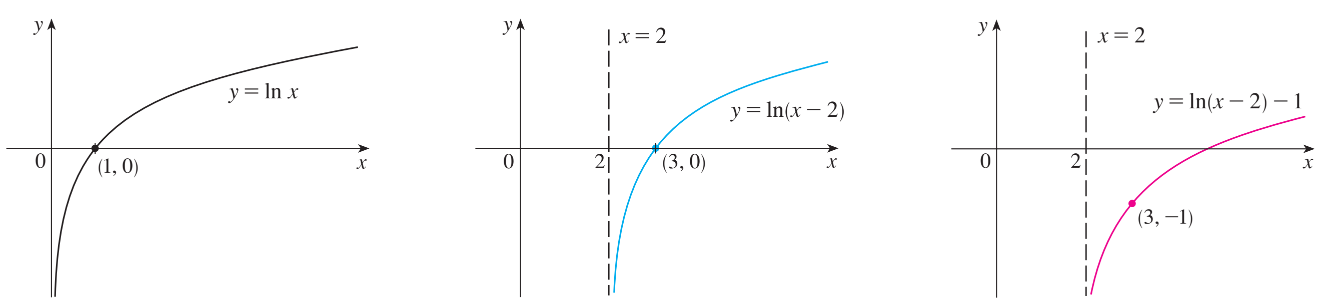


FIGURE 4

Notice that the line $x = 2$ is a vertical asymptote since

$$\lim_{x \to 2^+} [\ln(x - 2) - 1] = -\infty$$

■

We have seen that $\ln x \to \infty$ as $x \to \infty$. But this happens *very* slowly. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 5 and 6 on page 426. You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm. In fact, we will be able to show in Section 6.8 that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any positive power p. So for large x, the values of $\ln x$ are very small compared with x^p. (See Exercise 72.)

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\dfrac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

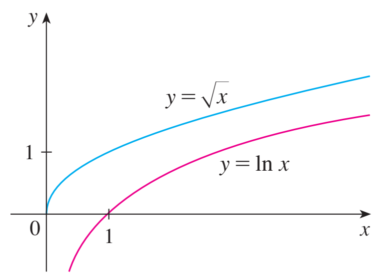


FIGURE 5

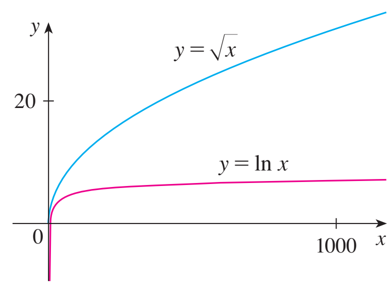


FIGURE 6

6.3 EXERCISES

1. (a) How is the logarithmic function $y = \log_b x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

2. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_2 32$ (b) $\log_8 2$

4. (a) $\log_5 \frac{1}{125}$ (b) $\ln(1/e^2)$

5. (a) $e^{\ln 4.5}$ (b) $\log_{10} 0.0001$

6. (a) $\log_{1.5} 2.25$ (b) $\log_5 4 - \log_5 500$

7. (a) $\log_{10} 40 + \log_{10} 2.5$
(b) $\log_8 60 - \log_8 3 - \log_8 5$

8. (a) $e^{-\ln 2}$ (b) $e^{\ln(\ln e^3)}$

9–12 Use the properties of logarithms to expand the quantity.

9. $\ln \sqrt{ab}$

10. $\log_{10} \sqrt{\dfrac{x-1}{x+1}}$

11. $\ln \dfrac{x^2}{y^3 z^4}$

12. $\ln\left(s^4 \sqrt{t\sqrt{u}}\right)$

13–18 Express the quantity as a single logarithm.

13. $2 \ln x + 3 \ln y - \ln z$

14. $\log_{10} 4 + \log_{10} a - \frac{1}{3} \log_{10}(a + 1)$

15. $\ln 10 + 2 \ln 5$

16. $\ln 3 + \frac{1}{3} \ln 8$

17. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2}[\ln x - \ln(x^2 + 3x + 2)^2]$

18. $\ln b + 2 \ln c - 3 \ln d$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
(a) $\log_5 10$ (b) $\log_3 57$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

21. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

22. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

25–26
(a) What are the domain and range of f?
(b) What is the x-intercept of the graph of f?
(c) Sketch the graph of f.

25. $f(x) = \ln x + 2$ **26.** $f(x) = \ln(x - 1) - 1$

27–36 Solve each equation for x.

27. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

28. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

29. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

30. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

31. $e - e^{-2x} = 1$ **32.** $10(1 + e^{-x})^{-1} = 3$

33. $\ln(\ln x) = 1$ **34.** $e^{e^x} = 10$

35. $e^{2x} - e^x - 6 = 0$ **36.** $\ln(2x + 1) = 2 - \ln x$

37–38 Find the solution of the equation correct to four decimal places.

37. (a) $\ln(1 + x^3) - 4 = 0$ (b) $2e^{1/x} = 42$

38. (a) $2^{1-3x} = 99$ (b) $\ln\left(\dfrac{x+1}{x}\right) = 2$

39–40 Solve each inequality for x.

39. (a) $\ln x < 0$ (b) $e^x > 5$

40. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

41. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

42. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.
(c) At what time is the velocity equal to half the initial velocity?

43. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

44. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

45. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

46. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

47–52 Find the limit.

47. $\lim_{x \to 3^+} \ln(x^2 - 9)$ **48.** $\lim_{x \to 2^-} \log_5(8x - x^4)$

49. $\lim_{x \to 0} \ln(\cos x)$ **50.** $\lim_{x \to 0^+} \ln(\sin x)$

51. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

52. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

53–54 Find the domain of the function.

53. $f(x) = \ln(4 - x^2)$ **54.** $g(x) = \log_2(x^2 + 3x)$

55–57 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$ **56.** $f(x) = \ln(2 + \ln x)$

57. $f(x) = \ln(e^x - 3)$

58. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

59–64 Find the inverse function.

59. $y = 2\ln(x - 1)$

60. $g(x) = \log_4(x^3 + 2)$

61. $f(x) = e^{x^3}$

62. $y = (\ln x)^2, \quad x \geqslant 1$

63. $y = 3^{2x-4}$

64. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

---

65. On what interval is the function $f(x) = e^{3x} - e^x$ increasing?

66. On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

67. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.
(b) Find the inverse function of f.

68. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

69. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?

70. Any function of the form $f(x) = [g(x)]^{h(x)}$, where $g(x) > 0$, can be analyzed as a power of e by writing $g(x) = e^{\ln g(x)}$ so that $f(x) = e^{h(x)\ln g(x)}$. Using this device, calculate each limit.

(a) $\lim\limits_{x \to \infty} x^{\ln x}$

(b) $\lim\limits_{x \to 0^+} x^{-\ln x}$

(c) $\lim\limits_{x \to 0^+} x^{1/x}$

(d) $\lim\limits_{x \to \infty} (\ln 2x)^{-\ln x}$

71. Let $b > 1$. Prove, using Definitions 3.4.6 and 3.4.7, that

(a) $\lim\limits_{x \to -\infty} b^x = 0$

(b) $\lim\limits_{x \to \infty} b^x = \infty$

72. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

73. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

74. A **prime number** is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... We denote by $\pi(n)$ the number of primes that are less than or equal to n. For instance, $\pi(15) = 6$ because there are six primes smaller than 15.
(a) Calculate the numbers $\pi(25)$ and $\pi(100)$.
[*Hint:* To find $\pi(100)$, first compile a list of the primes up to 100 using the *sieve of Eratosthenes:* Write the numbers from 2 to 100 and cross out all multiples of 2. Then cross out all multiples of 3. The next remaining number is 5, so cross out all remaining multiples of it, and so on.]
(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to n is approximately $n/\ln n$ when n is large. More precisely, he conjectured that

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1$$

This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the **Prime Number Theorem**. Provide evidence for the truth of this theorem by computing the ratio of $\pi(n)$ to $n/\ln n$ for $n = 100, 1000, 10^4, 10^5, 10^6$, and 10^7. Use the following data: $\pi(1000) = 168$, $\pi(10^4) = 1229$, $\pi(10^5) = 9592$, $\pi(10^6) = 78{,}498$, $\pi(10^7) = 664{,}579$.
(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.

6.4 Derivatives of Logarithmic Functions

In this section we find the derivatives of the logarithmic functions $y = \log_b x$ and the exponential functions $y = b^x$. We start with the natural logarithmic function $y = \ln x$. We know that it is differentiable because it is the inverse of the differentiable function $y = e^x$.

1
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

PROOF Let $y = \ln x$. Then

$$e^y = x$$

Differentiating this equation implicitly with respect to x, we get

$$e^y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$ ■

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$ ■

In general, if we combine Formula 1 with the Chain Rule as in Example 1, we get

2

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (2), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$ ■

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$ ■

Figure 1 shows the graph of the function f of Example 4 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

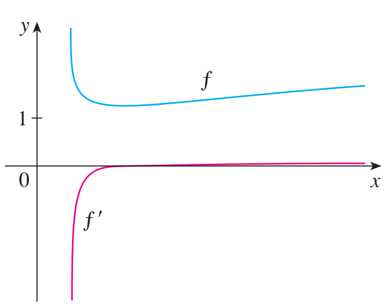


FIGURE 1

EXAMPLE 4 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x - 2 - \frac{1}{2}(x+1)}{(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

SOLUTION 2 If we first simplify the given function using the properties of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x+2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) ■

EXAMPLE 5 Find the absolute minimum value of $f(x) = x^2 \ln x$.

SOLUTION The domain is $(0, \infty)$ and the Product Rule gives

$$f'(x) = x^2 \cdot \frac{1}{x} + 2x \ln x = x(1 + 2\ln x)$$

Therefore $f'(x) = 0$ when $2\ln x = -1$, that is, $\ln x = -\frac{1}{2}$, or $x = e^{-1/2}$. Also, $f'(x) > 0$ when $x > e^{-1/2}$ and $f'(x) < 0$ for $0 < x < e^{-1/2}$. So, by the First Derivative Test for Absolute Extreme Values, $f\left(1/\sqrt{e}\right) = -1/(2e)$ is the absolute minimum. ■

EXAMPLE 6 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 3.5.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

by (6.3.8). Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.

$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 2.

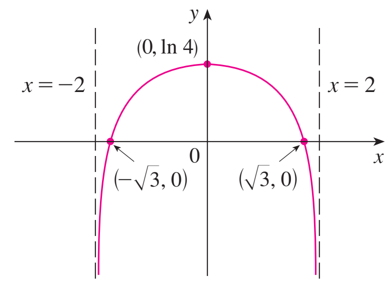


FIGURE 2
$y = \ln(4 - x^2)$

■

Figure 3 shows the graph of the function $f(x) = \ln|x|$ in Example 7 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

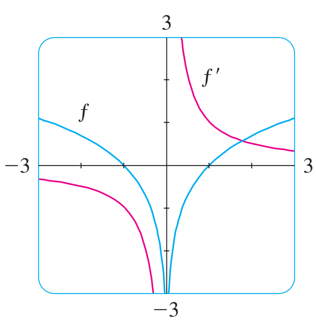


FIGURE 3

EXAMPLE 7 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$. ■

The result of Example 7 is worth remembering:

3
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

The corresponding integration formula is

4
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case $(n = -1)$ is supplied by Formula 4.

EXAMPLE 8 Find, correct to three decimal places, the area of the region under the hyperbola $xy = 1$ from $x = 1$ to $x = 2$.

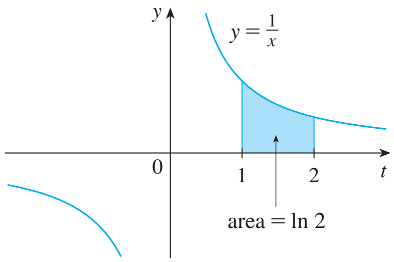


FIGURE 4

SOLUTION The given region is shown in Figure 4. Using Formula 4 (without the absolute value sign, since $x > 0$), we see that the area is

$$A = \int_1^2 \frac{1}{x}\, dx = \ln x \Big]_1^2$$
$$= \ln 2 - \ln 1 = \ln 2 \approx 0.693$$

EXAMPLE 9 Evaluate $\displaystyle\int \frac{x}{x^2 + 1}\, dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\, dx$ occurs (except for the constant factor 2). Thus $x\, dx = \frac{1}{2}\, du$ and

$$\int \frac{x}{x^2 + 1}\, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C$$
$$= \frac{1}{2} \ln |x^2 + 1| + C = \frac{1}{2} \ln(x^2 + 1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the properties of logarithms to write the answer as

$$\ln \sqrt{x^2 + 1} + C$$

but this isn't necessary.

Since the function $f(x) = (\ln x)/x$ in Example 10 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 5.

EXAMPLE 10 Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$

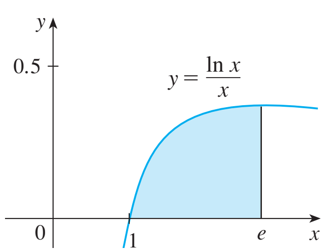


FIGURE 5

EXAMPLE 11 Calculate $\displaystyle\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du$$
$$= -\ln |u| + C = -\ln |\cos x| + C$$

Since $-\ln |\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln |\sec x|$, the result of Example 11 can also be written as

5 $$\int \tan x\, dx = \ln |\sec x| + C$$

General Logarithmic and Exponential Functions

Formula 6.3.7 expresses a logarithmic function with base b in terms of the natural logarithmic function:

$$\log_b x = \frac{\ln x}{\ln b}$$

Since $\ln b$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{x \ln b}$$

6

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

EXAMPLE 12 Using Formula 6 and the Chain Rule, we get

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) = \frac{\cos x}{(2 + \sin x)\ln 10}$$ ■

From Formula 6 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: the differentiation formula is simplest when $b = e$ because $\ln e = 1$.

Exponential Functions with Base b In Section 6.2 we showed that the derivative of the general exponential function $f(x) = b^x$, $b > 0$, is a constant multiple of itself:

$$f'(x) = f'(0)b^x \qquad \text{where} \qquad f'(0) = \lim_{h \to 0}\frac{b^h - 1}{h}$$

We are now in a position to show that the value of the constant is $f'(0) = \ln b$.

7

$$\frac{d}{dx}(b^x) = b^x \ln b$$

PROOF We use the fact that $e^{\ln b} = b$:

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{\ln b})^x = \frac{d}{dx}e^{(\ln b)x} = e^{(\ln b)x}\frac{d}{dx}(\ln b)x$$

$$= (e^{\ln b})^x(\ln b) = b^x \ln b$$ ■

In Example 2.7.6 we considered a population of bacteria cells that doubles every hour and we saw that the population after t hours is $n = n_0 2^t$, where n_0 is the initial population. Formula 7 enables us to find the growth rate:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

EXAMPLE 13 Combining Formula 7 with the Chain Rule, we have

$$\frac{d}{dx}(10^{x^2}) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2\ln 10)x10^{x^2}$$ ■

The integration formula that follows from Formula 7 is

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \qquad b \neq 1$$

EXAMPLE 14

$$\int_0^5 2^x\,dx = \frac{2^x}{\ln 2}\Big]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$$ ■

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 15 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn't used logarithmic differentiation in Example 15, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ ■

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can still use logarithmic differentiation by first writing $|y| = |f(x)|$ and then using Equation 3. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 2.3.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

PROOF Let $y = x^n$ and use logarithmic differentiation:

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$
■

You should distinguish carefully between the Power Rule $[(d/dx)x^n = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(d/dx)b^x = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\dfrac{d}{dx}(b^n) = 0 \qquad$ (b and n are constants)

Variable base, constant exponent

2. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

Constant base, variable exponent

3. $\dfrac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 16 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

Figure 6 illustrates Example 16 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

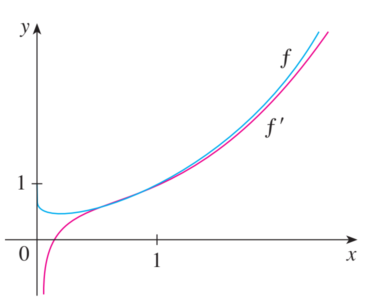


FIGURE 6

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\,\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$
■

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$
$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$
$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

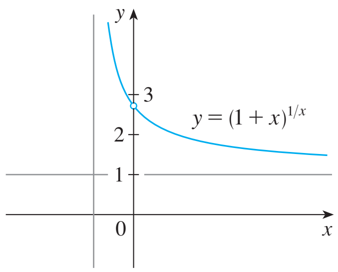


FIGURE 7

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 1.8.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}$$

8
$$e = \lim_{x \to 0} (1+x)^{1/x}$$

Formula 8 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 7 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9
$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

6.4 EXERCISES

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_b x$.

2–26 Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \ln \dfrac{1}{x}$

6. $y = \dfrac{1}{\ln x}$

7. $f(x) = \log_{10}(1 + \cos x)$

8. $f(x) = \log_{10} \sqrt{x}$

9. $g(x) = \ln(xe^{-2x})$

10. $g(t) = \sqrt{1 + \ln t}$

11. $F(t) = (\ln t)^2 \sin t$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $G(y) = \ln \dfrac{(2y+1)^5}{\sqrt{y^2+1}}$

14. $P(v) = \dfrac{\ln v}{1 - v}$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln|1 + t - t^3|$

17. $f(x) = x^5 + 5^x$

18. $g(x) = x \sin(2^x)$

19. $T(z) = 2^z \log_2 z$

20. $y = \ln(\csc x - \cot x)$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

23. $y = \tan[\ln(ax + b)]$

24. $y = \log_2(x \log_5 x)$

25. $G(x) = 4^{C/x}$

26. $F(t) = 3^{\cos 2t}$

27–30 Find y' and y''.

27. $y = \sqrt{x} \ln x$

28. $y = \dfrac{\ln x}{1 + \ln x}$

29. $y = \ln|\sec x|$

30. $y = \ln(1 + \ln x)$

31–34 Differentiate f and find the domain of f.

31. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

32. $f(x) = \sqrt{2 + \ln x}$

33. $f(x) = \ln(x^2 - 2x)$

34. $f(x) = \ln \ln \ln x$

35. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

36. If $f(x) = \cos(\ln x^2)$, find $f'(1)$.

37–38 Find an equation of the tangent line to the curve at the given point.

37. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

38. $y = x^2 \ln x$, $(1, 0)$

39. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

40. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

41. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

42. Let $f(x) = \log_b(3x^2 - 2)$. For what value of b is $f'(1) = 3$?

43–54 Use logarithmic differentiation to find the derivative of the function.

43. $y = (x^2 + 2)^2(x^4 + 4)^4$

44. $y = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}$

45. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

46. $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$

47. $y = x^x$

48. $y = x^{\cos x}$

49. $y = x^{\sin x}$

50. $y = (\sqrt{x})^x$

51. $y = (\cos x)^x$

52. $y = (\sin x)^{\ln x}$

53. $y = (\tan x)^{1/x}$

54. $y = (\ln x)^{\cos x}$

55. Find y' if $y = \ln(x^2 + y^2)$.

56. Find y' if $x^y = y^x$.

57. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

58. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

59–60 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

59. $(x - 4)^2 = \ln x$

60. $\ln(4 - x^2) = x$

61. Find the intervals of concavity and the inflection points of the function $f(x) = (\ln x)/\sqrt{x}$.

62. Find the absolute minimum value of the function $f(x) = x \ln x$.

63–66 Discuss the curve under the guidelines of Section 3.5.

63. $y = \ln(\sin x)$

64. $y = \ln(\tan^2 x)$

65. $y = \ln(1 + x^2)$

66. $y = \ln(1 + x^3)$

CAS **67.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

68. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

69. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.

(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 1.4.2.

70. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

71–82 Evaluate the integral.

71. $\int_2^4 \frac{3}{x}\,dx$

72. $\int_0^3 \frac{dx}{5x+1}$

73. $\int_1^2 \frac{dt}{8-3t}$

74. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

75. $\int_1^e \frac{x^2+x+1}{x}\,dx$

76. $\int \frac{\cos(\ln t)}{t}\,dt$

77. $\int \frac{(\ln x)^2}{x}\,dx$

78. $\int \frac{\cos x}{2+\sin x}\,dx$

79. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

80. $\int \frac{e^x}{e^x+1}\,dx$

81. $\int_0^4 2^s\,ds$

82. $\int x2^{x^2}\,dx$

83. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

84. Sketch the region enclosed by the curves

$$y = \frac{\ln x}{x} \quad \text{and} \quad y = \frac{(\ln x)^2}{x}$$

and find its area.

85. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

86. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

87. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 5.4.29.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

88. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

89. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

90. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

91. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

92. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

93. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

94. Show that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

6.2* The Natural Logarithmic Function

If your instructor has assigned Sections 6.2–6.4 (pp. 408–438), you need not read Sections 6.2*, 6.3*, and 6.4* (pp. 438–465).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

1 Definition The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

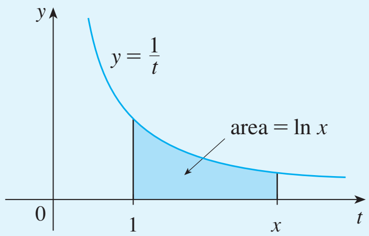


FIGURE 1

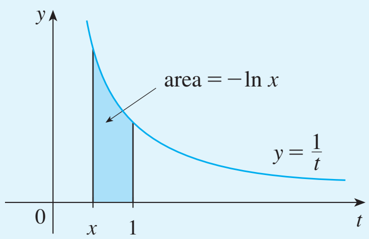


FIGURE 2

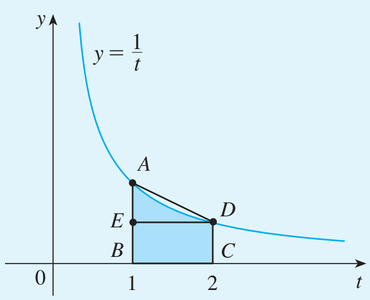


FIGURE 3

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

For $0 < x < 1$,
$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shaded in Figure 2.

EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.
(b) Use the Midpoint Rule with $n = 10$ to estimate the value of $\ln 2$.

SOLUTION

(a) We can interpret $\ln 2$ as the area under the curve $y = 1/t$ from 1 to 2. From Figure 3 we see that this area is larger than the area of rectangle $BCDE$ and smaller than the area of trapezoid $ABCD$. Thus we have

$$\tfrac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right)$$

$$\tfrac{1}{2} < \ln 2 < \tfrac{3}{4}$$

(b) If we use the Midpoint Rule with $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.1$, we get

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx (0.1)[f(1.05) + f(1.15) + \cdots + f(1.95)]$$

$$= (0.1)\left(\frac{1}{1.05} + \frac{1}{1.15} + \cdots + \frac{1}{1.95}\right) \approx 0.693$$ ■

Notice that the integral that defines $\ln x$ is exactly the type of integral discussed in Part 1 of the Fundamental Theorem of Calculus (see Section 4.3). In fact, using that theorem, we have

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

and so

2
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

We now use this differentiation rule to prove the following properties of the logarithm function.

3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r \ln x$

PROOF

1. Let $f(x) = \ln(ax)$, where a is a positive constant. Then, using Equation 2 and the Chain Rule, we have

$$f'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$$

Therefore $f(x)$ and $\ln x$ have the same derivative and so they must differ by a constant:

$$\ln(ax) = \ln x + C$$

Putting $x = 1$ in this equation, we get $\ln a = \ln 1 + C = 0 + C = C$. Thus

$$\ln(ax) = \ln x + \ln a$$

If we now replace the constant a by any number y, we have

$$\ln(xy) = \ln x + \ln y$$

2. Using Law 1 with $x = 1/y$, we have

$$\ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y}\cdot y\right) = \ln 1 = 0$$

and so
$$\ln\frac{1}{y} = -\ln y$$

Using Law 1 again, we have

$$\ln\left(\frac{x}{y}\right) = \ln\left(x\cdot\frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y$$

The proof of Law 3 is left as an exercise. ■

EXAMPLE 2 Expand the expression $\ln\dfrac{(x^2+5)^4\sin x}{x^3+1}$.

SOLUTION Using Laws 1, 2, and 3, we get

$$\begin{aligned}\ln\frac{(x^2+5)^4\sin x}{x^3+1} &= \ln(x^2+5)^4 + \ln\sin x - \ln(x^3+1)\\ &= 4\ln(x^2+5) + \ln\sin x - \ln(x^3+1)\end{aligned}$$ ■

EXAMPLE 3 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\begin{aligned}\ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2}\\ &= \ln a + \ln\sqrt{b}\\ &= \ln\left(a\sqrt{b}\right)\end{aligned}$$ ■

In order to graph $y = \ln x$, we first determine its limits:

4 (a) $\displaystyle\lim_{x\to\infty}\ln x = \infty$ (b) $\displaystyle\lim_{x\to 0^+}\ln x = -\infty$

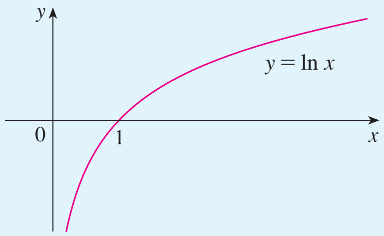


FIGURE 4

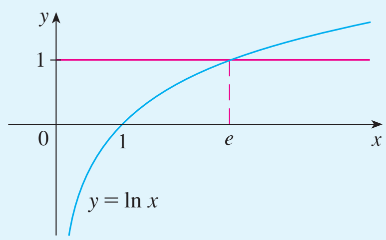


FIGURE 5

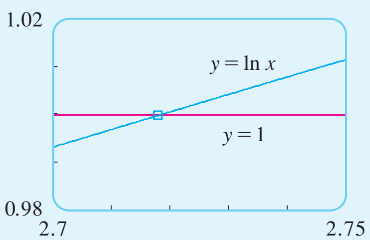


FIGURE 6

PROOF

(a) Using Law 3 with $x = 2$ and $r = n$ (where n is any positive integer), we have $\ln(2^n) = n \ln 2$. Now $\ln 2 > 0$, so this shows that $\ln(2^n) \to \infty$ as $n \to \infty$. But $\ln x$ is an increasing function since its derivative $1/x > 0$. Therefore $\ln x \to \infty$ as $x \to \infty$.

(b) If we let $t = 1/x$, then $t \to \infty$ as $x \to 0^+$. Thus, using (a), we have

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln\left(\frac{1}{t}\right) = \lim_{t \to \infty} (-\ln t) = -\infty$$ ■

If $y = \ln x$, $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} > 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$$

which shows that $\ln x$ is increasing and concave downward on $(0, \infty)$. Putting this information together with (4), we draw the graph of $y = \ln x$ in Figure 4.

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where $\ln x$ takes on the value 1. (See Figure 5.) This important number is denoted by e.

5 Definition e is the number such that $\ln e = 1$.

EXAMPLE 4 Use a graphing calculator or computer to estimate the value of e.

SOLUTION According to Definition 5, we estimate the value of e by graphing the curves $y = \ln x$ and $y = 1$ and determining the x-coordinate of the point of intersection. By zooming in repeatedly, as in Figure 6, we find that

$$e \approx 2.718$$ ■

With more sophisticated methods, it can be shown that the approximate value of e, to 20 decimal places, is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

Now let's use Formula 2 to differentiate functions that involve the natural logarithmic function.

EXAMPLE 5 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$ ■

In general, if we combine Formula 2 with the Chain Rule as in Example 5, we get

6 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 6 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (6), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x \qquad \blacksquare$$

EXAMPLE 7 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} = \frac{1}{2x\sqrt{\ln x}} \qquad \blacksquare$$

EXAMPLE 8 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

Figure 7 shows the graph of the function f of Example 8 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

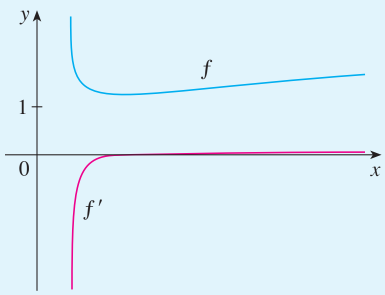


FIGURE 7

SOLUTION 2 If we first simplify the given function using the Laws of Logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x+2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) ■

EXAMPLE 9 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 3.5.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x\to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x\to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E. $$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G. $$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 8. ■

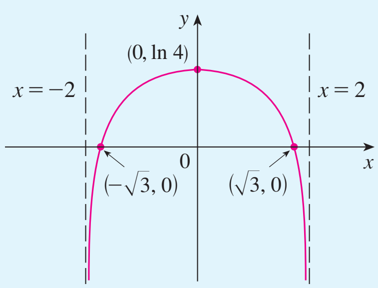


FIGURE 8
$y = \ln(4 - x^2)$

EXAMPLE 10 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$. ■

Figure 9 shows the graph of the function $f(x) = \ln|x|$ in Example 10 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

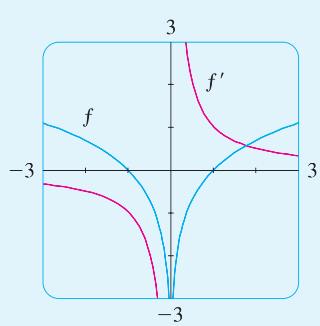


FIGURE 9

The result of Example 10 is worth remembering:

7 $$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

The corresponding integration formula is

8 $$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case ($n = -1$) is supplied by Formula 8.

EXAMPLE 11 Evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\,dx$ occurs (except for the constant factor 2). Thus $x\,dx = \frac{1}{2}\,du$ and

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the Laws of Logarithms to write the answer as

$$\ln\sqrt{x^2+1} + C$$

but this isn't necessary. ■

Since the function $f(x) = (\ln x)/x$ in Example 12 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 10.

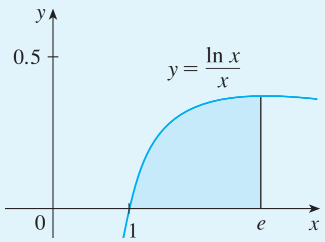


FIGURE 10

EXAMPLE 12 Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\bigg]_0^1 = \frac{1}{2}$$ ■

EXAMPLE 13 Calculate $\displaystyle\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$ ■

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 13 can also be written as

9 $$\int \tan x\,dx = \ln|\sec x| + C$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 14 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn't used logarithmic differentiation in Example 14, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

■

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can still use logarithmic differentiation by first writing $|y| = |f(x)|$ and then using Equation 7.

6.2* EXERCISES

1–4 Use the Laws of Logarithms to expand the quantity.

1. $\ln\sqrt{ab}$

2. $\ln\sqrt[3]{\dfrac{x-1}{x+1}}$

3. $\ln\dfrac{x^2}{y^3z^4}$

4. $\ln\left(s^4\sqrt{t\sqrt{u}}\right)$

5–10 Express the quantity as a single logarithm.

5. $2\ln x + 3\ln y - \ln z$

6. $\log_{10}4 + \log_{10}a - \frac{1}{3}\log_{10}(a+1)$

7. $\ln 10 + 2\ln 5$

8. $\ln 3 + \frac{1}{3}\ln 8$

9. $\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$

10. $\ln b + 2\ln c - 3\ln d$

11–14 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 4 and, if necessary, the transformations of Section 1.3.

11. $y = -\ln x$

12. $y = \ln|x|$

13. $y = \ln(x+3)$

14. $y = 1 + \ln(x-2)$

15–16 Find the limit.

15. $\lim_{x \to 3^+} \ln(x^2 - 9)$

16. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

17–36 Differentiate the function.

17. $f(x) = x^3 \ln x$

18. $f(x) = x \ln x - x$

19. $f(x) = \sin(\ln x)$

20. $f(x) = \ln(\sin^2 x)$

21. $f(x) = \ln \dfrac{1}{x}$

22. $y = \dfrac{1}{\ln x}$

23. $f(x) = \sin x \ln(5x)$

24. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

25. $g(x) = \ln \dfrac{a - x}{a + x}$

26. $g(t) = \sqrt{1 + \ln t}$

27. $G(y) = \ln \dfrac{(2y + 1)^5}{\sqrt{y^2 + 1}}$

28. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

29. $F(t) = (\ln t)^2 \sin t$

30. $P(v) = \dfrac{\ln v}{1 - v}$

31. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

32. $y = (\ln \tan x)^2$

33. $y = \ln |2 - x - 5x^2|$

34. $y = \ln \tan^2 x$

35. $y = \tan[\ln(ax + b)]$

36. $y = \ln(\csc x - \cot x)$

37–38 Find y' and y''.

37. $y = \sqrt{x} \ln x$

38. $y = \ln(1 + \ln x)$

39–42 Differentiate f and find the domain of f.

39. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

40. $f(x) = \ln(x^2 - 2x)$

41. $f(x) = \sqrt{1 - \ln x}$

42. $f(x) = \ln \ln \ln x$

43. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

44. If $f(x) = \dfrac{\ln x}{x}$, find $f''(e)$.

45–46 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

45. $f(x) = \sin x + \ln x$

46. $f(x) = \ln(x^2 + x + 1)$

47–48 Find an equation of the tangent line to the curve at the given point.

47. $y = \sin(2 \ln x)$, $(1, 0)$

48. $y = \ln(x^3 - 7)$, $(2, 0)$

49. Find y' if $y = \ln(x^2 + y^2)$.

50. Find y' if $\ln xy = y \sin x$.

51. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

52. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

53–54 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

53. $(x - 4)^2 = \ln x$

54. $\ln(4 - x^2) = x$

55–58 Discuss the curve under the guidelines of Section 3.5.

55. $y = \ln(\sin x)$

56. $y = \ln(\tan^2 x)$

57. $y = \ln(1 + x^2)$

58. $y = \ln(1 + x^3)$

CAS **59.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

60. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

61–64 Use logarithmic differentiation to find the derivative of the function.

61. $y = (x^2 + 2)^2(x^4 + 4)^4$

62. $y = \dfrac{(x + 1)^4(x - 5)^3}{(x - 3)^8}$

63. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

64. $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}}$

65–74 Evaluate the integral.

65. $\displaystyle\int_2^4 \frac{3}{x}\, dx$

66. $\displaystyle\int_0^3 \frac{dx}{5x + 1}$

67. $\displaystyle\int_1^2 \frac{dt}{8 - 3t}$

68. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

69. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\, dx$

70. $\displaystyle\int_e^6 \frac{dx}{x \ln x}$

71. $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

72. $\displaystyle\int \frac{\cos x}{2 + \sin x}\, dx$

73. $\displaystyle\int \frac{\sin 2x}{1 + \cos^2 x}\, dx$

74. $\displaystyle\int \frac{\cos(\ln t)}{t}\, dt$

75. Show that $\int \cot x\, dx = \ln |\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 13.

76. Sketch the region enclosed by the curves

$$y = \frac{\ln x}{x} \qquad \text{and} \qquad y = \frac{(\ln x)^2}{x}$$

and find its area.

77. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

78. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

79. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 5.4.29.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

80. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

81. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

82. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

83. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate ln 1.5.

84. Refer to Example 1.
(a) Find an equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.
(b) Use part (a) to show that $\ln 2 > 0.66$.

85. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

86. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

87. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

88. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad \frac{\ln x}{x^{0.1}} < 0.1$$

89. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

6.3* The Natural Exponential Function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

1
$$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

2
$$\exp(\ln x) = x \qquad \text{and} \qquad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$

$$\exp(1) = e \quad \text{since} \quad \ln e = 1$$

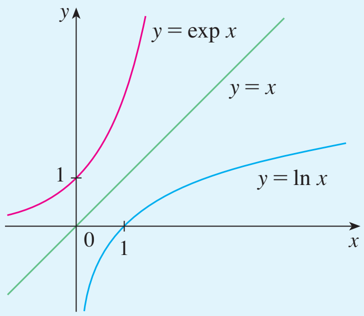


FIGURE 1

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ about the line $y = x$. (See Figure 1.) The domain of exp is the range of ln, that is, $(-\infty, \infty)$; the range of exp is the domain of ln, that is, $(0, \infty)$.

If r is any rational number, then the third law of logarithms gives

$$\ln(e^r) = r \ln e = r$$

Therefore, by (1), $$\exp(r) = e^r$$

Thus $\exp(x) = e^x$ whenever x is a rational number. This leads us to define e^x, even for irrational values of x, by the equation

$$e^x = \exp(x)$$

In other words, for the reasons given, we define e^x to be the inverse of the function $\ln x$. In this notation (1) becomes

3 $$e^x = y \iff \ln y = x$$

and the cancellation equations (2) become

4 $$e^{\ln x} = x \qquad x > 0$$

5 $$\ln(e^x) = x \qquad \text{for all } x$$

EXAMPLE 1 Find x if $\ln x = 5$.

SOLUTION 1 From (3) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But (4) says that $e^{\ln x} = x$. Therefore $x = e^5$. ■

EXAMPLE 2 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (5):

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

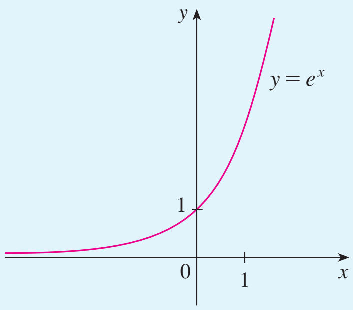


FIGURE 2
The natural exponential function

The exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 2) and its properties (which follow from the fact that it is the inverse of the natural logarithmic function).

6 Properties of the Natural Exponential Function The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

EXAMPLE 3 Find $\displaystyle\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1}$.

SOLUTION We divide numerator and denominator by e^{2x}:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \displaystyle\lim_{x \to \infty} e^{-2x}}$$

$$= \frac{1}{1 + 0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0$$ ■

We now verify that $f(x) = e^x$ has the other properties expected of an exponential function.

7 Laws of Exponents If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \dfrac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

PROOF OF LAW 1 Using the first law of logarithms and Equation 5, we have

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y})$$

Since ln is a one-to-one function, it follows that $e^x e^y = e^{x+y}$.

Laws 2 and 3 are proved similarly (see Exercises 107 and 108). As we will see in the next section, Law 3 actually holds when r is any real number. ■

Differentiation

The natural exponential function has the remarkable property that *it is its own derivative.*

TEC Visual 6.2/6.3* uses the slope-a-scope to illustrate this formula.

8
$$\frac{d}{dx}(e^x) = e^x$$

PROOF The function $y = e^x$ is differentiable because it is the inverse function of $y = \ln x$, which we know is differentiable with nonzero derivative. To find its derivative, we use the inverse function method. Let $y = e^x$. Then $\ln y = x$ and, differentiating this latter equation implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x$$ ■

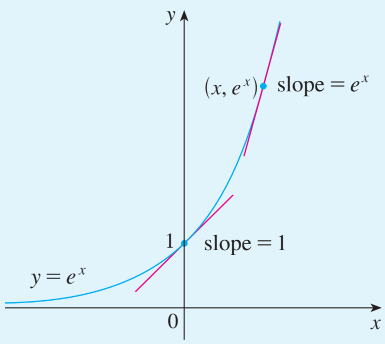


FIGURE 3

The geometric interpretation of Formula 8 is that the slope of a tangent line to the curve $y = e^x$ at any point is equal to the y-coordinate of the point (see Figure 3). This property implies that the exponential curve $y = e^x$ grows very rapidly (see Exercise 112).

EXAMPLE 4 Differentiate the function $y = e^{\tan x}$.

SOLUTION To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx} = e^{\tan x}\sec^2 x$$ ■

In general, if we combine Formula 8 with the Chain Rule, as in Example 4, we get

9
$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

EXAMPLE 5 Find y' if $y = e^{-4x}\sin 5x$.

SOLUTION Using Formula 9 and the Product Rule, we have

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5\cos 5x - 4\sin 5x)$$ ■

EXAMPLE 6 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

SOLUTION We differentiate to find any critical numbers:

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, when $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37$$ ■

EXAMPLE 7 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote (both to the left and right).

Now let's compute the derivative. The Chain Rule gives

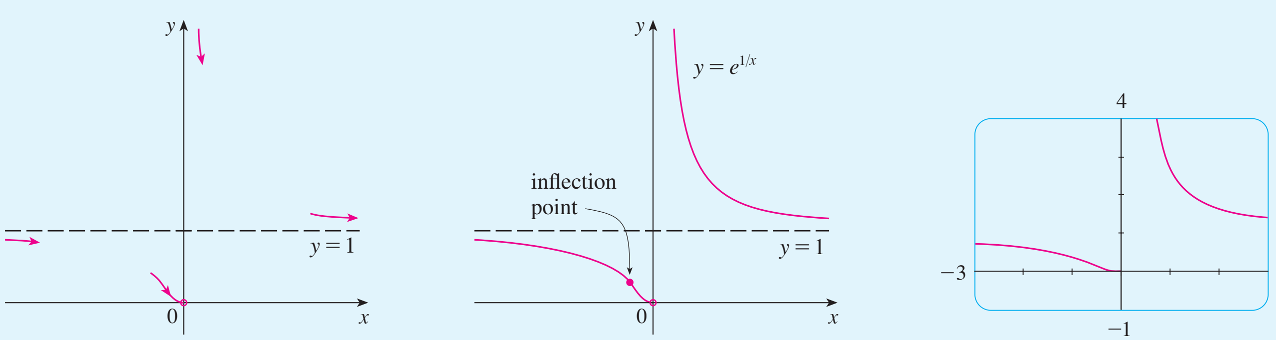
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no local maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 4(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 4(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 4(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

FIGURE 4 ■

Integration

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

10
$$\int e^x\,dx = e^x + C$$

EXAMPLE 8 Evaluate $\int x^2 e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \tfrac{1}{3}\int e^u\,du = \tfrac{1}{3}e^u + C = \tfrac{1}{3}e^{x^3} + C$$ ■

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x}\Big]_0^1 = \tfrac{1}{3}(1 - e^{-3})$$ ■

6.3* EXERCISES

1. Sketch, by hand, the graph of the function $f(x) = e^x$ with particular attention to how the graph crosses the y-axis. What fact allows you to do this?

2–4 Simplify each expression.

2. (a) $e^{\ln 15}$ (b) $\ln(1/e^2)$

3. (a) $e^{-\ln 2}$ (b) $e^{\ln(\ln e^3)}$

4. (a) $\ln e^{\sin x}$ (b) $e^{x + \ln x}$

5–12 Solve each equation for x.

5. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

6. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

7. (a) $e^{3x+1} = k$ (b) $\ln x + \ln(x - 1) = 1$

8. (a) $\ln(\ln x) = 1$ (b) $e^{e^x} = 10$

9. $e - e^{-2x} = 1$

10. $10(1 + e^{-x})^{-1} = 3$

11. $e^{2x} - e^x - 6 = 0$

12. $\ln(2x + 1) = 2 - \ln x$

13–14 Find the solution of the equation correct to four decimal places.

13. (a) $\ln(1 + x^3) - 4 = 0$ (b) $2e^{1/x} = 42$

14. (a) $\ln\left(\dfrac{x + 1}{x}\right) = 2$ (b) $e^{1/(x-4)} = 7$

15–16 Solve each inequality for x.

15. (a) $\ln x < 0$ (b) $e^x > 5$

16. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

17–20 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graph given in Figure 2 and, if necessary, the transformations of Section 1.3.

17. $y = e^{-x}$

18. $y = e^{|x|}$

19. $y = 1 - \frac{1}{2}e^{-x}$

20. $y = 2(1 - e^x)$

21–22 Find (a) the domain of f and (b) f^{-1} and its domain.

21. $f(x) = \sqrt{3 - e^{2x}}$

22. $f(x) = \ln(2 + \ln x)$

23–26 Find the inverse function.

23. $y = 2\ln(x - 1)$

24. $y = (\ln x)^2, \quad x \geqslant 1$

25. $f(x) = e^{x^3}$

26. $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$

27–32 Find the limit.

27. $\displaystyle\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

28. $\displaystyle\lim_{x\to\infty} e^{-x^2}$

29. $\lim_{x \to 2^+} e^{3/(2-x)}$ **30.** $\lim_{x \to 2^-} e^{3/(2-x)}$

31. $\lim_{x \to \infty} (e^{-2x} \cos x)$ **32.** $\lim_{x \to (\pi/2)^+} e^{\tan x}$

33–52 Differentiate the function.

33. $f(x) = e^5$ **34.** $k(r) = e^r + r^e$

35. $f(x) = (3x^2 - 5x)e^x$ **36.** $y = \dfrac{e^x}{1 - e^x}$

37. $y = e^{ax^3}$ **38.** $g(x) = e^{x^2 - x}$

39. $y = e^{\tan \theta}$ **40.** $V(t) = \dfrac{4 + t}{te^t}$

41. $f(x) = \dfrac{x^2 e^x}{x^2 + e^x}$ **42.** $y = x^2 e^{-1/x}$

43. $y = x^2 e^{-3x}$ **44.** $f(t) = \tan(1 + e^{2t})$

45. $f(t) = e^{at} \sin bt$ **46.** $f(z) = e^{z/(z-1)}$

47. $F(t) = e^{t \sin 2t}$ **48.** $y = e^{\sin 2x} + \sin(e^{2x})$

49. $g(u) = e^{\sqrt{\sec u^2}}$ **50.** $y = \sqrt{1 + xe^{-2x}}$

51. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$ **52.** $f(t) = \sin^2(e^{\sin^2 t})$

53–54 Find an equation of the tangent line to the curve at the given point.

53. $y = e^{2x} \cos \pi x$, $(0, 1)$ **54.** $y = \dfrac{e^x}{x}$, $(1, e)$

55. Find y' if $e^{x/y} = x - y$.

56. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

57. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

58. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

59. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' + 6y' + 8y = 0$?

60. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

61. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

62. Find the thousandth derivative of $f(x) = xe^{-x}$.

63. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

64. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

65. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable equation for $p(t)$.]
(a) Find $\lim_{t \to \infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

66. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

67. Find the absolute maximum value of the function $f(x) = x - e^x$.

68. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

69–70 Find the absolute maximum and absolute minimum values of f on the given interval.

69. $f(x) = xe^{-x^2/8}$, $[-1, 4]$

70. $f(x) = xe^{x/2}$, $[-3, 1]$

71–72 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

71. $f(x) = (1 - x)e^{-x}$ **72.** $f(x) = \dfrac{e^x}{x^2}$

73–75 Discuss the curve using the guidelines of Section 3.5.

73. $y = e^{-1/(x+1)}$ **74.** $y = e^{2x} - e^x$

75. $y = 1/(1 + e^{-x})$

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^pe^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

78. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in $\mu g/mL$. What is the maximum concentration of the antibiotic during the first 12 hours?

79. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the average BAC, measured in mg/mL, of a group of eight male subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

80–81 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

80. $f(x) = e^{\cos x}$

81. $f(x) = e^{x^3-x}$

82. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

83–94 Evaluate the integral.

83. $\int_0^1 (x^e + e^x)\,dx$

84. $\int_{-5}^5 e\,dx$

85. $\int_0^2 \frac{dx}{e^{\pi x}}$

86. $\int x^2e^{x^3}\,dx$

87. $\int e^x\sqrt{1+e^x}\,dx$

88. $\int \frac{(1+e^x)^2}{e^x}\,dx$

89. $\int (e^x + e^{-x})^2\,dx$

90. $\int e^x(4+e^x)^5\,dx$

91. $\int \frac{e^u}{(1-e^u)^2}\,du$

92. $\int e^{\sin\theta}\cos\theta\,d\theta$

93. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

94. $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx$

95. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

96. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

97. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

98. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

99. The **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering. Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

100. Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

101. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

102. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

103. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^{30} u(t)\,dt$ and interpret it.

104. The rate of growth of a fish population was modeled by the equation

$$G(t) = \frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}$$

where t is measured in years and G in kilograms per year. If the biomass was 25,000 kg in the year 2000, what is the predicted biomass for the year 2020?

105. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

106. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

107. Prove the second law of exponents [see (7)].

108. Prove the third law of exponents [see (7)].

109. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

110. (a) Use the inequality of Exercise 109(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 109(b).

111. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

112. This exercise illustrates Exercise 111(c) for the case $k = 10$.
(a) Compare the rates of growth of $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
(b) Find a viewing rectangle that shows how the function $h(x) = e^x/x^{10}$ behaves for large x.
(c) Find a number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad \frac{e^x}{x^{10}} > 10^{10}$$

6.4* General Logarithmic and Exponential Functions

In this section we use the natural exponential and logarithmic functions to study exponential and logarithmic functions with base $b > 0$.

General Exponential Functions

If $b > 0$ and r is any rational number, then by (4) and (7) in Section 6.3*,

$$b^r = (e^{\ln b})^r = e^{r \ln b}$$

Therefore, even for irrational numbers x, we *define*

1
$$b^x = e^{x \ln b}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = b^x$ is called the **exponential function with base b**. Notice that b^x is positive for all x because e^x is positive for all x.

Definition 1 allows us to extend one of the laws of logarithms. We already know that $\ln(b^r) = r \ln b$ when r is rational. But if we now let r be any real number we have, from Definition 1,

$$\ln b^r = \ln(e^{r \ln b}) = r \ln b$$

Thus

2 $$\ln b^r = r \ln b \qquad \text{for any real number } r$$

The general laws of exponents follow from Definition 1 together with the laws of exponents for e^x.

3 Laws of Exponents If x and y are real numbers and $a, b > 0$, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

PROOF

1. Using Definition 1 and the laws of exponents for e^x, we have

$$b^{x+y} = e^{(x+y)\ln b} = e^{x \ln b + y \ln b}$$
$$= e^{x \ln b} e^{y \ln b} = b^x b^y$$

3. Using Equation 2 we obtain

$$(b^x)^y = e^{y \ln(b^x)} = e^{yx \ln b} = e^{xy \ln b} = b^{xy}$$

The remaining proofs are left as exercises. ■

The differentiation formula for exponential functions is also a consequence of Definition 1:

4 $$\frac{d}{dx}(b^x) = b^x \ln b$$

PROOF

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x \ln b}) = e^{x \ln b} \frac{d}{dx}(x \ln b) = b^x \ln b$$ ■

Notice that if $b = e$, then $\ln e = 1$ and Formula 4 simplifies to a formula that we already know: $(d/dx)\, e^x = e^x$. In fact, the reason that the natural exponential function is used more often than other exponential functions is that its differentiation formula is simpler.

EXAMPLE 1 In Example 2.7.6 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after t hours is

$$n = n_0 2^t$$

where n_0 is the initial population. Now we can use (4) to compute the growth rate:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

For instance, if the initial population is $n_0 = 1000$ cells, then the growth rate after two hours is

$$\left.\frac{dn}{dt}\right|_{t=2} = (1000)2^t \ln 2\,\big|_{t=2}$$

$$= 4000 \ln 2 \approx 2773 \text{ cells/h}$$ ■

EXAMPLE 2 Combining Formula 4 with the Chain Rule, we have

$$\frac{d}{dx}\left(10^{x^2}\right) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2 \ln 10)x10^{x^2}$$ ■

Exponential Graphs

If $b > 1$, then $\ln b > 0$, so $(d/dx)\, b^x = b^x \ln b > 0$, which shows that $y = b^x$ is increasing (see Figure 1). If $0 < b < 1$, then $\ln b < 0$ and so $y = b^x$ is decreasing (see Figure 2).

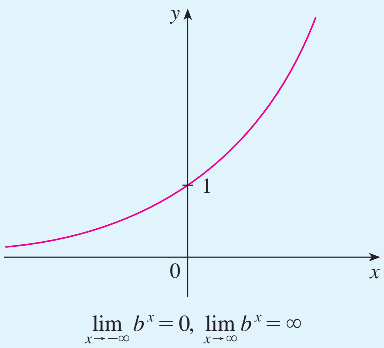


FIGURE 1 $y = b^x$, $b > 1$

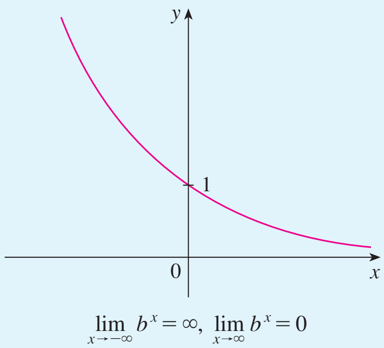


FIGURE 2 $y = b^x$, $0 < b < 1$

Notice from Figure 3 that as the base b gets larger, the exponential function grows more rapidly (for $x > 0$).

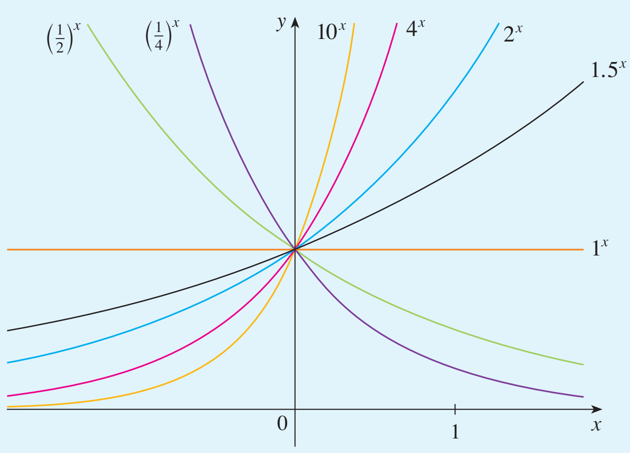


FIGURE 3

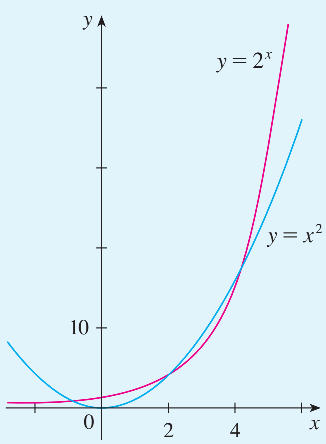


FIGURE 4

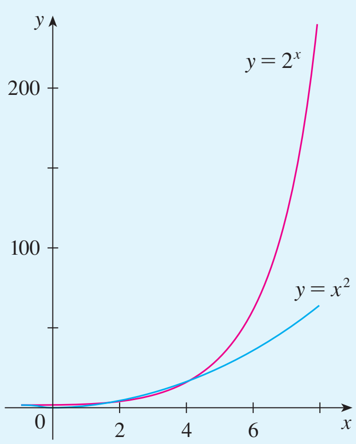


FIGURE 5

Figure 4 (on page 457) shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

In Section 6.5 we will show how exponential functions occur in the description of population growth and radioactive decay. Let's look at human population growth. Table 1 shows data for the population of the world in the 20th century, and Figure 6 shows the corresponding scatter plot.

Table 1

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

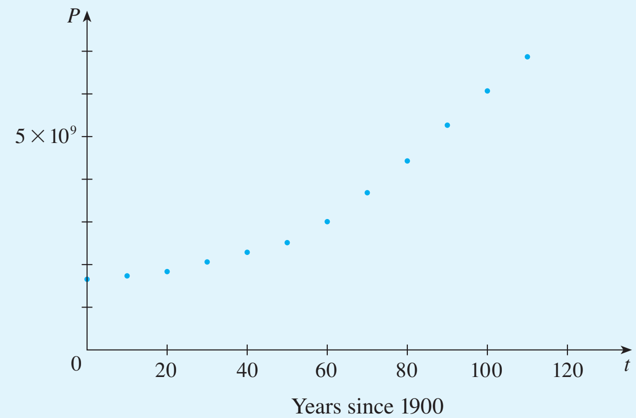


FIGURE 6
Scatter plot for world population growth

The pattern of the data points in Figure 6 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to the year 1900. Figure 7 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

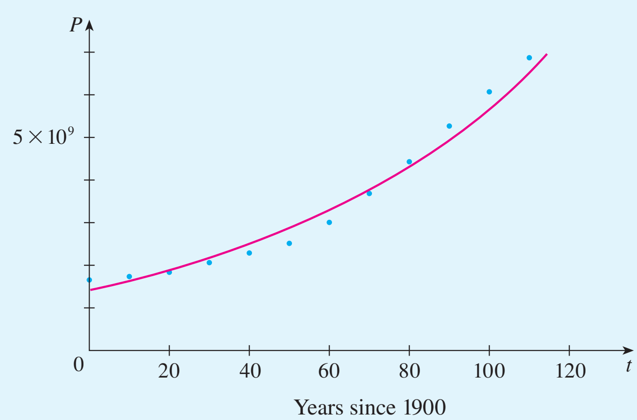


FIGURE 7
Exponential model for population growth

In 1995 a paper appeared detailing the effect of the protease inhibitor ABT-538 on the human immunodeficiency virus HIV-1.[1] Table 2 shows values of the plasma viral load

1. D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

$V(t)$ of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 8.

Table 2

t (days)	$V(t)$
1	76.0
4	53.0
8	18.0
11	9.4
15	5.2
22	3.6

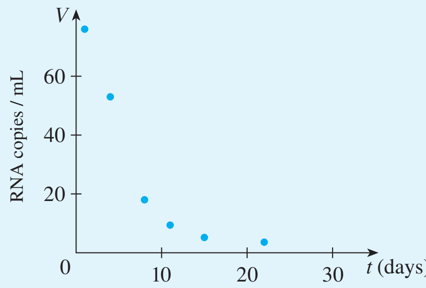


FIGURE 8 Plasma viral load in patient 303

The rather dramatic decline of the viral load that we see in Figure 8 reminds us of the graphs of the exponential function $y = b^x$ in Figures 2 and 3 for the case where the base b is less than 1. So let's model the function $V(t)$ by an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form $y = a \cdot b^t$, we obtain the model

$$V = 96.39785 \cdot (0.818656)^t$$

In Figure 9 we graph this exponential function with the data points and see that the model represents the viral load reasonably well for the first month of treatment.

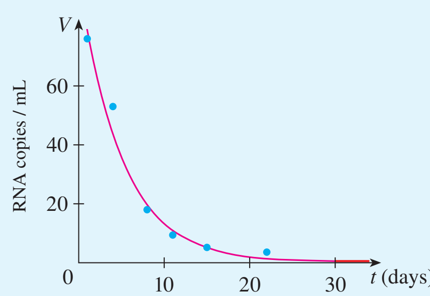


FIGURE 9
Exponential model for viral load

We could use the graph in Figure 9 to estimate the *half-life* of V, that is, the time required for the viral load to be reduced to half its initial value (see Exercise 63).

Exponential Integrals

The integration formula that follows from Formula 4 is

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \qquad b \neq 1$$

EXAMPLE 3 $\displaystyle\int_0^5 2^x\,dx = \frac{2^x}{\ln 2}\bigg]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$ ■

The Power Rule versus the Exponential Rule

Now that we have defined arbitrary powers of numbers, we are in a position to prove the general version of the Power Rule, as promised in Section 2.3.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$ ■

You should distinguish carefully between the Power Rule $[(d/dx)\,x^n = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(d/dx)\,b^x = b^x \ln b]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\dfrac{d}{dx}(b^n) = 0$ (b and n are constants)

Variable base, constant exponent

2. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

Constant base, variable exponent

3. $\dfrac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 4 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

Figure 10 illustrates Example 4 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

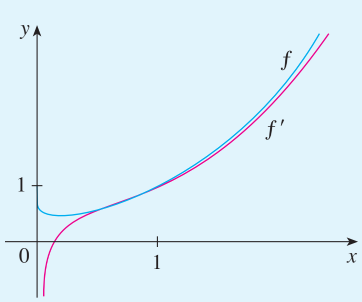


FIGURE 10

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$ ■

General Logarithmic Functions

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a one-to-one function. Its inverse function is called the **logarithmic function with base *b*** and is denoted by $\log_b$. Thus

5
$$\log_b x = y \iff b^y = x$$

In particular, we see that

$$\log_e x = \ln x$$

The cancellation equations for the inverse functions $\log_b x$ and b^x are

$$b^{\log_b x} = x \qquad \text{and} \qquad \log_b(b^x) = x$$

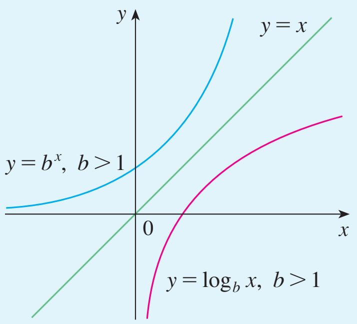


FIGURE 11

Figure 11 shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.

Figure 12 shows the graphs of $y = \log_b x$ with various values of the base $b > 1$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

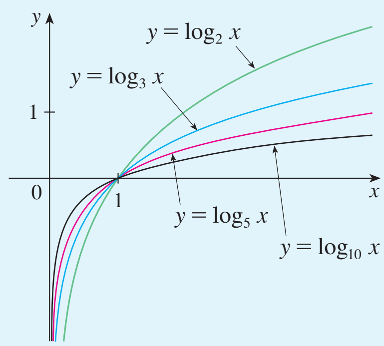


FIGURE 12

The Laws of Logarithms are similar to those for the natural logarithm and can be deduced from the Laws of Exponents (see Exercise 69).

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

6 Change of Base Formula For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

PROOF Let $y = \log_b x$. Then, from (5), we have $b^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln b = \ln x$. Therefore

$$y = \frac{\ln x}{\ln b}$$ ■

Scientific calculators have a key for natural logarithms, so Formula 6 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 6 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 14–16).

Notation for Logarithms
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

EXAMPLE 5 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 6 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

■

Formula 6 enables us to differentiate any logarithmic function. Since $\ln b$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln x) = \frac{1}{x \ln b}$$

7
$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

EXAMPLE 6 Using Formula 7 and the Chain Rule, we get

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x)\ln 10}$$

■

From Formula 7 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: the differentiation formula is simplest when $b = e$ because $\ln e = 1$.

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0}\frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0}\frac{f(1 + x) - f(1)}{x}$$

$$= \lim_{x \to 0}\frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0}\frac{1}{x}\ln(1 + x)$$

$$= \lim_{x \to 0}\ln(1 + x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0}\ln(1 + x)^{1/x} = 1$$

Then, by Theorem 1.8.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0}\ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0}(1 + x)^{1/x}$$

8
$$e = \lim_{x \to 0}(1 + x)^{1/x}$$

x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Formula 8 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 13 and a table of values for small values of x.

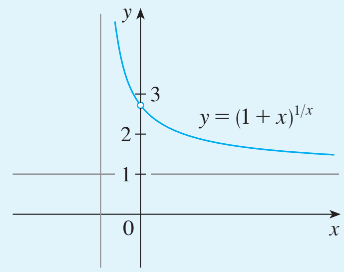


FIGURE 13

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

$$\boxed{9} \qquad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

6.4* EXERCISES

1. (a) Write an equation that defines b^x when b is a positive number and x is a real number.
(b) What is the domain of the function $f(x) = b^x$?
(c) If $b \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $b > 1$ (ii) $b = 1$ (iii) $0 < b < 1$

2. (a) If b is a positive number and $b \neq 1$, how is $\log_b x$ defined?
(b) What is the domain of the function $f(x) = \log_b x$?
(c) What is the range of this function?
(d) If $b > 1$, sketch the general shapes of the graphs of $y = \log_b x$ and $y = b^x$ with a common set of axes.

3–6 Write the expression as a power of e.

3. $4^{-\pi}$

4. $x^{\sqrt{5}}$

5. 10^{x^2}

6. $(\tan x)^{\sec x}$

7–10 Evaluate the expression.

7. (a) $\log_2 32$ (b) $\log_8 2$

8. (a) $\log_{10} \sqrt{10}$ (b) $\log_8 320 - \log_8 5$

9. (a) $\log_{10} 40 + \log_{10} 2.5$
(b) $\log_8 60 - \log_8 3 - \log_8 5$

10. (a) $\log_a \dfrac{1}{a}$ (b) $10^{(\log_{10} 4 + \log_{10} 7)}$

11–12 Graph the given functions on a common screen. How are these graphs related?

11. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

12. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

13. Use Formula 6 to evaluate each logarithm correct to six decimal places.
(a) $\log_5 10$ (b) $\log_3 57$ (c) $\log_2 \pi$

14–16 Use Formula 6 to graph the given functions on a common screen. How are these graphs related?

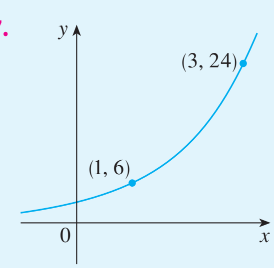
14. $y = \log_2 x, \quad y = \log_4 x, \quad y = \log_6 x, \quad y = \log_8 x$

15. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

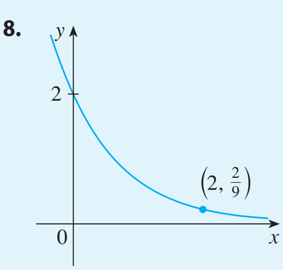
16. $y = \ln x, \quad y = \log_{10} x, \quad y = e^x, \quad y = 10^x$

17–18 Find the exponential function $f(x) = Cb^x$ whose graph is given.

17.



18.



19. (a) Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
(b) Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21–24 Find the limit.

21. $\lim_{x\to\infty} (1.001)^x$

22. $\lim_{x\to-\infty} (1.001)^x$

23. $\lim_{t\to\infty} 2^{-t^2}$

24. $\lim_{x\to3^+} \log_{10}(x^2 - 5x + 6)$

25–42 Differentiate the function.

25. $f(x) = x^5 + 5^x$

26. $g(x) = x\sin(2^x)$

27. $G(x) = 4^{C/x}$

28. $F(t) = 3^{\cos 2t}$

29. $L(v) = \tan(4^{v^2})$

30. $G(u) = (1 + 10^{\ln u})^6$

31. $f(x) = \log_2(1 - 3x)$

32. $f(x) = \log_{10}\sqrt{x}$

33. $y = x\log_4 \sin x$

34. $y = \log_2(x\log_5 x)$

35. $y = x^x$

36. $y = x^{\cos x}$

37. $y = x^{\sin x}$

38. $y = (\sqrt{x})^x$

39. $y = (\cos x)^x$

40. $y = (\sin x)^{\ln x}$

41. $y = (\tan x)^{1/x}$

42. $y = (\ln x)^{\cos x}$

43. Find an equation of the tangent line to the curve $y = 10^x$ at the point (1, 10).

44. If $f(x) = x^{\cos x}$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

45–50 Evaluate the integral.

45. $\int_0^4 2^s\,ds$

46. $\int (x^5 + 5^x)\,dx$

47. $\int \frac{\log_{10} x}{x}\,dx$

48. $\int x2^{x^2}\,dx$

49. $\int 3^{\sin\theta}\cos\theta\,d\theta$

50. $\int \frac{2^x}{2^x + 1}\,dx$

51. Find the area of the region bounded by the curves $y = 2^x$, $y = 5^x$, $x = -1$, and $x = 1$.

52. The region under the curve $y = 10^{-x}$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.

53. Use a graph to find the root of the equation $2^x = 1 + 3^{-x}$ correct to one decimal place. Then use this estimate as the initial approximation in Newton's method to find the root correct to six decimal places.

54. Find y' if $x^y = y^x$.

55. Find the inverse function of $g(x) = \log_4(x^3 + 2)$.

56. Calculate $\lim_{x\to0^+} x^{-\ln x}$.

57. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

58. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

59. Referring to Exercise 58, find the rate of change of the loudness with respect to the intensity when the sound is measured at 50 dB (the level of ordinary conversation).

60. According to the Beer-Lambert Law, the light intensity at a depth of x meters below the surface of the ocean is $I(x) = I_0 a^x$, where I_0 is the light intensity at the surface and a is a constant such that $0 < a < 1$.
(a) Express the rate of change of $I(x)$ with respect to x in terms of $I(x)$.
(b) If $I_0 = 8$ and $a = 0.38$, find the rate of change of intensity with respect to depth at a depth of 20 m.
(c) Using the values from part (b), find the average light intensity between the surface and a depth of 20 m.

61. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the average BAC, measured in mg/mL, of a group of eight male subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink).

What is the maximum average BAC during the first 3 hours? When does it occur?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

62. In this section we modeled the world population from 1900 to 2010 with the exponential function

$$P(t) = (1436.53) \cdot (1.01395)^t$$

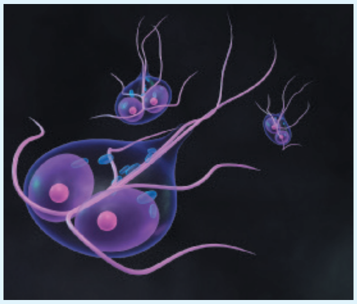
where $t = 0$ corresponds to the year 1900 and $P(t)$ is measured in millions. According to this model, what was the rate of increase of world population in 1920? In 1950? In 2000?

63. Use the graph of V in Figure 9 to estimate the half-life of the viral load of patient 303 during the first month of treatment.

64. A researcher is trying to determine the doubling time for a population of the bacterium *Giardia lamblia*. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Make a scatter plot of the data.
(b) Use a graphing calculator to find an exponential curve $f(t) = a \cdot b^t$ that models the bacteria population t hours later.
(c) Graph the model from part (b) together with the scatter plot in part (a). Use the TRACE feature to determine how long it takes for the bacteria count to double.



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G. lamblia

65. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.
(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 1.4.2.

66. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

67. Prove the second law of exponents [see (3)].

68. Prove the fourth law of exponents [see (3)].

69. Deduce the following laws of logarithms from (3):
(a) $\log_b(xy) = \log_b x + \log_b y$
(b) $\log_b(x/y) = \log_b x - \log_b y$
(c) $\log_b(x^y) = y \log_b x$

70. Show that $\lim_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

6.5 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1
$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt.

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$y'(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$$

We will see in Section 9.4 that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C, we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the initial value of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

What is the significance of the proportionality constant k? In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

3
$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth rate**.

According to (3), instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant." Then (2) says that a population with constant relative growth rate must grow exponentially. Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt}. For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year. If the population at time 0 is P_0, then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

SOLUTION We measure the time t in years and let $t = 0$ in the year 1950. We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$. Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

The model predicts that the population in 2020 will be

$$P(70) = 2560e^{0.017185(70)} \approx 8524 \text{ million}$$

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2020 is riskier.

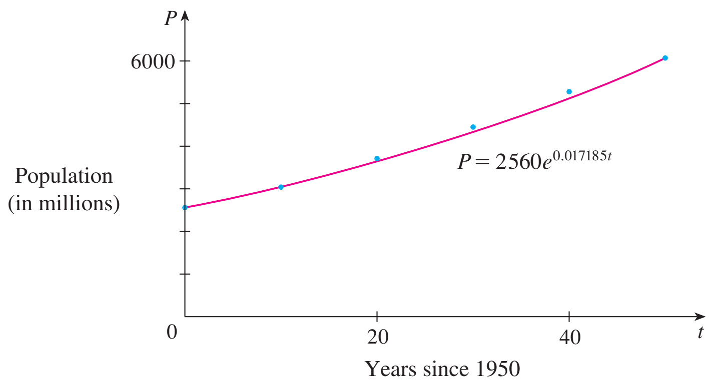


FIGURE 1
A model for world population growth in the second half of the 20th century

■

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use (2) to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

EXAMPLE 2 The half-life of radium-226 is 1590 years.
(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
(b) Find the mass after 1000 years correct to the nearest milligram.
(c) When will the mass be reduced to 30 mg?

SOLUTION

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. Then $dm/dt = km$ and $m(0) = 100$, so (2) gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k, we use the fact that $m(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \tfrac{1}{2}$$

and

$$1590k = \ln \tfrac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \qquad \text{or} \qquad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590}t = \ln 0.3$$

Thus
$$t = -1590\frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

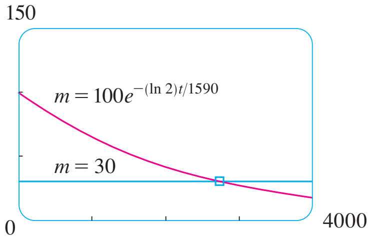


FIGURE 2

As a check on our work in Example 2, we use a graphing device to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c).

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use (2) to find an expression for y, from which we can find T.

EXAMPLE 3 A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.
(a) What is the temperature of the soda pop after another half hour?
(b) How long does it take for the soda pop to cool to 50°F?

SOLUTION

(a) Let $T(t)$ be the temperature of the soda after t minutes. The surrounding temperature is $T_s = 44°F$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \qquad y(0) = 28$$

and by (2) we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

We are given that $T(30) = 61$, so $y(30) = 61 - 44 = 17$ and

$$28e^{30k} = 17 \qquad e^{30k} = \tfrac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30} \approx -0.01663$$

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)} \approx 54.3$$

So after another half hour the pop has cooled to about 54°F.

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \tfrac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663} \approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes. ■

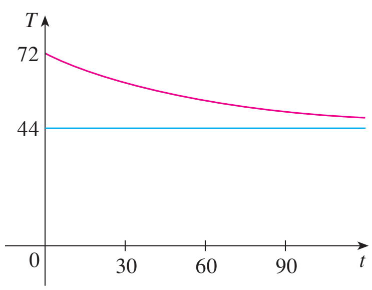


FIGURE 3

Notice that in Example 3, we have

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. The graph of the temperature function is shown in Figure 3.

Continuously Compounded Interest

EXAMPLE 4 If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth \$1000(1.06) = \$1060, after 2 years it's worth \$[1000(1.06)]1.06 = \$1123.60, and after t years it's worth $\$1000(1.06)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.06$ in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a \$1000 investment will be worth

$$\$1000(1.06)^3 = \$1191.02 \quad \text{with annual compounding}$$

$$\$1000(1.03)^6 = \$1194.05 \quad \text{with semiannual compounding}$$

$$\$1000(1.015)^{12} = \$1195.62 \quad \text{with quarterly compounding}$$

$$\$1000(1.005)^{36} = \$1196.68 \quad \text{with monthly compounding}$$

$$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20 \quad \text{with daily compounding}$$

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \to \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n\to\infty} A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \qquad \text{(where } m = n/r\text{)} \end{aligned}$$

But the limit in this expression is equal to the number e (see Equation 6.4.9 or 6.4*.9). So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of \$1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3} = \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, \$1197.20. But the amount is easier to compute if we use continuous compounding. ■

6.5 EXERCISES

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.
(a) Find the relative growth rate.
(b) Find an expression for the number of cells after t hours.
(c) Find the number of cells after 6 hours.
(d) Find the rate of growth after 6 hours.
(e) When will the population reach a million cells?

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.
(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 3 hours.
(c) Find the rate of growth after 3 hours.
(d) When will the population reach 10,000?

4. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.
(a) What is the relative growth rate? Express your answer as a percentage.
(b) What was the initial size of the culture?
(c) Find an expression for the number of bacteria after t hours.
(d) Find the number of cells after 4.5 hours.
(e) Find the rate of growth after 4.5 hours.
(f) When will the population reach 50,000?

5. The table gives estimates of the world population, in millions, from 1750 to 2000.

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

6. The table gives the population of Indonesia, in millions, for the second half of the 20th century.

Year	Population
1950	83
1960	100
1970	122
1980	150
1990	182
2000	214

(a) Assuming the population grows at a rate proportional to its size, use the census figures for 1950 and 1960 to predict the population in 1980. Compare with the actual figure.
(b) Use the census figures for 1960 and 1980 to predict the population in 2000. Compare with the actual population.
(c) Use the census figures for 1980 and 2000 to predict the population in 2010 and compare with the actual population of 243 million.
(d) Use the model in part (c) to predict the population in 2020. Do you think the prediction will be too high or too low? Why?

7. Experiments show that if the chemical reaction

$$N_2O_5 \longrightarrow 2NO_2 + \tfrac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(See Example 2.7.4.)
(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.
(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

8. Strontium-90 has a half-life of 28 days.
(a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.
(b) Find the mass remaining after 40 days.
(c) How long does it take the sample to decay to a mass of 2 mg?
(d) Sketch the graph of the mass function.

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
(a) Find the mass that remains after t years.
(b) How much of the sample remains after 100 years?
(c) After how long will only 1 mg remain?

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
(a) What is the half-life of tritium-3?
(b) How long would it take the sample to decay to 20% of its original amount?

11. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A discovery revealed a parchment fragment that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

12. Dinosaur fossils are too old to be reliably dated using carbon-14. (See Exercise 11.) Suppose we had a 68-million-year-old dinosaur fossil. What fraction of the living dinosaur's ^{14}C would be remaining today? Suppose the minimum detectable amount is 0.1%. What is the maximum age of a fossil that we could date using ^{14}C?

13. Dinosaur fossils are often dated by using an element other than carbon, such as potassium-40, that has a longer half-life (in this case, approximately 1.25 billion years). Suppose the minimum detectable amount is 0.1% and a dinosaur is dated

with ^{40}K to be 68 million years old. Is such a dating possible? In other words, what is the maximum age of a fossil that we could date using ^{40}K?

14. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

15. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.
(a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?
(b) When will the turkey have cooled to 100°F?

16. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

17. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.
(a) What is the temperature of the drink after 50 minutes?
(b) When will its temperature be 15°C?

18. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

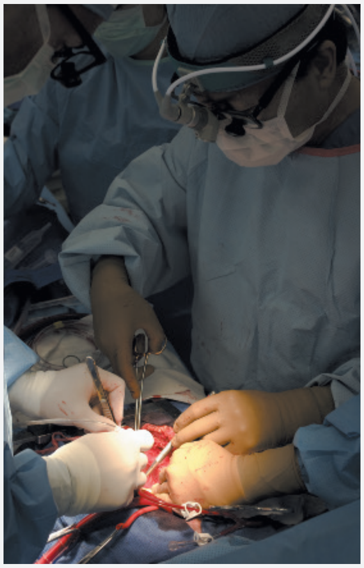
19. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.
(a) What is the pressure at an altitude of 3000 m?
(b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?

20. (a) If \$1000 is borrowed at 8% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (vii) continuously.
(b) Suppose \$1000 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leqslant t \leqslant 3$, graph $A(t)$ for each of the interest rates 6%, 8%, and 10% on a common screen.

21. (a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.

22. (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
(b) What is the equivalent annual interest rate?

APPLIED PROJECT CONTROLLING RED BLOOD CELL LOSS DURING SURGERY


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A typical volume of blood in the human body is about 5 L. A certain percentage of that volume (called the *hematocrit*) consists of red blood cells (RBCs); typically the hematocrit is about 45% in males. Suppose that a surgery takes four hours and a male patient bleeds 2.5 L of blood. During surgery the patient's blood volume is maintained at 5 L by injection of saline solution, which mixes quickly with the blood but dilutes it so that the hematocrit decreases as time passes.

1. Assuming that the rate of RBC loss is proportional to the volume of RBCs, determine the patient's volume of RBCs by the end of the operation.

2. A procedure called *acute normovolemic hemodilution* (ANH) has been developed to minimize RBC loss during surgery. In this procedure blood is extracted from the patient before the operation and replaced with saline solution. This dilutes the patient's blood, resulting in fewer RBCs being lost during the bleeding. The extracted blood is then returned to the patient after surgery. Only a certain amount of blood can be extracted, however, because the RBC concentration can never be allowed to drop below 25% during surgery. What is the maximum amount of blood that can be extracted in the ANH procedure for the surgery described in this project?

3. What is the RBC loss without the ANH procedure? What is the loss if the procedure is carried out with the volume calculated in Problem 2?

6.6 Inverse Trigonometric Functions

In this section we apply the ideas of Section 6.1 to find the derivatives of the so-called inverse trigonometric functions. We have a slight difficulty in this task: because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, *is* one-to-one (see Figure 2). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

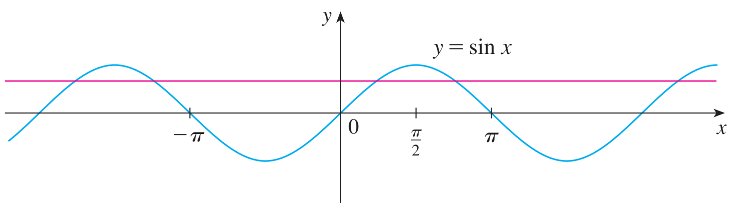


FIGURE 1

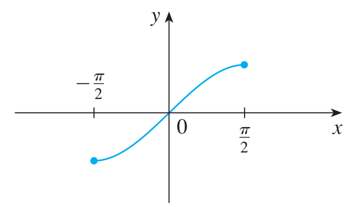


FIGURE 2
$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

1
$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

⊘ $\sin^{-1}x \neq \dfrac{1}{\sin x}$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 1 Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 3 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

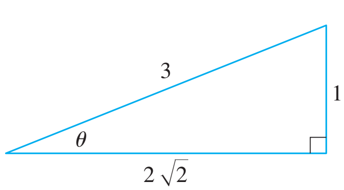


FIGURE 3

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$

■

The cancellation equations for inverse functions become, in this case,

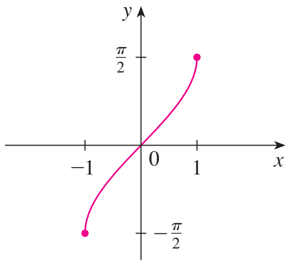


FIGURE 4
$y = \sin^{-1}x = \arcsin x$

2

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.

We know that the sine function f is continuous, so the inverse sine function is also continuous. We also know from Section 2.4 that the sine function is differentiable, so the inverse sine function is also differentiable. We could calculate the derivative of $\sin^{-1}$ by the formula in Theorem 6.1.7, but since we know that $\sin^{-1}$ is differentiable, we can just as easily calculate it by implicit differentiation as follows.

Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1$$

and

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$, since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

3

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

EXAMPLE 2 If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) the domain of f'.

SOLUTION

(a) Since the domain of the inverse sine function is $[-1, 1]$, the domain of f is

$$\{x \mid -1 \leqslant x^2 - 1 \leqslant 1\} = \{x \mid 0 \leqslant x^2 \leqslant 2\}$$

$$= \{x \mid |x| \leqslant \sqrt{2}\} = \left[-\sqrt{2}, \sqrt{2}\right]$$

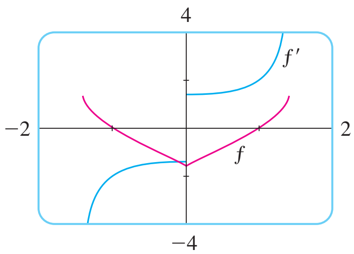


FIGURE 5

The graphs of the function f of Example 2 and its derivative are shown in Figure 5. Notice that f is not differentiable at 0 and this is consistent with the fact that the graph of f' makes a sudden jump at $x = 0$.

(b) Combining Formula 3 with the Chain Rule, we have

$$f'(x) = \frac{1}{\sqrt{1 - (x^2 - 1)^2}} \frac{d}{dx}(x^2 - 1)$$
$$= \frac{1}{\sqrt{1 - (x^4 - 2x^2 + 1)}} 2x = \frac{2x}{\sqrt{2x^2 - x^4}}$$

(c) The domain of f' is

$$\{x \mid -1 < x^2 - 1 < 1\} = \{x \mid 0 < x^2 < 2\}$$
$$= \{x \mid 0 < |x| < \sqrt{2}\} = (-\sqrt{2}, 0) \cup (0, \sqrt{2}) \qquad \blacksquare$$

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 6) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

4 $$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

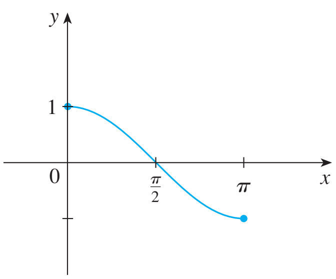


FIGURE 6
$y = \cos x$, $0 \leqslant x \leqslant \pi$

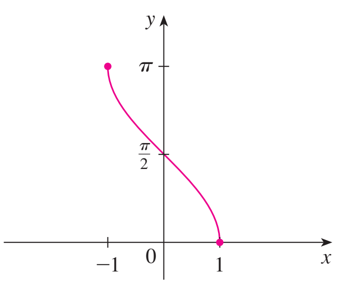


FIGURE 7
$y = \cos^{-1}x = \arccos x$

The cancellation equations are

5
$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$
$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$ and is a continuous function whose graph is shown in Figure 7. Its derivative is given by

6 $$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

Formula 6 can be proved by the same method as for Formula 3 and is left as Exercise 17.

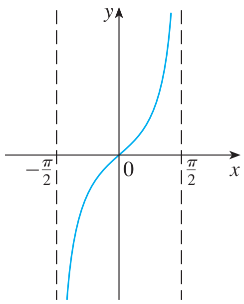


FIGURE 8
$y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 8.) It is denoted by $\tan^{-1}$ or arctan.

7 $$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE 3 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

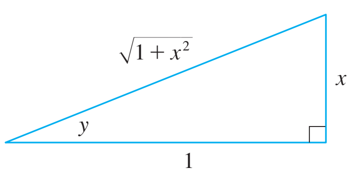


FIGURE 9

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 9 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ ■

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 10.

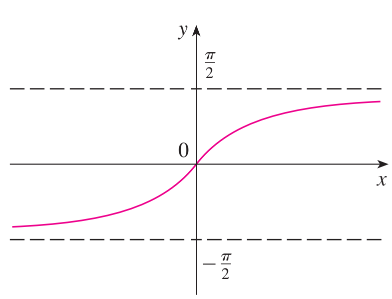


FIGURE 10
$y = \tan^{-1}x = \arctan x$

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the

line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$. This fact is expressed by the following limits:

8
$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

EXAMPLE 4 Evaluate $\displaystyle\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

SOLUTION If we let $t = 1/(x - 2)$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the second equation in (8), we have

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t = \frac{\pi}{2}$$ ■

Because tan is differentiable, $\tan^{-1}$ is also differentiable. To find its derivative, we let $y = \tan^{-1}x$. Then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

9
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

10
$$y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x \ \text{ and } \ y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
$$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x \ \text{ and } \ y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \ \text{ and } \ y \in (0, \pi)$$

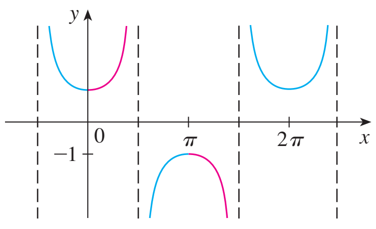


FIGURE 11
$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 11 that both this choice and the one in (10) will work.] The reason for the choice in (10) is that the differentiation formulas are simpler (see Exercise 79).

We collect in Table 11 the differentiation formulas for all of the inverse trigonometric functions. The proofs of the formulas for the derivatives of $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ are left as Exercises 19–21.

11 Table of Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Each of these formulas can be combined with the Chain Rule. For instance, if u is a differentiable function of x, then

$$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx} \qquad \text{and} \qquad \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1 + u^2}\frac{du}{dx}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1 - x^2}}$$

Recall that arctan x is an alternative notation for $\tan^{-1}x$.

(b)
$$f'(x) = x\frac{1}{1 + (\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1 + x)} + \arctan\sqrt{x}$$ ■

EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus isn't needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$ [because we can evaluate $f(1)$ exactly]. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$. ■

Each of the formulas in Table 11 gives rise to an integration formula. The two most useful of these are the following:

12
$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1}x + C$$

13
$$\int \frac{1}{x^2 + 1}\, dx = \tan^{-1}x + C$$

EXAMPLE 7 Find $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1 - 4x^2}}\, dx$.

SOLUTION If we write

$$\int_0^{1/4} \frac{1}{\sqrt{1 - 4x^2}}\, dx = \int_0^{1/4} \frac{1}{\sqrt{1 - (2x)^2}}\, dx$$

then the integral resembles Equation 12 and the substitution $u = 2x$ is suggested. This gives $du = 2\, dx$, so $dx = du/2$. When $x = 0$, $u = 0$; when $x = \frac{1}{4}$, $u = \frac{1}{2}$. So

$$\int_0^{1/4} \frac{1}{\sqrt{1 - 4x^2}}\, dx = \tfrac{1}{2} \int_0^{1/2} \frac{du}{\sqrt{1 - u^2}} = \tfrac{1}{2} \sin^{-1}u \Big]_0^{1/2}$$

$$= \tfrac{1}{2}\Big[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1} 0\Big] = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$ ■

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1}{x^2 + a^2}\, dx$.

SOLUTION To make the given integral more like Equation 13 we write

$$\int \frac{dx}{x^2 + a^2} = \int \frac{dx}{a^2\left(\dfrac{x^2}{a^2} + 1\right)} = \frac{1}{a^2} \int \frac{dx}{\left(\dfrac{x}{a}\right)^2 + 1}$$

This suggests that we substitute $u = x/a$. Then $du = dx/a$, $dx = a\, du$, and

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a^2} \int \frac{a\, du}{u^2 + 1} = \frac{1}{a} \int \frac{du}{u^2 + 1} = \frac{1}{a} \tan^{-1}u + C$$

Thus we have the formula

One of the main uses of inverse trigonometric functions in calculus is that they often arise when we integrate rational functions.

14
$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$ ■

EXAMPLE 9 Find $\int \frac{x}{x^4+9}\,dx$.

SOLUTION We substitute $u = x^2$ because then $du = 2x\,dx$ and we can use Equation 14 with $a = 3$:

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

6.6 EXERCISES

1–10 Find the exact value of each expression.

1. (a) $\sin^{-1}(0.5)$ (b) $\cos^{-1}(-1)$

2. (a) $\tan^{-1}\sqrt{3}$ (b) $\sec^{-1} 2$

3. (a) $\csc^{-1}\sqrt{2}$ (b) $\cos^{-1}(\sqrt{3}/2)$

4. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\arcsin 1$

5. (a) $\tan(\arctan 10)$ (b) $\arcsin(\sin(5\pi/4))$

6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos(\arcsin \frac{1}{2})$

7. $\tan(\sin^{-1}(\frac{2}{3}))$ **8.** $\csc(\arccos \frac{3}{5})$

9. $\cos(2\sin^{-1}(\frac{5}{13}))$ **10.** $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

11. Prove that $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

12–14 Simplify the expression.

12. $\tan(\sin^{-1} x)$ **13.** $\sin(\tan^{-1} x)$ **14.** $\sin(2\arccos x)$

15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$

16. $y = \tan x,\ -\pi/2 < x < \pi/2;\quad y = \tan^{-1}x;\quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$.

20. Prove that $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$.

21. Prove that $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$.

22–35 Find the derivative of the function. Simplify where possible.

22. $y = \tan^{-1}(x^2)$

23. $y = (\tan^{-1}x)^2$ **24.** $g(x) = \arccos\sqrt{x}$

25. $y = \sin^{-1}(2x+1)$ **26.** $R(t) = \arcsin(1/t)$

27. $y = x\sin^{-1}x + \sqrt{1-x^2}$ **28.** $y = \cos^{-1}(\sin^{-1}t)$

29. $F(x) = x\sec^{-1}(x^3)$ **30.** $y = \arctan\sqrt{\frac{1-x}{1+x}}$

31. $y = \arctan(\cos\theta)$

32. $y = \tan^{-1}(x - \sqrt{1+x^2})$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

34. $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

35. $y = \arccos\left(\frac{b + a\cos x}{a + b\cos x}\right),\quad 0 \leqslant x \leqslant \pi,\ a > b > 0$

36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$ **37.** $g(x) = \cos^{-1}(3-2x)$

38. Find y' if $\tan^{-1}(x^2y) = x + xy^2$.

39. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16-x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = \sqrt{1-x^2}\arcsin x$ **42.** $f(x) = \arctan(x^2 - x)$

43–46 Find the limit.

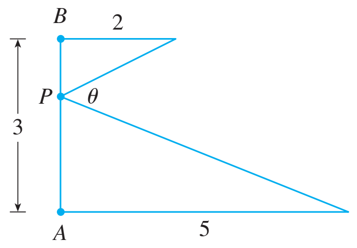
43. $\lim_{x \to -1^+} \sin^{-1}x$

44. $\lim_{x \to \infty} \arccos\left(\frac{1 + x^2}{1 + 2x^2}\right)$

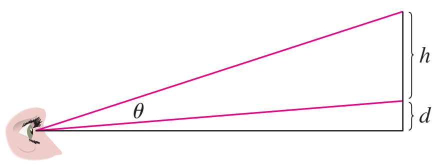
45. $\lim_{x \to \infty} \arctan(e^x)$

46. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



49. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 ft/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 Sketch the curve using the guidelines of Section 3.5.

51. $y = \sin^{-1}\left(\frac{x}{x + 1}\right)$

52. $y = \tan^{-1}\left(\frac{x - 1}{x + 1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = e^{\arctan x}$

CAS **55.** If $f(x) = \arctan(\cos(3 \arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c \sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.

57. Find the most general antiderivative of the function

$$f(x) = \frac{2x^2 + 5}{x^2 + 1}$$

58. Find $g(t)$ if $g'(t) = 2/\sqrt{1 - t^2}$ and $g(1) = 5$.

59–70 Evaluate the integral.

59. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2}\,dx$

60. $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{6}{\sqrt{1 - p^2}}\,dp$

61. $\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$

62. $\int_0^{\sqrt{3}/4} \frac{dx}{1 + 16x^2}$

63. $\int \frac{1 + x}{1 + x^2}\,dx$

64. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\,dx$

65. $\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1}x}$

66. $\int \frac{1}{x\sqrt{x^2 - 4}}\,dx$

67. $\int \frac{t^2}{\sqrt{1 - t^6}}\,dt$

68. $\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\,dx$

69. $\int \frac{dx}{\sqrt{x}\,(1 + x)}$

70. $\int \frac{x}{1 + x^4}\,dx$

71. Use the method of Example 8 to show that, if $a > 0$,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

72. The region under the curve $y = 1/\sqrt{x^2 + 4}$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume of the resulting solid.

73. Evaluate $\int_0^1 \sin^{-1}x\,dx$ by interpreting it as an area and integrating with respect to y instead of x.

74. Prove that, for $xy \neq 1$,

$$\arctan x + \arctan y = \arctan\frac{x + y}{1 - xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

75. Use the result of Exercise 74 to prove the following:

(a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \pi/4$

(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \pi/4$

76. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}x)$.

(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.

(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.

(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

77. Use the method of Example 6 to prove the identity

$$2\sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geqslant 0$$

78. Prove the identity

$$\arcsin\frac{x-1}{x+1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$$

79. Some authors define $y = \sec^{-1}x \iff \sec y = x$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$. Show that with this definition we have (instead of the formula given in Exercise 20)

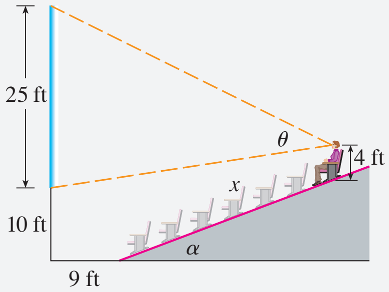
$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

80. Let $f(x) = x\arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) Is f continuous at 0?

(b) Is f differentiable at 0?

APPLIED PROJECT — CAS WHERE TO SIT AT THE MOVIES



A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leqslant x \leqslant 60$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4 ft above the floor, as shown in the figure. (In Exercise 6.6.48 we looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)

1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where $$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$

and $$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

2. Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row?

3. Use your computer algebra system to differentiate θ and find a numerical value for the root of the equation $d\theta/dx = 0$. Does this value confirm your result in Problem 2?

4. Use the graph of θ to estimate the average value of θ on the interval $0 \leqslant x \leqslant 60$. Then use your CAS to compute the average value. Compare with the maximum and minimum values of θ.

6.7 Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and hyperbolic cosine can be sketched using graphical addition as in Figures 1 and 2.

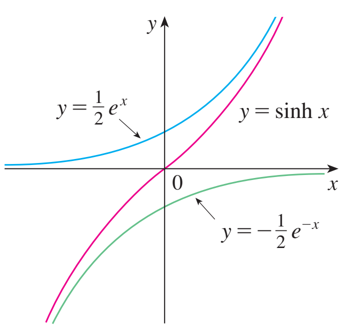


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

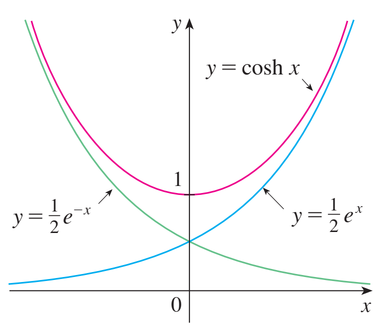


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

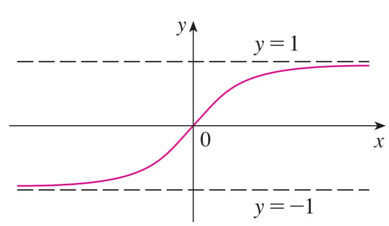


FIGURE 3
$y = \tanh x$

Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, while cosh has domain $\mathbb{R}$ and range $[1, \infty)$. The graph of tanh is shown in Figure 3. It has the horizontal asymptotes $y = \pm 1$. (See Exercise 23.)

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, because the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of a curve with equation $y = c + a\cosh(x/a)$ called a *catenary* (see Figure 4). (The Latin word *catena* means "chain.")

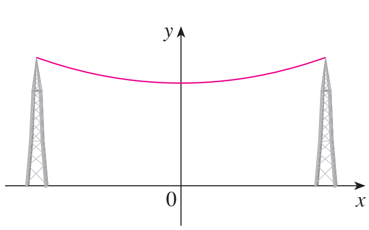


FIGURE 4
A catenary $y = c + a\cosh(x/a)$

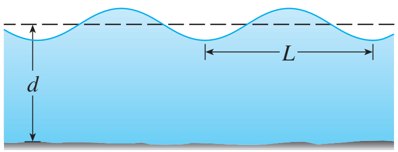


FIGURE 5
Idealized ocean wave

Another application of hyperbolic functions occurs in the description of ocean waves: the velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5 and Exercise 49.)

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.



Stockbyte / Getty Images

The Gateway Arch in St. Louis was designed using a hyperbolic cosine function (see Exercise 48).

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \text{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \text{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \text{sech}^2 x$$ ■

The identity proved in Example 1(a) gives a clue to the reason for the name "hyperbolic" functions:

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.

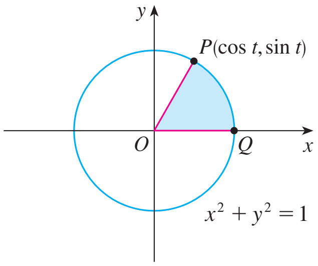


FIGURE 6

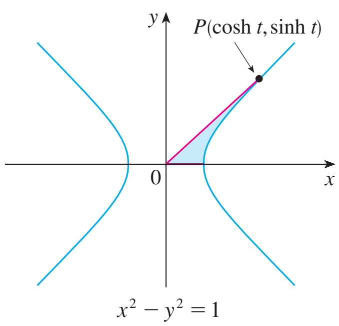


FIGURE 7

Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geqslant 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 6.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but beware that the signs are different in some cases.

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

EXAMPLE 2 Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}\left(\cosh\sqrt{x}\right) = \sinh\sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$ ■

Inverse Hyperbolic Functions

You can see from Figures 1 and 3 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Figure 2 shows that cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 28).

We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10 by using Figures 1, 2, and 3.

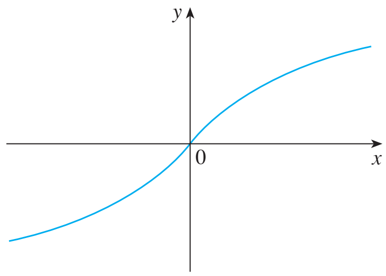


FIGURE 8 $y = \sinh^{-1}x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

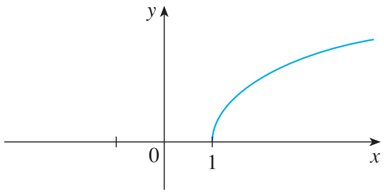


FIGURE 9 $y = \cosh^{-1}x$
domain $= [1, \infty)$ range $= [0, \infty)$

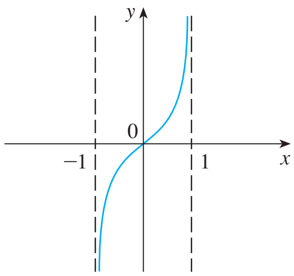


FIGURE 10 $y = \tanh^{-1}x$
domain $= (-1, 1)$ range $= \mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 26 and 27.

3 $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

4 $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

5 $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

EXAMPLE 3 Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so

$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ $\left(\text{because } x < \sqrt{x^2 + 1}\right)$. Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore $\qquad y = \ln(e^y) = \ln\left(x + \sqrt{x^2 + 1}\right)$

This shows that $\qquad \sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$

(See Exercise 25 for another method.) ■

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}
\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \frac{d}{dx}\left(x + \sqrt{x^2 + 1}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\
&= \frac{\sqrt{x^2 + 1} + x}{\left(x + \sqrt{x^2 + 1}\right)\sqrt{x^2 + 1}} \\
&= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned}$$

■

EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1 - (\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1 - \sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$

■

EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 + x^2}}$.

SOLUTION Using Table 6 (or Example 4) we know that an antiderivative of $1/\sqrt{1 + x^2}$ is $\sinh^{-1}x$. Therefore

$$\int_0^1 \frac{dx}{\sqrt{1 + x^2}} = \sinh^{-1}x\Big]_0^1$$

$$= \sinh^{-1} 1$$

$$= \ln\left(1 + \sqrt{2}\right) \qquad \text{(from Equation 3)}$$

■

6.7 EXERCISES

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$

2. (a) $\tanh 0$ (b) $\tanh 1$

3. (a) $\cosh(\ln 5)$ (b) $\cosh 5$

4. (a) $\sinh 4$ (b) $\sinh(\ln 4)$

5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$

6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)

9. $\cosh x + \sinh x = e^x$

10. $\cosh x - \sinh x = e^{-x}$

11. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$
(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$
(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$
(g) $\lim_{x\to0^+} \coth x$ (h) $\lim_{x\to0^-} \coth x$
(i) $\lim_{x\to-\infty} \operatorname{csch} x$ (j) $\lim_{x\to\infty} \dfrac{\sinh x}{e^x}$

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

28. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–45 Find the derivative. Simplify where possible.

30. $f(x) = e^x \cosh x$

31. $f(x) = \tanh \sqrt{x}$

32. $g(x) = \sinh^2 x$

33. $h(x) = \sinh(x^2)$

34. $F(t) = \ln(\sinh t)$

35. $G(t) = \sinh(\ln t)$

36. $y = \operatorname{sech} x\,(1 + \ln \operatorname{sech} x)$

37. $y = e^{\cosh 3x}$

38. $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$

39. $g(t) = t \coth \sqrt{t^2 + 1}$

40. $y = \sinh^{-1}(\tan x)$

41. $y = \cosh^{-1}\sqrt{x}$

42. $y = x \tanh^{-1}x + \ln \sqrt{1 - x^2}$

43. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

44. $y = \operatorname{sech}^{-1}(e^{-x})$

45. $y = \coth^{-1}(\sec x)$

46. Show that $\dfrac{d}{dx} \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2}e^{x/2}$.

47. Show that $\dfrac{d}{dx} \arctan(\tanh x) = \operatorname{sech} 2x$.

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

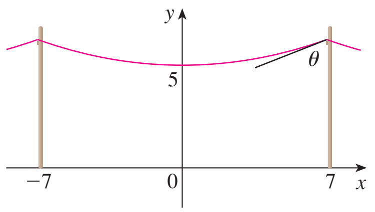
where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

50. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters.
(a) Find the slope of this curve where it meets the right pole.
(b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. A cable with linear density $\rho = 2$ kg/m is strung from the tops of two poles that are 200 m apart.
(a) Use Exercise 52 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?
(b) If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

54. A model for the velocity of a falling object after time t is

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(t\sqrt{\frac{gk}{m}}\right)$$

where m is the mass of the object, $g = 9.8$ m/s^2 is the

acceleration due to gravity, k is a constant, t is measured in seconds, and v in m/s.

(a) Calculate the terminal velocity of the object, that is, $\lim_{t\to\infty} v(t)$.

(b) If a person falls from a building, the value of the constant k depends on his or her position. For a "belly-to-earth" position, $k = 0.515$ kg/s, but for a "feet-first" position, $k = 0.067$ kg/s. If a 60-kg person falls in belly-to-earth position, what is the terminal velocity? What about feet-first?

Source: L. Long et al., "How Terminal Is Terminal Velocity?" *American Mathematical Monthly* 113 (2006): 752–55.

55. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

56. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

57. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

58. Investigate the family of functions

$$f_n(x) = \tanh(n \sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

59–67 Evaluate the integral.

59. $\int \sinh x \cosh^2 x\, dx$

60. $\int \sinh(1 + 4x)\, dx$

61. $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\, dx$

62. $\int \tanh x\, dx$

63. $\int \frac{\cosh x}{\cosh^2 x - 1}\, dx$

64. $\int \frac{\operatorname{sech}^2 x}{2 + \tanh x}\, dx$

65. $\int_4^6 \frac{1}{\sqrt{t^2 - 9}}\, dt$

66. $\int_0^1 \frac{1}{\sqrt{16t^2 + 1}}\, dt$

67. $\int \frac{e^x}{1 - e^{2x}}\, dx$

68. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

69. (a) Use Newton's method or a graphing device to find approximate solutions of the equation $\cosh 2x = 1 + \sinh x$.

(b) Estimate the area of the region bounded by the curves $y = \cosh 2x$ and $y = 1 + \sinh x$.

70. Show that the area of the shaded hyperbolic sector in Figure 7 is $A(t) = \frac{1}{2}t$. [*Hint:* First show that

$$A(t) = \tfrac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

and then verify that $A'(t) = \frac{1}{2}$.]

71. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha \sinh(x + \beta)$ or $\alpha \cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

6.8 Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x\to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 1.6) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**. We met some limits of this type in Chapter 1.

For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

2 $$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ (the numerator was increasing significantly faster than the denominator); if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞**. We saw in Section 3.4 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs in the denominator. For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

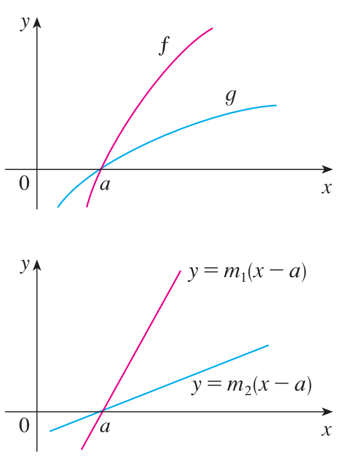


FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad [\text{since } f(a) = g(a) = 0]$$

L'Hospital

L'Hospital's Rule is named after a French nobleman, the Marquis de l'Hospital (1661–1704), but was discovered by a Swiss mathematician, John Bernoulli (1667–1748). You might sometimes see l'Hospital spelled as l'Hôpital, but he spelled his own name l'Hospital, as was common in the 17th century. See Exercise 95 for the example that the Marquis used to illustrate his rule. See the project on page 503 for further historical details.

The general version of l'Hospital's Rule for the indeterminate form $\frac{0}{0}$ is somewhat more difficult and its proof is deferred to the end of this section. The proof for the indeterminate form ∞/∞ can be found in more advanced books.

EXAMPLE 1 Find $\displaystyle\lim_{x \to 1} \frac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

the limit is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

■

Notice that when using l'Hospital's Rule we differentiate the numerator and denominator *separately*. We do *not* use the Quotient Rule.

The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 73.

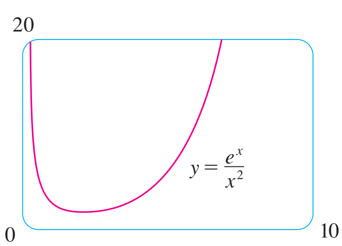


FIGURE 2

EXAMPLE 2 Calculate $\displaystyle\lim_{x \to \infty} \frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, so the limit is an indeterminate form of type ∞/∞, and l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\dfrac{d}{dx}(e^x)}{\dfrac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but

a second application of l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$ ■

EXAMPLE 3 Calculate $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$.

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \to \infty$. See also Exercise 74.

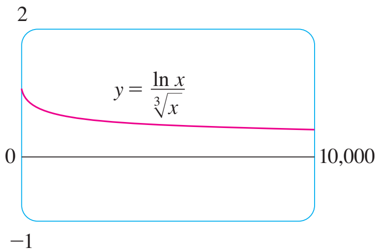


FIGURE 3

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0$$ ■

In both Examples 2 and 3 we evaluated limits of type ∞/∞, but we got two different results. In Example 2, the infinite limit tells us that the numerator e^x increases significantly faster than the denominator x^2, resulting in larger and larger ratios. In fact, $y = e^x$ grows more quickly than all the power functions $y = x^n$ (see Exercise 73). In Example 3 we have the opposite situation; the limit of 0 means that the denominator outpaces the numerator, and the ratio eventually approaches 0.

EXAMPLE 4 Find $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x^3}$. (See Exercise 1.5.44.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}$$

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 1.5.44(d).

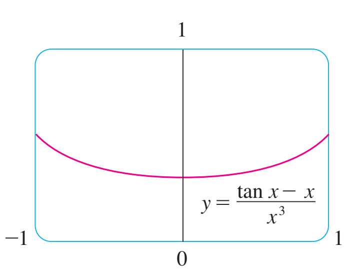


FIGURE 4

Because $\lim_{x \to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\sec^2 x}{1} = \frac{1}{3}$$ ■

EXAMPLE 5 Find $\displaystyle\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is wrong! Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$ ■

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Other limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 1.6.3, 1.6.5, and 3.4.3, and the discussion at the beginning of this section.) So when evaluating any limit, you should consider other methods before using l'Hospital's Rule.

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} [f(x)\,g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Figure 5 shows the graph of the function in Example 6. Notice that the function is undefined at $x = 0$; the graph approaches the origin but never quite reaches it.

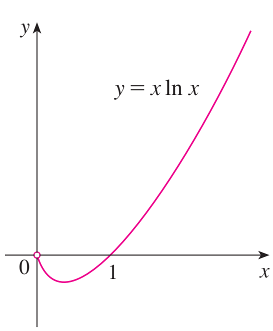


FIGURE 5

EXAMPLE 6 Evaluate $\displaystyle\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$ ■

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $\frac{0}{0}$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

EXAMPLE 7 Use l'Hospital's Rule to help sketch the graph of $f(x) = xe^x$.

SOLUTION Because both x and e^x become large as $x \to \infty$, we have $\lim_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus the x-axis is a horizontal asymptote.

We use the methods of Chapter 3 to gather other information concerning the graph. The derivative is

$$f'(x) = xe^x + e^x = (x + 1)e^x$$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$. Because $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1} \approx -0.37$ is a local (and absolute) minimum. The second derivative is

$$f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$$

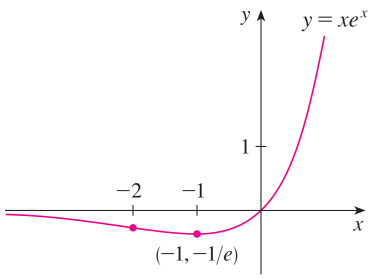


FIGURE 6

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2}) \approx (-2, -0.27)$.

We use this information to sketch the curve in Figure 6. ■

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

EXAMPLE 8 Compute $\displaystyle\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$. ■

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} = [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Although forms of the type 0^0, ∞^0, and 1^∞ are indeterminate, the form 0^∞ is not indeterminate. (See Exercise 98.)

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate (type 1^∞). Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$ ■

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 7. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 10.

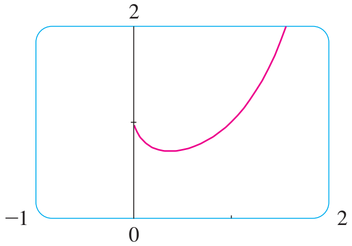


FIGURE 7

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. (Recall that 0^0 is undefined.) We could proceed as in Example 9 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$ ■

In order to give the promised proof of l'Hospital's Rule, we first need a generalization of the Mean Value Theorem. The following theorem is named after another French mathematician, Augustin-Louis Cauchy (1789–1857).

See the biographical sketch of Cauchy on page 77.

3 Cauchy's Mean Value Theorem Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Notice that if we take the special case in which $g(x) = x$, then $g'(c) = 1$ and Theorem 3 is just the ordinary Mean Value Theorem. Furthermore, Theorem 3 can be proved in a similar manner. You can verify that all we have to do is change the function h given by Equation 3.2.4 to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

and apply Rolle's Theorem as before.

PROOF OF L'HOSPITAL'S RULE We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\begin{aligned}
\lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} \\
&= \lim_{t \to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)} \\
&= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\end{aligned}$$

■

6.8 EXERCISES

1–4 Given that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 1$$

$$\lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x \to a} \frac{f(x)}{g(x)}$ (b) $\lim_{x \to a} \frac{f(x)}{p(x)}$

(c) $\lim_{x \to a} \frac{h(x)}{p(x)}$ (d) $\lim_{x \to a} \frac{p(x)}{f(x)}$

(e) $\lim_{x \to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x \to a} [f(x)p(x)]$ (b) $\lim_{x \to a} [h(x)p(x)]$

(c) $\lim_{x \to a} [p(x)q(x)]$

3. (a) $\lim_{x \to a} [f(x) - p(x)]$ (b) $\lim_{x \to a} [p(x) - q(x)]$

(c) $\lim_{x \to a} [p(x) + q(x)]$

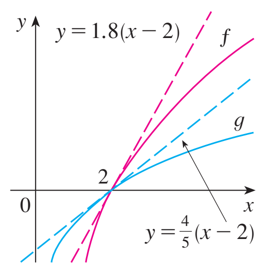
4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ (b) $\lim_{x \to a} [f(x)]^{p(x)}$

(c) $\lim_{x \to a} [h(x)]^{p(x)}$ (d) $\lim_{x \to a} [p(x)]^{f(x)}$

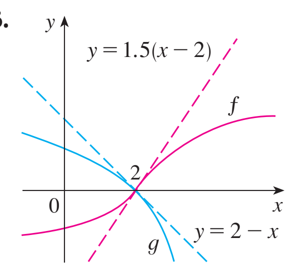
(e) $\lim_{x \to a} [p(x)]^{q(x)}$ (f) $\lim_{x \to a} \sqrt[q(x)]{p(x)}$

5–6 Use the graphs of f and g and their tangent lines at (2, 0) to find $\lim_{x \to 2} \frac{f(x)}{g(x)}$.

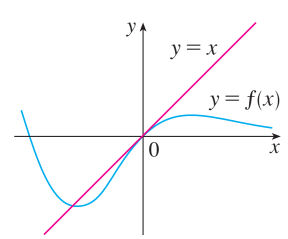
5.



6.



7. The graph of a function f and its tangent line at 0 are shown. What is the value of $\lim_{x \to 0} \frac{f(x)}{e^x - 1}$?



8–68 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

8. $\lim_{x \to 3} \frac{x - 3}{x^2 - 9}$

9. $\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$ **10.** $\lim_{x \to -2} \frac{x^3 + 8}{x + 2}$

11. $\displaystyle\lim_{x\to 1}\frac{x^3-2x^2+1}{x^3-1}$

12. $\displaystyle\lim_{x\to 1/2}\frac{6x^2+5x-4}{4x^2+16x-9}$

13. $\displaystyle\lim_{x\to(\pi/2)^+}\frac{\cos x}{1-\sin x}$

14. $\displaystyle\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}$

15. $\displaystyle\lim_{t\to 0}\frac{e^{2t}-1}{\sin t}$

16. $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x}$

17. $\displaystyle\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos 2\theta}$

18. $\displaystyle\lim_{\theta\to\pi}\frac{1+\cos\theta}{1-\cos\theta}$

19. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

20. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}$

21. $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{x}$

22. $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}$

23. $\displaystyle\lim_{t\to 1}\frac{t^8-1}{t^5-1}$

24. $\displaystyle\lim_{t\to 0}\frac{8^t-5^t}{t}$

25. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

26. $\displaystyle\lim_{u\to\infty}\frac{e^{u/10}}{u^3}$

27. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

28. $\displaystyle\lim_{x\to 0}\frac{\sinh x-x}{x^3}$

29. $\displaystyle\lim_{x\to 0}\frac{\tanh x}{\tan x}$

30. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

31. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

32. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

33. $\displaystyle\lim_{x\to 0}\frac{x3^x}{3^x-1}$

34. $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

35. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{\cos x+e^x-1}$

36. $\displaystyle\lim_{x\to 1}\frac{x\sin(x-1)}{2x^2-x-1}$

37. $\displaystyle\lim_{x\to 0^+}\frac{\arctan(2x)}{\ln x}$

38. $\displaystyle\lim_{x\to 0^+}\frac{x^x-1}{\ln x+x-1}$

39. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1},\ b\neq 0$

40. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

41. $\displaystyle\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}$

42. $\displaystyle\lim_{x\to a^+}\frac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$

43. $\displaystyle\lim_{x\to\infty}x\sin(\pi/x)$

44. $\displaystyle\lim_{x\to\infty}\sqrt{x}\,e^{-x/2}$

45. $\displaystyle\lim_{x\to 0}\sin 5x\csc 3x$

46. $\displaystyle\lim_{x\to-\infty}x\ln\left(1-\frac{1}{x}\right)$

47. $\displaystyle\lim_{x\to\infty}x^3e^{-x^2}$

48. $\displaystyle\lim_{x\to\infty}x^{3/2}\sin(1/x)$

49. $\displaystyle\lim_{x\to 1^+}\ln x\tan(\pi x/2)$

50. $\displaystyle\lim_{x\to(\pi/2)^-}\cos x\sec 5x$

51. $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

52. $\displaystyle\lim_{x\to 0}(\csc x-\cot x)$

53. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

54. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)$

55. $\displaystyle\lim_{x\to\infty}(x-\ln x)$

56. $\displaystyle\lim_{x\to 1^+}[\ln(x^7-1)-\ln(x^5-1)]$

57. $\displaystyle\lim_{x\to 0^+}x^{\sqrt{x}}$

58. $\displaystyle\lim_{x\to 0^+}(\tan 2x)^x$

59. $\displaystyle\lim_{x\to 0}(1-2x)^{1/x}$

60. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

61. $\displaystyle\lim_{x\to 1^+}x^{1/(1-x)}$

62. $\displaystyle\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}$

63. $\displaystyle\lim_{x\to\infty}x^{1/x}$

64. $\displaystyle\lim_{x\to\infty}x^{e^{-x}}$

65. $\displaystyle\lim_{x\to 0^+}(4x+1)^{\cot x}$

66. $\displaystyle\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

67. $\displaystyle\lim_{x\to 0^+}(1+\sin 3x)^{1/x}$

68. $\displaystyle\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

69–70 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

69. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

70. $\displaystyle\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$

71–72 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$ to see that these ratios have the same limit as $x\to 0$. Also, calculate the exact value of the limit.

71. $f(x)=e^x-1,\quad g(x)=x^3+4x$

72. $f(x)=2x\sin x,\quad g(x)=\sec x-1$

73. Prove that

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

74. Prove that

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=0$$

for any number $p>0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

75–76 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

75. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

76. $\displaystyle\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$

77–82 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 3.5.

77. $y = xe^{-x}$

78. $y = \dfrac{\ln x}{x^2}$

79. $y = xe^{-x^2}$

80. $y = e^x/x$

81. $y = \dfrac{1}{x} + \ln x$

82. $y = (x^2 - 3)e^{-x}$

CAS **83–85**

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x \to 0^+$ or as $x \to \infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

83. $f(x) = x^{-x}$

84. $f(x) = (\sin x)^{\sin x}$

85. $f(x) = x^{1/x}$

86. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

87. Investigate the family of curves $f(x) = e^x - cx$. In particular, find the limits as $x \to \pm\infty$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

88. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t\to\infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c\to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

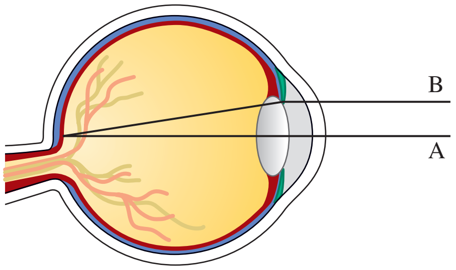
89. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

90. Light enters the eye through the pupil and strikes the retina, where photoreceptor cells sense light and color. W. Stanley Stiles and B. H. Crawford studied the phenomenon in which measured brightness decreases as light enters farther from the center of the pupil. (See the figure.)



A light beam A that enters through the center of the pupil measures brighter than a beam B entering near the edge of the pupil.

They detailed their findings of this phenomenon, known as the *Stiles–Crawford effect of the first kind*, in an important paper published in 1933. In particular, they observed that the amount of luminance sensed was *not* proportional to the area of the pupil as they expected. The percentage P of the total luminance entering a pupil of radius r mm that is sensed at the retina can be described by

$$P = \frac{1 - 10^{-\rho r^2}}{\rho r^2 \ln 10}$$

where ρ is an experimentally determined constant, typically about 0.05.

(a) What is the percentage of luminance sensed by a pupil of radius 3 mm? Use $\rho = 0.05$.

(b) Compute the percentage of luminance sensed by a pupil of radius 2 mm. Does it make sense that it is larger than the answer to part (a)?

(c) Compute $\lim_{r\to 0^+} P$. Is the result what you would expect? Is this result physically possible?

Source: Adapted from W. Stiles and B. Crawford, "The Luminous Efficiency of Rays Entering the Eye Pupil at Different Points." *Proceedings of the Royal Society of London, Series B: Biological Sciences* 112 (1933): 428–50.

91. Some populations initally grow exponentially but eventually level off. Equations of the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

where M, A, and k are positive constants, are called *logistic equations* and are often used to model such populations. (We will investigate these in detail in Chapter 9.) Here M is called the *carrying capacity* and represents the maximum population size that can be supported, and $A = \dfrac{M - P_0}{P_0}$, where P_0 is the initial population.

(a) Compute $\lim_{t \to \infty} P(t)$. Explain why your answer is to be expected.

(b) Compute $\lim_{M \to \infty} P(t)$. (Note that A is defined in terms of M.) What kind of function is your result?

92. A metal cable has radius r and is covered by insulation so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

93. In Section 4.3 we investigated the Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$, which arises in the study of the diffraction of light waves. Evaluate

$$\lim_{x \to 0} \frac{S(x)}{x^3}$$

94. Suppose that the temperature in a long thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\, du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

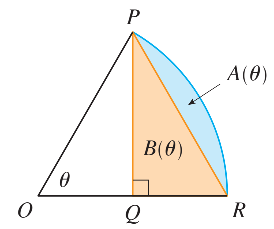
Use l'Hospital's Rule to find this limit.

95. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus textbook ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

96. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



97. Evaluate

$$\lim_{x \to \infty} \left[x - x^2 \ln\left(\frac{1 + x}{x}\right)\right]$$

98. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

99. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

100. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

101. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

102. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

103. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

104. Let

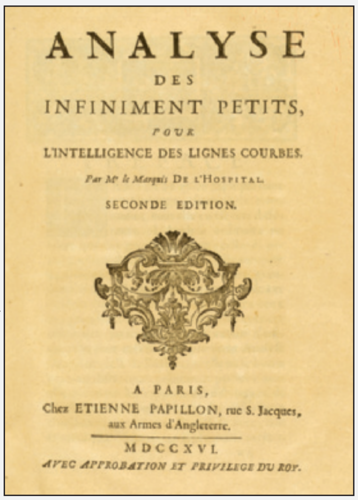
$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.

(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.

(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

WRITING PROJECT — THE ORIGINS OF L'HOSPITAL'S RULE



Thomas Fisher Rare Book Library

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits*, but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 6.8 and show that the two statements are essentially the same.

1. Howard Eves, *In Mathematical Circles (Volume 2: Quadrants III and IV)* (Boston: Prindle, Weber and Schmidt, 1969), pp. 20–22.

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.

3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), p. 484.

4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton University Press, 1969), pp. 315–16.

www.stewartcalculus.com
The Internet is another source of information for this project. Click on *History of Mathematics* for a list of reliable websites.

6 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?
(c) If f is a one-to-one function and $f'(f^{-1}(a)) \neq 0$, write a formula for $(f^{-1})'(a)$.

2. (a) What are the domain and range of the natural exponential function $f(x) = e^x$?
(b) What are the domain and range of the natural logarithmic function $g(x) = \ln x$?
(c) How are the functions $f(x) = e^x$ and $g(x) = \ln x$ related?
(d) How are the graphs of these functions related? Sketch these graphs by hand, using the same axes.
(e) If b is a positive number, $b \neq 1$, write an equation that expresses $\log_b x$ in terms of $\ln x$.

3. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?

(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?

(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range? Sketch its graph.

4. Write the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, and $\tanh x$.

5. State the derivative of each function.

(a) $y = e^x$ (b) $y = b^x$ (c) $y = \ln x$

(d) $y = \log_b x$ (e) $y = \sin^{-1}x$ (f) $y = \cos^{-1}x$

(g) $y = \tan^{-1}x$ (h) $y = \sinh x$ (i) $y = \cosh x$

(j) $y = \tanh x$ (k) $y = \sinh^{-1}x$ (l) $y = \cosh^{-1}x$

(m) $y = \tanh^{-1}x$

6. (a) How is the number e defined?

(b) Express e as a limit.

(c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = b^x$?

(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_b x$?

7. (a) Write a differential equation that expresses the law of natural growth.

(b) Under what circumstances is this an appropriate model for population growth?

(c) What are the solutions of this equation?

8. (a) What does l'Hospital's Rule say?

(b) How can you use l'Hospital's Rule if you have a product $f(x)\,g(x)$, where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?

(c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$, where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?

(d) How can you use l'Hospital's Rule if you have a power $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

9. State whether each of the following limit forms is indeterminate. Where possible, state the limit.

(a) $\dfrac{0}{0}$ (b) $\dfrac{\infty}{\infty}$ (c) $\dfrac{0}{\infty}$ (d) $\dfrac{\infty}{0}$

(e) $\infty + \infty$ (f) $\infty - \infty$ (g) $\infty \cdot \infty$ (h) $\infty \cdot 0$

(i) 0^0 (j) 0^∞ (k) ∞^0 (l) 1^∞

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$.

2. If f is one-to-one and differentiable, with domain $\mathbb{R}$, then $(f^{-1})'(6) = 1/f'(6)$.

3. The function $f(x) = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one.

4. $\tan^{-1}(-1) = 3\pi/4$

5. If $0 < a < b$, then $\ln a < \ln b$.

6. $\pi^{\sqrt{5}} = e^{\sqrt{5}\ln\pi}$

7. You can always divide by e^x.

8. If $a > 0$ and $b > 0$, then $\ln(a + b) = \ln a + \ln b$.

9. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

10. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

11. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

12. The inverse function of $y = e^{3x}$ is $y = \frac{1}{3}\ln x$.

13. $\cos^{-1}x = \dfrac{1}{\cos x}$

14. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

15. $\cosh x \geqslant 1$ for all x

16. $\ln \dfrac{1}{10} = -\displaystyle\int_1^{10} \frac{dx}{x}$

17. $\displaystyle\int_2^{16} \frac{dx}{x} = 3 \ln 2$

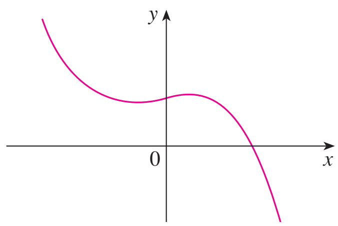
18. $\displaystyle\lim_{x\to\pi^-} \frac{\tan x}{1 - \cos x} = \lim_{x\to\pi^-} \frac{\sec^2 x}{\sin x} = \infty$

19. If $\displaystyle\lim_{x\to\infty} f(x) = 1$ and $\displaystyle\lim_{x\to\infty} g(x) = \infty$, then

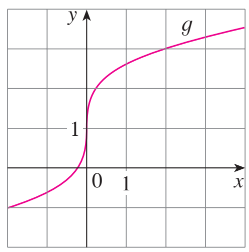
$$\lim_{x\to\infty} [f(x)]^{g(x)} = 1$$

EXERCISES

1. The graph of f is shown. Is f one-to-one? Explain.



2. The graph of g is given.
(a) Why is g one-to-one?
(b) Estimate the value of $g^{-1}(2)$.
(c) Estimate the domain of g^{-1}.
(d) Sketch the graph of g^{-1}.



3. Suppose f is one-to-one, $f(7) = 3$, and $f'(7) = 8$. Find (a) $f^{-1}(3)$ and (b) $(f^{-1})'(3)$.

4. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

5–9 Sketch a rough graph of the function without using a calculator.

5. $y = 5^x - 1$

6. $y = -e^{-x}$

7. $y = -\ln x$

8. $y = \ln(x + 1)$

9. $y = 2 \arctan x$

10. Let $b > 1$. For large values of x, which of the functions $y = x^b$, $y = b^x$, and $y = \log_b x$ has the largest values and which has the smallest values?

11–12 Find the exact value of each expression.

11. (a) $e^{2 \ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$

12. (a) $\ln e^{\pi}$ (b) $\tan\left(\arcsin \frac{1}{2}\right)$

13–20 Solve the equation for x.

13. $\ln x = \frac{1}{3}$

14. $e^x = \frac{1}{3}$

15. $e^{e^x} = 17$

16. $\ln(1 + e^{-x}) = 3$

17. $\ln(x + 1) + \ln(x - 1) = 1$

18. $\log_5(c^x) = d$

19. $\tan^{-1}x = 1$

20. $\sin x = 0.3$

21–47 Differentiate.

21. $f(t) = t^2 \ln t$

22. $g(t) = \dfrac{e^t}{1 + e^t}$

23. $h(\theta) = e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x|$

26. $y = x \cos^{-1}x$

27. $y = x \tan^{-1}(4x)$

28. $y = e^{mx} \cos nx$

29. $y = \ln(\sec^2 x)$

30. $y = \sqrt{t \ln(t^4)}$

31. $y = \dfrac{e^{1/x}}{x^2}$

32. $y = (\arcsin 2x)^2$

33. $y = 3^{x \ln x}$

34. $y = e^{\cos x} + \cos(e^x)$

35. $H(v) = v \tan^{-1}v$

36. $F(z) = \log_{10}(1 + z^2)$

37. $y = x \sinh(x^2)$

38. $y = (\cos x)^x$

39. $y = \ln \sin x - \frac{1}{2}\sin^2 x$

40. $y = \arctan\left(\arcsin \sqrt{x}\right)$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

42. $xe^y = y - 1$

43. $y = \ln(\cosh 3x)$

44. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x \tanh^{-1}\sqrt{x}$

47. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

48. Show that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x + 1)^2}{x^2 + 1}\right) = \frac{1}{(1 + x)(1 + x^2)}$$

49–52 Find f' in terms of g'.

49. $f(x) = e^{g(x)}$

50. $f(x) = g(e^x)$

51. $f(x) = \ln|g(x)|$

52. $f(x) = g(\ln x)$

53–54 Find $f^{(n)}(x)$.

53. $f(x) = 2^x$

54. $f(x) = \ln(2x)$

55. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

56. Find y' if $y = x + \arctan y$.

57–58 Find an equation of the tangent to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $y = x \ln x$, (e, e)

59. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

60. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

61. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

62. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate of change of drug concentration in the blood.
(c) When is this rate equal to 0?

63–78 Evaluate the limit.

63. $\displaystyle\lim_{x\to\infty} e^{-3x}$

64. $\displaystyle\lim_{x\to 10^-} \ln(100 - x^2)$

65. $\displaystyle\lim_{x\to 3^-} e^{2/(x-3)}$

66. $\displaystyle\lim_{x\to\infty} \arctan(x^3 - x)$

67. $\displaystyle\lim_{x\to 0^+} \ln(\sinh x)$

68. $\displaystyle\lim_{x\to\infty} e^{-x}\sin x$

69. $\displaystyle\lim_{x\to\infty} \frac{1 + 2^x}{1 - 2^x}$

70. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x$

71. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x}$

72. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2 + x}$

73. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

74. $\displaystyle\lim_{x\to\infty} \frac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

75. $\displaystyle\lim_{x\to -\infty} (x^2 - x^3)e^{2x}$

76. $\displaystyle\lim_{x\to 0^+} x^2 \ln x$

77. $\displaystyle\lim_{x\to 1^+} \left(\frac{x}{x - 1} - \frac{1}{\ln x}\right)$

78. $\displaystyle\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x}$

79–84 Sketch the curve using the guidelines of Section 3.5.

79. $y = e^x \sin x, \quad -\pi \leqslant x \leqslant \pi$

80. $y = \sin^{-1}(1/x)$

81. $y = x \ln x$

82. $y = e^{2x - x^2}$

83. $y = (x - 2)e^{-x}$

84. $y = x + \ln(x^2 + 1)$

85. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

86. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

87. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

88. (a) Show that there is exactly one root of the equation $\ln x = 3 - x$ and that it lies between 2 and e.
(b) Find the root of the equation in part (a) correct to four decimal places.

89. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

90. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

91. The biologist G. F. Gause conducted an experiment in the 1930s with the protozoan *Paramecium* and used the population function

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

to model his data, where t was measured in days. Use this model to determine when the population was increasing most rapidly.

92–105 Evaluate the integral.

92. $\displaystyle\int_0^4 \frac{1}{16 + t^2}\,dt$

93. $\displaystyle\int_0^1 ye^{-2y^2}\,dy$

94. $\displaystyle\int_2^5 \frac{dr}{1 + 2r}$

95. $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

96. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx$

97. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

98. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

99. $\displaystyle\int \frac{x + 1}{x^2 + 2x}\,dx$

100. $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\,dx$

101. $\displaystyle\int \tan x \ln(\cos x)\,dx$

102. $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx$

103. $\displaystyle\int 2^{\tan\theta}\sec^2\theta\,d\theta$

104. $\displaystyle\int \sinh au\,du$

105. $\displaystyle\int \left(\frac{1 - x}{x}\right)^2 dx$

106–108 Use properties of integrals to prove the inequality.

106. $\displaystyle\int_0^1 \sqrt{1 + e^{2x}}\,dx \geqslant e - 1$

107. $\displaystyle\int_0^1 e^x \cos x\,dx \leqslant e - 1$

108. $\displaystyle\int_0^1 x\sin^{-1}x\,dx \leqslant \pi/4$

109–110 Find $f'(x)$.

109. $f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\,ds$ **110.** $f(x) = \int_{\ln x}^{2x} e^{-t^2}\,dt$

111. Find the average value of the function $f(x) = 1/x$ on the interval $[1, 4]$.

112. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.

113. Find the volume of the solid obtained by rotating about the y-axis the region under the curve $y = 1/(1 + x^4)$ from $x = 0$ to $x = 1$.

114. If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

115. If $f(x) = \ln x + \tan^{-1}x$, find $(f^{-1})'(\pi/4)$.

116. What is the area of the largest rectangle in the first quadrant with two sides on the axes and one vertex on the curve $y = e^{-x}$?

117. What is the area of the largest triangle in the first quadrant with two sides on the axes and the third side tangent to the curve $y = e^{-x}$?

118. Evaluate $\int_0^1 e^x\,dx$ without using the Fundamental Theorem of Calculus. [*Hint:* Use the definition of a definite integral with right endpoints, sum a geometric series, and then use l'Hospital's Rule.]

119. If $F(x) = \int_a^b t^x\,dt$, where $a, b > 0$, then, by the Fundamental Theorem,

$$F(x) = \frac{b^{x+1} - a^{x+1}}{x + 1} \qquad x \neq -1$$

$$F(-1) = \ln b - \ln a$$

Use l'Hospital's Rule to show that F is continuous at -1.

120. Show that

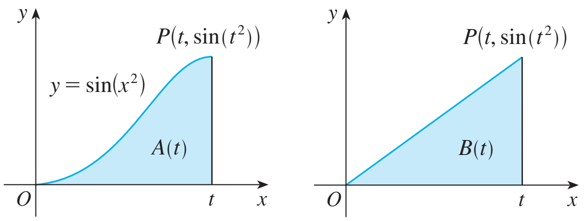
$$\cos\{\arctan[\sin(\operatorname{arccot} x)]\} = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$$

121. If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

122. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim_{t \to 0^+} [A(t)/B(t)]$.



Problems Plus

Cover up the solution to the example and try it yourself.

EXAMPLE For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: for what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c. We know that, for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

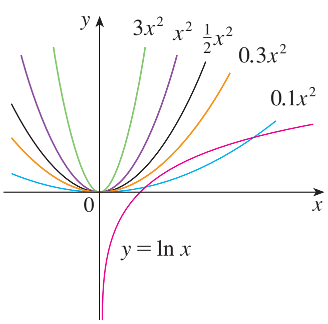


FIGURE 1

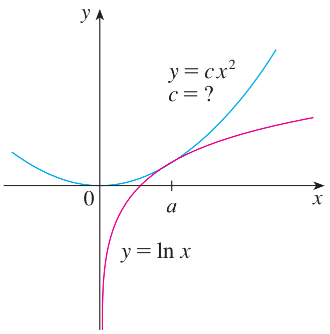


FIGURE 2

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

Solving the equations $\ln a = ca^2$ and $1/a = 2ca$, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus $a = e^{1/2}$ and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

For negative values of c we have the situation illustrated in Figure 3: all parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And let's not forget

about $c = 0$: the curve $y = 0x^2 = 0$ is just the x-axis, which intersects $y = \ln x$ exactly once.

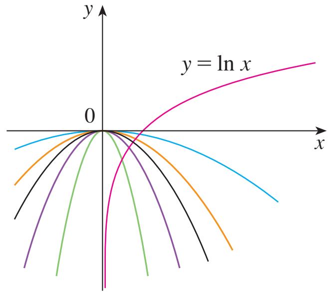


FIGURE 3

To summarize, the required values of c are $c = 1/(2e)$ and $c \leqslant 0$. ■

Problems

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Prove that $\log_2 5$ is an irrational number.

3. Does the function $f(x) = e^{10|x-2|-x^2}$ have an absolute maximum? If so, find it. What about an absolute minimum?

4. If $\int_0^4 e^{(x-2)^4}\,dx = k$, find the value of $\int_0^4 xe^{(x-2)^4}\,dx$.

5. Show that

$$\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

6. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

7. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

8. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\,dy$.

9. Show that $f(x) = \int_1^x \sqrt{1 + t^3}\,dt$ is one-to-one and find $(f^{-1})'(0)$.

10. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}}\arctan\frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \dfrac{1}{a + \cos x}$.

11. For what value of a is the following equation true?

$$\lim_{x\to\infty}\left(\frac{x + a}{x - a}\right)^x = e$$

12. Evaluate

$$\lim_{x\to\infty} \frac{(x + 2)^{1/x} - x^{1/x}}{(x + 3)^{1/x} - x^{1/x}}$$

13. Evaluate $\lim_{x\to 0} \frac{1}{x} \int_0^x (1 - \tan 2t)^{1/t}\, dt$.
[Assume that the integrand is defined and continuous at $t = 0$; see Exercise 4.3.64.]

14. Sketch the set of all points (x, y) such that $|x + y| \leqslant e^x$.

15. Prove that $\cosh(\sinh x) < \sinh(\cosh x)$ for all x.

16. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \geqslant e^2$$

17. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

18. For which positive numbers a is it true that $a^x \geqslant 1 + x$ for all x?

19. For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

20. For what values of c does the curve $y = cx^3 + e^x$ have inflection points?

7 Techniques of Integration



USDA

The photo shows a screw-worm fly, the first pest effectively eliminated from a region by the sterile insect technique without pesticides. The idea is to introduce into the population sterile males that mate with females but produce no offspring. In Exercise 7.4.67 you will evaluate an integral that relates the female insect population to time.

BECAUSE OF THE FUNDAMENTAL THEOREM of Calculus, we can integrate a function if we know an antiderivative, that is, an indefinite integral. We summarize here the most important integrals that we have learned so far.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

In this chapter we develop techniques for using these basic integration formulas to obtain indefinite integrals of more complicated functions. We learned the most important method of integration, the Substitution Rule, in Section 4.5. The other general technique, integration by parts, is presented in Section 7.1. Then we learn methods that are special to particular classes of functions, such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore we discuss a strategy for integration in Section 7.5.

7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts*.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

1
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

2
$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Find $\int x \sin x\,dx$.

SOLUTION USING FORMULA 1 Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}\int x \sin x\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= x(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

It is helpful to use the pattern:

$$u = \square \qquad dv = \square$$
$$du = \square \qquad v = \square$$

SOLUTION USING FORMULA 2 Let

$$u = x \qquad dv = \sin x \, dx$$

Then
$$du = dx \qquad v = -\cos x$$

and so

$$\int x \sin x \, dx = \int \overbrace{x}^{u} \, \overbrace{\sin x \, dx}^{dv} = \overbrace{x}^{u} \, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

■

NOTE Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x \, dx$ and expressed it in terms of the simpler integral $\int \cos x \, dx$. If we had instead chosen $u = \sin x$ and $dv = x \, dx$, then $du = \cos x \, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x \, dx = (\sin x) \frac{x^2}{2} - \frac{1}{2} \int x^2 \cos x \, dx$$

Although this is true, $\int x^2 \cos x \, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x) \, dx$ can be readily integrated to give v.

EXAMPLE 2 Evaluate $\int \ln x \, dx$.

SOLUTION Here we don't have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then
$$du = \frac{1}{x} \, dx \qquad v = x$$

Integrating by parts, we get

It's customary to write $\int 1 \, dx$ as $\int dx$.

Check the answer by differentiating it.

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f. ■

EXAMPLE 3 Find $\int t^2 e^t\,dt$.

SOLUTION Notice that t^2 becomes simpler when differentiated (whereas e^t is unchanged when differentiated or integrated), so we choose

$$u = t^2 \qquad dv = e^t\,dt$$

Then
$$du = 2t\,dt \qquad v = e^t$$

Integration by parts gives

$$\int t^2 e^t\,dt = t^2 e^t - 2\int te^t\,dt \tag{3}$$

The integral that we obtained, $\int te^t\,dt$, is simpler than the original integral but is still not obvious. Therefore we use integration by parts a second time, this time with $u = t$ and $dv = e^t\,dt$. Then $du = dt$, $v = e^t$, and

$$\begin{aligned}\int te^t\,dt &= te^t - \int e^t\,dt\\ &= te^t - e^t + C\end{aligned}$$

Putting this in Equation 3, we get

$$\begin{aligned}\int t^2 e^t\,dt &= t^2 e^t - 2\int te^t\,dt\\ &= t^2 e^t - 2(te^t - e^t + C)\\ &= t^2 e^t - 2te^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C\end{aligned}$$

■

EXAMPLE 4 Evaluate $\int e^x \sin x\,dx$.

An easier method, using complex numbers, is given in Exercise 50 in Appendix G.

SOLUTION Neither e^x nor $\sin x$ becomes simpler when differentiated, but we try choosing $u = e^x$ and $dv = \sin x\,dx$ anyway. Then $du = e^x\,dx$ and $v = -\cos x$, so integration by parts gives

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx \tag{4}$$

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. This time we use $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx \tag{5}$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x\,dx$, which is where we started. However, if we put the expression for $\int e^x \cos x\,dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

Figure 1 illustrates Example 4 by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2}e^x(\sin x - \cos x)$. As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.

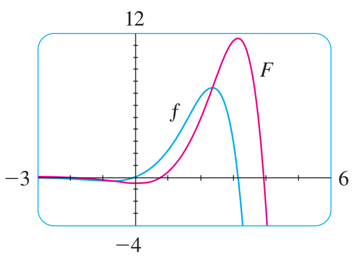


FIGURE 1

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x\,dx$ to both sides, we obtain

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$ ■

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

6 $$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

EXAMPLE 5 Calculate $\int_0^1 \tan^{-1}x\,dx$.

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then $$du = \frac{dx}{1 + x^2} \qquad v = x$$

So Formula 6 gives

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1 + x^2}\,dx$$

$$= 1 \cdot \tan^{-1}1 - 0 \cdot \tan^{-1}0 - \int_0^1 \frac{x}{1 + x^2}\,dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2}\,dx$$

Since $\tan^{-1}x \geqslant 0$ for $x \geqslant 0$, the integral in Example 5 can be interpreted as the area of the region shown in Figure 2.

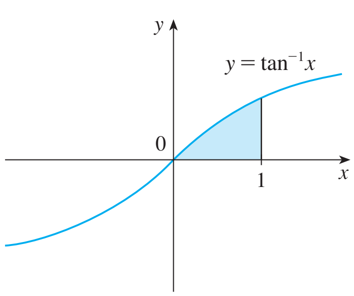


FIGURE 2

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\int_0^1 \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big]_1^2$$

$$= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2$$

Therefore $$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ ■

EXAMPLE 6 Prove the reduction formula

Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n - 1$ and $n - 2$.

7 $$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n - 1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let

$$u = \sin^{n-1}x \qquad dv = \sin x\,dx$$

Then $$du = (n-1)\sin^{n-2}x\cos x\,dx \qquad v = -\cos x$$

so integration by parts gives

$$\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or $$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$ ■

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

7.1 EXERCISES

1–2 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int xe^{2x}\,dx$; $u = x$, $dv = e^{2x}\,dx$

2. $\int \sqrt{x}\ln x\,dx$; $u = \ln x$, $dv = \sqrt{x}\,dx$

3–36 Evaluate the integral.

3. $\int x\cos 5x\,dx$

4. $\int ye^{0.2y}\,dy$

5. $\int te^{-3t}\,dt$

6. $\int (x-1)\sin \pi x\,dx$

7. $\int (x^2+2x)\cos x\,dx$

8. $\int t^2\sin \beta t\,dt$

9. $\int \cos^{-1}x\,dx$

10. $\int \ln\sqrt{x}\,dx$

11. $\int t^4\ln t\,dt$

12. $\int \tan^{-1}2y\,dy$

13. $\int t\csc^2 t\,dt$

14. $\int x\cosh ax\,dx$

15. $\int (\ln x)^2\,dx$

16. $\int \frac{z}{10^z}\,dz$

17. $\int e^{2\theta}\sin 3\theta\,d\theta$

18. $\int e^{-\theta}\cos 2\theta\,d\theta$

19. $\int z^3e^z\,dz$

20. $\int x\tan^2 x\,dx$

21. $\int \frac{xe^{2x}}{(1+2x)^2}\,dx$

22. $\int (\arcsin x)^2\,dx$

23. $\int_0^{1/2} x\cos \pi x\,dx$

24. $\int_0^1 (x^2+1)e^{-x}\,dx$

25. $\int_0^2 y\sinh y\,dy$

26. $\int_1^2 w^2\ln w\,dw$

27. $\int_1^5 \frac{\ln R}{R^2}\,dR$

28. $\int_0^{2\pi} t^2\sin 2t\,dt$

29. $\int_0^{\pi} x\sin x\cos x\,dx$

30. $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

31. $\displaystyle\int_1^5 \frac{M}{e^M}\,dM$

32. $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$

33. $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

34. $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

35. $\displaystyle\int_1^2 x^4(\ln x)^2\,dx$

36. $\displaystyle\int_0^t e^s \sin(t-s)\,ds$

37–42 First make a substitution and then use integration by parts to evaluate the integral.

37. $\displaystyle\int e^{\sqrt{x}}\,dx$

38. $\displaystyle\int \cos(\ln x)\,dx$

39. $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

40. $\displaystyle\int_0^{\pi} e^{\cos t} \sin 2t\,dt$

41. $\displaystyle\int x \ln(1+x)\,dx$

42. $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx$

43–46 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

43. $\displaystyle\int xe^{-2x}\,dx$

44. $\displaystyle\int x^{3/2} \ln x\,dx$

45. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

46. $\displaystyle\int x^2 \sin 2x\,dx$

47. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

48. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

49. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.
(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}$$

50. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\,\frac{\pi}{2}$$

51–54 Use integration by parts to prove the reduction formula.

51. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

52. $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

53. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$

54. $\displaystyle\int \sec^n x\,dx = \frac{\tan x \sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n \neq 1)$

55. Use Exercise 51 to find $\int (\ln x)^3\,dx$.

56. Use Exercise 52 to find $\int x^4 e^x\,dx$.

57–58 Find the area of the region bounded by the given curves.

57. $y = x^2 \ln x, \quad y = 4 \ln x$

58. $y = x^2 e^{-x}, \quad y = xe^{-x}$

59–60 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

59. $y = \arcsin\left(\frac{1}{2}x\right), \quad y = 2 - x^2$

60. $y = x\ln(x+1), \quad y = 3x - x^2$

61–64 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves about the given axis.

61. $y = \cos(\pi x/2),\ y = 0,\ 0 \leqslant x \leqslant 1;$ about the y-axis

62. $y = e^x,\ y = e^{-x},\ x = 1;$ about the y-axis

63. $y = e^{-x},\ y = 0,\ x = -1,\ x = 0;$ about $x = 1$

64. $y = e^x,\ x = 0,\ y = 3;$ about the x-axis

65. Calculate the volume generated by rotating the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 2$ about each axis.
(a) The y-axis
(b) The x-axis

66. Calculate the average value of $f(x) = x\sec^2 x$ on the interval $[0, \pi/4]$.

67. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right)dt$ was discussed in Example 4.3.3 and is used extensively in the theory of optics. Find $\int S(x)\,dx$. [Your answer will involve $S(x)$.]

68. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

69. A particle that moves along a straight line has velocity $v(t) = t^2e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

70. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

71. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

72. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]

(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

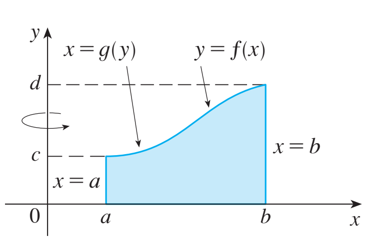
(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

73. We arrived at Formula 5.3.2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 5.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$



74. Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.

(b) Use Exercise 50 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.

(d) Use part (c) and Exercises 49 and 50 to show that

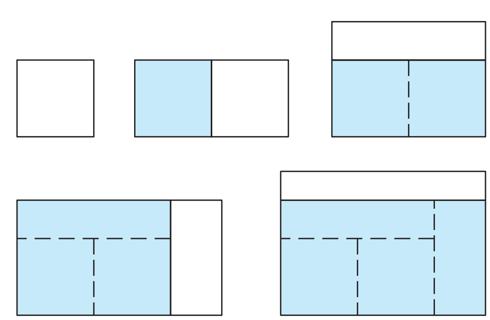
$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product*.

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.



7.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

EXAMPLE 1 Evaluate $\int \cos^3 x\, dx$.

SOLUTION Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x\, dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. Thus here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

We can then evaluate the integral by substituting $u = \sin x$, so $du = \cos x\, dx$ and

$$\begin{aligned}\int \cos^3 x\, dx &= \int \cos^2 x \cdot \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx \\ &= \int (1 - u^2)\, du = u - \tfrac{1}{3}u^3 + C \\ &= \sin x - \tfrac{1}{3}\sin^3 x + C\end{aligned}$$

■

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine). The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

EXAMPLE 2 Find $\int \sin^5 x \cos^2 x\, dx$.

SOLUTION We could convert $\cos^2 x$ to $1 - \sin^2 x$, but we would be left with an expression in terms of $\sin x$ with no extra $\cos x$ factor. Instead, we separate a single sine factor and rewrite the remaining $\sin^4 x$ factor in terms of $\cos x$:

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Figure 1 shows the graphs of the integrand $\sin^5 x \cos^2 x$ in Example 2 and its indefinite integral (with $C = 0$). Which is which?

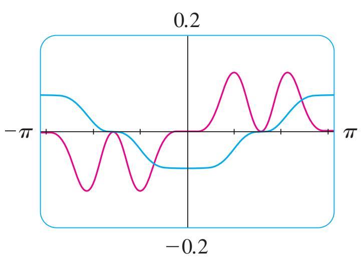


FIGURE 1

Substituting $u = \cos x$, we have $du = -\sin x\, dx$ and so

$$\begin{aligned}\int \sin^5 x \cos^2 x\, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x\, dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx \\ &= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6)\, du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C\end{aligned}$$

■

In the preceding examples, an odd power of sine or cosine enabled us to separate a single factor and convert the remaining even power. If the integrand contains even powers of both sine and cosine, this strategy fails. In this case, we can take advantage of the following half-angle identities (see Equations 17b and 17a in Appendix D):

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \text{and} \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

EXAMPLE 3 Evaluate $\int_0^\pi \sin^2 x\, dx$.

Example 3 shows that the area of the region shown in Figure 2 is $\pi/2$.

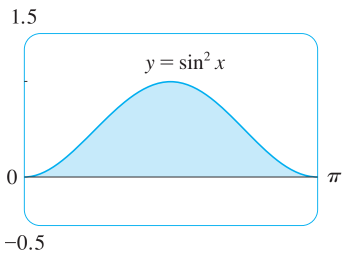


FIGURE 2

SOLUTION If we write $\sin^2 x = 1 - \cos^2 x$, the integral is no simpler to evaluate. Using the half-angle formula for $\sin^2 x$, however, we have

$$\begin{aligned}\int_0^\pi \sin^2 x\, dx &= \tfrac{1}{2}\int_0^\pi (1 - \cos 2x)\, dx \\ &= \left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right)\right]_0^\pi \\ &= \tfrac{1}{2}\left(\pi - \tfrac{1}{2}\sin 2\pi\right) - \tfrac{1}{2}\left(0 - \tfrac{1}{2}\sin 0\right) = \tfrac{1}{2}\pi\end{aligned}$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$. Another method for evaluating this integral was given in Exercise 7.1.47. ■

EXAMPLE 4 Find $\int \sin^4 x\, dx$.

SOLUTION We could evaluate this integral using the reduction formula for $\int \sin^n x\, dx$ (Equation 7.1.7) together with Example 3 (as in Exercise 7.1.47), but a better method is to write $\sin^4 x = (\sin^2 x)^2$ and use a half-angle formula:

$$\begin{aligned}\int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx \\ &= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx\end{aligned}$$

Since $\cos^2 2x$ occurs, we must use another half-angle formula

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\begin{aligned}\int \sin^4 x\, dx &= \tfrac{1}{4}\int \left[1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)\right] dx \\ &= \tfrac{1}{4}\int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{1}{4}\left(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right) + C\end{aligned}$$ ■

To summarize, we list guidelines to follow when evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$, where $m \geqslant 0$ and $n \geqslant 0$ are integers.

Strategy for Evaluating $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x\, dx &= \int \sin^m x\, (\cos^2 x)^k \cos x\, dx \\ &= \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx\end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x\, dx &= \int (\sin^2 x)^k \cos^n x\, \sin x\, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x\, \sin x\, dx\end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x\, dx$. Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$. Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

EXAMPLE 5 Evaluate $\int \tan^6 x \sec^4 x\, dx$.

SOLUTION If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$. We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x\, dx$:

$$\begin{aligned}\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x \sec^2 x \sec^2 x\, dx \\ &= \int \tan^6 x\, (1 + \tan^2 x) \sec^2 x\, dx \\ &= \int u^6 (1 + u^2)\, du = \int (u^6 + u^8)\, du \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C \\ &= \tfrac{1}{7} \tan^7 x + \tfrac{1}{9} \tan^9 x + C\end{aligned}$$

■

EXAMPLE 6 Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

SOLUTION If we separate a $\sec^2\theta$ factor, as in the preceding example, we are left with a $\sec^5\theta$ factor, which isn't easily converted to tangent. However, if we separate a $\sec\theta \tan\theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$. We can then evaluate the integral by substituting $u = \sec\theta$, so $du = \sec\theta \tan\theta \, d\theta$:

$$\begin{aligned}\int \tan^5\theta \sec^7\theta \, d\theta &= \int \tan^4\theta \sec^6\theta \sec\theta \tan\theta \, d\theta \\ &= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta \tan\theta \, d\theta \\ &= \int (u^2 - 1)^2 u^6 \, du \\ &= \int (u^{10} - 2u^8 + u^6) \, du \\ &= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \\ &= \tfrac{1}{11}\sec^{11}\theta - \tfrac{2}{9}\sec^9\theta + \tfrac{1}{7}\sec^7\theta + C\end{aligned}$$

■

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x \, dx &= \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx\end{aligned}$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \, dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx\end{aligned}$$

Then substitute $u = \sec x$.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity. We will sometimes need to be able to integrate $\tan x$ by using the formula established in Chapter 6:

$$\int \tan x \, dx = \ln|\sec x| + C$$

We will also need the indefinite integral of secant:

Formula 1 was discovered by James Gregory in 1668. (See his biography on page 153.) Gregory used this formula to solve a problem in constructing nautical tables.

1
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

We could verify Formula 1 by differentiating the right side, or as follows. First we multiply numerator and denominator by $\sec x + \tan x$:

$$\int \sec x\,dx = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)\,dx$, so the integral becomes $\int (1/u)\,du = \ln|u| + C$. Thus we have

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

EXAMPLE 7 Find $\int \tan^3 x\,dx$.

SOLUTION Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

In the first integral we mentally substituted $u = \tan x$ so that $du = \sec^2 x\,dx$. ■

If an even power of tangent appears with an odd power of secant, it is helpful to express the integrand completely in terms of $\sec x$. Powers of $\sec x$ may require integration by parts, as shown in the following example.

EXAMPLE 8 Find $\int \sec^3 x\,dx$.

SOLUTION Here we integrate by parts with

$$u = \sec x \qquad dv = \sec^2 x\,dx$$
$$du = \sec x \tan x\,dx \qquad v = \tan x$$

Then
$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\, dx = \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$ ■

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 8. Integrals of the form $\int \cot^m x \csc^n x\, dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Finally, we can make use of another set of trigonometric identities:

These product identities are discussed in Appendix D.

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x\, dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x\, dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x]\, dx$$

$$= \tfrac{1}{2}\int (-\sin x + \sin 9x)\, dx$$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{9}\cos 9x\right) + C$$ ■

7.2 EXERCISES

1–49 Evaluate the integral.

1. $\int \sin^2 x \cos^3 x\, dx$

2. $\int \sin^3\theta \cos^4\theta\, d\theta$

3. $\int_0^{\pi/2} \sin^7\theta \cos^5\theta\, d\theta$

4. $\int_0^{\pi/2} \sin^5 x\, dx$

5. $\int \sin^5(2t) \cos^2(2t)\, dt$

6. $\int t \cos^5(t^2)\, dt$

7. $\int_0^{\pi/2} \cos^2\theta\, d\theta$

8. $\int_0^{2\pi} \sin^2\left(\tfrac{1}{3}\theta\right) d\theta$

9. $\int_0^{\pi} \cos^4(2t)\, dt$

10. $\int_0^{\pi} \sin^2 t \cos^4 t\, dt$

11. $\int_0^{\pi/2} \sin^2 x \cos^2 x\, dx$

12. $\int_0^{\pi/2} (2 - \sin\theta)^2\, d\theta$

13. $\int \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$

14. $\int \frac{\sin^2(1/t)}{t^2}\, dt$

15. $\int \cot x \cos^2 x\, dx$

16. $\int \tan^2 x \cos^3 x\, dx$

17. $\int \sin^2 x \sin 2x\, dx$

18. $\int \sin x \cos\left(\tfrac{1}{2}x\right) dx$

19. $\int t \sin^2 t\, dt$

20. $\int x \sin^3 x\, dx$

21. $\int \tan x \sec^3 x\, dx$

22. $\int \tan^2\theta \sec^4\theta\, d\theta$

23. $\int \tan^2 x\, dx$

24. $\int (\tan^2 x + \tan^4 x)\, dx$

25. $\int \tan^4 x \sec^6 x\, dx$

26. $\int_0^{\pi/4} \sec^6\theta \tan^6\theta\, d\theta$

27. $\int \tan^3 x \sec x\, dx$

28. $\int \tan^5 x \sec^3 x\, dx$

29. $\int \tan^3 x \sec^6 x\, dx$

30. $\int_0^{\pi/4} \tan^4 t\, dt$

31. $\int \tan^5 x \, dx$

32. $\int \tan^2 x \sec x \, dx$

33. $\int x \sec x \tan x \, dx$

34. $\int \frac{\sin \phi}{\cos^3 \phi} \, d\phi$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x \, dx$

36. $\int_{\pi/4}^{\pi/2} \cot^3 x \, dx$

37. $\int_{\pi/4}^{\pi/2} \cot^5 \phi \csc^3 \phi \, d\phi$

38. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \cot^4 \theta \, d\theta$

39. $\int \csc x \, dx$

40. $\int_{\pi/6}^{\pi/3} \csc^3 x \, dx$

41. $\int \sin 8x \cos 5x \, dx$

42. $\int \sin 2\theta \sin 6\theta \, d\theta$

43. $\int_0^{\pi/2} \cos 5t \cos 10t \, dt$

44. $\int \sin x \sec^5 x \, dx$

45. $\int_0^{\pi/6} \sqrt{1 + \cos 2x} \, dx$

46. $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta$

47. $\int \frac{1 - \tan^2 x}{\sec^2 x} \, dx$

48. $\int \frac{dx}{\cos x - 1}$

49. $\int x \tan^2 x \, dx$

50. If $\int_0^{\pi/4} \tan^6 x \sec x \, dx = I$, express the value of $\int_0^{\pi/4} \tan^8 x \sec x \, dx$ in terms of I.

51–54 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the integrand and its antiderivative (taking $C = 0$).

51. $\int x \sin^2(x^2) \, dx$

52. $\int \sin^5 x \cos^3 x \, dx$

53. $\int \sin 3x \sin 6x \, dx$

54. $\int \sec^4\left(\frac{1}{2}x\right) dx$

55. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ on the interval $[-\pi, \pi]$.

56. Evaluate $\int \sin x \cos x \, dx$ by four methods:

(a) the substitution $u = \cos x$
(b) the substitution $u = \sin x$
(c) the identity $\sin 2x = 2 \sin x \cos x$
(d) integration by parts

Explain the different appearances of the answers.

57–58 Find the area of the region bounded by the given curves.

57. $y = \sin^2 x, \quad y = \sin^3 x, \quad 0 \leqslant x \leqslant \pi$

58. $y = \tan x, \quad y = \tan^2 x, \quad 0 \leqslant x \leqslant \pi/4$

59–60 Use a graph of the integrand to guess the value of the integral. Then use the methods of this section to prove that your guess is correct.

59. $\int_0^{2\pi} \cos^3 x \, dx$

60. $\int_0^2 \sin 2\pi x \cos 5\pi x \, dx$

61–64 Find the volume obtained by rotating the region bounded by the curves about the given axis.

61. $y = \sin x, \ y = 0, \ \pi/2 \leqslant x \leqslant \pi$; about the x-axis

62. $y = \sin^2 x, \ y = 0, \ 0 \leqslant x \leqslant \pi$; about the x-axis

63. $y = \sin x, \ y = \cos x, \ 0 \leqslant x \leqslant \pi/4$; about $y = 1$

64. $y = \sec x, \ y = \cos x, \ 0 \leqslant x \leqslant \pi/3$; about $y = -1$

65. A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

66. Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155 \sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.

(a) Calculate the RMS voltage of household current.
(b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = A \sin(120\pi t)$.

67–69 Prove the formula, where m and n are positive integers.

67. $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

68. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

69. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

70. A *finite Fourier series* is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n \sin nx$$
$$= a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

7.3 Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\,|\cos \theta\,|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin \theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule (Equation 4.5.4), we obtain

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution.*

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one. (These are the same intervals used in Section 6.6 in defining the inverse functions.)

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$.

SOLUTION Let $x = 3 \sin \theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$. Then $dx = 3 \cos \theta\, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2\theta} = \sqrt{9 \cos^2\theta} = 3\,|\cos \theta\,| = 3 \cos \theta$$

(Note that $\cos\theta \geqslant 0$ because $-\pi/2 \leqslant \theta \leqslant \pi/2$.) Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

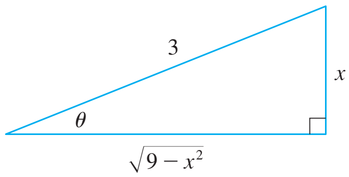


FIGURE 1

$\sin\theta = \dfrac{x}{3}$

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

$$\cot\theta = \frac{\sqrt{9 - x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$ ■

EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

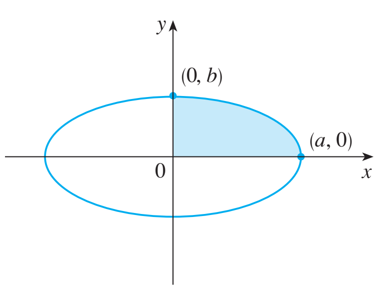


FIGURE 2

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

SOLUTION Solving the equation of the ellipse for y, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so
$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

To evaluate this integral we substitute $x = a\sin\theta$. Then $dx = a\cos\theta\,d\theta$. To change

the limits of integration we note that when $x = 0$, $\sin\theta = 0$, so $\theta = 0$; when $x = a$, $\sin\theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a|\cos\theta| = a\cos\theta$$

since $0 \leqslant \theta \leqslant \pi/2$. Therefore

$$\begin{aligned} A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= 2ab\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) = \pi ab \end{aligned}$$

We have shown that the area of an ellipse with semiaxes a and b is πab. In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2. ■

NOTE Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

EXAMPLE 3 Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx$.

SOLUTION Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2 + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

So we have

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2 + 4}} &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2} \\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C \\ &= -\frac{\csc\theta}{4} + C \end{aligned}$$

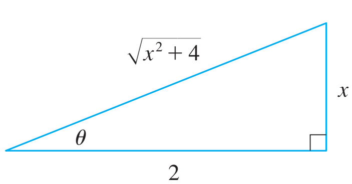


FIGURE 3

$\tan\theta = \dfrac{x}{2}$

We use Figure 3 to determine that $\csc\theta = \sqrt{x^2 + 4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2 + 4}} = -\frac{\sqrt{x^2 + 4}}{4x} + C$$ ■

EXAMPLE 4 Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,dx$.

SOLUTION It would be possible to use the trigonometric substitution $x = 2\tan\theta$ here (as in Example 3). But the direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C$$ ■

NOTE Example 4 illustrates the fact that even when trigonometric substitutions are possible, they may not give the easiest solution. You should look for a simpler method first.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

SOLUTION 1 We let $x = a\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a\sec\theta\,\tan\theta\,d\theta$ and

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\,|\tan\theta| = a\tan\theta$$

Therefore

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\,\tan\theta}{a\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

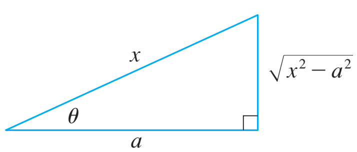


FIGURE 4

$\sec\theta = \dfrac{x}{a}$

The triangle in Figure 4 gives $\tan\theta = \sqrt{x^2-a^2}/a$, so we have

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C$$

Writing $C_1 = C - \ln a$, we have

1
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C_1$$

SOLUTION 2 For $x > 0$ the hyperbolic substitution $x = a\cosh t$ can also be used. Using the identity $\cosh^2 y - \sinh^2 y = 1$, we have

$$\sqrt{x^2-a^2} = \sqrt{a^2(\cosh^2 t - 1)} = \sqrt{a^2\sinh^2 t} = a\sinh t$$

Since $dx = a\sinh t\,dt$, we obtain

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh t\,dt}{a\sinh t} = \int dt = t + C$$

Since $\cosh t = x/a$, we have $t = \cosh^{-1}(x/a)$ and

2
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

Although Formulas 1 and 2 look quite different, they are actually equivalent by Formula 6.7.4. ■

NOTE As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers. But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

As Example 6 shows, trigonometric substitution is sometimes a good idea when $(x^2 + a^2)^{n/2}$ occurs in an integral, where n is any integer. The same is true when $(a^2 - x^2)^{n/2}$ or $(x^2 - a^2)^{n/2}$ occur.

EXAMPLE 6 Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx$.

SOLUTION First we note that $(4x^2 + 9)^{3/2} = (\sqrt{4x^2 + 9})^3$ so trigonometric substitution is appropriate. Although $\sqrt{4x^2 + 9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$. When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\,d\theta$ and

$$\sqrt{4x^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx &= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\tfrac{3}{2}\sec^2\theta\,d\theta \\
&= \tfrac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \tfrac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta \\
&= \tfrac{3}{16}\int_0^{\pi/3} \frac{1 - \cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta
\end{aligned}$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$. Therefore

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}}\,dx &= -\tfrac{3}{16}\int_1^{1/2} \frac{1 - u^2}{u^2}\,du \\
&= \tfrac{3}{16}\int_1^{1/2} (1 - u^{-2})\,du = \tfrac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} \\
&= \tfrac{3}{16}\left[\left(\tfrac{1}{2} + 2\right) - (1 + 1)\right] = \frac{3}{32}
\end{aligned}$$

■

EXAMPLE 7 Evaluate $\displaystyle\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx$.

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$\begin{aligned}
3 - 2x - x^2 &= 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) \\
&= 4 - (x + 1)^2
\end{aligned}$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\,du$$

Figure 5 shows the graphs of the integrand in Example 7 and its indefinite integral (with $C = 0$). Which is which?

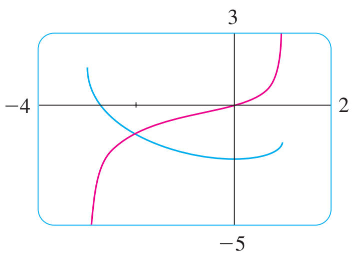


FIGURE 5

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\, d\theta$ and $\sqrt{4 - u^2} = 2\cos\theta$, so

$$\begin{aligned}
\int \frac{x}{\sqrt{3 - 2x - x^2}}\, dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\, 2\cos\theta\, d\theta \\
&= \int (2\sin\theta - 1)\, d\theta \\
&= -2\cos\theta - \theta + C \\
&= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\
&= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C
\end{aligned}$$

■

7.3 EXERCISES

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4 - x^2}}$ $x = 2\sin\theta$

2. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 4}}\, dx$ $x = 2\tan\theta$

3. $\displaystyle\int \frac{\sqrt{x^2 - 4}}{x}\, dx$ $x = 2\sec\theta$

4–30 Evaluate the integral.

4. $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$

5. $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x^4}\, dx$

6. $\displaystyle\int_0^3 \frac{x}{\sqrt{36 - x^2}}\, dx$

7. $\displaystyle\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}}$, $a > 0$

8. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2 - 16}}$

9. $\displaystyle\int_2^3 \frac{dx}{(x^2 - 1)^{3/2}}$

10. $\displaystyle\int_0^{2/3} \sqrt{4 - 9x^2}\, dx$

11. $\displaystyle\int_0^{1/2} x\sqrt{1 - 4x^2}\, dx$

12. $\displaystyle\int_0^2 \frac{dt}{\sqrt{4 + t^2}}$

13. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^3}\, dx$

14. $\displaystyle\int_0^1 \frac{dx}{(x^2 + 1)^2}$

15. $\displaystyle\int_0^a x^2\sqrt{a^2 - x^2}\, dx$

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2 - 1}}$

17. $\displaystyle\int \frac{x}{\sqrt{x^2 - 7}}\, dx$

18. $\displaystyle\int \frac{dx}{[(ax)^2 - b^2]^{3/2}}$

19. $\displaystyle\int \frac{\sqrt{1 + x^2}}{x}\, dx$

20. $\displaystyle\int \frac{x}{\sqrt{1 + x^2}}\, dx$

21. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9 - 25x^2}}\, dx$

22. $\displaystyle\int_0^1 \sqrt{x^2 + 1}\, dx$

23. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$

24. $\displaystyle\int_0^1 \sqrt{x - x^2}\, dx$

25. $\displaystyle\int x^2\sqrt{3 + 2x - x^2}\, dx$

26. $\displaystyle\int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}}\, dx$

27. $\displaystyle\int \sqrt{x^2 + 2x}\, dx$

28. $\displaystyle\int \frac{x^2 + 1}{(x^2 - 2x + 2)^2}\, dx$

29. $\displaystyle\int x\sqrt{1 - x^4}\, dx$

30. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1 + \sin^2 t}}\, dt$

31. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 6.7.3.

32. Evaluate

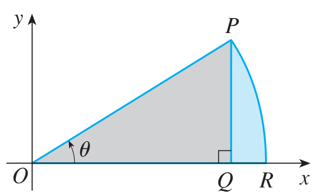
$$\int \frac{x^2}{(x^2 + a^2)^{3/2}}\,dx$$

(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a \sinh t$.

33. Find the average value of $f(x) = \sqrt{x^2 - 1}/x$, $1 \leqslant x \leqslant 7$.

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

35. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



36. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

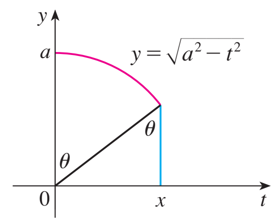
37. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the curves $y = 9/(x^2 + 9)$, $y = 0$, $x = 0$, and $x = 3$.

38. Find the volume of the solid obtained by rotating about the line $x = 1$ the region under the curve $y = x\sqrt{1 - x^2}$, $0 \leqslant x \leqslant 1$.

39. (a) Use trigonometric substitution to verify that

$$\int_0^x \sqrt{a^2 - t^2}\,dt = \tfrac{1}{2}a^2 \sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



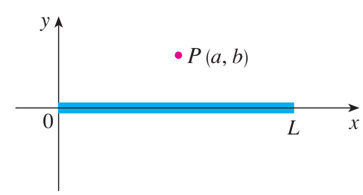
40. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \leqslant 8$ into two parts. Find the areas of both parts.

41. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

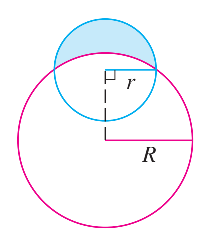
42. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\,dx$$

where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



43. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



44. A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?

7.4 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions,* that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*. Recall that if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, then the degree of P is n and we write $\deg(P) = n$.

If f is *improper*, that is, $\deg(P) \geqslant \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

1 $$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

EXAMPLE 1 Find $\displaystyle\int \frac{x^3 + x}{x - 1}\,dx$.

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x-1 & x^3 \qquad + x \\ & x^3 - x^2 \\ \hline & x^2 + x \\ & x^2 - x \\ \hline & 2x \\ & 2x - 2 \\ \hline & 2 \end{array}$$

SOLUTION Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$ ■

In the case of an Equation 1 whose denominator is more complicated, the next step is to factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

CASE I The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

2 $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

SOLUTION Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

3 $$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

Another method for finding A, B, and C is given in the note after this example.

To determine the values of A, B, and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

4 $$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

5 $$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side—namely, 1. Likewise, the coefficients of x are equal and the constant terms are equal. This gives the following system of equations for A, B, and C:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

We could check our work by taking the terms to a common denominator and adding them.

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2} \right) dx$$

$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K$$

Figure 1 shows the graphs of the integrand in Example 2 and its indefinite integral (with $K = 0$). Which is which?

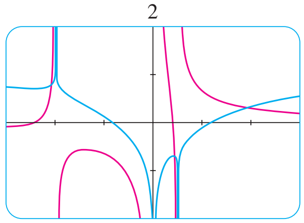


FIGURE 1

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2\,dx$ and $dx = \frac{1}{2}\,du$. ■

NOTE We can use an alternative method to find the coefficients A, B, and C in Example 2. Equation 4 is an identity; it is true for every value of x. Let's choose values of x that simplify the equation. If we put $x = 0$ in Equation 4, then the second and third terms on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$. Likewise, $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$. (You may object that Equation 3 is not valid for $x = 0, \frac{1}{2}$, or -2, so why should Equation 4 be valid for those values? In fact, Equation 4 is true for all values of x, even $x = 0, \frac{1}{2}$, and -2. See Exercise 73 for the reason.)

EXAMPLE 3 Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx$$

$$= \frac{1}{2a}\left(\ln|x - a| - \ln|x + a| \right) + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

6 $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

See Exercises 57–58 for ways of using Formula 6. ■

CASE II $Q(x)$ **is a product of linear factors, some of which are repeated.**
Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

7 $$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

By way of illustration, we could write

$$\frac{x^3 - x + 1}{x^2(x - 1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} + \frac{E}{(x - 1)^3}$$

but we prefer to work out in detail a simpler example.

EXAMPLE 4 Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$.

SOLUTION The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator $Q(x) = x^3 - x^2 - x + 1$. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) \\ &= (x - 1)^2(x + 1) \end{aligned}$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x - 1)^2(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$$

Multiplying by the least common denominator, $(x - 1)^2(x + 1)$, we get

8 $$\begin{aligned} 4x &= A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2 \\ &= (A + C)x^2 + (B - 2C)x + (-A + B + C) \end{aligned}$$

Now we equate coefficients:

$$\begin{aligned} A \qquad + \quad C &= 0 \\ B - 2C &= 4 \\ -A + B + \quad C &= 0 \end{aligned}$$

Another method for finding the coefficients:
Put $x = 1$ in (8): $B = 2$.
Put $x = -1$: $C = -1$.
Put $x = 0$: $A = B + C = 1$.

Solving, we obtain $A = 1$, $B = 2$, and $C = -1$, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right] dx$$

$$= \frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x-1} - \ln|x + 1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$

■

CASE III $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

9
$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. For instance, the function given by $f(x) = x/[(x-2)(x^2+1)(x^2+4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$

The term given in (9) can be integrated by completing the square (if necessary) and using the formula

10
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

SOLUTION Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$= (A + B)x^2 + Cx + 4A$$

Equating coefficients, we obtain

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Therefore $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x-1}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 10 with $a = 2$:

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}(x/2) + K \qquad \blacksquare$$

EXAMPLE 6 Evaluate $\displaystyle\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx$.

SOLUTION Since the degree of the numerator is *not less than* the degree of the denominator, we first divide and obtain

$$\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x-1}{4x^2 - 4x + 3}$$

Notice that the quadratic $4x^2 - 4x + 3$ is irreducible because its discriminant is $b^2 - 4ac = -32 < 0$. This means it can't be factored, so we don't need to use the partial fraction technique.

To integrate the given function we complete the square in the denominator:

$$4x^2 - 4x + 3 = (2x-1)^2 + 2$$

This suggests that we make the substitution $u = 2x - 1$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u+1)$, so

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\,dx = \int \left(1 + \frac{x-1}{4x^2 - 4x + 3}\right) dx$$

$$= x + \tfrac{1}{2}\int \frac{\frac{1}{2}(u+1) - 1}{u^2+2}\,du = x + \tfrac{1}{4}\int \frac{u-1}{u^2+2}\,du$$

$$= x + \tfrac{1}{4}\int \frac{u}{u^2+2}\,du - \tfrac{1}{4}\int \frac{1}{u^2+2}\,du$$

$$= x + \tfrac{1}{8}\ln(u^2+2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \tfrac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C \qquad \blacksquare$$

NOTE Example 6 illustrates the general procedure for integrating a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c} \qquad \text{where } b^2 - 4ac < 0$$

We complete the square in the denominator and then make a substitution that brings the integral into the form

$$\int \frac{Cu + D}{u^2 + a^2}\,du = C\int \frac{u}{u^2 + a^2}\,du + D\int \frac{1}{u^2 + a^2}\,du$$

Then the first integral is a logarithm and the second is expressed in terms of $\tan^{-1}$.

CASE IV ***Q*(*x*) contains a repeated irreducible quadratic factor.**

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (9), the sum

11
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms in (11) can be integrated by using a substitution or by first completing the square if necessary.

It would be extremely tedious to work out by hand the numerical values of the coefficients in Example 7. Most computer algebra systems, however, can find the numerical values very quickly. For instance, the Maple command

```
convert(f, parfrac, x)
```

or the Mathematica command

```
Apart[f]
```

gives the following values:

$A = -1, \quad B = \frac{1}{8}, \quad C = D = -1,$

$E = \frac{15}{8}, \quad F = -\frac{1}{8}, \quad G = H = \frac{3}{4},$

$I = -\frac{1}{2}, \quad J = \frac{1}{2}$

EXAMPLE 7 Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2} + \frac{Ix + J}{(x^2 + 1)^3}$$ ■

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx$.

SOLUTION The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned}
-x^3 + 2x^2 - x + 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\
&= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\
&= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A
\end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$. Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\,dx = \int \left(\frac{1}{x} - \frac{x + 1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right) dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2 + 1}\,dx - \int \frac{dx}{x^2 + 1} + \int \frac{x\,dx}{(x^2 + 1)^2}$$

$$= \ln|x| - \tfrac{1}{2}\ln(x^2 + 1) - \tan^{-1}x - \frac{1}{2(x^2 + 1)} + K \quad ■$$

In the second and fourth terms we made the mental substitution $u = x^2 + 1$.

NOTE Example 8 worked out rather nicely because the coefficient E turned out to be 0. In general, we might get a term of the form $1/(x^2 + 1)^2$. One way to integrate such a term is to make the substitution $x = \tan\theta$. Another method is to use the formula in Exercise 72.

Sometimes partial fractions can be avoided when integrating a rational function. For instance, although the integral

$$\int \frac{x^2 + 1}{x(x^2 + 3)}\,dx$$

could be evaluated by using the method of Case III, it's much easier to observe that if $u = x(x^2 + 3) = x^3 + 3x$, then $du = (3x^2 + 3)\,dx$ and so

$$\int \frac{x^2 + 1}{x(x^2 + 3)}\,dx = \tfrac{1}{3}\ln|x^3 + 3x| + C$$

Rationalizing Substitutions

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

EXAMPLE 9 Evaluate $\displaystyle\int \frac{\sqrt{x + 4}}{x}\,dx$.

SOLUTION Let $u = \sqrt{x + 4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x + 4}}{x}\,dx = \int \frac{u}{u^2 - 4}\,2u\,du = 2\int \frac{u^2}{u^2 - 4}\,du = 2\int \left(1 + \frac{4}{u^2 - 4}\right) du$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x + 4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2 - 4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln\left|\frac{u - 2}{u + 2}\right| + C$$

$$= 2\sqrt{x + 4} + 2\ln\left|\frac{\sqrt{x + 4} - 2}{\sqrt{x + 4} + 2}\right| + C \quad ■$$

7.4 EXERCISES

1–6 Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{4 + x}{(1 + 2x)(3 - x)}$ (b) $\dfrac{1 - x}{x^3 + x^4}$

2. (a) $\dfrac{x - 6}{x^2 + x - 6}$ (b) $\dfrac{x^2}{x^2 + x + 6}$

3. (a) $\dfrac{1}{x^2 + x^4}$ (b) $\dfrac{x^3 + 1}{x^3 - 3x^2 + 2x}$

4. (a) $\dfrac{x^4 - 2x^3 + x^2 + 2x - 1}{x^2 - 2x + 1}$ (b) $\dfrac{x^2 - 1}{x^3 + x^2 + x}$

5. (a) $\dfrac{x^6}{x^2 - 4}$ (b) $\dfrac{x^4}{(x^2 - x + 1)(x^2 + 2)^2}$

6. (a) $\dfrac{t^6 + 1}{t^6 + t^3}$ (b) $\dfrac{x^5 + 1}{(x^2 - x)(x^4 + 2x^2 + 1)}$

7–38 Evaluate the integral.

7. $\displaystyle\int \frac{x^4}{x - 1}\,dx$

8. $\displaystyle\int \frac{3t - 2}{t + 1}\,dt$

9. $\displaystyle\int \frac{5x + 1}{(2x + 1)(x - 1)}\,dx$

10. $\displaystyle\int \frac{y}{(y + 4)(2y - 1)}\,dy$

11. $\displaystyle\int_0^1 \frac{2}{2x^2 + 3x + 1}\,dx$

12. $\displaystyle\int_0^1 \frac{x - 4}{x^2 - 5x + 6}\,dx$

13. $\displaystyle\int \frac{ax}{x^2 - bx}\,dx$

14. $\displaystyle\int \frac{1}{(x + a)(x + b)}\,dx$

15. $\displaystyle\int_{-1}^0 \frac{x^3 - 4x + 1}{x^2 - 3x + 2}\,dx$

16. $\displaystyle\int_1^2 \frac{x^3 + 4x^2 + x - 1}{x^3 + x^2}\,dx$

17. $\displaystyle\int_1^2 \frac{4y^2 - 7y - 12}{y(y + 2)(y - 3)}\,dy$

18. $\displaystyle\int_1^2 \frac{3x^2 + 6x + 2}{x^2 + 3x + 2}\,dx$

19. $\displaystyle\int_0^1 \frac{x^2 + x + 1}{(x + 1)^2(x + 2)}\,dx$

20. $\displaystyle\int_2^3 \frac{x(3 - 5x)}{(3x - 1)(x - 1)^2}\,dx$

21. $\displaystyle\int \frac{dt}{(t^2 - 1)^2}$

22. $\displaystyle\int \frac{x^4 + 9x^2 + x + 2}{x^2 + 9}\,dx$

23. $\displaystyle\int \frac{10}{(x - 1)(x^2 + 9)}\,dx$

24. $\displaystyle\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx$

25. $\displaystyle\int \frac{4x}{x^3 + x^2 + x + 1}\,dx$

26. $\displaystyle\int \frac{x^2 + x + 1}{(x^2 + 1)^2}\,dx$

27. $\displaystyle\int \frac{x^3 + 4x + 3}{x^4 + 5x^2 + 4}\,dx$

28. $\displaystyle\int \frac{x^3 + 6x - 2}{x^4 + 6x^2}\,dx$

29. $\displaystyle\int \frac{x + 4}{x^2 + 2x + 5}\,dx$

30. $\displaystyle\int \frac{x^3 - 2x^2 + 2x - 5}{x^4 + 4x^2 + 3}\,dx$

31. $\displaystyle\int \frac{1}{x^3 - 1}\,dx$

32. $\displaystyle\int_0^1 \frac{x}{x^2 + 4x + 13}\,dx$

33. $\displaystyle\int_0^1 \frac{x^3 + 2x}{x^4 + 4x^2 + 3}\,dx$

34. $\displaystyle\int \frac{x^5 + x - 1}{x^3 + 1}\,dx$

35. $\displaystyle\int \frac{5x^4 + 7x^2 + x + 2}{x(x^2 + 1)^2}\,dx$

36. $\displaystyle\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}\,dx$

37. $\displaystyle\int \frac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2}\,dx$

38. $\displaystyle\int \frac{x^3 + 2x^2 + 3x - 2}{(x^2 + 2x + 2)^2}\,dx$

39–52 Make a substitution to express the integrand as a rational function and then evaluate the integral.

39. $\displaystyle\int \frac{dx}{x\sqrt{x - 1}}$

40. $\displaystyle\int \frac{dx}{2\sqrt{x + 3} + x}$

41. $\displaystyle\int \frac{dx}{x^2 + x\sqrt{x}}$

42. $\displaystyle\int_0^1 \frac{1}{1 + \sqrt[3]{x}}\,dx$

43. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2 + 1}}\,dx$

44. $\displaystyle\int \frac{dx}{(1 + \sqrt{x})^2}$

45. $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$ [*Hint:* Substitute $u = \sqrt[6]{x}$.]

46. $\displaystyle\int \frac{\sqrt{1 + \sqrt{x}}}{x}\,dx$

47. $\displaystyle\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx$

48. $\displaystyle\int \frac{\sin x}{\cos^2 x - 3\cos x}\,dx$

49. $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt$

50. $\displaystyle\int \frac{e^x}{(e^x - 2)(e^{2x} + 1)}\,dx$

51. $\displaystyle\int \frac{dx}{1 + e^x}$

52. $\displaystyle\int \frac{\cosh t}{\sinh^2 t + \sinh^4 t}\,dt$

53–54 Use integration by parts, together with the techniques of this section, to evaluate the integral.

53. $\displaystyle\int \ln(x^2 - x + 2)\,dx$

54. $\displaystyle\int x\tan^{-1}x\,dx$

55. Use a graph of $f(x) = 1/(x^2 - 2x - 3)$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

56. Evaluate

$$\int \frac{1}{x^2 + k}\,dx$$

by considering several cases for the constant k.

57–58 Evaluate the integral by completing the square and using Formula 6.

57. $\displaystyle\int \frac{dx}{x^2 - 2x}$

58. $\displaystyle\int \frac{2x + 1}{4x^2 + 12x - 7}\, dx$

59. The German mathematician Karl Weierstrass (1815–1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t.

(a) If $t = \tan(x/2)$, $-\pi < x < \pi$, sketch a right triangle or use trigonometric identities to show that

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1 + t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1 + t^2}}$$

(b) Show that

$$\cos x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad \sin x = \frac{2t}{1 + t^2}$$

(c) Show that

$$dx = \frac{2}{1 + t^2}\, dt$$

60–63 Use the substitution in Exercise 59 to transform the integrand into a rational function of t and then evaluate the integral.

60. $\displaystyle\int \frac{dx}{1 - \cos x}$

61. $\displaystyle\int \frac{1}{3 \sin x - 4 \cos x}\, dx$

62. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x - \cos x}\, dx$

63. $\displaystyle\int_{0}^{\pi/2} \frac{\sin 2x}{2 + \cos x}\, dx$

64–65 Find the area of the region under the given curve from 1 to 2.

64. $\displaystyle y = \frac{1}{x^3 + x}$

65. $\displaystyle y = \frac{x^2 + 1}{3x - x^2}$

66. Find the volume of the resulting solid if the region under the curve $y = 1/(x^2 + 3x + 2)$ from $x = 0$ to $x = 1$ is rotated about (a) the x-axis and (b) the y-axis.

67. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. (The photo shows a screw-worm fly, the first pest effectively eliminated from a region by this method.)



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Let P represent the number of female insects in a population and S the number of sterile males introduced each generation. Let r be the per capita rate of production of females by females, provided their chosen mate is not sterile. Then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r - 1)P - S]}\, dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 1.1$ and 900 sterile males are added initially. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P.)

68. Factor $x^4 + 1$ as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4 + 1)\, dx$.

69. (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\, dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

CAS **70.** (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\, dx$ and graph f and its indefinite integral on the same screen.

(c) Use the graph of f to discover the main features of the graph of $\int f(x)\, dx$.

71. The rational number $\frac{22}{7}$ has been used as an approximation to the number π since the time of Archimedes. Show that

$$\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\, dx = \frac{22}{7} - \pi$$

72. (a) Use integration by parts to show that, for any positive integer n,

$$\int \frac{dx}{(x^2 + a^2)^n}\, dx = \frac{x}{2a^2(n - 1)(x^2 + a^2)^{n-1}} + \frac{2n - 3}{2a^2(n - 1)} \int \frac{dx}{(x^2 + a^2)^{n-1}}$$

(b) Use part (a) to evaluate

$$\int \frac{dx}{(x^2 + 1)^2} \quad \text{and} \quad \int \frac{dx}{(x^2 + 1)^3}$$

73. Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x. [*Hint:* Use continuity.]

74. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx$$

is a rational function, find the value of $f'(0)$.

75. If $a \neq 0$ and n is a positive integer, find the partial fraction decomposition of

$$f(x) = \frac{1}{x^n(x-a)}$$

[*Hint:* First find the coefficient of $1/(x-a)$. Then subtract the resulting term and simplify what is left.]

7.5 Strategy for Integration

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Until now individual techniques have been applied in each section. For instance, we usually used substitution in Exercises 4.5, integration by parts in Exercises 7.1, and partial fractions in Exercises 7.4. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for applying a strategy is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter.

Table of Integration Formulas Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$	**2.** $\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert$
3. $\displaystyle\int e^x\,dx = e^x$	**4.** $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$
5. $\displaystyle\int \sin x\,dx = -\cos x$	**6.** $\displaystyle\int \cos x\,dx = \sin x$
7. $\displaystyle\int \sec^2 x\,dx = \tan x$	**8.** $\displaystyle\int \csc^2 x\,dx = -\cot x$
9. $\displaystyle\int \sec x \tan x\,dx = \sec x$	**10.** $\displaystyle\int \csc x \cot x\,dx = -\csc x$
11. $\displaystyle\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$	**12.** $\displaystyle\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$
13. $\displaystyle\int \tan x\,dx = \ln\lvert \sec x\rvert$	**14.** $\displaystyle\int \cot x\,dx = \ln\lvert \sin x\rvert$
15. $\displaystyle\int \sinh x\,dx = \cosh x$	**16.** $\displaystyle\int \cosh x\,dx = \sinh x$
17. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	**18.** $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$
***19.** $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	***20.** $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert$

Most of these formulas should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be memorized since they are easily derived. Formula 19 can be avoided by using partial fractions, and trigonometric substitutions can be used in place of Formula 20.

Once you are armed with these basic integration formulas, if you don't immediately see how to attack a given integral, you might try the following four-step strategy.

1. Simplify the Integrand if Possible Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}\,(1 + \sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta$$

$$= \int \sin\theta\,\cos\theta\,d\theta = \tfrac{1}{2}\int \sin 2\theta\,d\theta$$

$$\int (\sin x + \cos x)^2\,dx = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,dx$$

$$= \int (1 + 2\sin x\cos x)\,dx$$

2. Look for an Obvious Substitution Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\,dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{x^2 - 1}\,dx$$

we notice that if $u = x^2 - 1$, then $du = 2x\,dx$. Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. Classify the Integrand According to Its Form If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

(a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or of $\cot x$ and $\csc x$, then we use the substitutions recommended in Section 7.2.

(b) *Rational functions.* If f is a rational function, we use the procedure of Section 7.4 involving partial fractions.

(c) *Integration by parts.* If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv according to the advice given in Section 7.1. If you look at the functions in Exercises 7.1, you will see that most of them are the type just described.

(d) *Radicals.* Particular kinds of substitutions are recommended when certain radicals appear.

(i) If $\sqrt{\pm x^2 \pm a^2}$ occurs, we use a trigonometric substitution according to the table in Section 7.3.

(ii) If $\sqrt[n]{ax + b}$ occurs, we use the rationalizing substitution $u = \sqrt[n]{ax + b}$. More generally, this sometimes works for $\sqrt[n]{g(x)}$.

4. Try Again If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

(a) *Try substitution.* Even if no substitution is obvious (Step 2), some inspiration or ingenuity (or even desperation) may suggest an appropriate substitution.

(b) *Try parts.* Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions. Looking at Section 7.1, we see that it works on $\tan^{-1}x$, $\sin^{-1}x$, and $\ln x$, and these are all inverse functions.

(c) *Manipulate the integrand.* Algebraic manipulations (perhaps rationalizing the denominator or using trigonometric identities) may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here is an example:

$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\, dx = \int \frac{1 + \cos x}{1 - \cos^2 x}\, dx$$

$$= \int \frac{1 + \cos x}{\sin^2 x}\, dx = \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x} \right) dx$$

(d) *Relate the problem to previous problems.* When you have built up some experience in integration, you may be able to use a method on a given integral that is similar to a method you have already used on a previous integral. Or you may even be able to express the given integral in terms of a previous one. For instance, $\int \tan^2 x \sec x\, dx$ is a challenging integral, but if we make use of the identity $\tan^2 x = \sec^2 x - 1$, we can write

$$\int \tan^2 x \sec x\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$$

and if $\int \sec^3 x\, dx$ has previously been evaluated (see Example 7.2.8), then that calculation can be used in the present problem.

(e) *Use several methods.* Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions.

In the following examples we indicate a method of attack but do not fully work out the integral.

EXAMPLE 1 $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\, dx$

In Step 1 we rewrite the integral:

$$\int \frac{\tan^3 x}{\cos^3 x}\, dx = \int \tan^3 x \sec^3 x\, dx$$

The integral is now of the form $\int \tan^m x \sec^n x\, dx$ with m odd, so we can use the advice in Section 7.2.

Alternatively, if in Step 1 we had written

$$\int \frac{\tan^3 x}{\cos^3 x}\, dx = \int \frac{\sin^3 x}{\cos^3 x} \frac{1}{\cos^3 x}\, dx = \int \frac{\sin^3 x}{\cos^6 x}\, dx$$

then we could have continued as follows with the substitution $u = \cos x$:

$$\int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{1 - \cos^2 x}{\cos^6 x}\sin x\,dx = \int \frac{1 - u^2}{u^6}(-du)$$

$$= \int \frac{u^2 - 1}{u^6}\,du = \int (u^{-4} - u^{-6})\,du$$ ■

EXAMPLE 2 $\int e^{\sqrt{x}}\,dx$

According to (ii) in Step 3(d), we substitute $u = \sqrt{x}$. Then $x = u^2$, so $dx = 2u\,du$ and

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du$$

The integrand is now a product of u and the transcendental function e^u so it can be integrated by parts. ■

EXAMPLE 3 $\displaystyle\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\,dx$

No algebraic simplification or substitution is obvious, so Steps 1 and 2 don't apply here. The integrand is a rational function so we apply the procedure of Section 7.4, remembering that the first step is to divide. ■

EXAMPLE 4 $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

Here Step 2 is all that is needed. We substitute $u = \ln x$ because its differential is $du = dx/x$, which occurs in the integral. ■

EXAMPLE 5 $\displaystyle\int \sqrt{\frac{1 - x}{1 + x}}\,dx$

Although the rationalizing substitution

$$u = \sqrt{\frac{1 - x}{1 + x}}$$

works here [(ii) in Step 3(d)], it leads to a very complicated rational function. An easier method is to do some algebraic manipulation [either as Step 1 or as Step 4(c)]. Multiplying numerator and denominator by $\sqrt{1 - x}$, we have

$$\int \sqrt{\frac{1 - x}{1 + x}}\,dx = \int \frac{1 - x}{\sqrt{1 - x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1 - x^2}}\,dx - \int \frac{x}{\sqrt{1 - x^2}}\,dx$$

$$= \sin^{-1}x + \sqrt{1 - x^2} + C$$ ■

Can We Integrate All Continuous Functions?

The question arises: Will our strategy for integration enable us to find the integral of every continuous function? For example, can we use it to evaluate $\int e^{x^2}\,dx$? The answer is No, at least not in terms of the functions that we are familiar with.

The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^n), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\,\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4 + 2}\,dt$

6. $\displaystyle\int_0^1 \frac{x}{(2x + 1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\,dy$

8. $\displaystyle\int t \sin t \cos t\,dt$

9. $\displaystyle\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

11. $\displaystyle\int \frac{1}{x^3\sqrt{x^2 - 1}}\,dx$

12. $\displaystyle\int \frac{2x - 3}{x^3 + 3x}\,dx$

13. $\displaystyle\int \sin^5 t \cos^4 t\,dt$

14. $\displaystyle\int \ln(1 + x^2)\,dx$

15. $\displaystyle\int x \sec x \tan x\,dx$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

17. $\displaystyle\int_0^{\pi} t \cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x + e^x}\,dx$

20. $\displaystyle\int e^2\,dx$

21. $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

23. $\int_0^1 (1 + \sqrt{x})^8\, dx$

24. $\int (1 + \tan x)^2 \sec x\, dx$

25. $\int_0^1 \frac{1 + 12t}{1 + 3t}\, dt$

26. $\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1}\, dx$

27. $\int \frac{dx}{1 + e^x}$

28. $\int \sin \sqrt{at}\, dt$

29. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$

30. $\int_{-1}^2 | e^x - 1 |\, dx$

31. $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$

32. $\int_1^3 \frac{e^{3/x}}{x^2}\, dx$

33. $\int \sqrt{3 - 2x - x^2}\, dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1 + 4 \cot x}{4 - \cot x}\, dx$

35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x}\, dx$

36. $\int \frac{1 + \sin x}{1 + \cos x}\, dx$

37. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cot\theta}{\sec\theta}\, d\theta$

39. $\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\, d\theta$

40. $\int_0^{\pi} \sin 6x \cos 3x\, dx$

41. $\int \theta \tan^2\theta\, d\theta$

42. $\int \frac{\tan^{-1} x}{x^2}\, dx$

43. $\int \frac{\sqrt{x}}{1 + x^3}\, dx$

44. $\int \sqrt{1 + e^x}\, dx$

45. $\int x^5 e^{-x^3}\, dx$

46. $\int \frac{(x - 1)e^x}{x^2}\, dx$

47. $\int x^3 (x - 1)^{-4}\, dx$

48. $\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\, dx$

49. $\int \frac{1}{x\sqrt{4x + 1}}\, dx$

50. $\int \frac{1}{x^2\sqrt{4x + 1}}\, dx$

51. $\int \frac{1}{x\sqrt{4x^2 + 1}}\, dx$

52. $\int \frac{dx}{x(x^4 + 1)}$

53. $\int x^2 \sinh mx\, dx$

54. $\int (x + \sin x)^2\, dx$

55. $\int \frac{dx}{x + x\sqrt{x}}$

56. $\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$

57. $\int x\sqrt[3]{x + c}\, dx$

58. $\int \frac{x \ln x}{\sqrt{x^2 - 1}}\, dx$

59. $\int \frac{dx}{x^4 - 16}$

60. $\int \frac{dx}{x^2\sqrt{4x^2 - 1}}$

61. $\int \frac{d\theta}{1 + \cos\theta}$

62. $\int \frac{d\theta}{1 + \cos^2\theta}$

63. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

64. $\int \frac{1}{\sqrt{\sqrt{x} + 1}}\, dx$

65. $\int \frac{\sin 2x}{1 + \cos^4 x}\, dx$

66. $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\, dx$

67. $\int \frac{1}{\sqrt{x + 1} + \sqrt{x}}\, dx$

68. $\int \frac{x^2}{x^6 + 3x^3 + 2}\, dx$

69. $\int_1^{\sqrt{3}} \frac{\sqrt{1 + x^2}}{x^2}\, dx$

70. $\int \frac{1}{1 + 2e^x - e^{-x}}\, dx$

71. $\int \frac{e^{2x}}{1 + e^x}\, dx$

72. $\int \frac{\ln(x + 1)}{x^2}\, dx$

73. $\int \frac{x + \arcsin x}{\sqrt{1 - x^2}}\, dx$

74. $\int \frac{4^x + 10^x}{2^x}\, dx$

75. $\int \frac{dx}{x \ln x - x}$

76. $\int \frac{x^2}{\sqrt{x^2 + 1}}\, dx$

77. $\int \frac{xe^x}{\sqrt{1 + e^x}}\, dx$

78. $\int \frac{1 + \sin x}{1 - \sin x}\, dx$

79. $\int x \sin^2 x \cos x\, dx$

80. $\int \frac{\sec x \cos 2x}{\sin x + \sec x}\, dx$

81. $\int \sqrt{1 - \sin x}\, dx$

82. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

83. The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2 + 1)e^{x^2}$ does. Evaluate $\int (2x^2 + 1)e^{x^2}\, dx$.

84. We know that $F(x) = \int_0^x e^{e^t}\, dt$ is a continuous function by FTC1, though it is not an elementary function. The functions

$$\int \frac{e^x}{x}\, dx \qquad \text{and} \qquad \int \frac{1}{\ln x}\, dx$$

are not elementary either, but they can be expressed in terms of F. Evaluate the following integrals in terms of F.

(a) $\int_1^2 \frac{e^x}{x}\, dx$ (b) $\int_2^3 \frac{1}{\ln x}\, dx$

7.6 Integration Using Tables and Computer Algebra Systems

In this section we describe how to use tables and computer algebra systems to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even

the most powerful computer algebra systems can't find explicit formulas for the antiderivatives of functions like e^{x^2} or the other functions described at the end of Section 7.5.

Tables of Integrals

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don't have access to a computer algebra system. A relatively brief table of 120 integrals, categorized by form, is provided on the Reference Pages at the back of the book. More extensive tables are available in the *CRC Standard Mathematical Tables and Formulae*, 31st ed. by Daniel Zwillinger (Boca Raton, FL, 2002) (709 entries) or in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products,* 7e (San Diego, 2007), which contains hundreds of pages of integrals. It should be remembered, however, that integrals do not often occur in exactly the form listed in a table. Usually we need to use the Substitution Rule or algebraic manipulation to transform a given integral into one of the forms in the table.

EXAMPLE 1 The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

SOLUTION Using the method of cylindrical shells, we see that the volume is

$$V = \int_0^1 2\pi x \arctan x \, dx$$

The Table of Integrals appears on Reference Pages 6–10 at the back of the book.

In the section of the Table of Integrals titled *Inverse Trigonometric Forms* we locate Formula 92:

$$\int u \tan^{-1}u \, du = \frac{u^2 + 1}{2} \tan^{-1}u - \frac{u}{2} + C$$

So the volume is

$$V = 2\pi \int_0^1 x \tan^{-1}x \, dx = 2\pi \left[\frac{x^2 + 1}{2} \tan^{-1}x - \frac{x}{2} \right]_0^1$$

$$= \pi \left[(x^2 + 1) \tan^{-1}x - x \right]_0^1 = \pi (2 \tan^{-1}1 - 1)$$

$$= \pi [2(\pi/4) - 1] = \tfrac{1}{2}\pi^2 - \pi$$

■

EXAMPLE 2 Use the Table of Integrals to find $\displaystyle\int \frac{x^2}{\sqrt{5 - 4x^2}} \, dx$.

SOLUTION If we look at the section of the table titled *Forms Involving* $\sqrt{a^2 - u^2}$, we see that the closest entry is number 34:

$$\int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{u}{a}\right) + C$$

This is not exactly what we have, but we will be able to use it if we first make the substitution $u = 2x$:

$$\int \frac{x^2}{\sqrt{5 - 4x^2}} \, dx = \int \frac{(u/2)^2}{\sqrt{5 - u^2}} \frac{du}{2} = \frac{1}{8} \int \frac{u^2}{\sqrt{5 - u^2}} \, du$$

Then we use Formula 34 with $a^2 = 5$ (so $a = \sqrt{5}$):

$$\int \frac{x^2}{\sqrt{5 - 4x^2}}\,dx = \frac{1}{8}\int \frac{u^2}{\sqrt{5 - u^2}}\,du = \frac{1}{8}\left(-\frac{u}{2}\sqrt{5 - u^2} + \frac{5}{2}\sin^{-1}\frac{u}{\sqrt{5}}\right) + C$$

$$= -\frac{x}{8}\sqrt{5 - 4x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + C$$ ■

EXAMPLE 3 Use the Table of Integrals to evaluate $\int x^3 \sin x\,dx$.

SOLUTION If we look in the section called *Trigonometric Forms*, we see that none of the entries explicitly includes a u^3 factor. However, we can use the reduction formula in entry 84 with $n = 3$:

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3\int x^2 \cos x\,dx$$

85. $\int u^n \cos u\,du$
$= u^n \sin u - n\int u^{n-1} \sin u\,du$

We now need to evaluate $\int x^2 \cos x\,dx$. We can use the reduction formula in entry 85 with $n = 2$, followed by entry 82:

$$\int x^2 \cos x\,dx = x^2 \sin x - 2\int x \sin x\,dx$$

$$= x^2 \sin x - 2(\sin x - x\cos x) + K$$

Combining these calculations, we get

$$\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

where $C = 3K$. ■

EXAMPLE 4 Use the Table of Integrals to find $\int x\sqrt{x^2 + 2x + 4}\,dx$.

SOLUTION Since the table gives forms involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$, but not $\sqrt{ax^2 + bx + c}$, we first complete the square:

$$x^2 + 2x + 4 = (x + 1)^2 + 3$$

If we make the substitution $u = x + 1$ (so $x = u - 1$), the integrand will involve the pattern $\sqrt{a^2 + u^2}$:

$$\int x\sqrt{x^2 + 2x + 4}\,dx = \int (u - 1)\sqrt{u^2 + 3}\,du$$

$$= \int u\sqrt{u^2 + 3}\,du - \int \sqrt{u^2 + 3}\,du$$

The first integral is evaluated using the substitution $t = u^2 + 3$:

$$\int u\sqrt{u^2 + 3}\,du = \tfrac{1}{2}\int \sqrt{t}\,dt = \tfrac{1}{2}\cdot\tfrac{2}{3}t^{3/2} = \tfrac{1}{3}(u^2 + 3)^{3/2}$$

21. $\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2}$
$+ \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

For the second integral we use Formula 21 with $a = \sqrt{3}$:

$$\int \sqrt{u^2 + 3}\,du = \frac{u}{2}\sqrt{u^2 + 3} + \tfrac{3}{2}\ln\left(u + \sqrt{u^2 + 3}\right)$$

Therefore

$$\int x\sqrt{x^2 + 2x + 4}\, dx$$

$$= \tfrac{1}{3}(x^2 + 2x + 4)^{3/2} - \frac{x + 1}{2}\sqrt{x^2 + 2x + 4} - \tfrac{3}{2}\ln\left(x + 1 + \sqrt{x^2 + 2x + 4}\right) + C$$

■

■ Computer Algebra Systems

We have seen that the use of tables involves matching the form of the given integrand with the forms of the integrands in the tables. Computers are particularly good at matching patterns. And just as we used substitutions in conjunction with tables, a CAS can perform substitutions that transform a given integral into one that occurs in its stored formulas. So it isn't surprising that computer algebra systems excel at integration. That doesn't mean that integration by hand is an obsolete skill. We will see that a hand computation sometimes produces an indefinite integral in a form that is more convenient than a machine answer.

To begin, let's see what happens when we ask a machine to integrate the relatively simple function $y = 1/(3x - 2)$. Using the substitution $u = 3x - 2$, an easy calculation by hand gives

$$\int \frac{1}{3x - 2}\, dx = \tfrac{1}{3}\ln|3x - 2| + C$$

whereas Mathematica and Maple both return the answer

$$\tfrac{1}{3}\ln(3x - 2)$$

The first thing to notice is that computer algebra systems omit the constant of integration. In other words, they produce a *particular* antiderivative, not the most general one. Therefore, when making use of a machine integration, we might have to add a constant. Second, the absolute value signs are omitted in the machine answer. That is fine if our problem is concerned only with values of x greater than $\frac{2}{3}$. But if we are interested in other values of x, then we need to insert the absolute value symbol.

In the next example we reconsider the integral of Example 4, but this time we ask a machine for the answer.

EXAMPLE 5 Use a computer algebra system to find $\int x\sqrt{x^2 + 2x + 4}\, dx$.

SOLUTION Maple responds with the answer

$$\tfrac{1}{3}(x^2 + 2x + 4)^{3/2} - \tfrac{1}{4}(2x + 2)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1 + x)$$

This looks different from the answer we found in Example 4, but it is equivalent because the third term can be rewritten using the identity

This is equation 6.7.3.

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Thus

$$\operatorname{arcsinh}\frac{\sqrt{3}}{3}(1 + x) = \ln\left[\frac{\sqrt{3}}{3}(1 + x) + \sqrt{\tfrac{1}{3}(1 + x)^2 + 1}\right]$$

$$= \ln\frac{1}{\sqrt{3}}\left[1 + x + \sqrt{(1 + x)^2 + 3}\right]$$

$$= \ln\frac{1}{\sqrt{3}} + \ln\left(x + 1 + \sqrt{x^2 + 2x + 4}\right)$$

The resulting extra term $-\frac{3}{2}\ln(1/\sqrt{3})$ can be absorbed into the constant of integration.

Mathematica gives the answer

$$\left(\frac{5}{6} + \frac{x}{6} + \frac{x^2}{3}\right)\sqrt{x^2 + 2x + 4} - \frac{3}{2}\operatorname{arcsinh}\left(\frac{1 + x}{\sqrt{3}}\right)$$

Mathematica combined the first two terms of Example 4 (and the Maple result) into a single term by factoring. ■

EXAMPLE 6 Use a CAS to evaluate $\int x(x^2 + 5)^8\,dx$.

SOLUTION Maple and Mathematica give the same answer:

$$\tfrac{1}{18}x^{18} + \tfrac{5}{2}x^{16} + 50x^{14} + \tfrac{1750}{3}x^{12} + 4375x^{10} + 21875x^8 + \tfrac{218750}{3}x^6 + 156250x^4 + \tfrac{390625}{2}x^2$$

It's clear that both systems must have expanded $(x^2 + 5)^8$ by the Binomial Theorem and then integrated each term.

If we integrate by hand instead, using the substitution $u = x^2 + 5$, we get

The TI-89 also produces this answer.

$$\int x(x^2 + 5)^8\,dx = \tfrac{1}{18}(x^2 + 5)^9 + C$$

For most purposes, this is a more convenient form of the answer. ■

EXAMPLE 7 Use a CAS to find $\int \sin^5 x\,\cos^2 x\,dx$.

SOLUTION In Example 7.2.2 we found that

$$\boxed{1} \qquad \int \sin^5 x\,\cos^2 x\,dx = -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C$$

Maple and the TI-89 report the answer

$$-\tfrac{1}{7}\sin^4 x\,\cos^3 x - \tfrac{4}{35}\sin^2 x\,\cos^3 x - \tfrac{8}{105}\cos^3 x$$

whereas Mathematica produces

$$-\tfrac{5}{64}\cos x - \tfrac{1}{192}\cos 3x + \tfrac{3}{320}\cos 5x - \tfrac{1}{448}\cos 7x$$

We suspect that there are trigonometric identities which show that these three answers are equivalent. Indeed, if we ask Maple and Mathematica to simplify their expressions using trigonometric identities, they ultimately produce the same form of the answer as in Equation 1. ■

7.6 EXERCISES

1–4 Use the indicated entry in the Table of Integrals on the Reference Pages to evaluate the integral.

1. $\int_0^{\pi/2} \cos 5x \cos 2x\,dx$; entry 80
2. $\int_0^1 \sqrt{x - x^2}\,dx$; entry 113
3. $\int_1^2 \sqrt{4x^2 - 3}\,dx$; entry 39
4. $\int_0^1 \tan^3(\pi x/6)\,dx$; entry 69

5–32 Use the Table of Integrals on Reference Pages 6–10 to evaluate the integral.

5. $\displaystyle\int_0^{\pi/8} \arctan 2x\,dx$

6. $\displaystyle\int_0^2 x^2\sqrt{4 - x^2}\,dx$

7. $\displaystyle\int \frac{\cos x}{\sin^2 x - 9}\,dx$

8. $\displaystyle\int \frac{e^x}{4 - e^{2x}}\,dx$

9. $\displaystyle\int \frac{\sqrt{9x^2 + 4}}{x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{2y^2 - 3}}{y^2}\,dy$

11. $\displaystyle\int_0^{\pi} \cos^6\theta\,d\theta$

12. $\displaystyle\int x\sqrt{2 + x^4}\,dx$

13. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$

14. $\displaystyle\int_0^{\pi} x^3 \sin x\,dx$

15. $\displaystyle\int \frac{\coth(1/y)}{y^2}\,dy$

16. $\displaystyle\int \frac{e^{3t}}{\sqrt{e^{2t} - 1}}\,dt$

17. $\displaystyle\int y\sqrt{6 + 4y - 4y^2}\,dy$

18. $\displaystyle\int \frac{dx}{2x^3 - 3x^2}$

19. $\displaystyle\int \sin^2 x \cos x \ln(\sin x)\,dx$

20. $\displaystyle\int \frac{\sin 2\theta}{\sqrt{5 - \sin\theta}}\,d\theta$

21. $\displaystyle\int \frac{e^x}{3 - e^{2x}}\,dx$

22. $\displaystyle\int_0^2 x^3\sqrt{4x^2 - x^4}\,dx$

23. $\displaystyle\int \sec^5 x\,dx$

24. $\displaystyle\int x^3 \arcsin(x^2)\,dx$

25. $\displaystyle\int \frac{\sqrt{4 + (\ln x)^2}}{x}\,dx$

26. $\displaystyle\int_0^1 x^4 e^{-x}\,dx$

27. $\displaystyle\int \frac{\cos^{-1}(x^{-2})}{x^3}\,dx$

28. $\displaystyle\int \frac{dx}{\sqrt{1 - e^{2x}}}$

29. $\displaystyle\int \sqrt{e^{2x} - 1}\,dx$

30. $\displaystyle\int e^t \sin(\alpha t - 3)\,dt$

31. $\displaystyle\int \frac{x^4\,dx}{\sqrt{x^{10} - 2}}$

32. $\displaystyle\int \frac{\sec^2\theta\,\tan^2\theta}{\sqrt{9 - \tan^2\theta}}\,d\theta$

33. The region under the curve $y = \sin^2 x$ from 0 to π is rotated about the x-axis. Find the volume of the resulting solid.

34. Find the volume of the solid obtained when the region under the curve $y = \arcsin x$, $x \geqslant 0$, is rotated about the y-axis.

35. Verify Formula 53 in the Table of Integrals (a) by differentiation and (b) by using the substitution $t = a + bu$.

36. Verify Formula 31 (a) by differentiation and (b) by substituting $u = a \sin\theta$.

CAS **37–44** Use a computer algebra system to evaluate the integral. Compare the answer with the result of using tables. If the answers are not the same, show that they are equivalent.

37. $\displaystyle\int \sec^4 x\,dx$

38. $\displaystyle\int \csc^5 x\,dx$

39. $\displaystyle\int x^2\sqrt{x^2 + 4}\,dx$

40. $\displaystyle\int \frac{dx}{e^x(3e^x + 2)}$

41. $\displaystyle\int \cos^4 x\,dx$

42. $\displaystyle\int x^2\sqrt{1 - x^2}\,dx$

43. $\displaystyle\int \tan^5 x\,dx$

44. $\displaystyle\int \frac{1}{\sqrt{1 + \sqrt[3]{x}}}\,dx$

CAS **45.** (a) Use the table of integrals to evaluate $F(x) = \int f(x)\,dx$, where

$$f(x) = \frac{1}{x\sqrt{1 - x^2}}$$

What is the domain of f and F?

(b) Use a CAS to evaluate $F(x)$. What is the domain of the function F that the CAS produces? Is there a discrepancy between this domain and the domain of the function F that you found in part (a)?

CAS **46.** Computer algebra systems sometimes need a helping hand from human beings. Try to evaluate

$$\int (1 + \ln x)\sqrt{1 + (x \ln x)^2}\,dx$$

with a computer algebra system. If it doesn't return an answer, make a substitution that changes the integral into one that the CAS *can* evaluate.

DISCOVERY PROJECT CAS PATTERNS IN INTEGRALS

In this project a computer algebra system is used to investigate indefinite integrals of families of functions. By observing the patterns that occur in the integrals of several members of the family, you will first guess, and then prove, a general formula for the integral of any member of the family.

1. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\displaystyle\int \frac{1}{(x + 2)(x + 3)}\,dx$ (ii) $\displaystyle\int \frac{1}{(x + 1)(x + 5)}\,dx$

(iii) $\displaystyle\int \frac{1}{(x + 2)(x - 5)}\,dx$ (iv) $\displaystyle\int \frac{1}{(x + 2)^2}\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \frac{1}{(x+a)(x+b)}\,dx$$

if $a \neq b$. What if $a = b$?

(c) Check your guess by asking your CAS to evaluate the integral in part (b). Then prove it using partial fractions.

2. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\int \sin x \cos 2x\,dx$ (ii) $\int \sin 3x \cos 7x\,dx$ (iii) $\int \sin 8x \cos 3x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

$$\int \sin ax \cos bx\,dx$$

(c) Check your guess with a CAS. Then prove it using the techniques of Section 7.2. For what values of a and b is it valid?

3. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\int \ln x\,dx$ (ii) $\int x \ln x\,dx$ (iii) $\int x^2 \ln x\,dx$

(iv) $\int x^3 \ln x\,dx$ (v) $\int x^7 \ln x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of

$$\int x^n \ln x\,dx$$

(c) Use integration by parts to prove the conjecture that you made in part (b). For what values of n is it valid?

4. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\int xe^x\,dx$ (ii) $\int x^2e^x\,dx$ (iii) $\int x^3e^x\,dx$

(iv) $\int x^4e^x\,dx$ (v) $\int x^5e^x\,dx$

(b) Based on the pattern of your responses in part (a), guess the value of $\int x^6e^x\,dx$. Then use your CAS to check your guess.

(c) Based on the patterns in parts (a) and (b), make a conjecture as to the value of the integral

$$\int x^ne^x\,dx$$

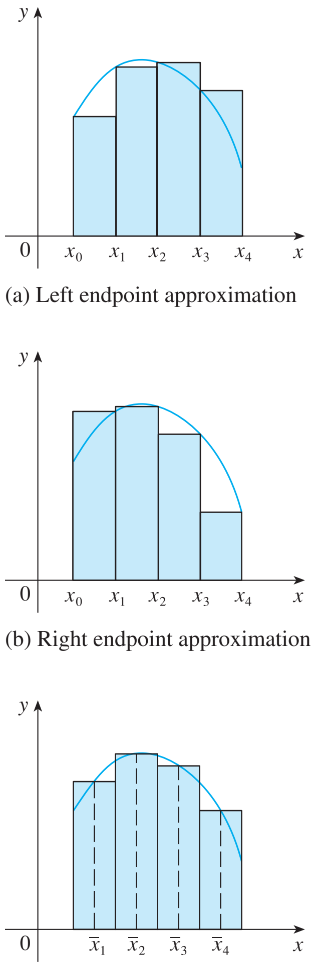
when n is a positive integer.

(d) Use mathematical induction to prove the conjecture you made in part (c).

7.7 Approximate Integration

There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus we need to know an antiderivative of f. Sometimes, however, it is difficult, or even impossible, to find an antiderivative (see Section 7.5). For



(a) Left endpoint approximation

(b) Right endpoint approximation

(c) Midpoint approximation

FIGURE 1

example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2} dx \qquad \int_{-1}^1 \sqrt{1 + x^3}\, dx$$

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function (see Example 5).

In both cases we need to find approximate values of definite integrals. We already know one such method. Recall that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral: If we divide $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$, then we have

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

where x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$. If x_i^* is chosen to be the left endpoint of the interval, then $x_i^* = x_{i-1}$ and we have

1
$$\int_a^b f(x)\, dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\, \Delta x$$

If $f(x) \geqslant 0$, then the integral represents an area and (1) represents an approximation of this area by the rectangles shown in Figure 1(a). If we choose x_i^* to be the right endpoint, then $x_i^* = x_i$ and we have

2
$$\int_a^b f(x)\, dx \approx R_n = \sum_{i=1}^{n} f(x_i)\, \Delta x$$

[See Figure 1(b).] The approximations L_n and R_n defined by Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.

In Section 4.2 we also considered the case where x_i^* is chosen to be the midpoint $\bar{x}_i$ of the subinterval $[x_{i-1}, x_i]$. Figure 1(c) shows the midpoint approximation M_n, which appears to be better than either L_n or R_n.

Midpoint Rule

$$\int_a^b f(x)\, dx \approx M_n = \Delta x\, [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\qquad \Delta x = \dfrac{b - a}{n}$

and $\qquad \bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations 1 and 2:

$$\int_a^b f(x)\, dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\, \Delta x + \sum_{i=1}^{n} f(x_i)\, \Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n} \big(f(x_{i-1}) + f(x_i)\big)\right]$$

$$= \frac{\Delta x}{2}\Big[\big(f(x_0) + f(x_1)\big) + \big(f(x_1) + f(x_2)\big) + \cdots + \big(f(x_{n-1}) + f(x_n)\big)\Big]$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

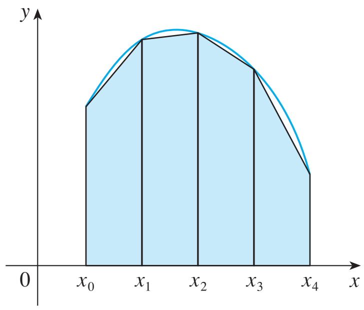


FIGURE 2
Trapezoidal approximation

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

The reason for the name Trapezoidal Rule can be seen from Figure 2, which illustrates the case with $f(x) \geqslant 0$ and $n = 4$. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) = \frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)]$$

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

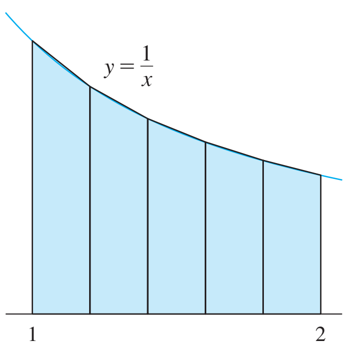


FIGURE 3

EXAMPLE 1 Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\,dx$.

SOLUTION

(a) With $n = 5$, $a = 1$, and $b = 2$, we have $\Delta x = (2 - 1)/5 = 0.2$, and so the Trapezoidal Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\approx 0.695635$$

This approximation is illustrated in Figure 3.

(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

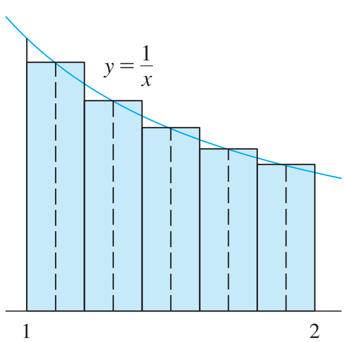


FIGURE 4

This approximation is illustrated in Figure 4. ■

In Example 1 we deliberately chose an integral whose value can be computed explicitly so that we can see how accurate the Trapezoidal and Midpoint Rules are. By the Fundamental Theorem of Calculus,

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2 = \ln 2 = 0.693147\ldots$$

$\int_a^b f(x)\,dx$ = approximation + error

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact. From the values in Example 1 we see that the

errors in the Trapezoidal and Midpoint Rule approximations for $n = 5$ are

$$E_T \approx -0.002488 \qquad \text{and} \qquad E_M \approx 0.001239$$

In general, we have

$$E_T = \int_a^b f(x)\,dx - T_n \qquad \text{and} \qquad E_M = \int_a^b f(x)\,dx - M_n$$

TEC Module 4.2/7.7 allows you to compare approximation methods.

The following tables show the results of calculations similar to those in Example 1, but for $n = 5$, 10, and 20 and for the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rules.

Approximations to $\int_1^2 \frac{1}{x}\,dx$

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

Corresponding errors

n	E_L	E_R	E_T	E_M
5	-0.052488	0.047512	-0.002488	0.001239
10	-0.025624	0.024376	-0.000624	0.000312
20	-0.012656	0.012344	-0.000156	0.000078

It turns out that these observations are true in most cases.

We can make several observations from these tables:

1. In all of the methods we get more accurate approximations when we increase the value of n. (But very large values of n result in so many arithmetic operations that we have to beware of accumulated round-off error.)
2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of n.
3. The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.
4. The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of n.
5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

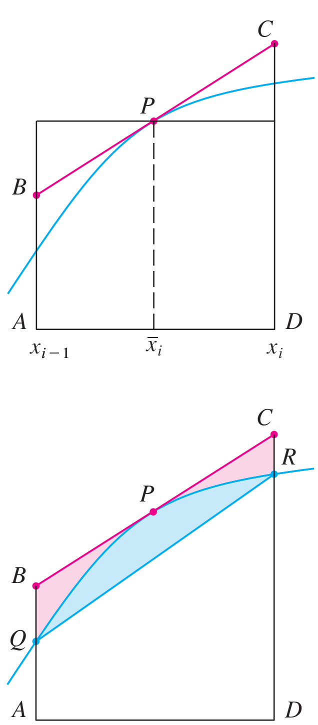


FIGURE 5

Figure 5 shows why we can usually expect the Midpoint Rule to be more accurate than the Trapezoidal Rule. The area of a typical rectangle in the Midpoint Rule is the same as the area of the trapezoid $ABCD$ whose upper side is tangent to the graph at P. The area of this trapezoid is closer to the area under the graph than is the area of the trapezoid $AQRD$ used in the Trapezoidal Rule. [The midpoint error (shaded red) is smaller than the trapezoidal error (shaded blue).]

These observations are corroborated in the following error estimates, which are proved in books on numerical analysis. Notice that Observation 4 corresponds to the n^2 in each denominator because $(2n)^2 = 4n^2$. The fact that the estimates depend on the size of the second derivative is not surprising if you look at Figure 5, because $f''(x)$ measures how much the graph is curved. [Recall that $f''(x)$ measures how fast the slope of $y = f(x)$ changes.]

3 Error Bounds Suppose $|f''(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leqslant \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leqslant \frac{K(b-a)^3}{24n^2}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1. If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. Because $1 \leqslant x \leqslant 2$, we have $1/x \leqslant 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \leqslant \frac{2}{1^3} = 2$$

Therefore, taking $K = 2$, $a = 1$, $b = 2$, and $n = 5$ in the error estimate (3), we see that

K can be any number larger than all the values of $|f''(x)|$, but smaller values of K give better error bounds.

$$|E_T| \leqslant \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Comparing this error estimate of 0.006667 with the actual error of about 0.002488, we see that it can happen that the actual error is substantially less than the upper bound for the error given by (3).

EXAMPLE 2 How large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

SOLUTION We saw in the preceding calculation that $|f''(x)| \leqslant 2$ for $1 \leqslant x \leqslant 2$, so we can take $K = 2$, $a = 1$, and $b = 2$ in (3). Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore we choose n so that

$$\frac{2(1)^3}{12n^2} < 0.0001$$

Solving the inequality for n, we get

$$n^2 > \frac{2}{12(0.0001)}$$

or

$$n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

It's quite possible that a lower value for n would suffice, but 41 is the smallest value for which the error bound formula can *guarantee* us accuracy to within 0.0001.

Thus $n = 41$ will ensure the desired accuracy.

For the same accuracy with the Midpoint Rule we choose n so that

$$\frac{2(1)^3}{24n^2} < 0.0001 \qquad \text{and so} \qquad n > \frac{1}{\sqrt{0.0012}} \approx 29$$ ■

EXAMPLE 3
(a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
(b) Give an upper bound for the error involved in this approximation.

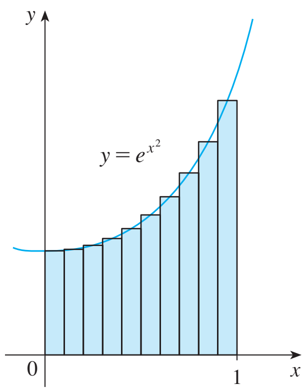


FIGURE 6

SOLUTION
(a) Since $a = 0$, $b = 1$, and $n = 10$, the Midpoint Rule gives

$$\int_0^1 e^{x^2}\,dx \approx \Delta x\,[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)]$$

$$= 0.1[e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025} + e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025}]$$

$$\approx 1.460393$$

Figure 6 illustrates this approximation.

(b) Since $f(x) = e^{x^2}$, we have $f'(x) = 2xe^{x^2}$ and $f''(x) = (2 + 4x^2)e^{x^2}$. Also, since $0 \leqslant x \leqslant 1$, we have $x^2 \leqslant 1$ and so

$$0 \leqslant f''(x) = (2 + 4x^2)e^{x^2} \leqslant 6e$$

Error estimates give upper bounds for the error. They are theoretical, worst-case scenarios. The actual error in this case turns out to be about 0.0023.

Taking $K = 6e$, $a = 0$, $b = 1$, and $n = 10$ in the error estimate (3), we see that an upper bound for the error is

$$\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007$$ ■

Simpson's Rule

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve. As before, we divide $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we assume that n is an *even* number. Then on each consecutive pair of intervals we approximate the curve $y = f(x) \geqslant 0$ by a parabola as shown in Figure 7. If $y_i = f(x_i)$, then $P_i(x_i, y_i)$ is the point on the curve lying above x_i. A typical parabola passes through three consecutive points P_i, P_{i+1}, and P_{i+2}.

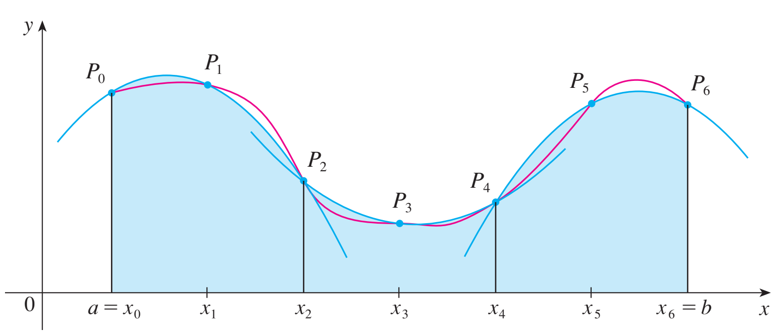


FIGURE 7

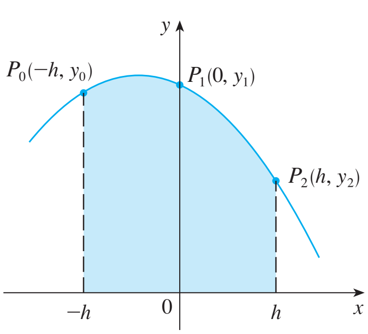


FIGURE 8

To simplify our calculations, we first consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. (See Figure 8.) We know that the equation of the parabola through P_0, P_1, and

P_2 is of the form $y = Ax^2 + Bx + C$ and so the area under the parabola from $x = -h$ to $x = h$ is

Here we have used Theorem 4.5.6. Notice that $Ax^2 + C$ is even and Bx is odd.

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_0^h (Ax^2 + C)\,dx = 2\left[A\frac{x^3}{3} + Cx\right]_0^h$$

$$= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}(2Ah^2 + 6C)$$

But, since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

and therefore $\qquad y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$

Thus we can rewrite the area under the parabola as

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Now by shifting this parabola horizontally we do not change the area under it. This means that the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ in Figure 7 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Although we have derived this approximation for the case in which $f(x) \geqslant 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761). Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Simpson

Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson's Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his book *Mathematical Dissertations* (1743).

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$.

EXAMPLE 4 Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$.

SOLUTION Putting $f(x) = 1/x$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain

$$\int_1^2 \frac{1}{x}\,dx \approx S_{10}$$

$$= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)]$$

$$= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right)$$

$$\approx 0.693150$$ ■

Notice that, in Example 4, Simpson's Rule gives us a *much* better approximation ($S_{10} \approx 0.693150$) to the true value of the integral ($\ln 2 \approx 0.693147\ldots$) than does the Trapezoidal Rule ($T_{10} \approx 0.693771$) or the Midpoint Rule ($M_{10} \approx 0.692835$). It turns out (see Exercise 50) that the approximations in Simpson's Rule are weighted averages of those in the Trapezoidal and Midpoint Rules:

$$S_{2n} = \tfrac{1}{3}T_n + \tfrac{2}{3}M_n$$

(Recall that E_T and E_M usually have opposite signs and $|E_M|$ is about half the size of $|E_T|$.)

In many applications of calculus we need to evaluate an integral even if no explicit formula is known for y as a function of x. A function may be given graphically or as a table of values of collected data. If there is evidence that the values are not changing rapidly, then the Trapezoidal Rule or Simpson's Rule can still be used to find an approximate value for $\int_a^b y\,dx$, the integral of y with respect to x.

EXAMPLE 5 Figure 9 shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Use Simpson's Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

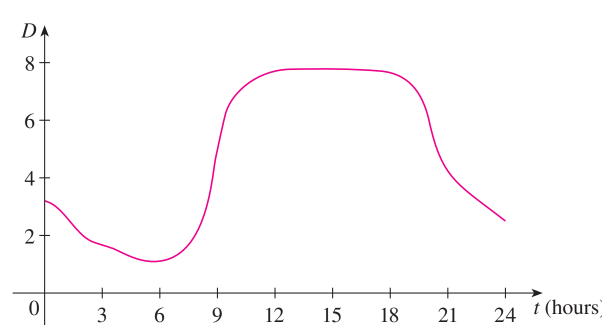


FIGURE 9

SOLUTION Because we want the units to be consistent and $D(t)$ is measured in megabits per second, we convert the units for t from hours to seconds. If we let $A(t)$ be the amount of data (in megabits) transmitted by time t, where t is measured in seconds, then $A'(t) = D(t)$. So, by the Net Change Theorem (see Section 4.4), the total amount

of data transmitted by noon (when $t = 12 \times 60^2 = 43{,}200$) is

$$A(43{,}200) = \int_0^{43{,}200} D(t)\,dt$$

We estimate the values of $D(t)$ at hourly intervals from the graph and compile them in the table.

t (hours)	t (seconds)	$D(t)$	t (hours)	t (seconds)	$D(t)$
0	0	3.2	7	25,200	1.3
1	3,600	2.7	8	28,800	2.8
2	7,200	1.9	9	32,400	5.7
3	10,800	1.7	10	36,000	7.1
4	14,400	1.3	11	39,600	7.7
5	18,000	1.0	12	43,200	7.9
6	21,600	1.1			

Then we use Simpson's Rule with $n = 12$ and $\Delta t = 3600$ to estimate the integral:

$$\int_0^{43{,}200} A(t)\,dt \approx \frac{\Delta t}{3}[D(0) + 4D(3600) + 2D(7200) + \cdots + 4D(39{,}600) + D(43{,}200)]$$

$$\approx \frac{3600}{3}[3.2 + 4(2.7) + 2(1.9) + 4(1.7) + 2(1.3) + 4(1.0)$$
$$+ 2(1.1) + 4(1.3) + 2(2.8) + 4(5.7) + 2(7.1) + 4(7.7) + 7.9]$$

$$= 143{,}880$$

Thus the total amount of data transmitted from midnight to noon is about 144,000 megabits, or 144 gigabits. ■

n	M_n	S_n
4	0.69121989	0.69315453
8	0.69266055	0.69314765
16	0.69302521	0.69314721

n	E_M	E_S
4	0.00192729	−0.00000735
8	0.00048663	−0.00000047
16	0.00012197	−0.00000003

The table in the margin shows how Simpson's Rule compares with the Midpoint Rule for the integral $\int_1^2 (1/x)\,dx$, whose value is about 0.69314718. The second table shows how the error E_S in Simpson's Rule decreases by a factor of about 16 when n is doubled. (In Exercises 27 and 28 you are asked to verify this for two additional integrals.) That is consistent with the appearance of n^4 in the denominator of the following error estimate for Simpson's Rule. It is similar to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of f.

4 Error Bound for Simpson's Rule Suppose that $|f^{(4)}(x)| \leqslant K$ for $a \leqslant x \leqslant b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leqslant \frac{K(b-a)^5}{180n^4}$$

EXAMPLE 6 How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?

SOLUTION If $f(x) = 1/x$, then $f^{(4)}(x) = 24/x^5$. Since $x \geqslant 1$, we have $1/x \leqslant 1$ and so

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \leqslant 24$$

Many calculators and computer algebra systems have a built-in algorithm that computes an approximation of a definite integral. Some of these machines use Simpson's Rule; others use more sophisticated techniques such as *adaptive* numerical integration. This means that if a function fluctuates much more on a certain part of the interval than it does elsewhere, then that part gets divided into more subintervals. This strategy reduces the number of calculations required to achieve a prescribed accuracy.

Therefore we can take $K = 24$ in (4). Thus, for an error less than 0.0001, we should choose n so that

$$\frac{24(1)^5}{180n^4} < 0.0001$$

This gives
$$n^4 > \frac{24}{180(0.0001)}$$

or
$$n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$$

Therefore $n = 8$ (n must be even) gives the desired accuracy. (Compare this with Example 2, where we obtained $n = 41$ for the Trapezoidal Rule and $n = 29$ for the Midpoint Rule.) ■

EXAMPLE 7
(a) Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
(b) Estimate the error involved in this approximation.

SOLUTION
(a) If $n = 10$, then $\Delta x = 0.1$ and Simpson's Rule gives

Figure 10 illustrates the calculation in Example 7. Notice that the parabolic arcs are so close to the graph of $y = e^{x^2}$ that they are practically indistinguishable from it.

$$\int_0^1 e^{x^2}\,dx \approx \frac{\Delta x}{3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1)]$$

$$= \frac{0.1}{3}[e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1]$$

$$\approx 1.462681$$

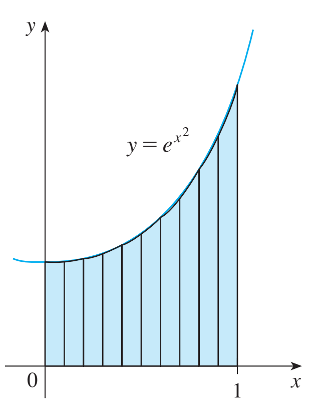


FIGURE 10

(b) The fourth derivative of $f(x) = e^{x^2}$ is

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

and so, since $0 \leqslant x \leqslant 1$, we have

$$0 \leqslant f^{(4)}(x) \leqslant (12 + 48 + 16)e^1 = 76e$$

Therefore, putting $K = 76e$, $a = 0$, $b = 1$, and $n = 10$ in (4), we see that the error is at most

$$\frac{76e(1)^5}{180(10)^4} \approx 0.000115$$

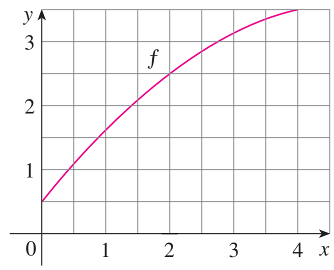
(Compare this with Example 3.) Thus, correct to three decimal places, we have

$$\int_0^1 e^{x^2}\,dx \approx 1.463$$ ■

7.7 EXERCISES

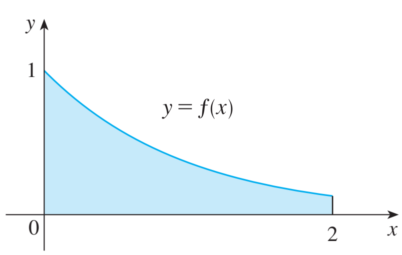
1. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown.

(a) Use the graph to find L_2, R_2, and M_2.
(b) Are these underestimates or overestimates of I?
(c) Use the graph to find T_2. How does it compare with I?
(d) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.



2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of subintervals were used in each case.

(a) Which rule produced which estimate?
(b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin(\frac{1}{2}x^2)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.

(a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
(b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
(c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5–6 Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

5. $\displaystyle\int_0^2 \frac{x}{1 + x^2}\,dx, \quad n = 10$

6. $\displaystyle\int_0^\pi x\cos x\,dx, \quad n = 4$

7–18 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)

7. $\displaystyle\int_1^2 \sqrt{x^3 - 1}\,dx, \quad n = 10$

8. $\displaystyle\int_0^2 \frac{1}{1 + x^6}\,dx, \quad n = 8$

9. $\displaystyle\int_0^2 \frac{e^x}{1 + x^2}\,dx, \quad n = 10$

10. $\displaystyle\int_0^{\pi/2} \sqrt[3]{1 + \cos x}\,dx, \quad n = 4$

11. $\displaystyle\int_0^4 x^3 \sin x\,dx, \quad n = 8$

12. $\displaystyle\int_1^3 e^{1/x}\,dx, \quad n = 8$

13. $\displaystyle\int_0^4 \sqrt{y}\cos y\,dy, \quad n = 8$

14. $\displaystyle\int_2^3 \frac{1}{\ln t}\,dt, \quad n = 10$

15. $\displaystyle\int_0^1 \frac{x^2}{1 + x^4}\,dx, \quad n = 10$

16. $\displaystyle\int_1^3 \frac{\sin t}{t}\,dt, \quad n = 4$

17. $\displaystyle\int_0^4 \ln(1 + e^x)\,dx, \quad n = 8$

18. $\displaystyle\int_0^1 \sqrt{x + x^3}\,dx, \quad n = 10$

19. (a) Find the approximations T_8 and M_8 for the integral $\int_0^1 \cos(x^2)\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

20. (a) Find the approximations T_{10} and M_{10} for $\int_1^2 e^{1/x}\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

21. (a) Find the approximations T_{10}, M_{10}, and S_{10} for $\int_0^\pi \sin x\,dx$ and the corresponding errors E_T, E_M, and E_S.
(b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
(c) How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral in part (a) are accurate to within 0.00001?

22. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

CAS **23.** The trouble with the error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound K for $|f^{(4)}(x)|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x)\,dx$, where $f(x) = e^{\cos x}$.
(a) Use a graph to get a good upper bound for $|f''(x)|$.
(b) Use M_{10} to approximate I.
(c) Use part (a) to estimate the error in part (b).
(d) Use the built-in numerical integration capability of your CAS to approximate I.
(e) How does the actual error compare with the error estimate in part (c)?
(f) Use a graph to get a good upper bound for $|f^{(4)}(x)|$.
(g) Use S_{10} to approximate I.
(h) Use part (f) to estimate the error in part (g).
(i) How does the actual error compare with the error estimate in part (h)?
(j) How large should n be to guarantee that the size of the error in using S_n is less than 0.0001?

CAS **24.** Repeat Exercise 23 for the integral $\int_{-1}^{1} \sqrt{4 - x^3}\,dx$.

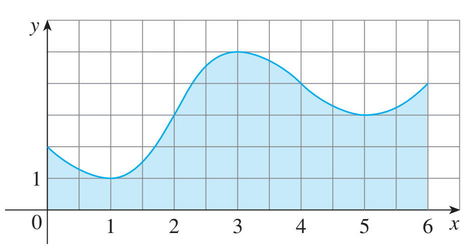
25–26 Find the approximations L_n, R_n, T_n, and M_n for $n = 5$, 10, and 20. Then compute the corresponding errors E_L, E_R, E_T, and E_M. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

25. $\int_0^1 xe^x\,dx$ **26.** $\int_1^2 \frac{1}{x^2}\,dx$

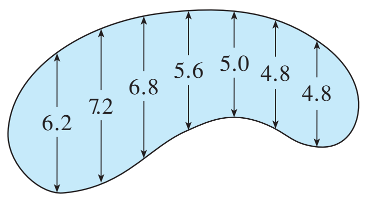
27–28 Find the approximations T_n, M_n, and S_n for $n = 6$ and 12. Then compute the corresponding errors E_T, E_M, and E_S. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

27. $\int_0^2 x^4\,dx$ **28.** $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

29. Estimate the area under the graph in the figure by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule, each with $n = 6$.



30. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson's Rule to estimate the area of the pool.



31. (a) Use the Midpoint Rule and the given data to estimate the value of the integral $\int_1^5 f(x)\,dx$.

x	$f(x)$	x	$f(x)$
1.0	2.4	3.5	4.0
1.5	2.9	4.0	4.1
2.0	3.3	4.5	3.9
2.5	3.6	5.0	3.5
3.0	3.8		

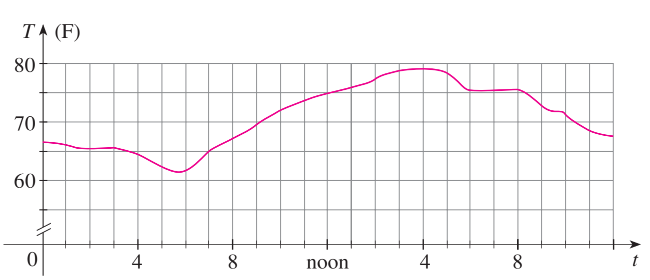
(b) If it is known that $-2 \leqslant f''(x) \leqslant 3$ for all x, estimate the error involved in the approximation in part (a).

32. (a) A table of values of a function g is given. Use Simpson's Rule to estimate $\int_0^{1.6} g(x)\,dx$.

x	$g(x)$	x	$g(x)$
0.0	12.1	1.0	12.2
0.2	11.6	1.2	12.6
0.4	11.3	1.4	13.0
0.6	11.1	1.6	13.2
0.8	11.7		

(b) If $-5 \leqslant g^{(4)}(x) \leqslant 2$ for $0 \leqslant x \leqslant 1.6$, estimate the error involved in the approximation in part (a).

33. A graph of the temperature in Boston on August 11, 2013, is shown. Use Simpson's Rule with $n = 12$ to estimate the average temperature on that day.

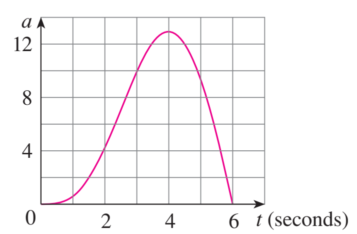


34. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's

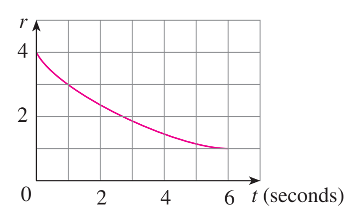
Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

35. The graph of the acceleration $a(t)$ of a car measured in ft/s^2 is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



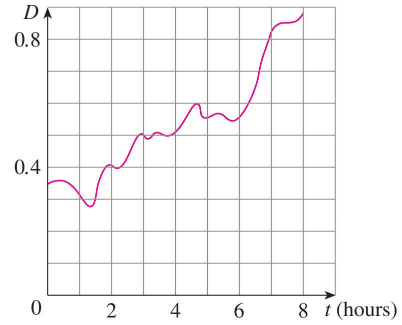
36. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first 6 hours.



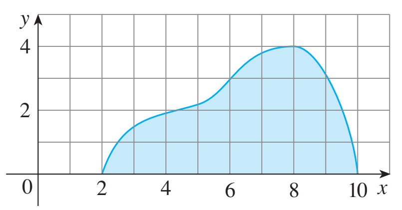
37. The table (supplied by San Diego Gas and Electric) gives the power consumption P in megawatts in San Diego County from midnight to 6:00 AM on a day in December. Use Simpson's Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

t	P	t	P
0:00	1814	3:30	1611
0:30	1735	4:00	1621
1:00	1686	4:30	1666
1:30	1646	5:00	1745
2:00	1637	5:30	1886
2:30	1609	6:00	2052
3:00	1604		

38. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use Simpson's Rule to estimate the total amount of data transmitted during that time period.



39. Use Simpson's Rule with $n = 8$ to estimate the volume of the solid obtained by rotating the region shown in the figure about (a) the x-axis and (b) the y-axis.



40. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use Simpson's Rule to estimate the work done by the force in moving an object a distance of 18 m.

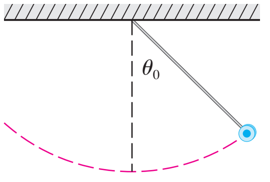
x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

41. The region bounded by the curve $y = 1/(1 + e^{-x})$, the x- and y-axes, and the line $x = 10$ is rotated about the x-axis. Use Simpson's Rule with $n = 10$ to estimate the volume of the resulting solid.

CAS **42.** The figure shows a pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law, it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.



43. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by $I(\theta) = N^2 \sin^2 k/k^2$, where $k = (\pi N d \sin\theta)/\lambda$ and d is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-6}}^{10^{-6}} I(\theta)\, d\theta$ emerging from the grating.

44. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)\, dx$. Compare your result to the actual value. Can you explain the discrepancy?

45. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

46. Sketch the graph of a continuous function on $[0, 2]$ for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

47. If f is a positive function and $f''(x) < 0$ for $a \leqslant x \leqslant b$, show that

$$T_n < \int_a^b f(x)\, dx < M_n$$

48. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)\, dx$.

49. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

50. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x)\, dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity (see Section 4.2). In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral. One of the most important applications of this idea, probability distributions, will be studied in Section 8.5.

Type 1: Infinite Intervals

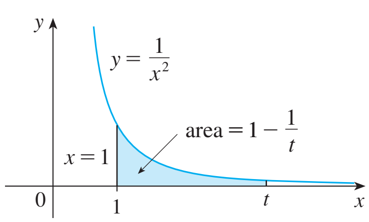


FIGURE 1

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ no matter how large t is chosen.

We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = 1$$

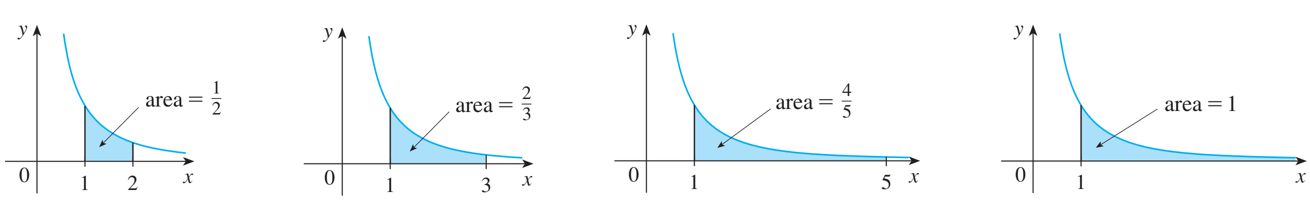


FIGURE 2

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Any of the improper integrals in Definition 1 can be interpreted as an area provided that f is a positive function. For instance, in case (a) if $f(x) \geqslant 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region $S = \{(x, y) \mid x \geqslant a, 0 \leqslant y \leqslant f(x)\}$ in Figure 3 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.

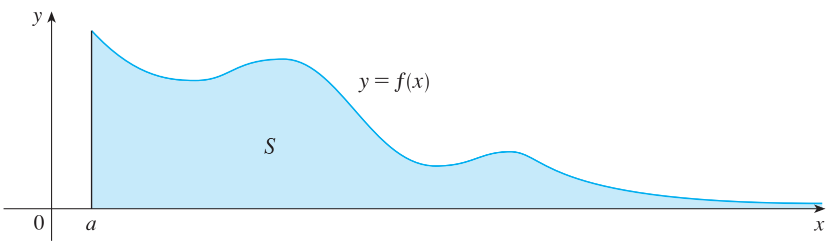


FIGURE 3

EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1) = \lim_{t\to\infty} \ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty (1/x)\,dx$ is divergent. ■

Let's compare the result of Example 1 with the example given at the beginning of this section:

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ (the shaded region in Figure 4) has finite area whereas the corresponding region under $y = 1/x$ (in Figure 5) has infinite area. Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

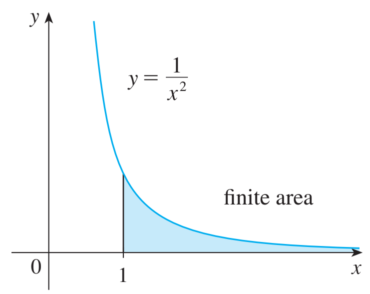


FIGURE 4
$\int_1^\infty (1/x^2)\,dx$ converges

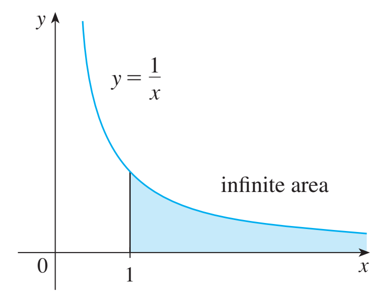


FIGURE 5
$\int_1^\infty (1/x)\,dx$ diverges

EXAMPLE 2 Evaluate $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$

$$= -te^t - 1 + e^t$$

TEC In Module 7.8 you can investigate visually and numerically whether several improper integrals are convergent or divergent.

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} (-e^t) = 0$$

Therefore

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} (-te^t - 1 + e^t)$$

$$= -0 - 1 + 0 = -1$$ ■

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

SOLUTION It's convenient to choose $a = 0$ in Definition 1(c):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately:

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty} \tan^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1}0 - \tan^{-1}t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

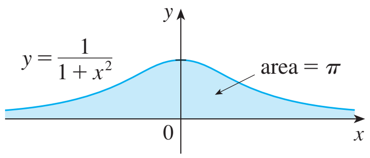


FIGURE 6

Since $1/(1+x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1+x^2)$ and above the x-axis (see Figure 6). ■

EXAMPLE 4 For what values of p is the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

convergent?

SOLUTION We know from Example 1 that if $p = 1$, then the integral is divergent, so let's assume that $p \neq 1$. Then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$. Therefore

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} \qquad \text{if } p > 1$$

and so the integral converges. But if $p < 1$, then $p - 1 < 0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \qquad \text{as } t \to \infty$$

and the integral diverges. ■

We summarize the result of Example 4 for future reference:

2 $$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

Type 2: Discontinuous Integrands

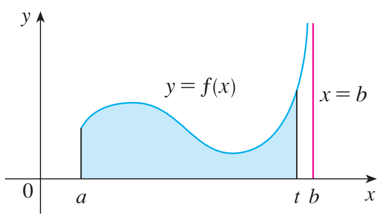


FIGURE 7

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x)\,dx$$

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

Parts (b) and (c) of Definition 3 are illustrated in Figures 8 and 9 for the case where $f(x) \geqslant 0$ and f has vertical asymptotes at a and c, respectively.

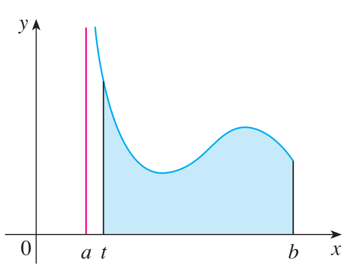


FIGURE 8

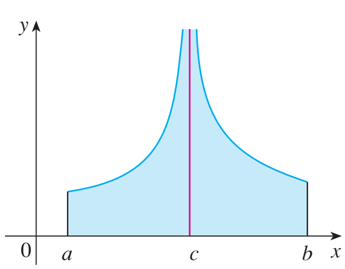


FIGURE 9

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 5 Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

SOLUTION We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left

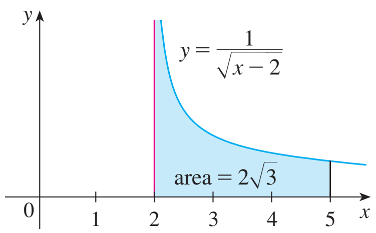


FIGURE 10

endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t \to 2^+} 2\sqrt{x-2}\Big]_t^5$$

$$= \lim_{t \to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$$

Thus the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 10. ■

EXAMPLE 6 Determine whether $\int_0^{\pi/2} \sec x\, dx$ converges or diverges.

SOLUTION Note that the given integral is improper because $\lim_{x \to (\pi/2)^-} \sec x = \infty$. Using part (a) of Definition 3 and Formula 14 from the Table of Integrals, we have

$$\int_0^{\pi/2} \sec x\, dx = \lim_{t \to (\pi/2)^-} \int_0^t \sec x\, dx = \lim_{t \to (\pi/2)^-} \ln|\sec x + \tan x|\Big]_0^t$$

$$= \lim_{t \to (\pi/2)^-} [\ln(\sec t + \tan t) - \ln 1] = \infty$$

because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$. Thus the given improper integral is divergent. ■

EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

SOLUTION Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|x-1|\Big]_0^t$$

$$= \lim_{t \to 1^-} \left(\ln|t-1| - \ln|-1|\right) = \lim_{t \to 1^-} \ln(1-t) = -\infty$$

because $1 - t \to 0^+$ as $t \to 1^-$. Thus $\int_0^1 dx/(x-1)$ is divergent. This implies that $\int_0^3 dx/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 dx/(x-1)$.] ■

WARNING If we had not noticed the asymptote $x = 1$ in Example 7 and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

From now on, whenever you meet the symbol $\int_a^b f(x)\, dx$ you must decide, by looking at the function f on $[a, b]$, whether it is an ordinary definite integral or an improper integral.

EXAMPLE 8 $\int_0^1 \ln x\, dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since

$\lim_{x \to 0^+} \ln x = -\infty$. Thus the given integral is improper and we have

$$\int_0^1 \ln x\,dx = \lim_{t \to 0^+} \int_t^1 \ln x\,dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\,dx = x \ln x\Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t) = -t \ln t - 1 + t$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} = \lim_{t \to 0^+} \frac{1/t}{-1/t^2} = \lim_{t \to 0^+} (-t) = 0$$

Therefore $$\int_0^1 \ln x\,dx = \lim_{t \to 0^+} (-t \ln t - 1 + t) = -0 - 1 + 0 = -1$$

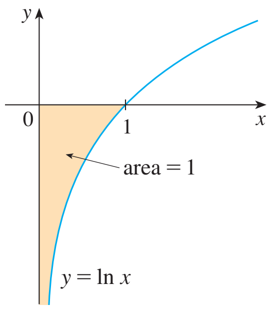


FIGURE 11

Figure 11 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1. ■

A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

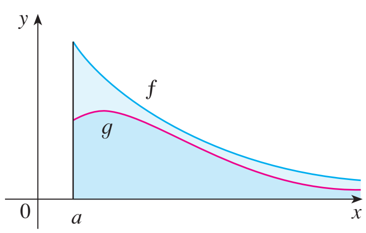


FIGURE 12

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible. If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$. And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. [Note that the reverse is not necessarily true: If $\int_a^\infty g(x)\,dx$ is convergent, $\int_a^\infty f(x)\,dx$ may or may not be convergent, and if $\int_a^\infty f(x)\,dx$ is divergent, $\int_a^\infty g(x)\,dx$ may or may not be divergent.]

EXAMPLE 9 Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

SOLUTION We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5). We write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

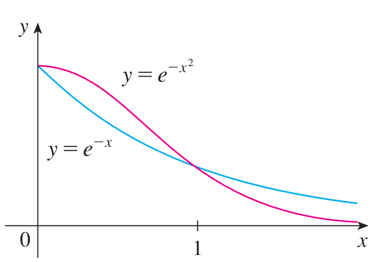


FIGURE 13

and observe that the first integral on the right-hand side is just an ordinary definite integral. In the second integral we use the fact that for $x \geqslant 1$ we have $x^2 \geqslant x$, so $-x^2 \leqslant -x$ and therefore $e^{-x^2} \leqslant e^{-x}$. (See Figure 13.) The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t \to \infty} \int_1^t e^{-x}\,dx = \lim_{t \to \infty} (e^{-1} - e^{-t}) = e^{-1}$$

Therefore, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent. ■

Table 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

In Example 9 we showed that $\int_0^\infty e^{-x^2}\,dx$ is convergent without computing its value. In Exercise 72 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral, as we will see in Section 8.5; using the methods of multivariable calculus it can be shown that the exact value is $\sqrt{\pi}/2$. Table 1 illustrates the definition of an improper integral by showing how the (computer-generated) values of $\int_0^t e^{-x^2}\,dx$ approach $\sqrt{\pi}/2$ as t becomes large. In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

Table 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

Table 2 illustrates the divergence of the integral in Example 10. It appears that the values are not approaching any fixed number.

7.8 EXERCISES

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^2 \frac{x}{x - 1}\,dx$ (b) $\displaystyle\int_0^\infty \frac{1}{1 + x^3}\,dx$

(c) $\displaystyle\int_{-\infty}^\infty x^2 e^{-x^2}\,dx$ (d) $\displaystyle\int_0^{\pi/4} \cot x\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_0^{\pi/4} \tan x\,dx$ (b) $\displaystyle\int_0^\pi \tan x\,dx$

(c) $\displaystyle\int_{-1}^1 \frac{dx}{x^2 - x - 2}$ (d) $\displaystyle\int_0^\infty e^{-x^3}\,dx$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.

(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.

(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–40 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\displaystyle\int_3^\infty \frac{1}{(x - 2)^{3/2}}\,dx$

6. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1 + x}}\,dx$

7. $\displaystyle\int_{-\infty}^0 \frac{1}{3 - 4x}\,dx$

8. $\displaystyle\int_1^\infty \frac{1}{(2x + 1)^3}\,dx$

9. $\displaystyle\int_2^\infty e^{-5p}\,dp$

10. $\displaystyle\int_{-\infty}^0 2^r\,dr$

11. $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{1 + x^3}}\,dx$

12. $\displaystyle\int_{-\infty}^\infty (y^3 - 3y^2)\,dy$

13. $\displaystyle\int_{-\infty}^\infty x e^{-x^2}\,dx$

14. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

15. $\displaystyle\int_0^\infty \sin^2\alpha\,d\alpha$

16. $\displaystyle\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$

17. $\displaystyle\int_1^\infty \frac{1}{x^2 + x}\,dx$

18. $\displaystyle\int_2^\infty \frac{dv}{v^2 + 2v - 3}$

19. $\displaystyle\int_{-\infty}^0 z e^{2z}\,dz$

20. $\displaystyle\int_2^\infty y e^{-3y}\,dy$

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

22. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

23. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4 + 4}\,dz$

24. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

25. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$

26. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}}$

27. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

28. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5 - x}}\,dx$

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x + 2}}$

30. $\displaystyle\int_{-1}^2 \frac{x}{(x + 1)^2}\,dx$

31. $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

33. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

34. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

35. $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

36. $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

37. $\displaystyle\int_0^1 r\ln r\,dr$

38. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

39. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

40. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

41–46 Sketch the region and find its area (if the area is finite).

41. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant e^{-x}\}$

42. $S = \{(x, y) \mid x \leqslant 0,\ 0 \leqslant y \leqslant e^{x}\}$

43. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/(x^3 + x)\}$

44. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant xe^{-x}\}$

45. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

46. $S = \{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\}$

47. (a) If $g(x) = (\sin^2 x)/x^2$, use your calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2, 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

48. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use your calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?
(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

55. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

56. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

by the same method as in Exercise 55.

57–59 Find the values of p for which the integral converges and evaluate the integral for those values of p.

57. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

58. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

59. $\displaystyle\int_0^1 x^p \ln x\,dx$

60. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.
(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.
(c) Prove your guess using mathematical induction.

61. (a) Show that $\int_{-\infty}^\infty x\,dx$ is divergent.
(b) Show that

$$\lim_{t\to\infty} \int_{-t}^t x\,dx = 0$$

This shows that we can't define

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty} \int_{-t}^t f(x)\,dx$$

62. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

63. We know from Example 1 that the region $\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid with finite volume.

64. Use the information and data in Exercise 5.4.33 to find the work required to propel a 1000-kg space vehicle out of the earth's gravitational field.

65. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 5.4.33) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

66. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}} x(r)\, dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

67. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
(a) Make a rough sketch of what you think the graph of F might look like.
(b) What is the meaning of the derivative $r(t) = F'(t)$?
(c) What is the value of $\int_0^\infty r(t)\, dt$? Why?

68. As we saw in Section 6.5, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life* M of an atom in the substance is

$$M = -k \int_0^\infty te^{kt}\, dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

69. In a study of the spread of illicit drug use from an enthusiastic user to a population of N users, the authors model the number of expected new users by the equation

$$\gamma = \int_0^\infty \frac{cN(1 - e^{-kt})}{k} e^{-\lambda t}\, dt$$

where c, k and λ are positive constants. Evaluate this integral to express γ in terms of c, N, k, and λ.

Source: F. Hoppensteadt et al., "Threshold Analysis of a Drug Use Epidemic Model," *Mathematical Biosciences* 53 (1981): 79–87.

70. Dialysis treatment removes urea and other waste products from a patient's blood by diverting some of the bloodflow externally through a machine called a dialyzer. The rate at which urea is removed from the blood (in mg/min) is often well described by the equation

$$u(t) = \frac{r}{V} C_0 e^{-rt/V}$$

where r is the rate of flow of blood through the dialyzer (in mL/min), V is the volume of the patient's blood (in mL), and C_0 is the amount of urea in the blood (in mg) at time $t = 0$. Evaluate the integral $\int_0^\infty u(t)$ and interpret it.

71. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2 + 1}\, dx < 0.001$$

72. Estimate the numerical value of $\int_0^\infty e^{-x^2}\, dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\, dx$ and $\int_4^\infty e^{-x^2}\, dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\, dx$, which is less than 0.0000001.

73. If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges. Find the Laplace transforms of the following functions.
(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

74. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

75. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

76. If $\int_{-\infty}^\infty f(x)\, dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx = \int_{-\infty}^b f(x)\, dx + \int_b^\infty f(x)\, dx$$

77. Show that $\int_0^\infty x^2 e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty e^{-x^2}\, dx$.

78. Show that $\int_0^\infty e^{-x^2} dx = \int_0^1 \sqrt{-\ln y}\; dy$ by interpreting the integrals as areas.

79. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) dx$$

converges. Evaluate the integral for this value of C.

80. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx$$

converges. Evaluate the integral for this value of C.

81. Suppose f is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\, dx$ is convergent?

82. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1 + x^b}\, dx$$

7 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\, dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\, dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
   (a) $\displaystyle\int_a^\infty f(x)\, dx$ (b) $\displaystyle\int_{-\infty}^b f(x)\, dx$ (c) $\displaystyle\int_{-\infty}^\infty f(x)\, dx$

7. Define the improper integral $\int_a^b f(x)\, dx$ for each of the following cases.
   (a) f has an infinite discontinuity at a.
   (b) f has an infinite discontinuity at b.
   (c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\dfrac{x(x^2+4)}{x^2-4}$ can be put in the form $\dfrac{A}{x+2} + \dfrac{B}{x-2}$.

2. $\dfrac{x^2+4}{x(x^2-4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$.

3. $\dfrac{x^2+4}{x^2(x-4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x-4}$.

4. $\dfrac{x^2-4}{x(x^2+4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2+4}$.

5. $\displaystyle\int_0^4 \frac{x}{x^2-1}\, dx = \tfrac{1}{2}\ln 15$

6. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\, dx$ is convergent.

7. If f is continuous, then $\int_{-\infty}^\infty f(x)\, dx = \lim_{t\to\infty} \int_{-t}^t f(x)\, dx$.

8. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

9. (a) Every elementary function has an elementary derivative.
   (b) Every elementary function has an elementary antiderivative.

10. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\, dx$ is convergent, then $\int_0^\infty f(x)\, dx$ is convergent.

11. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\, dx$ is convergent.

12. If $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\, dx$ is convergent.

13. If $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\, dx$ is divergent.

14. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\, dx$ diverges, then $\int_0^\infty f(x)\, dx$ also diverges.

EXERCISES

Note: Additional practice in techniques of integration is provided in Exercises 7.5.

1–40 Evaluate the integral.

1. $\displaystyle\int_1^2 \frac{(x+1)^2}{x}\, dx$

2. $\displaystyle\int_1^2 \frac{x}{(x+1)^2}\, dx$

3. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\, dx$

4. $\displaystyle\int_0^{\pi/6} t \sin 2t\, dt$

5. $\displaystyle\int \frac{dt}{2t^2 + 3t + 1}$

6. $\displaystyle\int_1^2 x^5 \ln x\, dx$

7. $\displaystyle\int_0^{\pi/2} \sin^3\theta \cos^2\theta\, d\theta$

8. $\displaystyle\int \frac{dx}{\sqrt{e^x - 1}}$

9. $\int \frac{\sin(\ln t)}{t}\,dt$

10. $\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx$

11. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

12. $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

13. $\int e^{\sqrt[3]{x}}\,dx$

14. $\int \frac{x^2+2}{x+2}\,dx$

15. $\int \frac{x-1}{x^2+2x}\,dx$

16. $\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

17. $\int x\cosh x\,dx$

18. $\int \frac{x^2+8x-3}{x^3+3x^2}\,dx$

19. $\int \frac{x+1}{9x^2+6x+5}\,dx$

20. $\int \tan^5\theta\,\sec^3\theta\,d\theta$

21. $\int \frac{dx}{\sqrt{x^2-4x}}$

22. $\int \cos\sqrt{t}\,dt$

23. $\int \frac{dx}{x\sqrt{x^2+1}}$

24. $\int e^x\cos x\,dx$

25. $\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx$

26. $\int x\sin x\cos x\,dx$

27. $\int_0^{\pi/2}\cos^3x\,\sin 2x\,dx$

28. $\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$

29. $\int_{-3}^3 \frac{x}{1+|x|}\,dx$

30. $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

31. $\int_0^{\ln 10}\frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx$

32. $\int_0^{\pi/4}\frac{x\sin x}{\cos^3x}\,dx$

33. $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

34. $\int (\arcsin x)^2\,dx$

35. $\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx$

36. $\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta$

37. $\int (\cos x+\sin x)^2\cos 2x\,dx$

38. $\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

39. $\int_0^{1/2}\frac{xe^{2x}}{(1+2x)^2}\,dx$

40. $\int_{\pi/4}^{\pi/3}\frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

41–50 Evaluate the integral or show that it is divergent.

41. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

42. $\int_1^\infty \frac{\ln x}{x^4}\,dx$

43. $\int_2^\infty \frac{dx}{x\ln x}$

44. $\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$

45. $\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$

46. $\int_0^1 \frac{1}{2-3x}\,dx$

47. $\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$

48. $\int_{-1}^1 \frac{dx}{x^2-2x}$

49. $\int_{-\infty}^{\infty}\frac{dx}{4x^2+4x+5}$

50. $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$

51–52 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C=0$).

51. $\int \ln(x^2+2x+2)\,dx$

52. $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

53. Graph the function $f(x)=\cos^2x\,\sin^3x$ and use the graph to guess the value of the integral $\int_0^{2\pi}f(x)\,dx$. Then evaluate the integral to confirm your guess.

CAS 54. (a) How would you evaluate $\int x^5e^{-2x}\,dx$ by hand? (Don't actually carry out the integration.)
(b) How would you evaluate $\int x^5e^{-2x}\,dx$ using tables? (Don't actually do it.)
(c) Use a CAS to evaluate $\int x^5e^{-2x}\,dx$.
(d) Graph the integrand and the indefinite integral on the same screen.

55–58 Use the Table of Integrals on the Reference Pages to evaluate the integral.

55. $\int \sqrt{4x^2-4x-3}\,dx$

56. $\int \csc^5t\,dt$

57. $\int \cos x\sqrt{4+\sin^2x}\,dx$

58. $\int \frac{\cot x}{\sqrt{1+2\sin x}}\,dx$

59. Verify Formula 33 in the Table of Integrals (a) by differentiation and (b) by using a trigonometric substitution.

60. Verify Formula 62 in the Table of Integrals.

61. Is it possible to find a number n such that $\int_0^\infty x^n\,dx$ is convergent?

62. For what values of a is $\int_0^\infty e^{ax}\cos x\,dx$ convergent? Evaluate the integral for those values of a.

63–64 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n=10$ to approximate the given integral. Round your answers to six decimal places.

63. $\int_2^4 \frac{1}{\ln x}\,dx$

64. $\int_1^4 \sqrt{x}\cos x\,dx$

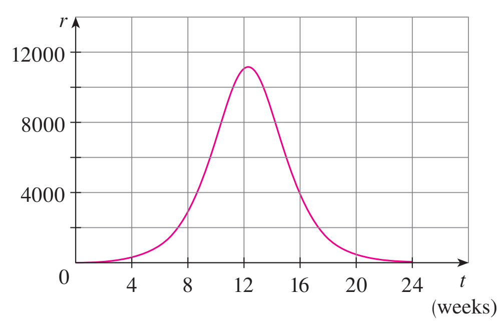
65. Estimate the errors involved in Exercise 63, parts (a) and (b). How large should n be in each case to guarantee an error of less than 0.00001?

66. Use Simpson's Rule with $n=6$ to estimate the area under the curve $y=e^x/x$ from $x=1$ to $x=4$.

67. The speedometer reading (v) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

t (min)	v (mi/h)	t (min)	v (mi/h)
0	40	6	56
1	42	7	57
2	45	8	57
3	49	9	55
4	52	10	56
5	54		

68. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.

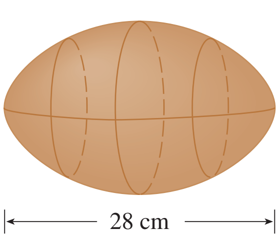


CAS **69.** (a) If $f(x) = \sin(\sin x)$, use a graph to find an upper bound for $|f^{(4)}(x)|$.

(b) Use Simpson's Rule with $n = 10$ to approximate $\int_0^{\pi} f(x)\,dx$ and use part (a) to estimate the error.

(c) How large should n be to guarantee that the size of the error in using S_n is less than 0.00001?

70. Suppose you are asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to make your estimate.



71. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\displaystyle\int_1^{\infty} \frac{2 + \sin x}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1 + x^4}}\,dx$

72. Find the area of the region bounded by the hyperbola $y^2 - x^2 = 1$ and the line $y = 3$.

73. Find the area bounded by the curves $y = \cos x$ and $y = \cos^2 x$ between $x = 0$ and $x = \pi$.

74. Find the area of the region bounded by the curves $y = 1/(2 + \sqrt{x})$, $y = 1/(2 - \sqrt{x})$, and $x = 1$.

75. The region under the curve $y = \cos^2 x$, $0 \leqslant x \leqslant \pi/2$, is rotated about the x-axis. Find the volume of the resulting solid.

76. The region in Exercise 75 is rotated about the y-axis. Find the volume of the resulting solid.

77. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, show that

$$\int_0^{\infty} f'(x)\,dx = -f(0)$$

78. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of f on the interval $[a, \infty)$ to be

$$\lim_{t\to\infty} \frac{1}{t - a} \int_a^t f(x)\,dx$$

(a) Find the average value of $y = \tan^{-1}x$ on the interval $[0, \infty)$.

(b) If $f(x) \geqslant 0$ and $\int_a^{\infty} f(x)\,dx$ is divergent, show that the average value of f on the interval $[a, \infty)$ is $\lim_{x\to\infty} f(x)$, if this limit exists.

(c) If $\int_a^{\infty} f(x)\,dx$ is convergent, what is the average value of f on the interval $[a, \infty)$?

(d) Find the average value of $y = \sin x$ on the interval $[0, \infty)$.

79. Use the substitution $u = 1/x$ to show that

$$\int_0^{\infty} \frac{\ln x}{1 + x^2}\,dx = 0$$

80. The magnitude of the repulsive force between two point charges with the same sign, one of size 1 and the other of size q, is

$$F = \frac{q}{4\pi\varepsilon_0 r^2}$$

where r is the distance between the charges and ε_0 is a constant. The *potential* V at a point P due to the charge q is defined to be the work expended in bringing a unit charge to P from infinity along the straight line that joins q and P. Find a formula for V.

Problems Plus

Cover up the solution to the example and try it yourself first.

EXAMPLE

(a) Prove that if f is a continuous function, then

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(b) Use part (a) to show that

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4}$$

for all positive numbers n.

SOLUTION

(a) At first sight, the given equation may appear somewhat baffling. How is it possible to connect the left side to the right side? Connections can often be made through one of the principles of problem solving: *introduce something extra*. Here the extra ingredient is a new variable. We often think of introducing a new variable when we use the Substitution Rule to integrate a specific function. But that technique is still useful in the present circumstance in which we have a general function f.

PS The principles of problem solving are discussed on page 98.

Once we think of making a substitution, the form of the right side suggests that it should be $u = a - x$. Then $du = -dx$. When $x = 0$, $u = a$; when $x = a$, $u = 0$. So

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du = \int_0^a f(u)\,du$$

But this integral on the right side is just another way of writing $\int_0^a f(x)\,dx$. So the given equation is proved.

(b) If we let the given integral be I and apply part (a) with $a = \pi/2$, we get

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} \frac{\sin^n(\pi/2 - x)}{\sin^n(\pi/2 - x) + \cos^n(\pi/2 - x)}\,dx$$

A well-known trigonometric identity tells us that $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$, so we get

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx$$

The computer graphs in Figure 1 make it seem plausible that all of the integrals in the example have the same value. The graph of each integrand is labeled with the corresponding value of n.

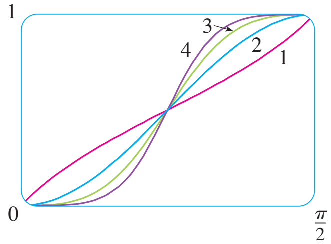


FIGURE 1

Notice that the two expressions for I are very similar. In fact, the integrands have the same denominator. This suggests that we should add the two expressions. If we do so, we get

$$2I = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

Therefore $I = \pi/4$. ■

Problems

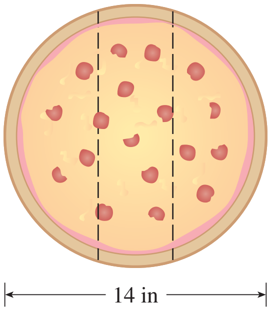


FIGURE FOR PROBLEM 1

1. Three mathematics students have ordered a 14-inch pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?

2. Evaluate

$$\int \frac{1}{x^7 - x}\,dx$$

The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

3. Evaluate $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx$.

4. The centers of two disks with radius 1 are one unit apart. Find the area of the union of the two disks.

5. An ellipse is cut out of a circle with radius a. The major axis of the ellipse coincides with a diameter of the circle and the minor axis has length $2b$. Prove that the area of the remaining part of the circle is the same as the area of an ellipse with semiaxes a and $a - b$.

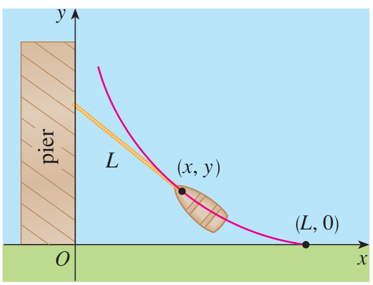


FIGURE FOR PROBLEM 6

6. A man initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The man keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix* and it has the property that the rope is always tangent to the curve (see the figure).

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

(b) Determine the function $y = f(x)$.

7. A function f is defined by

$$f(x) = \int_0^{\pi} \cos t \cos(x - t)\,dt \qquad 0 \leqslant x \leqslant 2\pi$$

Find the minimum value of f.

8. If n is a positive integer, prove that

$$\int_0^1 (\ln x)^n\,dx = (-1)^n n!$$

9. Show that

$$\int_0^1 (1 - x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n + 1)!}$$

Hint: Start by showing that if I_n denotes the integral, then

$$I_{k+1} = \frac{2k + 2}{2k + 3} I_k$$

10. Suppose that f is a positive function such that f' is continuous.

(a) How is the graph of $y = f(x) \sin nx$ related to the graph of $y = f(x)$? What happens as $n \to \infty$?

(b) Make a guess as to the value of the limit

$$\lim_{n\to\infty} \int_0^1 f(x) \sin nx \, dx$$

based on graphs of the integrand.

(c) Using integration by parts, confirm the guess that you made in part (b). [Use the fact that, since f' is continuous, there is a constant M such that $|f'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$.]

11. If $0 < a < b$, find

$$\lim_{t\to 0} \left\{ \int_0^1 [bx + a(1 - x)]^t \, dx \right\}^{1/t}$$

12. Graph $f(x) = \sin(e^x)$ and use the graph to estimate the value of t such that $\int_t^{t+1} f(x)\, dx$ is a maximum. Then find the exact value of t that maximizes this integral.

13. Evaluate $\displaystyle\int_{-1}^{\infty} \left(\frac{x^4}{1 + x^6} \right)^2 dx$.

14. Evaluate $\displaystyle\int \sqrt{\tan x}\, dx$.

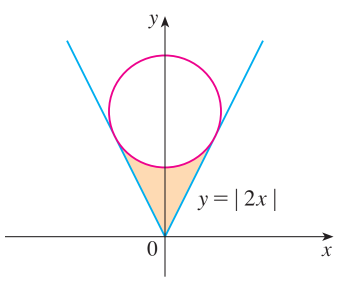


FIGURE FOR PROBLEM 15

15. The circle with radius 1 shown in the figure touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.

16. A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M \frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$M \frac{dv}{dt} - ub = -Mg \tag{1}$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2/b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.

(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.

(c) Determine the height of the rocket $y = y(t)$ at the burnout time.

(d) Find the height of the rocket at any time t.

8 Further Applications of Integration



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The Gateway Arch in St. Louis, Missouri, stands 630 feet high and was completed in 1965. The arch was designed by Eero Saarinen using an equation involving the hyperbolic cosine function. In Exercise 8.1.42 you are asked to compute the length of the curve that he used.

WE LOOKED AT SOME APPLICATIONS of integrals in Chapter 5: areas, volumes, work, and average values. Here we explore some of the many other geometric applications of integration—the length of a curve, the area of a surface—as well as quantities of interest in physics, engineering, biology, economics, and statistics. For instance, we will investigate the center of gravity of a plate, the force exerted by water pressure on a dam, the flow of blood from the human heart, and the average time spent on hold during a customer support telephone call.

8.1 Arc Length

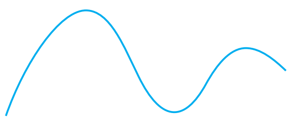


FIGURE 1

TEC Visual 8.1 shows an animation of Figure 2.

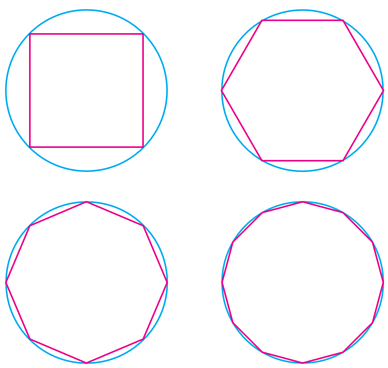


FIGURE 2

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve. We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for the concepts of area and volume.

If the curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment.) We are going to define the length of a general curve by first approximating it by a polygon and then taking a limit as the number of segments of the polygon is increased. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

Now suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \leqslant x \leqslant b$. We obtain a polygonal approximation to C by dividing the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$, illustrated in Figure 3, is an approximation to C.

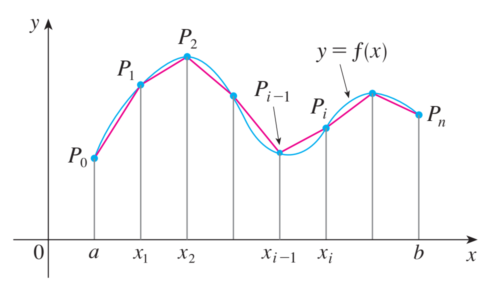


FIGURE 3

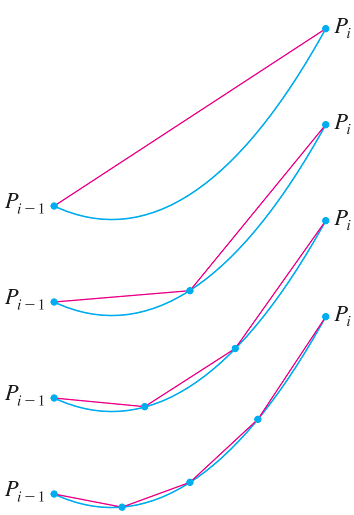


FIGURE 4

The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. (See Figure 4, where the arc of the curve between P_{i-1} and P_i has been magnified and approximations with successively smaller values of Δx are shown.) Therefore we define the **length** L of the curve C with equation $y = f(x)$, $a \leqslant x \leqslant b$, as the limit of the lengths of these inscribed polygons (if the limit exists):

1
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$.

The definition of arc length given by Equation 1 is not very convenient for computational purposes, but we can derive an integral formula for L in the case where f has a continuous derivative. [Such a function f is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

By applying the Mean Value Theorem to f on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

that is,
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Thus we have

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\,\Delta x]^2}$$

$$= \sqrt{1 + [f'(x_i^*)]^2}\,\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x \qquad (\text{since } \Delta x > 0)$$

Therefore, by Definition 1,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

We recognize this expression as being equal to

$$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

by the definition of a definite integral. We know that this integral exists because the function $g(x) = \sqrt{1 + [f'(x)]^2}$ is continuous. Thus we have proved the following theorem:

2 The Arc Length Formula If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

3
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

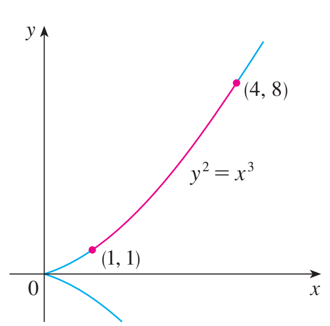


FIGURE 5

EXAMPLE 1 Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$. (See Figure 5.)

SOLUTION For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \tfrac{3}{2}x^{1/2}$$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\,dx$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}\,dx$. When $x = 1$, $u = \frac{13}{4}$; when $x = 4$, $u = 10$.

As a check on our answer to Example 1, notice from Figure 5 that the arc length ought to be slightly larger than the distance from (1, 1) to (4, 8), which is

$$\sqrt{58} \approx 7.615773$$

According to our calculation in Example 1, we have

$$L = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right) \approx 7.633705$$

Sure enough, this is a bit greater than the length of the line segment.

Therefore

$$L = \tfrac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du = \tfrac{4}{9} \cdot \tfrac{2}{3} u^{3/2}\Big]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right] = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right) \qquad \blacksquare$$

If a curve has the equation $x = g(y)$, $c \leqslant y \leqslant d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in Formula 2 or Equation 3, we obtain the following formula for its length:

4

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

EXAMPLE 2 Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

SOLUTION Since $x = y^2$, we have $dx/dy = 2y$, and Formula 4 gives

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta\, d\theta$ and $\sqrt{1 + 4y^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$. When $y = 0$, $\tan\theta = 0$, so $\theta = 0$; when $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1} 2 = \alpha$, say. Thus

$$L = \int_0^\alpha \sec\theta \cdot \tfrac{1}{2}\sec^2\theta\, d\theta = \tfrac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^\alpha \qquad \text{(from Example 7.2.8)}$$

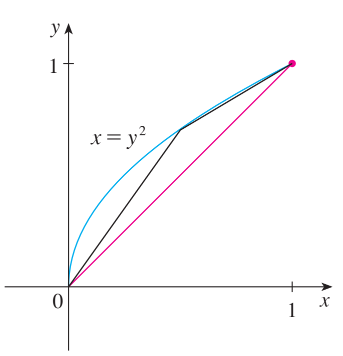
$$= \tfrac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

(We could have used Formula 21 in the Table of Integrals.) Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5} + 2\right)}{4} \qquad \blacksquare$$

Figure 6 shows the arc of the parabola whose length is computed in Example 2, together with polygonal approximations having $n = 1$ and $n = 2$ line segments, respectively. For $n = 1$ the approximate length is $L_1 = \sqrt{2}$, the diagonal of a square. The table shows the approximations L_n that we get by dividing [0, 1] into n equal subintervals. Notice that each time we double the number of sides of the polygon, we get closer to the exact length, which is

$$L = \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5} + 2\right)}{4} \approx 1.478943$$



n	L_n
1	1.414
2	1.445
4	1.464
8	1.472
16	1.476
32	1.478
64	1.479

FIGURE 6

Because of the presence of the square root sign in Formulas 2 and 4, the calculation of an arc length often leads to an integral that is very difficult or even impossible to evaluate explicitly. Thus we sometimes have to be content with finding an approximation to the length of a curve, as in the following example.

EXAMPLE 3
(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $\left(2, \frac{1}{2}\right)$.
(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

SOLUTION
(a) We have

$$y = \frac{1}{x} \qquad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so the arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx = \int_1^2 \frac{\sqrt{x^4 + 1}}{x^2}\, dx$$

(b) Using Simpson's Rule (see Section 7.7) with $a = 1$, $b = 2$, $n = 10$, $\Delta x = 0.1$, and $f(x) = \sqrt{1 + 1/x^4}$, we have

Checking the value of the definite integral with a more accurate approximation produced by a computing device, we see that the approximation using Simpson's Rule is accurate to four decimal places.

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \\ &\approx \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] \\ &\approx 1.1321 \end{aligned}$$

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. Thus if a smooth curve C has the equation $y = f(x)$, $a \leqslant x \leqslant b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called the **arc length function**, and, by Formula 2,

5 $$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

(We have replaced the variable of integration by t so that x does not have two meanings.) We can use Part 1 of the Fundamental Theorem of Calculus to differentiate Equation 5 (since the integrand is continuous):

6 $$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Equation 6 shows that the rate of change of s with respect to x is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0. The differential of arc length is

7 $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

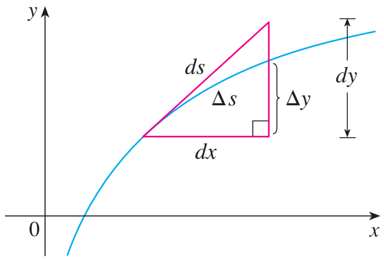


FIGURE 7

and this equation is sometimes written in the symmetric form

8 $$(ds)^2 = (dx)^2 + (dy)^2$$

The geometric interpretation of Equation 8 is shown in Figure 7. It can be used as a mnemonic device for remembering both of the Formulas 3 and 4. If we write $L = \int ds$, then from Equation 8 either we can solve to get (7), which gives (3), or we can solve to get

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

which gives (4).

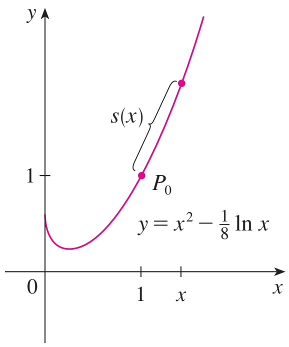


FIGURE 8

Figure 8 shows the interpretation of the arc length function in Example 4. Figure 9 shows the graph of this arc length function. Why is $s(x)$ negative when x is less than 1?

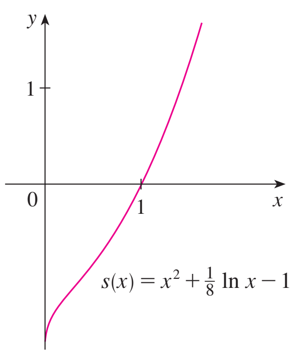


FIGURE 9

EXAMPLE 4 Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

SOLUTION If $f(x) = x^2 - \frac{1}{8}\ln x$, then

$$f'(x) = 2x - \frac{1}{8x}$$

$$1 + [f'(x)]^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x + \frac{1}{8x} \qquad \text{(since } x > 0\text{)}$$

Thus the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = t^2 + \tfrac{1}{8}\ln t\Big]_1^x$$

$$= x^2 + \tfrac{1}{8}\ln x - 1$$

For instance, the arc length along the curve from $(1, 1)$ to $(3, f(3))$ is

$$s(3) = 3^2 + \tfrac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$ ■

8.1 EXERCISES

1. Use the arc length formula (3) to find the length of the curve $y = 2x - 5$, $-1 \leqslant x \leqslant 3$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.

2. Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leqslant x \leqslant 1$. Check your answer by noting that the curve is part of a circle.

3–8 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

3. $y = \sin x,\ 0 \leqslant x \leqslant \pi$

4. $y = xe^{-x},\ 0 \leqslant x \leqslant 2$

5. $y = x - \ln x,\ 1 \leqslant x \leqslant 4$

6. $x = y^2 - 2y,\ 0 \leqslant y \leqslant 2$

7. $x = \sqrt{y} - y,\ 1 \leqslant y \leqslant 4$

8. $y^2 = \ln x,\ -1 \leqslant y \leqslant 1$

9–20 Find the exact length of the curve.

9. $y = 1 + 6x^{3/2},\quad 0 \leqslant x \leqslant 1$

10. $36y^2 = (x^2 - 4)^3,\quad 2 \leqslant x \leqslant 3,\quad y \geqslant 0$

11. $y = \dfrac{x^3}{3} + \dfrac{1}{4x},\quad 1 \leqslant x \leqslant 2$

12. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2},\quad 1 \leqslant y \leqslant 2$

13. $x = \frac{1}{3}\sqrt{y}\,(y - 3),\quad 1 \leqslant y \leqslant 9$

14. $y = \ln(\cos x),\quad 0 \leqslant x \leqslant \pi/3$

15. $y = \ln(\sec x),\quad 0 \leqslant x \leqslant \pi/4$

16. $y = 3 + \frac{1}{2}\cosh 2x,\quad 0 \leqslant x \leqslant 1$

17. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x,\quad 1 \leqslant x \leqslant 2$

18. $y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right)$

19. $y = \ln(1 - x^2),\quad 0 \leqslant x \leqslant \frac{1}{2}$

20. $y = 1 - e^{-x},\quad 0 \leqslant x \leqslant 2$

21–22 Find the length of the arc of the curve from point P to point Q.

21. $y = \frac{1}{2}x^2,\quad P\left(-1, \frac{1}{2}\right),\quad Q\left(1, \frac{1}{2}\right)$

22. $x^2 = (y - 4)^3,\quad P(1, 5),\quad Q(8, 8)$

23–24 Graph the curve and visually estimate its length. Then use your calculator to find the length correct to four decimal places.

23. $y = x^2 + x^3,\quad 1 \leqslant x \leqslant 2$

24. $y = x + \cos x,\quad 0 \leqslant x \leqslant \pi/2$

25–28 Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral produced by a calculator.

25. $y = x\sin x,\ 0 \leqslant x \leqslant 2\pi$

26. $y = \sqrt[3]{x},\ 1 \leqslant x \leqslant 6$

27. $y = \ln(1 + x^3),\ 0 \leqslant x \leqslant 5$

28. $y = e^{-x^2},\ 0 \leqslant x \leqslant 2$

29. (a) Graph the curve $y = x\sqrt[3]{4 - x},\ 0 \leqslant x \leqslant 4$.
(b) Compute the lengths of inscribed polygons with $n = 1$, 2, and 4 sides. (Divide the interval into equal subintervals.) Illustrate by sketching these polygons (as in Figure 6).
(c) Set up an integral for the length of the curve.
(d) Use your calculator to find the length of the curve to four decimal places. Compare with the approximations in part (b).

30. Repeat Exercise 29 for the curve

$$y = x + \sin x \qquad 0 \leqslant x \leqslant 2\pi$$

CAS **31.** Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = e^x$ that lies between the points (0, 1) and $(2, e^2)$.

CAS **32.** Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = x^{4/3}$ that lies between the points (0, 0) and (1, 1). If your CAS has trouble evaluating the integral, make a substitution that changes the integral into one that the CAS can evaluate.

33. Sketch the curve with equation $x^{2/3} + y^{2/3} = 1$ and use symmetry to find its length.

34. (a) Sketch the curve $y^3 = x^2$.
(b) Use Formulas 3 and 4 to set up two integrals for the arc length from (0, 0) to (1, 1). Observe that one of these is an improper integral and evaluate both of them.
(c) Find the length of the arc of this curve from $(-1, 1)$ to (8, 4).

35. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

36. (a) Find the arc length function for the curve $y = \ln(\sin x)$, $0 < x < \pi$, with starting point $(\pi/2, 0)$.
(b) Graph both the curve and its arc length function on the same screen.

37. Find the arc length function for the curve $y = \sin^{-1}x + \sqrt{1 - x^2}$ with starting point (0, 1).

38. The arc length function for a curve $y = f(x)$, where f is an increasing function, is $s(x) = \int_0^x \sqrt{3t + 5}\, dt$.
(a) If f has y-intercept 2, find an equation for f.
(b) What point on the graph of f is 3 units along the curve from the y-intercept? State your answer rounded to 3 decimal places.

39. For the function $f(x) = \frac{1}{4}e^x + e^{-x}$, prove that the arc length on any interval has the same value as the area under the curve.

40. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ ft is given by $y = 150 - \frac{1}{40}(x - 50)^2$. Find the distance traveled by the kite.

41. A hawk flying at 15 m/s at an altitude of 180 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

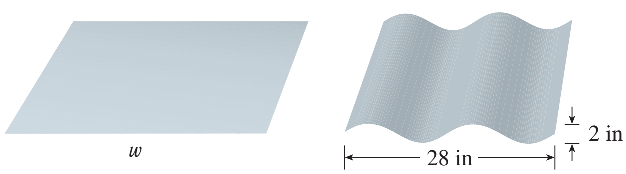
until it hits the ground, where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.

42. The Gateway Arch in St. Louis (see the photo on page 583) was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

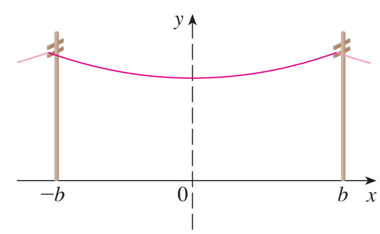
for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$. Set up an integral for the length of the arch and use your calculator to estimate the length correct to the nearest meter.

43. A manufacturer of corrugated metal roofing wants to produce panels that are 28 in. wide and 2 in. high by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = \sin(\pi x/7)$ and find the width w of a flat metal sheet that is needed to make a 28-inch panel. (Use your calculator to evaluate the integral correct to four significant digits.)



44. (a) The figure shows a telephone wire hanging between two poles at $x = -b$ and $x = b$. It takes the shape of a catenary with equation $y = c + a \cosh(x/a)$. Find the length of the wire.

(b) Suppose two telephone poles are 50 ft apart and the length of the wire between the poles is 51 ft. If the lowest point of the wire must be 20 ft above the ground, how high up on each pole should the wire be attached?



45. Find the length of the curve

$$y = \int_1^x \sqrt{t^3 - 1}\, dt \qquad 1 \leqslant x \leqslant 4$$

46. The curves with equations $x^n + y^n = 1$, $n = 4, 6, 8, \ldots$, are called **fat circles**. Graph the curves with $n = 2, 4, 6, 8$, and 10 to see why. Set up an integral for the length L_{2k} of the fat circle with $n = 2k$. Without attempting to evaluate this integral, state the value of $\lim_{k \to \infty} L_{2k}$.

DISCOVERY PROJECT ARC LENGTH CONTEST

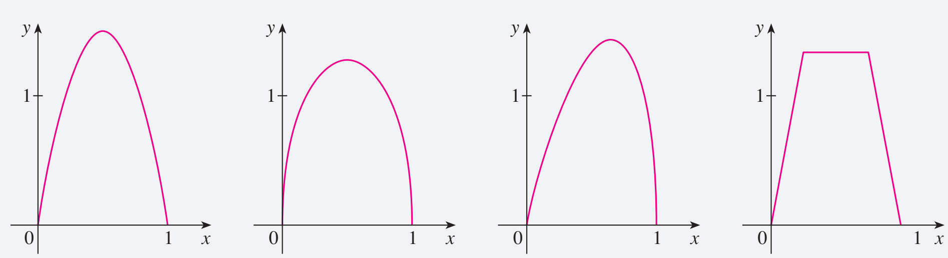
The curves shown are all examples of graphs of continuous functions f that have the following properties.

1. $f(0) = 0$ and $f(1) = 0$.

2. $f(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$.

3. The area under the graph of f from 0 to 1 is equal to 1.

The lengths L of these curves, however, are different.



$L \approx 3.249$ $L \approx 2.919$ $L \approx 3.152$ $L \approx 3.213$

Try to discover formulas for two functions that satisfy the given conditions 1, 2, and 3. (Your graphs might be similar to the ones shown or could look quite different.) Then calculate the arc length of each graph. The winning entry will be the one with the smallest arc length.

8.2 Area of a Surface of Revolution

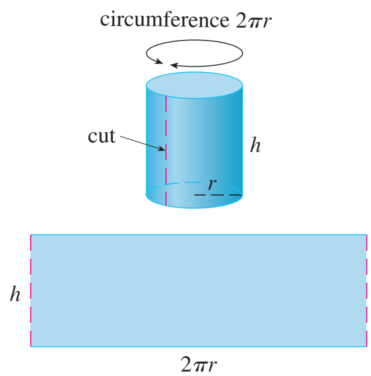


FIGURE 1

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution of the type discussed in Sections 5.2 and 5.3.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition. If the surface area is A, we can imagine that painting the surface would require the same amount of paint as does a flat region with area A.

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius r and height h is taken to be $A = 2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and h.

Likewise, we can take a circular cone with base radius r and slant height l, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius l and central angle $\theta = 2\pi r/l$. We know that, in general, the area of a sector of a circle with radius l and angle θ is $\frac{1}{2}l^2\theta$ (see Exercise 7.3.35) and so in this case the area is

$$A = \tfrac{1}{2}l^2\theta = \tfrac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi rl$$

Therefore we define the lateral surface area of a cone to be $A = \pi rl$.

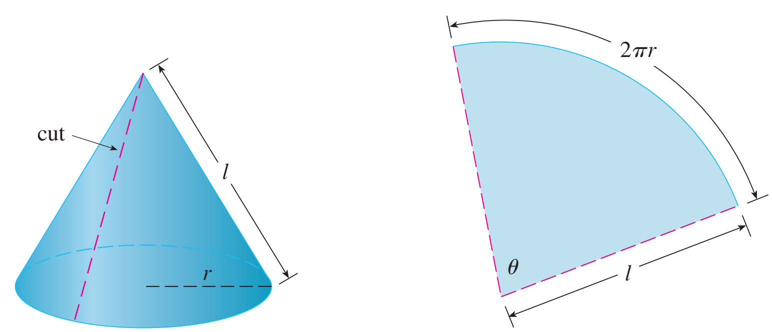


FIGURE 2

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon. When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of *bands,* each formed by rotating a line segment about an axis. To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3. The area of the band (or frustum of a cone) with slant height l and upper and lower radii r_1 and r_2 is found by subtracting the areas of two cones:

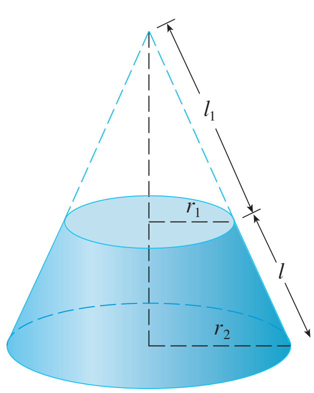


FIGURE 3

1 $$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \qquad \text{or} \qquad (r_2 - r_1)l_1 = r_1 l$$

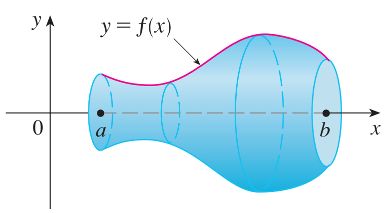
Putting this in Equation 1, we get

$$A = \pi(r_1 l + r_2 l)$$

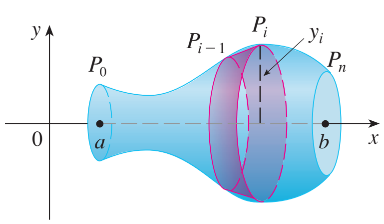
or

2
$$A = 2\pi r l$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.



(a) Surface of revolution



(b) Approximating band

FIGURE 4

Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where f is positive and has a continuous derivative. In order to define its surface area, we divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as we did in determining arc length. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

As in the proof of Theorem 8.1.2, we have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$. When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous. Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

3
$$\sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

This approximation appears to become better as $n \to \infty$ and, recognizing (3) as a Riemann sum for the function $g(x) = 2\pi f(x) \sqrt{1 + [f'(x)]^2}$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Therefore, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis as

4
$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

With the Leibniz notation for derivatives, this formula becomes

5 $$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y)$, $c \leqslant y \leqslant d$, then the formula for surface area becomes

6 $$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

and both Formulas 5 and 6 can be summarized symbolically, using the notation for arc length given in Section 8.1, as

7 $$S = \int 2\pi y\, ds$$

For rotation about the y-axis, the surface area formula becomes

8 $$S = \int 2\pi x\, ds$$

where, as before, we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

These formulas can be remembered by thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis, respectively (see Figure 5).

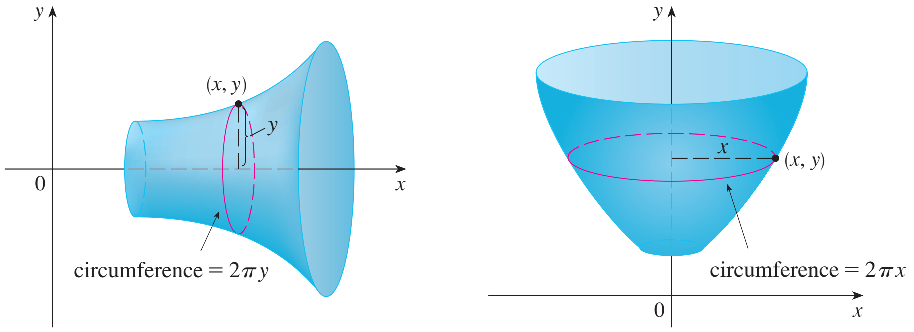


FIGURE 5 (a) Rotation about x-axis: $S = \int 2\pi y\, ds$ (b) Rotation about y-axis: $S = \int 2\pi x\, ds$

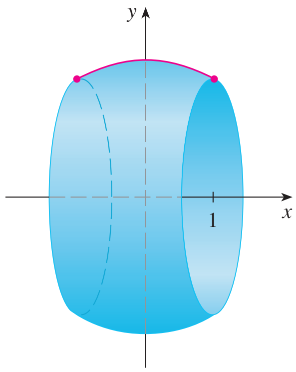


FIGURE 6

Figure 6 shows the portion of the sphere whose surface area is computed in Example 1.

EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

SOLUTION We have

$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$\begin{aligned} S &= \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\, \frac{2}{\sqrt{4 - x^2}}\, dx = 4\pi \int_{-1}^{1} 1\, dx = 4\pi(2) = 8\pi \end{aligned}$$

■

Figure 7 shows the surface of revolution whose area is computed in Example 2.

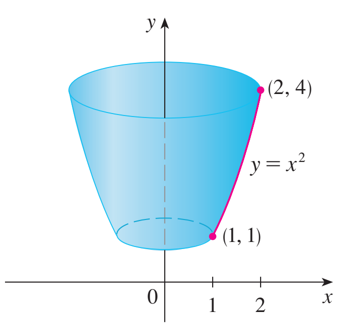


FIGURE 7

EXAMPLE 2 The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

SOLUTION 1 Using

$$y = x^2 \qquad \text{and} \qquad \frac{dy}{dx} = 2x$$

we have, from Formula 8,

$$\begin{aligned} S &= \int 2\pi x\, ds \\ &= \int_{1}^{2} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{1}^{2} x \sqrt{1 + 4x^2}\, dx \end{aligned}$$

Substituting $u = 1 + 4x^2$, we have $du = 8x\, dx$. Remembering to change the limits of integration, we have

$$\begin{aligned} S &= 2\pi \int_{5}^{17} \sqrt{u} \cdot \tfrac{1}{8}\, du \\ &= \frac{\pi}{4} \int_{5}^{17} u^{1/2}\, du = \frac{\pi}{4} \left[\tfrac{2}{3} u^{3/2}\right]_{5}^{17} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

SOLUTION 2 Using

$$x = \sqrt{y} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

As a check on our answer to Example 2, notice from Figure 7 that the surface area should be close to that of a circular cylinder with the same height and radius halfway between the upper and lower radius of the surface: $2\pi(1.5)(3) \approx 28.27$. We computed that the surface area was

$$\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.85$$

which seems reasonable. Alternatively, the surface area should be slightly larger than the area of a frustum of a cone with the same top and bottom edges. From Equation 2, this is $2\pi(1.5)(\sqrt{10}) \approx 29.80$.

we have

$$S = \int 2\pi x\, ds = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy = 2\pi \int_1^4 \sqrt{y + \tfrac{1}{4}}\, dy = \pi \int_1^4 \sqrt{4y + 1}\, dy$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du \qquad \text{(where } u = 1 + 4y\text{)}$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \qquad \text{(as in Solution 1)}$$

■

EXAMPLE 3 Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis.

SOLUTION Using Formula 5 with

$$y = e^x \qquad \text{and} \qquad \frac{dy}{dx} = e^x$$

Another method: Use Formula 6 with $x = \ln y$.

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du \qquad \text{(where } u = e^x\text{)}$$

$$= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta \qquad \text{(where } u = \tan\theta \text{ and } \alpha = \tan^{-1}e\text{)}$$

Or use Formula 21 in the Table of Integrals.

$$= 2\pi \cdot \tfrac{1}{2}\Big[\sec\theta\, \tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\alpha} \qquad \text{(by Example 7.2.8)}$$

$$= \pi\left[\sec\alpha\, \tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right]$$

Since $\tan\alpha = e$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\left[e\sqrt{1 + e^2} + \ln\left(e + \sqrt{1 + e^2}\right) - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right]$$

■

8.2 EXERCISES

1–6
(a) Set up an integral for the area of the surface obtained by rotating the curve about (i) the x-axis and (ii) the y-axis.
(b) Use the numerical integration capability of a calculator to evaluate the surface areas correct to four decimal places.

1. $y = \tan x$, $0 \leqslant x \leqslant \pi/3$

2. $y = x^{-2}$, $1 \leqslant x \leqslant 2$

3. $y = e^{-x^2}$, $-1 \leqslant x \leqslant 1$

4. $x = \ln(2y + 1)$, $0 \leqslant y \leqslant 1$

5. $x = y + y^3$, $0 \leqslant y \leqslant 1$

6. $y = \tan^{-1}x$, $0 \leqslant x \leqslant 2$

7–14 Find the exact area of the surface obtained by rotating the curve about the x-axis.

7. $y = x^3$, $0 \leqslant x \leqslant 2$

8. $y = \sqrt{5 - x}$, $3 \leqslant x \leqslant 5$

9. $y^2 = x + 1$, $0 \leqslant x \leqslant 3$

10. $y = \sqrt{1 + e^x}$, $0 \leqslant x \leqslant 1$

11. $y = \cos\left(\tfrac{1}{2}x\right)$, $0 \leqslant x \leqslant \pi$

12. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $\tfrac{1}{2} \leqslant x \leqslant 1$

13. $x = \tfrac{1}{3}(y^2 + 2)^{3/2}$, $1 \leqslant y \leqslant 2$

14. $x = 1 + 2y^2$, $1 \leqslant y \leqslant 2$

15–18 The given curve is rotated about the y-axis. Find the area of the resulting surface.

15. $y = \frac{1}{3}x^{3/2}$, $0 \leqslant x \leqslant 12$

16. $x^{2/3} + y^{2/3} = 1$, $0 \leqslant y \leqslant 1$

17. $x = \sqrt{a^2 - y^2}$, $0 \leqslant y \leqslant a/2$

18. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \leqslant x \leqslant 2$

19–22 Use Simpson's Rule with $n = 10$ to approximate the area of the surface obtained by rotating the curve about the x-axis. Compare your answer with the value of the integral produced by a calculator.

19. $y = \frac{1}{5}x^5$, $0 \leqslant x \leqslant 5$

20. $y = x + x^2$, $0 \leqslant x \leqslant 1$

21. $y = xe^x$, $0 \leqslant x \leqslant 1$

22. $y = x \ln x$, $1 \leqslant x \leqslant 2$

CAS **23–24** Use either a CAS or a table of integrals to find the exact area of the surface obtained by rotating the given curve about the x-axis.

23. $y = 1/x$, $1 \leqslant x \leqslant 2$

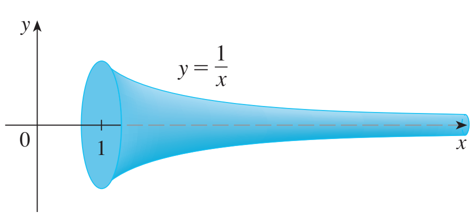
24. $y = \sqrt{x^2 + 1}$, $0 \leqslant x \leqslant 3$

CAS **25–26** Use a CAS to find the exact area of the surface obtained by rotating the curve about the y-axis. If your CAS has trouble evaluating the integral, express the surface area as an integral in the other variable.

25. $y = x^3$, $0 \leqslant y \leqslant 1$

26. $y = \ln(x + 1)$, $0 \leqslant x \leqslant 1$

27. If the region $\mathscr{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ is rotated about the x-axis, the volume of the resulting solid is finite (see Exercise 7.8.63). Show that the surface area is infinite. (The surface is shown in the figure and is known as **Gabriel's horn**.)



28. If the infinite curve $y = e^{-x}$, $x \geqslant 0$, is rotated about the x-axis, find the area of the resulting surface.

29. (a) If $a > 0$, find the area of the surface generated by rotating the loop of the curve $3ay^2 = x(a - x)^2$ about the x-axis.

(b) Find the surface area if the loop is rotated about the y-axis.

30. A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 10-ft diameter and a maximum depth of 2 ft, find the value of a and the surface area of the dish.

31. (a) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

is rotated about the x-axis to form a surface called an *ellipsoid*, or *prolate spheroid*. Find the surface area of this ellipsoid.

(b) If the ellipse in part (a) is rotated about its minor axis (the y-axis), the resulting ellipsoid is called an *oblate spheroid*. Find the surface area of this ellipsoid.

32. Find the surface area of the torus in Exercise 5.2.63.

33. If the curve $y = f(x)$, $a \leqslant x \leqslant b$, is rotated about the horizontal line $y = c$, where $f(x) \leqslant c$, find a formula for the area of the resulting surface.

CAS **34.** Use the result of Exercise 33 to set up an integral to find the area of the surface generated by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = 4$. Then use a CAS to evaluate the integral.

35. Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.

36. (a) Show that the surface area of a zone of a sphere that lies between two parallel planes is $S = 2\pi Rh$, where R is the radius of the sphere and h is the distance between the planes. (Notice that S depends only on the distance between the planes and not on their location, provided that both planes intersect the sphere.)

(b) Show that the surface area of a zone of a *cylinder* with radius R and height h is the same as the surface area of the zone of a *sphere* in part (a).

37. Show that if we rotate the curve $y = e^{x/2} + e^{-x/2}$ about the x-axis, the area of the resulting surface is the same value as the enclosed volume for any interval $a \leqslant x \leqslant b$.

38. Let L be the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, where f is positive and has a continuous derivative. Let S_f be the surface area generated by rotating the curve about the x-axis. If c is a positive constant, define $g(x) = f(x) + c$ and let S_g be the corresponding surface area generated by the curve $y = g(x)$, $a \leqslant x \leqslant b$. Express S_g in terms of S_f and L.

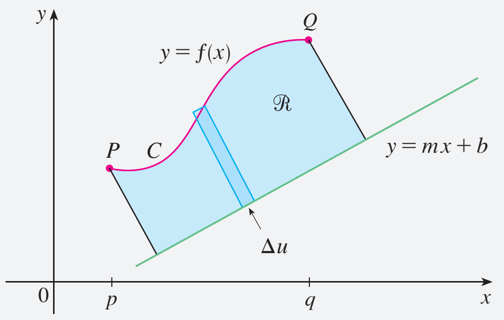
39. Formula 4 is valid only when $f(x) \geqslant 0$. Show that when $f(x)$ is not necessarily positive, the formula for surface area becomes

$$S = \int_a^b 2\pi\,|\,f(x)\,|\sqrt{1 + [f'(x)]^2}\,dx$$

DISCOVERY PROJECT ROTATING ON A SLANT

We know how to find the volume of a solid of revolution obtained by rotating a region about a horizontal or vertical line (see Section 5.2). We also know how to find the surface area of a surface of revolution if we rotate a curve about a horizontal or vertical line (see Section 8.2). But what if we rotate about a slanted line, that is, a line that is neither horizontal nor vertical? In this project you are asked to discover formulas for the volume of a solid of revolution and for the area of a surface of revolution when the axis of rotation is a slanted line.

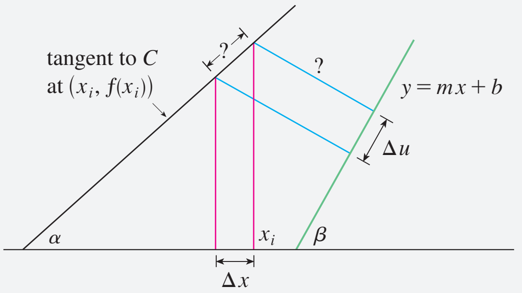
Let C be the arc of the curve $y = f(x)$ between the points $P(p, f(p))$ and $Q(q, f(q))$ and let $\mathcal{R}$ be the region bounded by C, by the line $y = mx + b$ (which lies entirely below C), and by the perpendiculars to the line from P and Q.

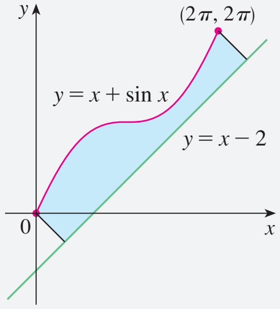


1. Show that the area of $\mathcal{R}$ is

$$\frac{1}{1 + m^2} \int_p^q [f(x) - mx - b][1 + mf'(x)]\, dx$$

[*Hint:* This formula can be verified by subtracting areas, but it will be helpful throughout the project to derive it by first approximating the area using rectangles perpendicular to the line, as shown in the following figure. Use the figure to help express Δu in terms of Δx.]





2. Find the area of the region shown in the figure at the left.

3. Find a formula (similar to the one in Problem 1) for the volume of the solid obtained by rotating $\mathcal{R}$ about the line $y = mx + b$.

4. Find the volume of the solid obtained by rotating the region of Problem 2 about the line $y = x - 2$.

5. Find a formula for the area of the surface obtained by rotating C about the line $y = mx + b$.

CAS **6.** Use a computer algebra system to find the exact area of the surface obtained by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = \frac{1}{2}x$. Then approximate your result to three decimal places.

8.3 Applications to Physics and Engineering

Among the many applications of integral calculus to physics and engineering, we consider two here: force due to water pressure and centers of mass. As with our previous applications to geometry (areas, volumes, and lengths) and to work, our strategy is to break up the physical quantity into a large number of small parts, approximate each small part, add the results (giving a Riemann sum), take the limit, and then evaluate the resulting integral.

Hydrostatic Pressure and Force

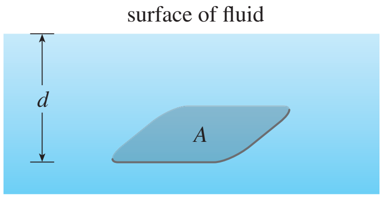


FIGURE 1

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid as in Figure 1. The fluid directly above the plate (think of a column of liquid) has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho gAd$$

where g is the acceleration due to gravity. The **pressure** P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho gd$$

When using US Customary units, we write $P = \rho gd = \delta d$, where $\delta = \rho g$ is the *weight density* (as opposed to ρ, which is the *mass density*). For instance, the weight density of water is $\delta = 62.5 \text{ lb/ft}^3$.

The SI unit for measuring pressure is a newton per square meter, which is called a pascal (abbreviation: $1 \text{ N/m}^2 = 1 \text{ Pa}$). Since this is a small unit, the kilopascal (kPa) is often used. For instance, because the density of water is $\rho = 1000 \text{ kg/m}^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$\begin{aligned} P = \rho gd &= 1000 \text{ kg/m}^3 \times 9.8 \text{ m/s}^2 \times 2 \text{ m} \\ &= 19{,}600 \text{ Pa} = 19.6 \text{ kPa} \end{aligned}$$

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.* (A diver feels the same pressure on nose and both ears.) Thus the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

$$\boxed{1} \qquad P = \rho gd = \delta d$$

This helps us determine the hydrostatic force (the force exerted by a fluid at rest) against a *vertical* plate or wall or dam. This is not a straightforward problem because the pressure is not constant but increases as the depth increases.

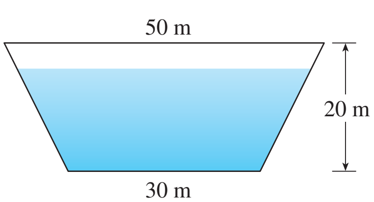


FIGURE 2

EXAMPLE 1 A dam has the shape of the trapezoid shown in Figure 2. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

SOLUTION We choose a vertical x-axis with origin at the surface of the water and directed downward as in Figure 3(a). The depth of the water is 16 m, so we divide the

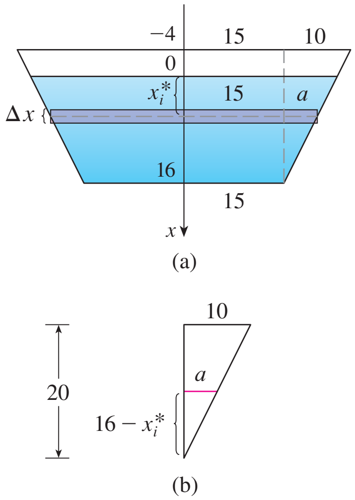


FIGURE 3

interval $[0, 16]$ into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$. The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles in Figure 3(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20} \qquad \text{or} \qquad a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

and so $$w_i = 2(15 + a) = 2\left(15 + 8 - \tfrac{1}{2}x_i^*\right) = 46 - x_i^*$$

If A_i is the area of the ith strip, then

$$A_i \approx w_i \,\Delta x = (46 - x_i^*)\,\Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 1 to write

$$P_i \approx 1000 g x_i^*$$

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i \approx 1000 g x_i^* (46 - x_i^*)\,\Delta x$$

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g x_i^*(46 - x_i^*)\,\Delta x = \int_0^{16} 1000 g x(46 - x)\,dx$$

$$= 1000(9.8)\int_0^{16} (46x - x^2)\,dx = 9800\left[23x^2 - \frac{x^3}{3}\right]_0^{16}$$

$$\approx 4.43 \times 10^7 \text{ N}$$

■

EXAMPLE 2 Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

SOLUTION In this example it is convenient to choose the axes as in Figure 4 so that the origin is placed at the center of the drum. Then the circle has a simple equation, $x^2 + y^2 = 9$. As in Example 1 we divide the circular region into horizontal strips of equal width. From the equation of the circle, we see that the length of the ith strip is $2\sqrt{9 - (y_i^*)^2}$ and so its area is

$$A_i = 2\sqrt{9 - (y_i^*)^2}\,\Delta y$$

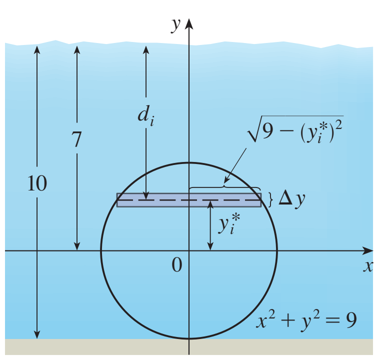


FIGURE 4

Because the weight density of water is $\delta = 62.5 \text{ lb/ft}^3$, the pressure on this strip is approximately

$$\delta d_i = 62.5(7 - y_i^*)$$

and so the force on the strip is approximately

$$\delta d_i A_i = 62.5(7 - y_i^*)\, 2\sqrt{9 - (y_i^*)^2}\,\Delta y$$

The total force is obtained by adding the forces on all the strips and taking the limit:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 62.5(7 - y_i^*)\, 2\sqrt{9 - (y_i^*)^2}\, \Delta y \\ &= 125 \int_{-3}^{3} (7 - y)\sqrt{9 - y^2}\, dy \\ &= 125 \cdot 7 \int_{-3}^{3} \sqrt{9 - y^2}\, dy - 125 \int_{-3}^{3} y\sqrt{9 - y^2}\, dy \end{aligned}$$

The second integral is 0 because the integrand is an odd function (see Theorem 4.5.6). The first integral can be evaluated using the trigonometric substitution $y = 3\sin\theta$, but it's simpler to observe that it is the area of a semicircular disk with radius 3. Thus

$$\begin{aligned} F &= 875 \int_{-3}^{3} \sqrt{9 - y^2}\, dy = 875 \cdot \tfrac{1}{2}\pi(3)^2 \\ &= \frac{7875\pi}{2} \approx 12{,}370 \text{ lb} \end{aligned}$$

■

Moments and Centers of Mass

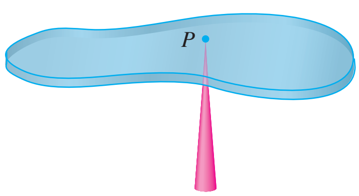


FIGURE 5

Our main objective here is to find the point P on which a thin plate of any given shape balances horizontally as in Figure 5. This point is called the **center of mass** (or center of gravity) of the plate.

We first consider the simpler situation illustrated in Figure 6, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum. The rod will balance if

$$\boxed{2} \qquad m_1 d_1 = m_2 d_2$$

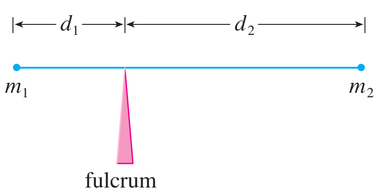


FIGURE 6

This is an experimental fact discovered by Archimedes and called the Law of the Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

Now suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\bar{x}$. If we compare Figures 6 and 7, we see that $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$ and so Equation 2 gives

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\boxed{3} \qquad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The numbers $m_1 x_1$ and $m_2 x_2$ are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 3 says that the center of mass $\bar{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

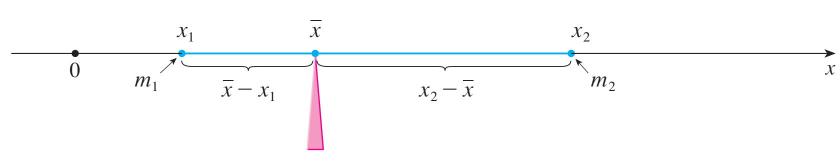


FIGURE 7

In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass

of the system is located at

4
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \Sigma m_i$ is the total mass of the system, and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the **moment of the system about the origin**. Then Equation 4 could be rewritten as $m\bar{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane as shown in Figure 8. By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be

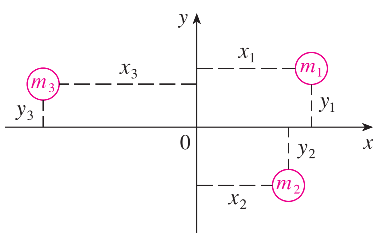


FIGURE 8

5
$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** as

6
$$M_x = \sum_{i=1}^{n} m_i y_i$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

As in the one-dimensional case, the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas

7
$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \Sigma m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

EXAMPLE 3 Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1)$, $(2, -1)$, and $(3, 2)$, respectively.

SOLUTION We use Equations 5 and 6 to compute the moments:

$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

Since $m = 3 + 4 + 8 = 15$, we use Equations 7 to obtain

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

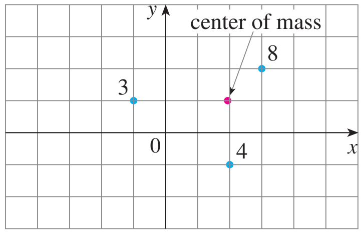


FIGURE 9

Thus the center of mass is $\left(1\frac{14}{15}, 1\right)$. (See Figure 9.) ■

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathcal{R}$ of the plane. We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathcal{R}$. In doing so we use the following physical principles: The **symmetry principle** says that if $\mathcal{R}$ is symmetric about a line l, then the centroid of $\mathcal{R}$ lies on l. (If $\mathcal{R}$ is reflected about l, then $\mathcal{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on l.) Thus the centroid of a rectangle is its center. Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged. Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

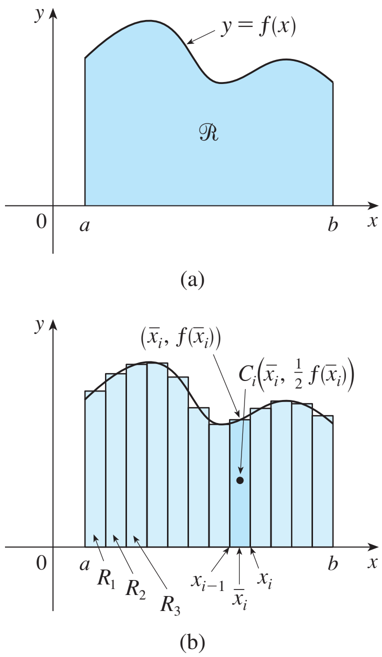


FIGURE 10

Suppose that the region $\mathcal{R}$ is of the type shown in Figure 10(a); that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of f, where f is a continuous function. We divide the interval $[a, b]$ into n subintervals with endpoints x_0, $x_1, \ldots, x_n$ and equal width Δx. We choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the ith subinterval, that is, $\bar{x}_i = (x_{i-1} + x_i)/2$. This determines the polygonal approximation to $\mathcal{R}$ shown in Figure 10(b). The centroid of the ith approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2} f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\,\Delta x$, so its mass is

$$\rho f(\bar{x}_i)\,\Delta x$$

The moment of R_i about the y-axis is the product of its mass and the distance from C_i to the y-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho\,\bar{x}_i f(\bar{x}_i)\,\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathcal{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathcal{R}$ itself about the y-axis:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\bar{x}_i f(\bar{x}_i)\,\Delta x = \rho \int_a^b x f(x)\,dx$$

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis (which is half the height of R_i):

$$M_x(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\tfrac{1}{2} f(\bar{x}_i) = \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathcal{R}$ about the x-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\,dx$$

Just as for systems of particles, the center of mass of the plate is defined so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)\,dx$$

and so

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Notice the cancellation of the ρ's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) is located at the point $(\bar{x}, \bar{y})$, where

8 $$\bar{x} = \frac{1}{A}\int_a^b xf(x)\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$$

EXAMPLE 4 Find the center of mass of a semicircular plate of radius r.

SOLUTION In order to use (8) we place the semicircle as in Figure 11 so that $f(x) = \sqrt{r^2 - x^2}$ and $a = -r$, $b = r$. Here there is no need to use the formula to calculate $\bar{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis, so $\bar{x} = 0$. The area of the semicircle is $A = \frac{1}{2}\pi r^2$, so

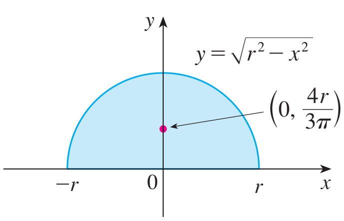


FIGURE 11

$$\begin{aligned}
\bar{y} &= \frac{1}{A}\int_{-r}^{r} \frac{1}{2}[f(x)]^2\,dx \\
&= \frac{1}{\frac{1}{2}\pi r^2}\cdot\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx \\
&= \frac{2}{\pi r^2}\int_0^r (r^2 - x^2)\,dx \qquad \text{(since the integrand is even)} \\
&= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \\
&= \frac{2}{\pi r^2}\,\frac{2r^3}{3} = \frac{4r}{3\pi}
\end{aligned}$$

The center of mass is located at the point $(0, 4r/(3\pi))$. ■

EXAMPLE 5 Find the centroid of the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

SOLUTION The area of the region is

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = 1$$

so Formulas 8 give

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} xf(x)\,dx = \int_0^{\pi/2} x\cos x\,dx$$

$$= x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx \qquad \text{(by integration by parts)}$$

$$= \frac{\pi}{2} - 1$$

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \tfrac{1}{2}[f(x)]^2\,dx = \tfrac{1}{2}\int_0^{\pi/2}\cos^2 x\,dx$$

$$= \tfrac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \tfrac{1}{4}\Big[x + \tfrac{1}{2}\sin 2x\Big]_0^{\pi/2} = \frac{\pi}{8}$$

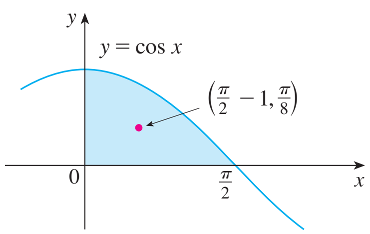


FIGURE 12

The centroid is $(\frac{1}{2}\pi - 1, \frac{1}{8}\pi)$ and is shown in Figure 12. ■

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$, as illustrated in Figure 13, then the same sort of argument that led to Formulas 8 can be used to show that the centroid of $\mathcal{R}$ is $(\bar{x}, \bar{y})$, where

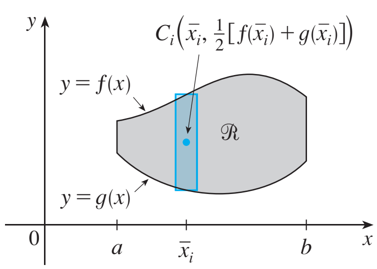


FIGURE 13

9
$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx$$

(See Exercise 51.)

EXAMPLE 6 Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

SOLUTION The region is sketched in Figure 14. We take $f(x) = x$, $g(x) = x^2$, $a = 0$, and $b = 1$ in Formulas 9. First we note that the area of the region is

$$A = \int_0^1 (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1 = \frac{1}{6}$$

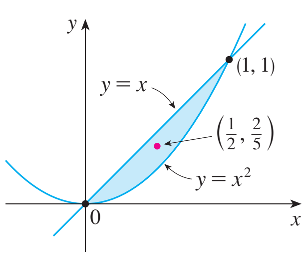


FIGURE 14

Therefore

$$\bar{x} = \frac{1}{A}\int_0^1 x[f(x) - g(x)]\,dx = \frac{1}{\frac{1}{6}}\int_0^1 x(x - x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_0^1 \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx = \frac{1}{\frac{1}{6}}\int_0^1 \tfrac{1}{2}(x^2 - x^4)\,dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

The centroid is $(\frac{1}{2}, \frac{2}{5})$. ■

We end this section by showing a surprising connection between centroids and volumes of revolution.

This theorem is named after the Greek mathematician Pappus of Alexandria, who lived in the fourth century AD.

Theorem of Pappus Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

PROOF We give the proof for the special case in which the region lies between $y = f(x)$ and $y = g(x)$ as in Figure 13 and the line l is the y-axis. Using the method of cylindrical shells (see Section 5.3), we have

$$\begin{aligned} V &= \int_a^b 2\pi x[f(x) - g(x)]\,dx \\ &= 2\pi \int_a^b x[f(x) - g(x)]\,dx \\ &= 2\pi(\bar{x}A) \qquad \text{(by Formulas 9)} \\ &= (2\pi\bar{x})A = Ad \end{aligned}$$

where $d = 2\pi\bar{x}$ is the distance traveled by the centroid during one rotation about the y-axis. ■

EXAMPLE 7 A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R ($> r$) from the center of the circle. Find the volume of the torus.

SOLUTION The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$$ ■

The method of Example 7 should be compared with the method of Exercise 5.2.63.

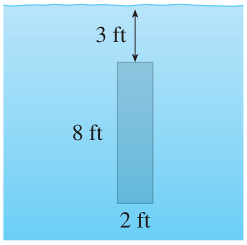
8.3 EXERCISES

1. An aquarium 5 ft long, 2 ft wide, and 3 ft deep is full of water. Find (a) the hydrostatic pressure on the bottom of the aquarium, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the aquarium.

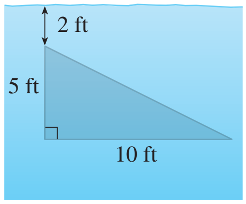
2. A tank is 8 m long, 4 m wide, 2 m high, and contains kerosene with density 820 kg/m^3 to a depth of 1.5 m. Find (a) the hydrostatic pressure on the bottom of the tank, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the tank.

3–11 A vertical plate is submerged (or partially submerged) in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

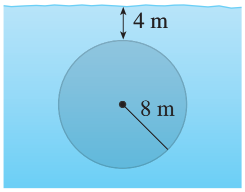
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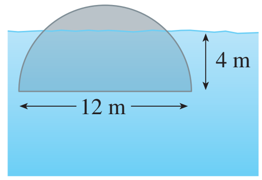
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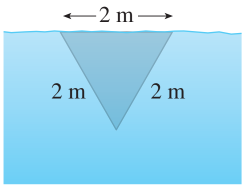
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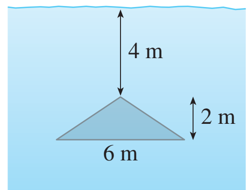
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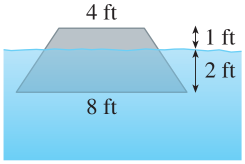
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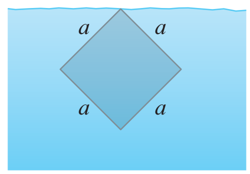
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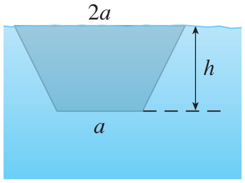
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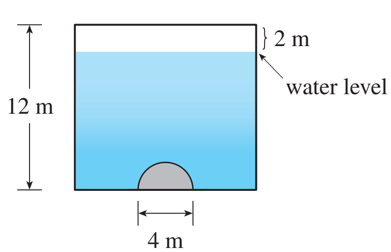
11.



12. A milk truck carries milk with density 64.6 lb/ft^3 in a horizontal cylindrical tank with diameter 6 ft.
(a) Find the force exerted by the milk on one end of the tank when the tank is full.
(b) What if the tank is half full?

13. A trough is filled with a liquid of density 840 kg/m^3. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. Find the hydrostatic force on one end of the trough.

14. A vertical dam has a semicircular gate as shown in the figure. Find the hydrostatic force against the gate.



15. A cube with 20-cm-long sides is sitting on the bottom of an aquarium in which the water is one meter deep. Find the hydrostatic force on (a) the top of the cube and (b) one of the sides of the cube.

16. A dam is inclined at an angle of 30° from the vertical and has the shape of an isosceles trapezoid 100 ft wide at the top and 50 ft wide at the bottom and with a slant height of 70 ft. Find the hydrostatic force on the dam when it is full of water.

17. A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, find the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.

18. Suppose that a plate is immersed vertically in a fluid with density ρ and the width of the plate is $w(x)$ at a depth of x meters beneath the surface of the fluid. If the top of the plate is at depth a and the bottom is at depth b, show that the hydrostatic force on one side of the plate is

$$F = \int_a^b \rho g x\, w(x)\, dx$$

19. A metal plate was found submerged vertically in seawater, which has density 64 lb/ft^3. Measurements of the width of the plate were taken at the indicated depths. Use Simpson's Rule to estimate the force of the water against the plate.

Depth (ft)	7.0	7.4	7.8	8.2	8.6	9.0	9.4
Plate width (ft)	1.2	1.8	2.9	3.8	3.6	4.2	4.4

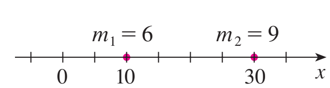
20. (a) Use the formula of Exercise 18 to show that

$$F = (\rho g \bar{x})A$$

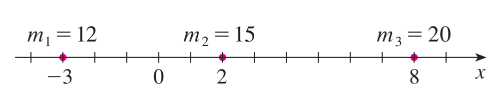
where $\bar{x}$ is the x-coordinate of the centroid of the plate and A is its area. This equation shows that the hydrostatic force against a vertical plane region is the same as if the region were horizontal at the depth of the centroid of the region.
(b) Use the result of part (a) to give another solution to Exercise 10.

21–22 Point-masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\bar{x}$.

21.



22.



23–24 The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

23. $m_1 = 4,\ m_2 = 2,\ m_3 = 4;$
$P_1(2, -3),\ P_2(-3, 1),\ P_3(3, 5)$

24. $m_1 = 5,\ m_2 = 4,\ m_3 = 3,\ m_4 = 6;$
$P_1(-4, 2),\ P_2(0, 5),\ P_3(3, 2),\ P_4(1, -2)$

25–28 Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

25. $y = 2x, \quad y = 0, \quad x = 1$

26. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

27. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

28. $y = \sin x, \quad y = 0, \quad 0 \leqslant x \leqslant \pi$

29–33 Find the centroid of the region bounded by the given curves.

29. $y = x^2, \quad x = y^2$

30. $y = 2 - x^2, \quad y = x$

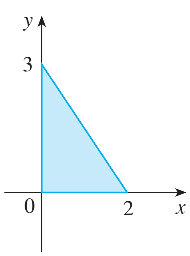
31. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \pi/4$

32. $y = x^3, \quad x + y = 2, \quad y = 0$

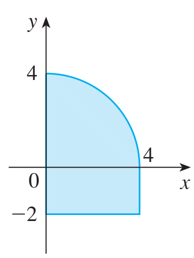
33. $x + y = 2, \quad x = y^2$

34–35 Calculate the moments M_x and M_y and the center of mass of a lamina with the given density and shape.

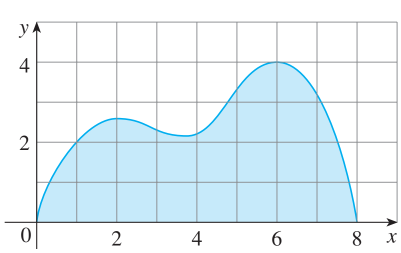
34. $\rho = 4$



35. $\rho = 6$



36. Use Simpson's Rule to estimate the centroid of the region shown.



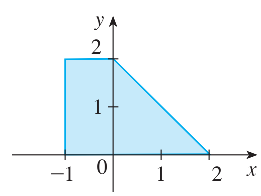
37. Find the centroid of the region bounded by the curves $y = x^3 - x$ and $y = x^2 - 1$. Sketch the region and plot the centroid to see if your answer is reasonable.

38. Use a graph to find approximate x-coordinates of the points of intersection of the curves $y = e^x$ and $y = 2 - x^2$. Then find (approximately) the centroid of the region bounded by these curves.

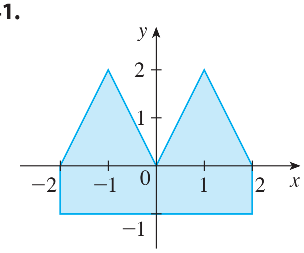
39. Prove that the centroid of any triangle is located at the point of intersection of the medians. [*Hints*: Place the axes so that the vertices are $(a, 0)$, $(0, b)$, and $(c, 0)$. Recall that a median is a line segment from a vertex to the midpoint of the opposite side. Recall also that the medians intersect at a point two-thirds of the way from each vertex (along the median) to the opposite side.]

40–41 Find the centroid of the region shown, not by integration, but by locating the centroids of the rectangles and triangles (from Exercise 39) and using additivity of moments.

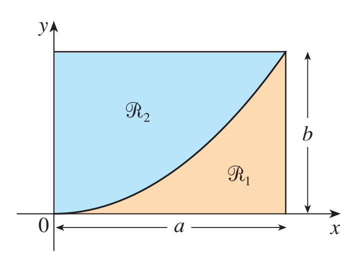
40.



41.



42. A rectangle $\mathcal{R}$ with sides a and b is divided into two parts $\mathcal{R}_1$ and $\mathcal{R}_2$ by an arc of a parabola that has its vertex at one corner of $\mathcal{R}$ and passes through the opposite corner. Find the centroids of both $\mathcal{R}_1$ and $\mathcal{R}_2$.



43. If $\bar{x}$ is the x-coordinate of the centroid of the region that lies under the graph of a continuous function f, where $a \leqslant x \leqslant b$, show that

$$\int_a^b (cx + d)f(x)\,dx = (c\bar{x} + d)\int_a^b f(x)\,dx$$

44–46 Use the Theorem of Pappus to find the volume of the given solid.

44. A sphere of radius r (Use Example 4.)

45. A cone with height h and base radius r

46. The solid obtained by rotating the triangle with vertices (2, 3), (2, 5), and (5, 4) about the x-axis

47. The centroid of a *curve* can be found by a process similar to the one we used for finding the centroid of a region. If C is a curve with length L, then the centroid is $(\bar{x}, \bar{y})$ where $\bar{x} = (1/L)\int x\,ds$ and $\bar{y} = (1/L)\int y\,ds$. Here we assign appropriate limits of integration, and ds is as defined in Sections 8.1 and 8.2. (The centroid often doesn't lie on the curve itself. If the curve were made of wire and placed on a weightless board, the centroid would be the balance point on the board.) Find the centroid of the quarter-circle $y = \sqrt{16 - x^2}$, $0 \leqslant x \leqslant 4$.

48. The *Second Theorem of Pappus* is in the same spirit as Pappus's Theorem on page 605, but for surface area rather than volume: Let C be a curve that lies entirely on one side of a line l in the plane. If C is rotated about l, then the area of the resulting surface is the product of the arc length of C and the distance traveled by the centroid of C (see Exercise 47).

(a) Prove the Second Theorem of Pappus for the case where C is given by $y = f(x)$, $f(x) \geqslant 0$, and C is rotated about the x-axis.

(b) Use the Second Theorem of Pappus to compute the surface area of the half-sphere obtained by rotating the curve from Exercise 47 about the x-axis. Does your answer agree with the one given by geometric formulas?

49. Use the Second Theorem of Pappus described in Exercise 48 to find the surface area of the torus in Example 7.

50. Let $\mathcal{R}$ be the region that lies between the curves

$$y = x^m \qquad y = x^n \qquad 0 \leqslant x \leqslant 1$$

where m and n are integers with $0 \leqslant n < m$.

(a) Sketch the region $\mathcal{R}$.

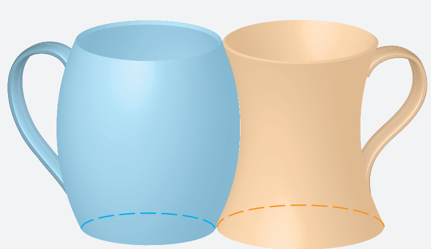
(b) Find the coordinates of the centroid of $\mathcal{R}$.

(c) Try to find values of m and n such that the centroid lies *outside* $\mathcal{R}$.

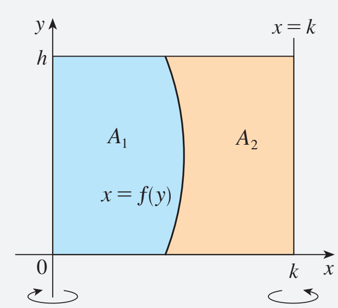
51. Prove Formulas 9.

DISCOVERY PROJECT COMPLEMENTARY COFFEE CUPS

Suppose you have a choice of two coffee cups of the type shown, one that bends outward and one inward, and you notice that they have the same height and their shapes fit together snugly. You wonder which cup holds more coffee. Of course you could fill one cup with water and pour it into the other one but, being a calculus student, you decide on a more mathematical approach. Ignoring the handles, you observe that both cups are surfaces of revolution, so you can think of the coffee as a volume of revolution.



Cup A Cup B



1. Suppose the cups have height h, cup A is formed by rotating the curve $x = f(y)$ about the y-axis, and cup B is formed by rotating the same curve about the line $x = k$. Find the value of k such that the two cups hold the same amount of coffee.

2. What does your result from Problem 1 say about the areas A_1 and A_2 shown in the figure?

3. Use Pappus's Theorem to explain your result in Problems 1 and 2.

4. Based on your own measurements and observations, suggest a value for h and an equation for $x = f(y)$ and calculate the amount of coffee that each cup holds.

8.4 Applications to Economics and Biology

In this section we consider some applications of integration to economics (consumer surplus) and biology (blood flow, cardiac output). Others are described in the exercises.

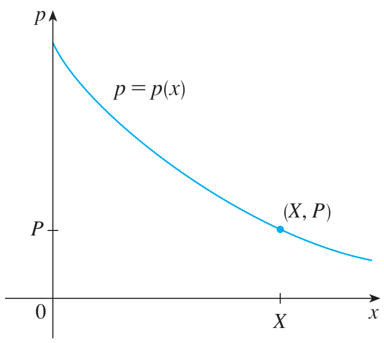


FIGURE 1
A typical demand curve

Consumer Surplus

Recall from Section 3.7 that the demand function $p(x)$ is the price that a company has to charge in order to sell x units of a commodity. Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function. The graph of a typical demand function, called a **demand curve**, is shown in Figure 1. If X is the amount of the commodity that can currently be sold, then $P = p(X)$ is the current selling price.

At a given price, some consumers who buy a good would be willing to pay more; they benefit by not having to. The difference between what a consumer is willing to pay and what the consumer actually pays for a good is called the *consumer surplus*. By finding the total consumer surplus among all purchasers of a good, economists can assess the overall benefit of a market to society.

To determine the total consumer surplus, we look at the demand curve and divide the interval $[0, X]$ into n subintervals, each of length $\Delta x = X/n$, and let $x_i^* = x_i$ be the right endpoint of the ith subinterval, as in Figure 2. According to the demand curve, x_{i-1} units would be purchased at a price of $p(x_{i-1})$ dollars per unit. To increase sales to x_i units, the price would have to be lowered to $p(x_i)$ dollars. In this case, an additional Δx units would be sold (but no more). In general, the consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them. So in paying only P dollars they have saved an amount of

$$(\text{savings per unit})(\text{number of units}) = [p(x_i) - P]\,\Delta x$$

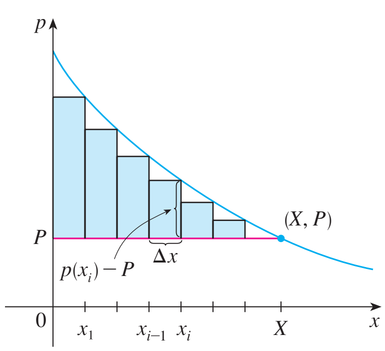


FIGURE 2

Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

$$\sum_{i=1}^{n} [p(x_i) - P]\,\Delta x$$

(This sum corresponds to the area enclosed by the rectangles in Figure 2.) If we let $n \to \infty$, this Riemann sum approaches the integral

$$\boxed{1} \qquad \int_0^X [p(x) - P]\,dx$$

which economists call the **consumer surplus** for the commodity.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X. Figure 3 shows the interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

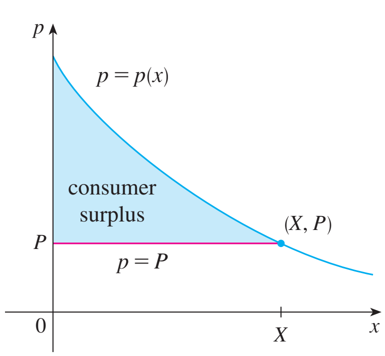


FIGURE 3

EXAMPLE 1 The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

SOLUTION Since the number of products sold is $X = 500$, the corresponding price is

$$P = 1200 - (0.2)(500) - (0.0001)(500)^2 = 1075$$

Therefore, from Definition 1, the consumer surplus is

$$\int_0^{500} [p(x) - P]\,dx = \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)\,dx$$

$$= \int_0^{500} (125 - 0.2x - 0.0001x^2)\,dx$$

$$= 125x - 0.1x^2 - (0.0001)\left(\frac{x^3}{3}\right)\Bigg]_0^{500}$$

$$= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3}$$

$$= \$33{,}333.33$$

Blood Flow

In Example 2.7.7 we discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Now, in order to compute the rate of blood flow, or *flux* (volume per unit time), we consider smaller, equally spaced radii $r_1, r_2, \ldots$. The approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i\,\Delta r \qquad \text{where} \qquad \Delta r = r_i - r_{i-1}$$

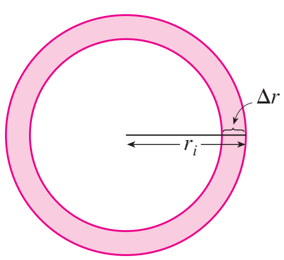


FIGURE 4

(See Figure 4.) If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$. Thus the volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i\,\Delta r)\,v(r_i) = 2\pi r_i\,v(r_i)\,\Delta r$$

and the total volume of blood that flows across a cross-section per unit time is about

$$\sum_{i=1}^{n} 2\pi r_i\,v(r_i)\,\Delta r$$

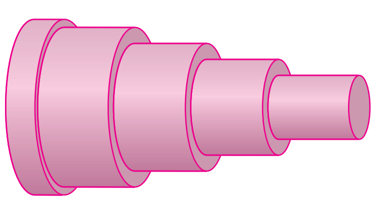


FIGURE 5

This approximation is illustrated in Figure 5. Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel. The approximation gets better as n increases. When we take the limit we get the exact value of the **flux** (or *discharge*), which is the volume of blood that passes a cross-section per unit time:

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i\,v(r_i)\,\Delta r = \int_0^R 2\pi r\,v(r)\,dr$$

$$= \int_0^R 2\pi r\,\frac{P}{4\eta l}(R^2 - r^2)\,dr$$

$$= \frac{\pi P}{2\eta l}\int_0^R (R^2 r - r^3)\,dr = \frac{\pi P}{2\eta l}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R}$$

$$= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta l}$$

The resulting equation

$$F = \frac{\pi P R^4}{8\eta l} \tag{2}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

Cardiac Output

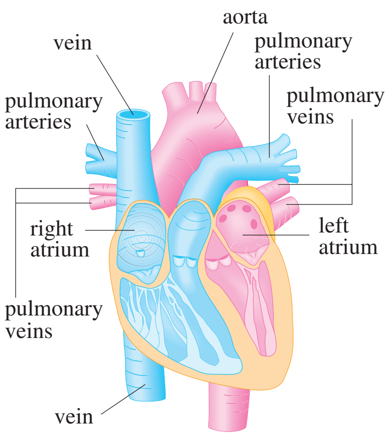


FIGURE 6

Figure 6 shows the human cardiovascular system. Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation. It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta. The **cardiac output** of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The *dye dilution method* is used to measure the cardiac output. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared. Let $c(t)$ be the concentration of the dye at time t. If we divide $[0, T]$ into subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately

$$\text{(concentration)(volume)} = c(t_i)(F\,\Delta t)$$

where F is the rate of flow that we are trying to determine. Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i) F\,\Delta t = F \sum_{i=1}^{n} c(t_i)\,\Delta t$$

and, letting $n \to \infty$, we find that the amount of dye is

$$A = F \int_0^T c(t)\,dt$$

Thus the cardiac output is given by

$$F = \frac{A}{\int_0^T c(t)\,dt} \tag{3}$$

where the amount of dye A is known and the integral can be approximated from the concentration readings.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

EXAMPLE 2 A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the table. Estimate the cardiac output.

SOLUTION Here $A = 5$, $\Delta t = 1$, and $T = 10$. We use Simpson's Rule to approximate the integral of the concentration:

$$\begin{aligned}\int_0^{10} c(t)\,dt &\approx \tfrac{1}{3}[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)\\ &\quad + 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0]\\ &\approx 41.87\end{aligned}$$

Thus Formula 3 gives the cardiac output to be

$$F = \frac{A}{\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87} \approx 0.12 \text{ L/s} = 7.2 \text{ L/min}$$

8.4 EXERCISES

1. The marginal cost function $C'(x)$ was defined to be the derivative of the cost function. (See Sections 2.7 and 3.7.) The marginal cost of producing x gallons of orange juice is

$$C'(x) = 0.82 - 0.00003x + 0.000000003x^2$$

(measured in dollars per gallon). The fixed start-up cost is $C(0) = \$18{,}000$. Use the Net Change Theorem to find the cost of producing the first 4000 gallons of juice.

2. A company estimates that the marginal revenue (in dollars per unit) realized by selling x units of a product is $48 - 0.0012x$. Assuming the estimate is accurate, find the increase in revenue if sales increase from 5000 units to 10,000 units.

3. A mining company estimates that the marginal cost of extracting x tons of copper ore from a mine is $0.6 + 0.008x$, measured in thousands of dollars per ton. Start-up costs are \$100,000. What is the cost of extracting the first 50 tons of copper? What about the next 50 tons?

4. The demand function for a particular vacation package is $p(x) = 2000 - 46\sqrt{x}$. Find the consumer surplus when the sales level for the packages is 400. Illustrate by drawing the demand curve and identifying the consumer surplus as an area.

5. A demand curve is given by $p = 450/(x + 8)$. Find the consumer surplus when the selling price is \$10.

6. The **supply function** $p_S(x)$ for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, manufacturers will produce more units, so p_S is an increasing function of x. Let X be the amount of the commodity currently produced and let $P = p_S(X)$ be the current price. Some producers would be willing to make and sell the commodity for a lower selling price and are therefore receiving more than their minimal price. The excess is called the **producer surplus**. An argument similar to that for consumer surplus shows that the surplus is given by the integral

$$\int_0^X [P - p_S(x)]\,dx$$

Calculate the producer surplus for the supply function $p_S(x) = 3 + 0.01x^2$ at the sales level $X = 10$. Illustrate by drawing the supply curve and identifying the producer surplus as an area.

7. If a supply curve is modeled by the equation $p = 125 + 0.002x^2$, find the producer surplus when the selling price is \$625.

8. In a purely competitive market, the price of a good is naturally driven to the value where the quantity demanded by consumers matches the quantity made by producers, and the market is said to be in *equilibrium*. These values are the coordinates of the point of intersection of the supply and demand curves.

(a) Given the demand curve $p = 50 - \frac{1}{20}x$ and the supply curve $p = 20 + \frac{1}{10}x$ for a good, at what quantity and price is the market for the good in equilibrium?

(b) Find the consumer surplus and the producer surplus when the market is in equilibrium. Illustrate by sketching the supply and demand curves and identifying the surpluses as areas.

9. The sum of consumer surplus and producer surplus is called the *total surplus*; it is one measure economists use as an indicator of the economic health of a society. Total surplus is maximized when the market for a good is in equilibrium.

(a) The demand function for an electronics company's car stereos is $p(x) = 228.4 - 18x$ and the supply function is $p_S(x) = 27x + 57.4$, where x is measured in thousands. At what quantity is the market for the stereos in equilibrium?

(b) Compute the maximum total surplus for the stereos.

10. A camera company estimates that the demand function for its new digital camera is $p(x) = 312e^{-0.14x}$ and the supply function is estimated to be $p_S(x) = 26e^{0.2x}$, where x is measured in thousands. Compute the maximum total surplus.

11. A company modeled the demand curve for its product (in dollars) by the equation

$$p = \frac{800{,}000e^{-x/5000}}{x + 20{,}000}$$

Use a graph to estimate the sales level when the selling price is \$16. Then find (approximately) the consumer surplus for this sales level.

12. A movie theater has been charging \$10.00 per person and selling about 500 tickets on a typical weeknight. After surveying their customers, the theater management estimates that for every 50 cents that they lower the price, the number

of moviegoers will increase by 50 per night. Find the demand function and calculate the consumer surplus when the tickets are priced at \$8.00.

13. If the amount of capital that a company has at time t is $f(t)$, then the derivative, $f'(t)$, is called the *net investment flow.* Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in years). Find the increase in capital (the *capital formation*) from the fourth year to the eighth year.

14. If revenue flows into a company at a rate of $f(t) = 9000\sqrt{1 + 2t}$, where t is measured in years and $f(t)$ is measured in dollars per year, find the total revenue obtained in the first four years.

15. If income is continuously collected at a rate of $f(t)$ dollars per year and will be invested at a constant interest rate r (compounded continuously) for a period of T years, then the *future value* of the income is given by $\int_0^T f(t)\, e^{r(T-t)}\, dt$. Compute the future value after 6 years for income received at a rate of $f(t) = 8000\, e^{0.04t}$ dollars per year and invested at 6.2% interest.

16. The *present value* of an income stream is the amount that would need to be invested now to match the future value as described in Exercise 15 and is given by $\int_0^T f(t)\, e^{-rt}\, dt$. Find the present value of the income stream in Exercise 15.

17. *Pareto's Law of Income* states that the number of people with incomes between $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}\, dx$, where A and k are constants with $A > 0$ and $k > 1$. The average income of these people is

$$\bar{x} = \frac{1}{N} \int_a^b Ax^{1-k}\, dx$$

Calculate $\bar{x}$.

18. A hot, wet summer is causing a mosquito population explosion in a lake resort area. The number of mosquitoes is increasing at an estimated rate of $2200 + 10e^{0.8t}$ per week (where t is measured in weeks). By how much does the mosquito population increase between the fifth and ninth weeks of summer?

19. Use Poiseuille's Law to calculate the rate of flow in a small human artery where we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2.

20. High blood pressure results from constriction of the arteries. To maintain a normal flow rate (flux), the heart has to pump harder, thus increasing the blood pressure. Use Poiseuille's Law to show that if R_0 and P_0 are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by the equation

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$$

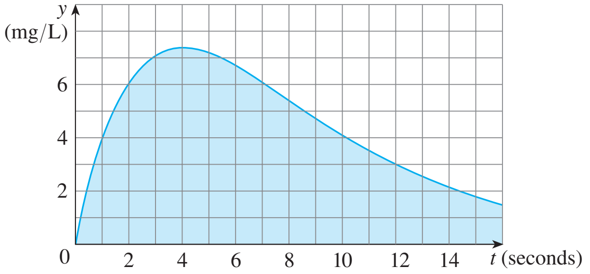
Deduce that if the radius of an artery is reduced to three-fourths of its former value, then the pressure is more than tripled.

21. The dye dilution method is used to measure cardiac output with 6 mg of dye. The dye concentrations, in mg/L, are modeled by $c(t) = 20te^{-0.6t}$, $0 \leqslant t \leqslant 10$, where t is measured in seconds. Find the cardiac output.

22. After a 5.5-mg injection of dye, the readings of dye concentration, in mg/L, at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0.0	10	4.3
2	4.1	12	2.5
4	8.9	14	1.2
6	8.5	16	0.2
8	6.7		

23. The graph of the concentration function $c(t)$ is shown after a 7-mg injection of dye into a heart. Use Simpson's Rule to estimate the cardiac output.



8.5 Probability

Calculus plays a role in the analysis of random behavior. Suppose we consider the cholesterol level of a person chosen at random from a certain age group, or the height of an adult female chosen at random, or the lifetime of a randomly chosen battery of a certain type. Such quantities are called **continuous random variables** because their values actually range over an interval of real numbers, although they might be measured or recorded only to the nearest integer. We might want to know the probability that a blood cholesterol level is greater than 250, or the probability that the height of an adult

female is between 60 and 70 inches, or the probability that the battery we are buying lasts between 100 and 200 hours. If X represents the lifetime of that type of battery, we denote this last probability as follows:

$$P(100 \leqslant X \leqslant 200)$$

According to the frequency interpretation of probability, this number is the long-run proportion of all batteries of the specified type whose lifetimes are between 100 and 200 hours. Since it represents a proportion, the probability naturally falls between 0 and 1.

Note that we always use *intervals* of values when working with probability density functions. We wouldn't, for instance, use a density function to find the probability that X *equals* a.

Every continuous random variable X has a **probability density function** f. This means that the probability that X lies between a and b is found by integrating f from a to b:

1 $$P(a \leqslant X \leqslant b) = \int_a^b f(x)\, dx$$

For example, Figure 1 shows the graph of a model for the probability density function f for a random variable X defined to be the height in inches of an adult female in the United States (according to data from the National Health Survey). The probability that the height of a woman chosen at random from this population is between 60 and 70 inches is equal to the area under the graph of f from 60 to 70.

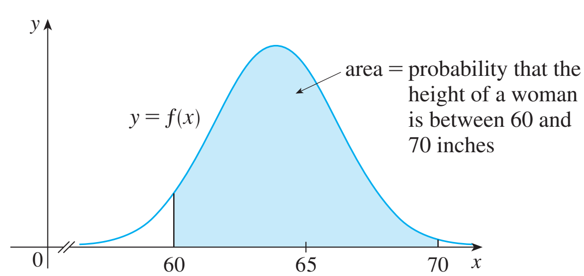


FIGURE 1
Probability density function for the height of an adult female

In general, the probability density function f of a random variable X satisfies the condition $f(x) \geqslant 0$ for all x. Because probabilities are measured on a scale from 0 to 1, it follows that

2 $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

EXAMPLE 1 Let $f(x) = 0.006x(10 - x)$ for $0 \leqslant x \leqslant 10$ and $f(x) = 0$ for all other values of x.
(a) Verify that f is a probability density function.
(b) Find $P(4 \leqslant X \leqslant 8)$.

SOLUTION
(a) For $0 \leqslant x \leqslant 10$ we have $0.006x(10 - x) \geqslant 0$, so $f(x) \geqslant 0$ for all x. We also need to check that Equation 2 is satisfied:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{10} 0.006x(10 - x)\, dx = 0.006 \int_0^{10} (10x - x^2)\, dx$$

$$= 0.006\Big[5x^2 - \tfrac{1}{3}x^3\Big]_0^{10} = 0.006\left(500 - \tfrac{1000}{3}\right) = 1$$

Therefore f is a probability density function.

(b) The probability that X lies between 4 and 8 is

$$P(4 \leqslant X \leqslant 8) = \int_4^8 f(x)\,dx = 0.006 \int_4^8 (10x - x^2)\,dx$$

$$= 0.006\left[5x^2 - \tfrac{1}{3}x^3\right]_4^8 = 0.544$$ ■

EXAMPLE 2 Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions. Find the exact form of such a function.

SOLUTION Think of the random variable as being the time you wait on hold before an agent of a company you're telephoning answers your call. So instead of x, let's use t to represent time, in minutes. If f is the probability density function and you call at time $t = 0$, then, from Definition 1, $\int_0^2 f(t)\,dt$ represents the probability that an agent answers within the first two minutes and $\int_4^5 f(t)\,dt$ is the probability that your call is answered during the fifth minute.

It's clear that $f(t) = 0$ for $t < 0$ (the agent can't answer before you place the call). For $t > 0$ we are told to use an exponentially decreasing function, that is, a function of the form $f(t) = Ae^{-ct}$, where A and c are positive constants. Thus

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ Ae^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

We use Equation 2 to determine the value of A:

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^{\infty} f(t)\,dt$$

$$= \int_0^{\infty} Ae^{-ct}dt = \lim_{x\to\infty} \int_0^x Ae^{-ct}\,dt$$

$$= \lim_{x\to\infty}\left[-\frac{A}{c}e^{-ct}\right]_0^x = \lim_{x\to\infty}\frac{A}{c}(1 - e^{-cx})$$

$$= \frac{A}{c}$$

Therefore $A/c = 1$ and so $A = c$. Thus every exponential density function has the form

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

A typical graph is shown in Figure 2. ■

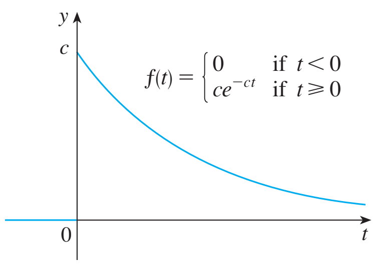


FIGURE 2
An exponential density function

Average Values

Suppose you're waiting for a company to answer your phone call and you wonder how long, on average, you can expect to wait. Let $f(t)$ be the corresponding density function, where t is measured in minutes, and think of a sample of N people who have called this company. Most likely, none of them had to wait more than an hour, so let's restrict our attention to the interval $0 \leqslant t \leqslant 60$. Let's divide that interval into n intervals of length

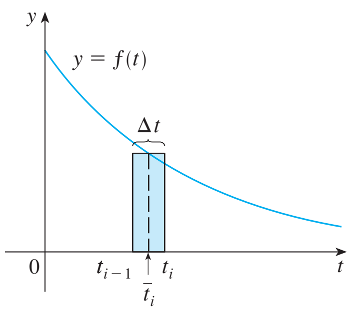


FIGURE 3

Δt and endpoints $0, t_1, t_2, \ldots, t_n = 60$. (Think of Δt as lasting a minute, or half a minute, or 10 seconds, or even a second.) The probability that somebody's call gets answered during the time period from t_{i-1} to t_i is the area under the curve $y = f(t)$ from t_{i-1} to t_i, which is approximately equal to $f(\bar{t}_i)\,\Delta t$. (This is the area of the approximating rectangle in Figure 3, where $\bar{t}_i$ is the midpoint of the interval.)

Since the long-run proportion of calls that get answered in the time period from t_{i-1} to t_i is $f(\bar{t}_i)\,\Delta t$, we expect that, out of our sample of N callers, the number whose call was answered in that time period is approximately $Nf(\bar{t}_i)\,\Delta t$ and the time that each waited is about $\bar{t}_i$. Therefore the total time they waited is the product of these numbers: approximately $\bar{t}_i[Nf(\bar{t}_i)\,\Delta t]$. Adding over all such intervals, we get the approximate total of everybody's waiting times:

$$\sum_{i=1}^{n} N\bar{t}_i\,f(\bar{t}_i)\,\Delta t$$

If we now divide by the number of callers N, we get the approximate *average* waiting time:

$$\sum_{i=1}^{n} \bar{t}_i\,f(\bar{t}_i)\,\Delta t$$

We recognize this as a Riemann sum for the function $t\,f(t)$. As the time interval shrinks (that is, $\Delta t \to 0$ and $n \to \infty$), this Riemann sum approaches the integral

$$\int_0^{60} t\,f(t)\,dt$$

This integral is called the *mean waiting time*.

In general, the **mean** of any probability density function f is defined to be

It is traditional to denote the mean by the Greek letter μ (mu).

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

The mean can be interpreted as the long-run average value of the random variable X. It can also be interpreted as a measure of centrality of the probability density function.

The expression for the mean resembles an integral we have seen before. If $\mathcal{R}$ is the region that lies under the graph of f, we know from Formula 8.3.8 that the x-coordinate of the centroid of $\mathcal{R}$ is

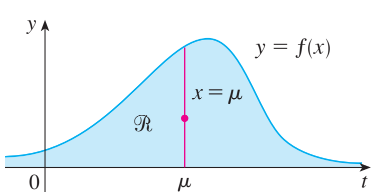


FIGURE 4
$\mathcal{R}$ balances at a point on the line $x = \mu$

$$\bar{x} = \frac{\displaystyle\int_{-\infty}^{\infty} x\,f(x)\,dx}{\displaystyle\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} x\,f(x)\,dx = \mu$$

because of Equation 2. So a thin plate in the shape of $\mathcal{R}$ balances at a point on the vertical line $x = \mu$. (See Figure 4.)

EXAMPLE 3 Find the mean of the exponential distribution of Example 2:

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ ce^{-ct} & \text{if } t \geqslant 0 \end{cases}$$

SOLUTION According to the definition of a mean, we have

$$\mu = \int_{-\infty}^{\infty} t\,f(t)\,dt = \int_0^{\infty} tce^{-ct}\,dt$$

To evaluate this integral we use integration by parts, with $u = t$ and $dv = ce^{-ct}\,dt$, so $du = dt$ and $v = -e^{-ct}$:

$$\int_0^\infty tce^{-ct}\,dt = \lim_{x\to\infty}\int_0^x tce^{-ct}\,dt = \lim_{x\to\infty}\left(-te^{-ct}\Big]_0^x + \int_0^x e^{-ct}\,dt\right)$$

The limit of the first term is 0 by l'Hospital's Rule.

$$= \lim_{x\to\infty}\left(-xe^{-cx} + \frac{1}{c} - \frac{e^{-cx}}{c}\right) = \frac{1}{c}$$

The mean is $\mu = 1/c$, so we can rewrite the probability density function as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

■

EXAMPLE 4 Suppose the average waiting time for a customer's call to be answered by a company representative is five minutes.
(a) Find the probability that a call is answered during the first minute, assuming that an exponential distribution is appropriate.
(b) Find the probability that a customer waits more than five minutes to be answered.

SOLUTION
(a) We are given that the mean of the exponential distribution is $\mu = 5$ min and so, from the result of Example 3, we know that the probability density function is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.2e^{-t/5} & \text{if } t \geqslant 0 \end{cases}$$

where t is measured in minutes. Thus the probability that a call is answered during the first minute is

$$P(0 \leqslant T \leqslant 1) = \int_0^1 f(t)\,dt$$

$$= \int_0^1 0.2e^{-t/5}\,dt = 0.2(-5)e^{-t/5}\Big]_0^1$$

$$= 1 - e^{-1/5} \approx 0.1813$$

So about 18% of customers' calls are answered during the first minute.

(b) The probability that a customer waits more than five minutes is

$$P(T > 5) = \int_5^\infty f(t)\,dt = \int_5^\infty 0.2e^{-t/5}\,dt$$

$$= \lim_{x\to\infty}\int_5^x 0.2e^{-t/5}\,dt = \lim_{x\to\infty}\left(e^{-1} - e^{-x/5}\right)$$

$$= \frac{1}{e} - 0 \approx 0.368$$

About 37% of customers wait more than five minutes before their calls are answered.

■

Notice the result of Example 4(b): Even though the mean waiting time is 5 minutes, only 37% of callers wait more than 5 minutes. The reason is that some callers have to wait much longer (maybe 10 or 15 minutes), and this brings up the average.

Another measure of centrality of a probability density function is the *median*. That is a number m such that half the callers have a waiting time less than m and the other callers have a waiting time longer than m. In general, the **median** of a probability density function is the number m such that

$$\int_m^\infty f(x)\,dx = \tfrac{1}{2}$$

This means that half the area under the graph of f lies to the right of m. In Exercise 9 you are asked to show that the median waiting time for the company described in Example 4 is approximately 3.5 minutes.

Normal Distributions

Many important random phenomena—such as test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location—are modeled by a **normal distribution**. This means that the probability density function of the random variable X is a member of the family of functions

3
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/(2\sigma^2)}$$

The standard deviation is denoted by the lowercase Greek letter σ (sigma).

You can verify that the mean for this function is μ. The positive constant σ is called the **standard deviation**; it measures how spread out the values of X are. From the bell-shaped graphs of members of the family in Figure 5, we see that for small values of σ the values of X are clustered about the mean, whereas for larger values of σ the values of X are more spread out. Statisticians have methods for using sets of data to estimate μ and σ.

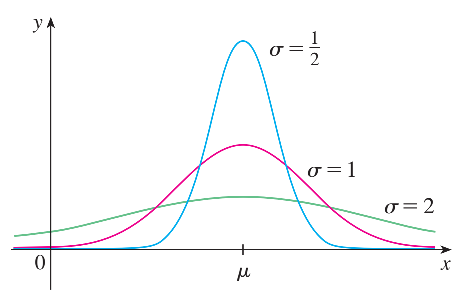


FIGURE 5
Normal distributions

The factor $1/(\sigma\sqrt{2\pi})$ is needed to make f a probability density function. In fact, it can be verified using the methods of multivariable calculus that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/(2\sigma^2)}\,dx = 1$$

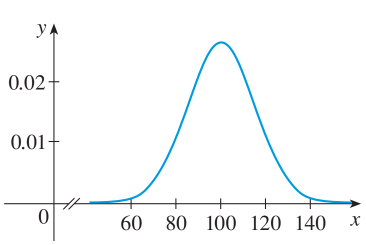


FIGURE 6

EXAMPLE 5 Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure 6 shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

SOLUTION

(a) Since IQ scores are normally distributed, we use the probability density function given by Equation 3 with $\mu = 100$ and $\sigma = 15$:

$$P(85 \leqslant X \leqslant 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2\cdot 15^2)}\, dx$$

Recall from Section 7.5 that the function $y = e^{-x^2}$ doesn't have an elementary antiderivative, so we can't evaluate the integral exactly. But we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule or Simpson's Rule) to estimate the integral. Doing so, we find that

$$P(85 \leqslant X \leqslant 115) \approx 0.68$$

So about 68% of the population has an IQ score between 85 and 115, that is, within one standard deviation of the mean.

(b) The probability that the IQ score of a person chosen at random is more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\, dx$$

To avoid the improper integral we could approximate it by the integral from 140 to 200. (It's quite safe to say that people with an IQ over 200 are extremely rare.) Then

$$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\, dx \approx 0.0038$$

Therefore about 0.4% of the population has an IQ score over 140. ■

8.5 EXERCISES

1. Let $f(x)$ be the probability density function for the lifetime of a manufacturer's highest quality car tire, where x is measured in miles. Explain the meaning of each integral.

(a) $\int_{30,000}^{40,000} f(x)\, dx$ (b) $\int_{25,000}^{\infty} f(x)\, dx$

2. Let $f(t)$ be the probability density function for the time it takes you to drive to school in the morning, where t is measured in minutes. Express the following probabilities as integrals.

(a) The probability that you drive to school in less than 15 minutes

(b) The probability that it takes you more than half an hour to get to school

3. Let $f(x) = 30x^2(1 - x)^2$ for $0 \leqslant x \leqslant 1$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P\left(X \leqslant \frac{1}{3}\right)$.

4. The density function

$$f(x) = \frac{e^{3-x}}{(1 + e^{3-x})^2}$$

is an example of a *logistic distribution*.

(a) Verify that f is a probability density function.

(b) Find $P(3 \leqslant X \leqslant 4)$.

(c) Graph f. What does the mean appear to be? What about the median?

5. Let $f(x) = c/(1 + x^2)$.

(a) For what value of c is f a probability density function?

(b) For that value of c, find $P(-1 < X < 1)$.

6. Let $f(x) = k(3x - x^2)$ if $0 \leqslant x \leqslant 3$ and $f(x) = 0$ if $x < 0$ or $x > 3$.

(a) For what value of k is f a probability density function?

(b) For that value of k, find $P(X > 1)$.

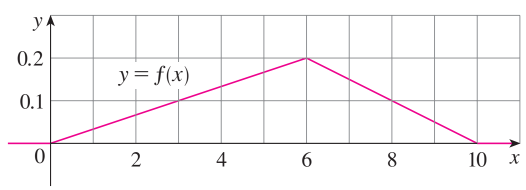
(c) Find the mean.

7. A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.
(a) Explain why the function

$$f(x) = \begin{cases} 0.1 & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.
(b) What does your intuition tell you about the value of the mean? Check your guess by evaluating an integral.

8. (a) Explain why the function whose graph is shown is a probability density function.
(b) Use the graph to find the following probabilities:
(i) $P(X < 3)$ (ii) $P(3 \leqslant X \leqslant 8)$
(c) Calculate the mean.



9. Show that the median waiting time for a phone call to the company described in Example 4 is about 3.5 minutes.

10. (a) A type of light bulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$. Use this model to find the probability that a bulb
(i) fails within the first 200 hours,
(ii) burns for more than 800 hours.
(b) What is the median lifetime of these light bulbs?

11. An online retailer has determined that the average time for credit card transactions to be electronically approved is 1.6 seconds.
(a) Use an exponential density function to find the probability that a customer waits less than a second for credit card approval.
(b) Find the probability that a customer waits more than 3 seconds.
(c) What is the minimum approval time for the slowest 5% of transactions?

12. The time between infection and the display of symptoms for streptococcal sore throat is a random variable whose probabililty density function can be approximated by $f(t) = \frac{1}{15{,}676}\, t^2 e^{-0.05t}$ if $0 \leqslant t \leqslant 150$ and $f(t) = 0$ otherwise (t measured in hours).
(a) What is the probability that an infected patient will display symptoms within the first 48 hours?
(b) What is the probability that an infected patient will not display symptoms until after 36 hours?

Source: Adapted from P. Sartwell, "The Distribution of Incubation Periods of Infectious Disease," *American Journal of Epidemiology* 141 (1995): 386–94.

13. REM sleep is the phase of sleep when most active dreaming occurs. In a study, the amount of REM sleep during the first four hours of sleep was described by a random variable T with probability density function

$$f(t) = \begin{cases} \frac{1}{1600}t & \text{if } 0 \leqslant t \leqslant 40 \\ \frac{1}{20} - \frac{1}{1600}t & \text{if } 40 < t \leqslant 80 \\ 0 & \text{otherwise} \end{cases}$$

where t is measured in minutes.
(a) What is the probability that the amount of REM sleep is between 30 and 60 minutes?
(b) Find the mean amount of REM sleep.

14. According to the National Health Survey, the heights of adult males in the United States are normally distributed with mean 69.0 inches and standard deviation 2.8 inches.
(a) What is the probability that an adult male chosen at random is between 65 inches and 73 inches tall?
(b) What percentage of the adult male population is more than 6 feet tall?

15. The "Garbage Project" at the University of Arizona reports that the amount of paper discarded by households per week is normally distributed with mean 9.4 lb and standard deviation 4.2 lb. What percentage of households throw out at least 10 lb of paper a week?

16. Boxes are labeled as containing 500 g of cereal. The machine filling the boxes produces weights that are normally distributed with standard deviation 12 g.
(a) If the target weight is 500 g, what is the probability that the machine produces a box with less than 480 g of cereal?
(b) Suppose a law states that no more than 5% of a manufacturer's cereal boxes can contain less than the stated weight of 500 g. At what target weight should the manufacturer set its filling machine?

17. The speeds of vehicles on a highway with speed limit 100 km/h are normally distributed with mean 112 km/h and standard deviation 8 km/h.
(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed?
(b) If police are instructed to ticket motorists driving 125 km/h or more, what percentage of motorists are targeted?

18. Show that the probability density function for a normally distributed random variable has inflection points at $x = \mu \pm \sigma$.

19. For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean.

20. The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx\right]^{1/2}$$

Find the standard deviation for an exponential density function with mean μ.

21. The hydrogen atom is composed of one proton in the nucleus and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a well-defined orbit. Instead, it occupies a state known as an *orbital*, which may be thought of as a "cloud" of negative charge surrounding the nucleus. At the state of lowest energy, called the *ground state*, or *1s-orbital*, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \qquad r \geqslant 0$$

where a_0 is the *Bohr radius* ($a_0 \approx 5.59 \times 10^{-11}$ m). The integral

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds$$

gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.

(a) Verify that $p(r)$ is a probability density function.

(b) Find $\lim_{r\to\infty} p(r)$. For what value of r does $p(r)$ have its maximum value?

(c) Graph the density function.

(d) Find the probability that the electron will be within the sphere of radius $4a_0$ centered at the nucleus.

(e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.

8 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) How is the length of a curve defined?
(b) Write an expression for the length of a smooth curve given by $y = f(x)$, $a \leqslant x \leqslant b$.
(c) What if x is given as a function of y?

2. (a) Write an expression for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis.
(b) What if x is given as a function of y?
(c) What if the curve is rotated about the y-axis?

3. Describe how we can find the hydrostatic force against a vertical wall submersed in a fluid.

4. (a) What is the physical significance of the center of mass of a thin plate?
(b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leqslant x \leqslant b$, write expressions for the coordinates of the center of mass.

5. What does the Theorem of Pappus say?

6. Given a demand function $p(x)$, explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.

7. (a) What is the cardiac output of the heart?
(b) Explain how the cardiac output can be measured by the dye dilution method.

8. What is a probability density function? What properties does such a function have?

9. Suppose $f(x)$ is the probability density function for the weight of a female college student, where x is measured in pounds.
(a) What is the meaning of the integral $\int_0^{130} f(x)\,dx$?
(b) Write an expression for the mean of this density function.
(c) How can we find the median of this density function?

10. What is a normal distribution? What is the significance of the standard deviation?

EXERCISES

1–3 Find the length of the curve.

1. $y = 4(x - 1)^{3/2}, \quad 1 \leqslant x \leqslant 4$

2. $y = 2\ln\left(\sin \tfrac{1}{2}x\right), \quad \pi/3 \leqslant x \leqslant \pi$

3. $12x = 4y^3 + 3y^{-1}, \quad 1 \leqslant y \leqslant 3$

4. (a) Find the length of the curve

$$y = \frac{x^4}{16} + \frac{1}{2x^2} \qquad 1 \leqslant x \leqslant 2$$

(b) Find the area of the surface obtained by rotating the curve in part (a) about the y-axis.

5. Let C be the arc of the curve $y = 2/(x + 1)$ from the point $(0, 2)$ to $\left(3, \frac{1}{2}\right)$. Use a calculator or other device to find the value of each of the following, correct to four decimal places.
(a) The length of C
(b) The area of the surface obtained by rotating C about the x-axis
(c) The area of the surface obtained by rotating C about the y-axis

6. (a) The curve $y = x^2$, $0 \leqslant x \leqslant 1$, is rotated about the y-axis. Find the area of the resulting surface.
(b) Find the area of the surface obtained by rotating the curve in part (a) about the x-axis.

7. Use Simpson's Rule with $n = 10$ to estimate the length of the sine curve $y = \sin x$, $0 \leqslant x \leqslant \pi$.

8. Use Simpson's Rule with $n = 10$ to estimate the area of the surface obtained by rotating the sine curve in Exercise 7 about the x-axis.

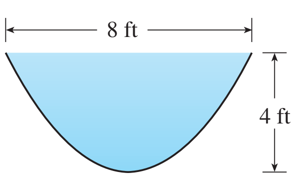
9. Find the length of the curve

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \qquad 1 \leqslant x \leqslant 16$$

10. Find the area of the surface obtained by rotating the curve in Exercise 9 about the y-axis.

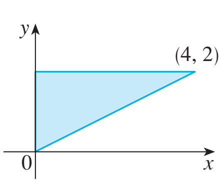
11. A gate in an irrigation canal is constructed in the form of a trapezoid 3 ft wide at the bottom, 5 ft wide at the top, and 2 ft high. It is placed vertically in the canal so that the water just covers the gate. Find the hydrostatic force on one side of the gate.

12. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure. Find the hydrostatic force on one end of the trough.

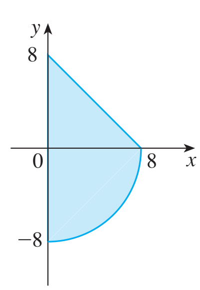


13–14 Find the centroid of the region shown.

13.



14.



15–16 Find the centroid of the region bounded by the given curves.

15. $y = \frac{1}{2}x, \quad y = \sqrt{x}$

16. $y = \sin x, \quad y = 0, \quad x = \pi/4, \quad x = 3\pi/4$

17. Find the volume obtained when the circle of radius 1 with center $(1, 0)$ is rotated about the y-axis.

18. Use the Theorem of Pappus and the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ to find the centroid of the semicircular region bounded by the curve $y = \sqrt{r^2 - x^2}$ and the x-axis.

19. The demand function for a commodity is given by

$$p = 2000 - 0.1x - 0.01x^2$$

Find the consumer surplus when the sales level is 100.

20. After a 6-mg injection of dye into a heart, the readings of dye concentration at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	14	4.7
2	1.9	16	3.3
4	3.3	18	2.1
6	5.1	20	1.1
8	7.6	22	0.5
10	7.1	24	0
12	5.8		

21. (a) Explain why the function

$$f(x) = \begin{cases} \dfrac{\pi}{20} \sin\left(\dfrac{\pi x}{10}\right) & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function.
(b) Find $P(X < 4)$.
(c) Calculate the mean. Is the value what you would expect?

22. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 days and 280 days?

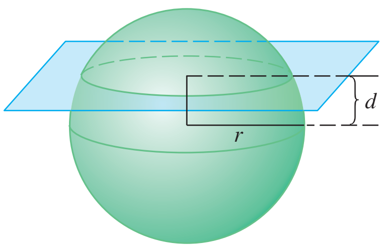
23. The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.
(a) What is the probability that a customer is served in the first 3 minutes?
(b) What is the probability that a customer has to wait more than 10 minutes?
(c) What is the median waiting time?

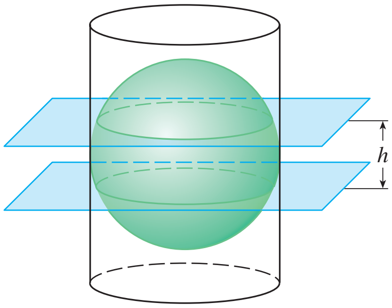
Problems Plus

1. Find the area of the region $S = \{(x, y) \mid x \geqslant 0,\ y \leqslant 1,\ x^2 + y^2 \leqslant 4y\}$.

2. Find the centroid of the region enclosed by the loop of the curve $y^2 = x^3 - x^4$.

3. If a sphere of radius r is sliced by a plane whose distance from the center of the sphere is d, then the sphere is divided into two pieces called segments of one base (see the first figure). The corresponding surfaces are called *spherical zones of one base.*
   (a) Determine the surface areas of the two spherical zones indicated in the figure.
   (b) Determine the approximate area of the Arctic Ocean by assuming that it is approximately circular in shape, with center at the North Pole and "circumference" at 75° north latitude. Use $r = 3960$ mi for the radius of the earth.
   (c) A sphere of radius r is inscribed in a right circular cylinder of radius r. Two planes perpendicular to the central axis of the cylinder and a distance h apart cut off a *spherical zone of two bases* on the sphere (see the second figure). Show that the surface area of the spherical zone equals the surface area of the region that the two planes cut off on the cylinder.
   (d) The *Torrid Zone* is the region on the surface of the earth that is between the Tropic of Cancer (23.45° north latitude) and the Tropic of Capricorn (23.45° south latitude). What is the area of the Torrid Zone?





4. (a) Show that an observer at height H above the north pole of a sphere of radius r can see a part of the sphere that has area

$$\frac{2\pi r^2 H}{r + H}$$

   (b) Two spheres with radii r and R are placed so that the distance between their centers is d, where $d > r + R$. Where should a light be placed on the line joining the centers of the spheres in order to illuminate the largest total surface?

5. Suppose that the density of seawater, $\rho = \rho(z)$, varies with the depth z below the surface.
   (a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

   where g is the acceleration due to gravity. Let P_0 and ρ_0 be the pressure and density at $z = 0$. Express the pressure at depth z as an integral.
   (b) Suppose the density of seawater at depth z is given by $\rho = \rho_0 e^{z/H}$, where H is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius r whose center is located at a distance $L > r$ below the surface.

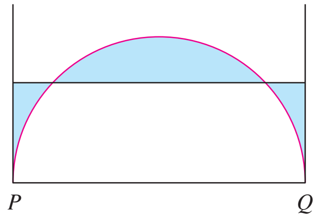


FIGURE FOR PROBLEM 6

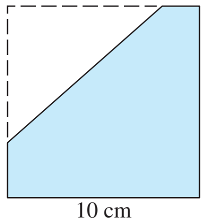
6. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q. At what height above the diameter should the horizontal line be placed so as to minimize the shaded area?

7. Let P be a pyramid with a square base of side $2b$ and suppose that S is a sphere with its center on the base of P and S is tangent to all eight edges of P. Find the height of P. Then find the volume of the intersection of S and P.

8. Consider a flat metal plate to be placed vertically underwater with its top 2 m below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.

9. A uniform disk with radius 1 m is to be cut by a line so that the center of mass of the smaller piece lies halfway along a radius. How close to the center of the disk should the cut be made? (Express your answer correct to two decimal places.)

10. A triangle with area 30 cm^2 is cut from a corner of a square with side 10 cm, as shown in the figure. If the centroid of the remaining region is 4 cm from the right side of the square, how far is it from the bottom of the square?



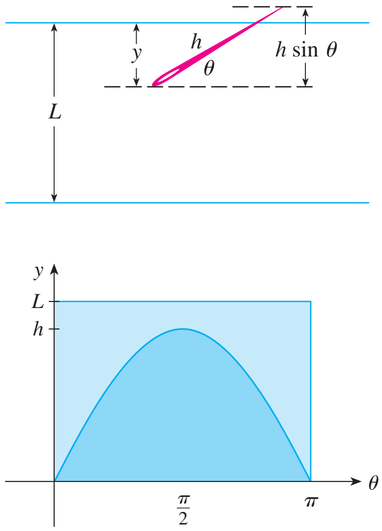


FIGURE FOR PROBLEM 11

11. In a famous 18th-century problem, known as *Buffon's needle problem*, a needle of length h is dropped onto a flat surface (for example, a table) on which parallel lines L units apart, $L \geqslant h$, have been drawn. The problem is to determine the probability that the needle will come to rest intersecting one of the lines. Assume that the lines run east-west, parallel to the x-axis in a rectangular coordinate system (as in the figure). Let y be the distance from the "southern" end of the needle to the nearest line to the north. (If the needle's southern end lies on a line, let $y = 0$. If the needle happens to lie east-west, let the "western" end be the "southern" end.) Let θ be the angle that the needle makes with a ray extending eastward from the "southern" end. Then $0 \leqslant y \leqslant L$ and $0 \leqslant \theta \leqslant \pi$. Note that the needle intersects one of the lines only when $y < h \sin \theta$. The total set of possibilities for the needle can be identified with the rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$, and the proportion of times that the needle intersects a line is the ratio

$$\frac{\text{area under } y = h \sin \theta}{\text{area of rectangle}}$$

This ratio is the probability that the needle intersects a line. Find the probability that the needle will intersect a line if $h = L$. What if $h = \frac{1}{2}L$?

12. If the needle in Problem 11 has length $h > L$, it's possible for the needle to intersect more than one line.

(a) If $L = 4$, find the probability that a needle of length 7 will intersect at least one line. [*Hint:* Proceed as in Problem 11. Define y as before; then the total set of possibilities for the needle can be identified with the same rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$. What portion of the rectangle corresponds to the needle intersecting a line?]

(b) If $L = 4$, find the probability that a needle of length 7 will intersect *two* lines.

(c) If $2L < h \leqslant 3L$, find a general formula for the probability that the needle intersects three lines.

13. Find the centroid of the region enclosed by the ellipse $x^2 + (x + y + 1)^2 = 1$.

9 Differential Equations

In the last section of this chapter we use pairs of differential equations to investigate the relationship between populations of predators and prey, such as jaguars and wart hogs, wolves and rabbits, lynx and hares, and ladybugs and aphids.



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PERHAPS THE MOST IMPORTANT of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.

9.1 Modeling with Differential Equations

Now is a good time to read (or reread) the discussion of mathematical modeling on page 23.

In describing the process of modeling in Section 1.2, we talked about formulating a mathematical model of a real-world problem either through intuitive reasoning about the phenomenon or from a physical law based on evidence from experiments. The mathematical model often takes the form of a *differential equation,* that is, an equation that contains an unknown function and some of its derivatives. This is not surprising because in a real-world problem we often notice that changes occur and we want to predict future behavior on the basis of how current values change. Let's begin by examining several examples of how differential equations arise when we model physical phenomena.

Models for Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model:

$t =$ time (the independent variable)

$P =$ the number of individuals in the population (the dependent variable)

The rate of growth of the population is the derivative dP/dt. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP \tag{1}$$

where k is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function P and its derivative dP/dt.

Having formulated a model, let's look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all t. So, if $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all t. This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that dP/dt becomes larger. In other words, the growth rate increases as the population increases.

Let's try to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We know from Chapter 6 that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Thus any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1. In Section 9.4, we will see that there is no other solution.

Allowing C to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 1. But populations have only positive values and so we are interested only in the solutions with $C > 0$. And we are probably

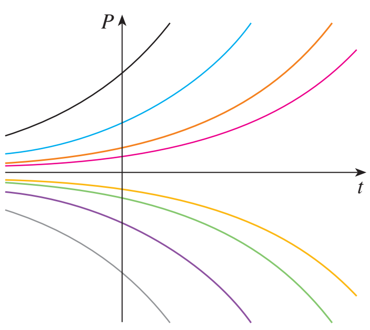


FIGURE 1
The family of solutions of $dP/dt = kP$

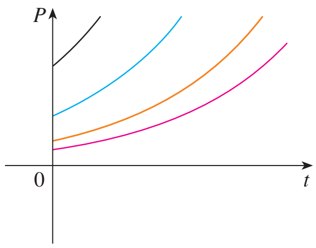


FIGURE 2
The family of solutions $P(t) = Ce^{kt}$ with $C > 0$ and $t \geqslant 0$

concerned only with values of t greater than the initial time $t = 0$. Figure 2 shows the physically meaningful solutions. Putting $t = 0$, we get $P(0) = Ce^{k(0)} = C$, so the constant C turns out to be the initial population, $P(0)$.

Equation 1 is appropriate for modeling population growth under ideal conditions, but we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* M (or decreases toward M if it ever exceeds M). For a model to take into account both trends, we make two assumptions:

- $\dfrac{dP}{dt} \approx kP$ if P is small (Initially, the growth rate is proportional to P.)
- $\dfrac{dP}{dt} < 0$ if $P > M$ (P decreases if it ever exceeds M.)

A simple expression that incorporates both assumptions is given by the equation

2
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Notice that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and so $dP/dt < 0$.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s as a model for world population growth. We will develop techniques that enable us to find explicit solutions of the logistic equation in Section 9.4, but for now we can deduce qualitative characteristics of the solutions directly from Equation 2. We first observe that the constant functions $P(t) = 0$ and $P(t) = M$ are solutions because, in either case, one of the factors on the right side of Equation 2 is zero. (This certainly makes physical sense: If the population is ever either 0 or at the carrying capacity, it stays that way.) These two constant solutions are called **equilibrium solutions**.

If the initial population $P(0)$ lies between 0 and M, then the right side of Equation 2 is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases. Notice that, in either case, if the population approaches the carrying capacity ($P \to M$), then $dP/dt \to 0$, which means the population levels off. So we expect that the solutions of the logistic differential equation have graphs that look something like the ones in Figure 3. Notice that the graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.

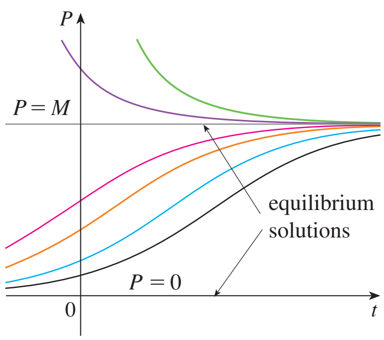


FIGURE 3
Solutions of the logistic equation

A Model for the Motion of a Spring

Let's now look at an example of a model from the physical sciences. We consider the motion of an object with mass m at the end of a vertical spring (as in Figure 4). In Section 5.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the *spring constant*). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law

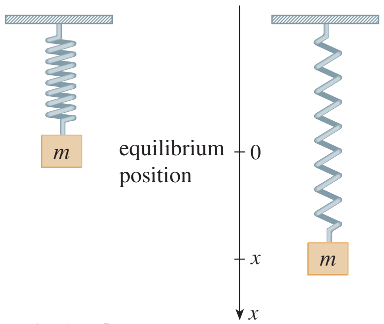


FIGURE 4

(force equals mass times acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx \tag{3}$$

This is an example of what is called a *second-order differential equation* because it involves second derivatives. Let's see what we can guess about the form of the solution directly from the equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which says that the second derivative of x is proportional to x but has the opposite sign. We know two functions with this property, the sine and cosine functions. In fact, it turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions (see Exercise 4). This is not surprising; we expect the spring to oscillate about its equilibrium position and so it is natural to think that trigonometric functions are involved.

General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus Equations 1 and 2 are first-order equations and Equation 3 is a second-order equation. In all three of those equations the independent variable is called t and represents time, but in general the independent variable doesn't have to represent time. For example, when we consider the differential equation

$$y' = xy \tag{4}$$

it is understood that y is an unknown function of x.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus f is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some interval.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple differential equations, namely, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

But, in general, solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations. In Section 9.2, how-

ever, we will see how to draw rough graphs of solutions even when we have no explicit formula. We will also learn how to find numerical approximations to solutions.

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

SOLUTION We use the Quotient Rule to differentiate the expression for y:

$$y' = \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2}$$

Figure 5 shows graphs of seven members of the family in Example 1. The differential equation shows that if $y \approx \pm 1$, then $y' \approx 0$. That is borne out by the flatness of the graphs near $y = 1$ and $y = -1$.

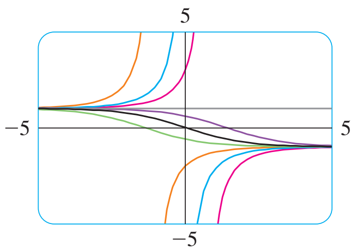


FIGURE 5

The right side of the differential equation becomes

$$\frac{1}{2}(y^2 - 1) = \frac{1}{2}\left[\left(\frac{1 + ce^t}{1 - ce^t}\right)^2 - 1\right]$$

$$= \frac{1}{2}\left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2}\right]$$

$$= \frac{1}{2}\,\frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2}$$

Therefore, for every value of c, the given function is a solution of the differential equation. ■

When applying differential equations, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (t_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict the future behavior of the system.

EXAMPLE 2 Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

SOLUTION Substituting the values $t = 0$ and $y = 2$ into the formula

$$y = \frac{1 + ce^t}{1 - ce^t}$$

from Example 1, we get

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c}$$

Solving this equation for c, we get $2 - 2c = 1 + c$, which gives $c = \frac{1}{3}$. So the solution of the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$ ■

9.1 EXERCISES

1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

2. Verify that $y = -t \cos t - t$ is a solution of the initial-value problem

$$t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$$

3. (a) For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?
(b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

4. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?
(b) For those values of k, verify that every member of the family of functions $y = A \sin kt + B \cos kt$ is also a solution.

5. Which of the following functions are solutions of the differential equation $y'' + y = \sin x$?
(a) $y = \sin x$ (b) $y = \cos x$
(c) $y = \frac{1}{2}x \sin x$ (d) $y = -\frac{1}{2}x \cos x$

6. (a) Show that every member of the family of functions $y = (\ln x + C)/x$ is a solution of the differential equation $x^2y' + xy = 1$.
(b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.
(c) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.
(d) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

7. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?
(b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
(c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?
(d) Find a solution of the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

8. (a) What can you say about the graph of a solution of the equation $y' = xy^3$ when x is close to 0? What if x is large?
(b) Verify that all members of the family $y = (c - x^2)^{-1/2}$ are solutions of the differential equation $y' = xy^3$.
(c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?
(d) Find a solution of the initial-value problem

$$y' = xy^3 \qquad y(0) = 2$$

9. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing?
(b) For what values of P is the population decreasing?
(c) What are the equilibrium solutions?

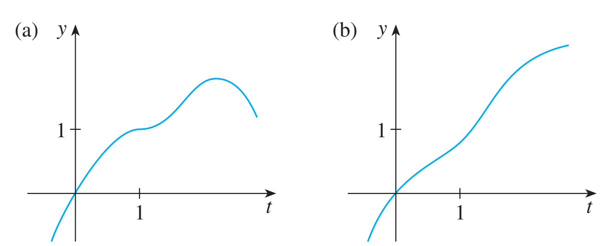
10. The Fitzhugh-Nagumo model for the electrical impulse in a neuron states that, in the absence of relaxation effects, the electrical potential in a neuron $v(t)$ obeys the differential equation

$$\frac{dv}{dt} = -v[v^2 - (1 + a)v + a]$$

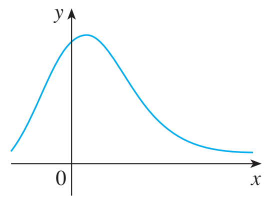
where a is a positive constant such that $0 < a < 1$.
(a) For what values of v is v unchanging (that is, $dv/dt = 0$)?
(b) For what values of v is v increasing?
(c) For what values of v is v decreasing?

11. Explain why the functions with the given graphs *can't* be solutions of the differential equation

$$\frac{dy}{dt} = e^t(y - 1)^2$$



12. The function with the given graph is a solution of one of the following differential equations. Decide which is the correct equation and justify your answer.

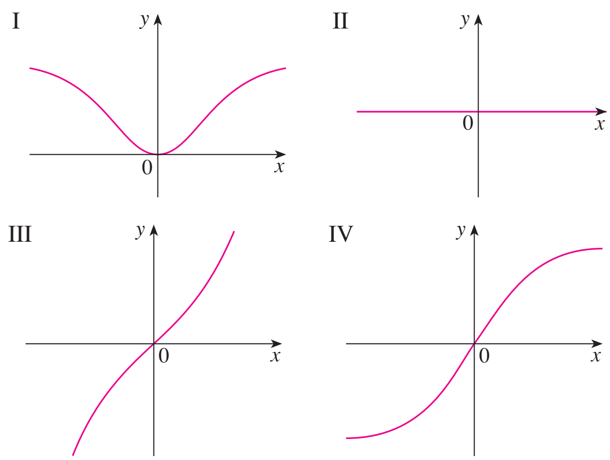


A. $y' = 1 + xy$ **B.** $y' = -2xy$ **C.** $y' = 1 - 2xy$

13. Match the differential equations with the solution graphs labeled I–IV. Give reasons for your choices.

(a) $y' = 1 + x^2 + y^2$ (b) $y' = xe^{-x^2-y^2}$

(c) $y' = \dfrac{1}{1 + e^{x^2+y^2}}$ (d) $y' = \sin(xy)\cos(xy)$



14. Suppose you have just poured a cup of freshly brewed coffee with temperature 95°C in a room where the temperature is 20°C.

(a) When do you think the coffee cools most quickly? What happens to the rate of cooling as time goes by? Explain.

(b) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition? In view of your answer to part (a), do you think this differential equation is an appropriate model for cooling?

(c) Make a rough sketch of the graph of the solution of the initial-value problem in part (b).

15. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.

(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

(c) Make a rough sketch of a possible solution of this differential equation.

16. Von Bertalanffy's equation states that the rate of growth in length of an individual fish is proportional to the difference between the current length L and the asymptotic length L_∞ (in centimeters).

(a) Write a differential equation that expresses this idea.

(b) Make a rough sketch of the graph of a solution of a typical initial-value problem for this differential equation.

17. Differential equations have been used extensively in the study of drug dissolution for patients given oral medications. One such equation is the Weibull equation for the concentration $c(t)$ of the drug:

$$\frac{dc}{dt} = \frac{k}{t^b}(c_s - c)$$

where k and c_s are positive constants and $0 < b < 1$. Verify that

$$c(t) = c_s\left(1 - e^{-\alpha t^{1-b}}\right)$$

is a solution of the Weibull equation for $t > 0$, where $\alpha = k/(1 - b)$. What does the differential equation say about how drug dissolution occurs?

9.2 Direction Fields and Euler's Method

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method).

Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x- and y-coordinates of the point (see Figure 1). In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1. (See Figure 2.)

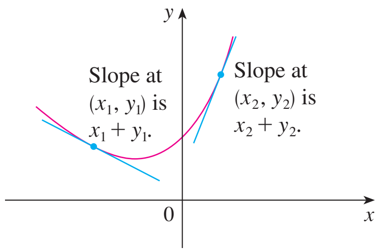


FIGURE 1
A solution of $y' = x + y$

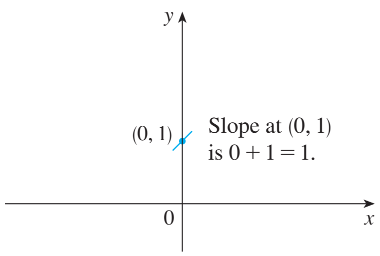


FIGURE 2
Beginning of the solution curve through (0,1)

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 3. For instance, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

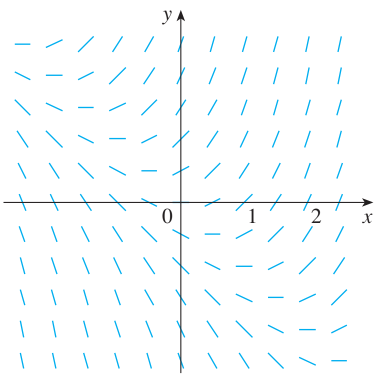


FIGURE 3
Direction field for $y' = x + y$

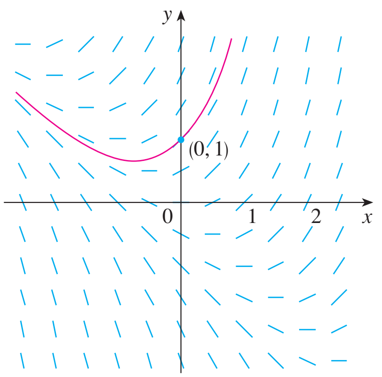


FIGURE 4
The solution curve through (0,1)

TEC Module 9.2A shows direction fields and solution curves for a variety of differential equations.

Now we can sketch the solution curve through the point $(0, 1)$ by following the direction field as in Figure 4. Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x, y)$ is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

EXAMPLE 1

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
(b) Use part (a) to sketch the solution curve that passes through the origin.

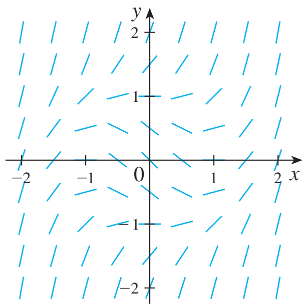


FIGURE 5

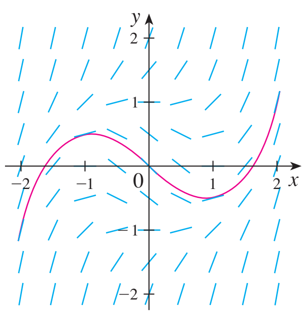


FIGURE 6

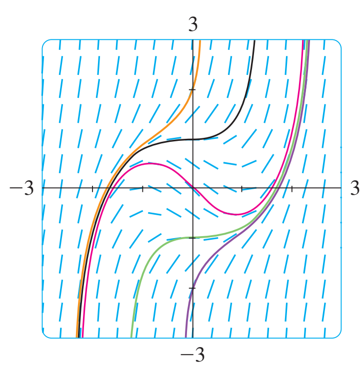


FIGURE 7

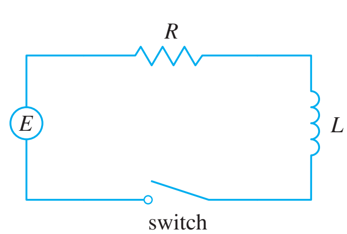


FIGURE 8

SOLUTION

(a) We start by computing the slope at several points in the following chart:

x	-2	-1	0	1	2	-2	-1	0	1	2	. . .
y	0	0	0	0	0	1	1	1	1	1	. . .
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	. . .

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

(b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6. Returning to the origin, we draw the solution curve to the left as well. ■

The more line segments we draw in a direction field, the clearer the picture becomes. Of course, it's tedious to compute slopes and draw line segments for a huge number of points by hand, but computers are well suited for this task. Figure 7 shows a more detailed, computer-drawn direction field for the differential equation in Example 1. It enables us to draw, with reasonable accuracy, the solution curves with y-intercepts -2, -1, 0, 1, and 2.

Now let's see how direction fields give insight into physical situations. The simple electric circuit shown in Figure 8 contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

1
$$L\frac{dI}{dt} + RI = E(t)$$

which is a first-order differential equation that models the current I at time t.

EXAMPLE 2 Suppose that in the simple circuit of Figure 8 the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V.

(a) Draw a direction field for Equation 1 with these values.
(b) What can you say about the limiting value of the current?
(c) Identify any equilibrium solutions.
(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the direction field to sketch the solution curve.

SOLUTION

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 1, we get

$$4\frac{dI}{dt} + 12I = 60 \qquad \text{or} \qquad \frac{dI}{dt} = 15 - 3I$$

The direction field for this differential equation is shown in Figure 9.

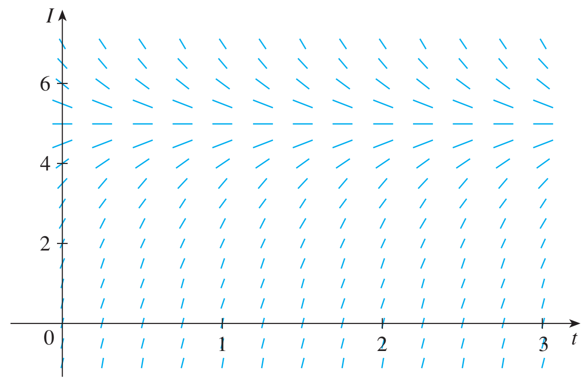


FIGURE 9

(b) It appears from the direction field that all solutions approach the value 5 A, that is,

$$\lim_{t \to \infty} I(t) = 5$$

(c) It appears that the constant function $I(t) = 5$ is an equilibrium solution. Indeed, we can verify this directly from the differential equation $dI/dt = 15 - 3I$. If $I(t) = 5$, then the left side is $dI/dt = 0$ and the right side is $15 - 3(5) = 0$.

(d) We use the direction field to sketch the solution curve that passes through (0, 0), as shown in red in Figure 10.

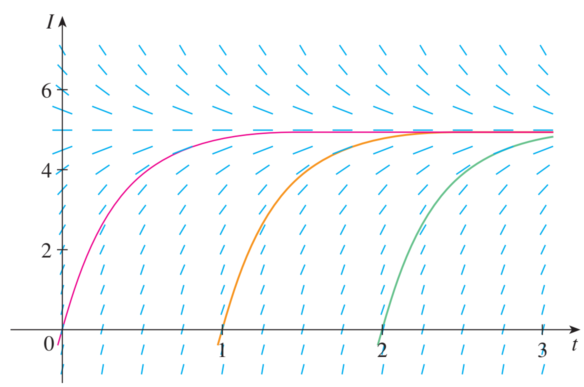


FIGURE 10

■

Notice from Figure 9 that the line segments along any horizontal line are parallel. That is because the independent variable t does not occur on the right side of the equation $I' = 15 - 3I$. In general, a differential equation of the form

$$y' = f(y)$$

in which the independent variable is missing from the right side, is called **autonomous**. For such an equation, the slopes corresponding to two different points with the same y-coordinate must be equal. This means that if we know one solution to an autonomous differential equation, then we can obtain infinitely many others just by shifting the graph of the known solution to the right or left. In Figure 10 we have shown the solutions that result from shifting the solution curve of Example 2 one and two time units (namely, seconds) to the right. They correspond to closing the switch when $t = 1$ or $t = 2$.

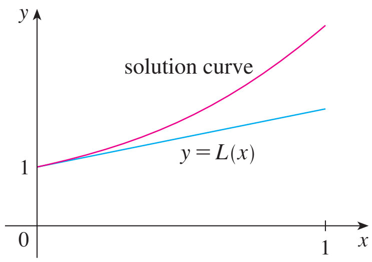


FIGURE 11
First Euler approximation

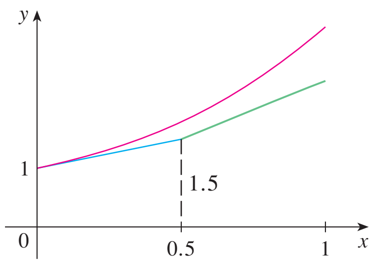


FIGURE 12
Euler approximation with step size 0.5

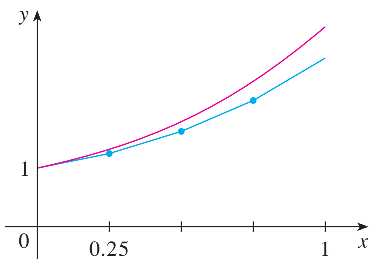


FIGURE 13
Euler approximation with step size 0.25

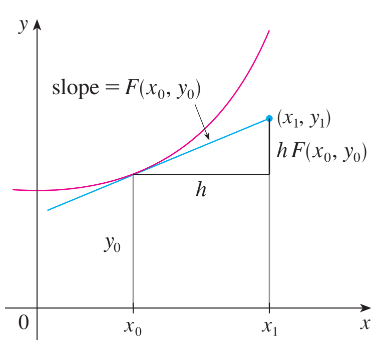


FIGURE 14

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the method on the initial-value problem that we used to introduce direction fields:

$$y' = x + y \qquad y(0) = 1$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $L(x) = x + 1$. In other words, we could use the tangent line at $(0, 1)$ as a rough approximation to the solution curve (see Figure 11).

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field. Figure 12 shows what happens if we start out along the tangent line but stop when $x = 0.5$. (This horizontal distance traveled is called the *step size*.) Since $L(0.5) = 1.5$, we have $y(0.5) \approx 1.5$ and we take $(0.5, 1.5)$ as the starting point for a new line segment. The differential equation tells us that $y'(0.5) = 0.5 + 1.5 = 2$, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for $x > 0.5$ (the green segment in Figure 12). If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 13.

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce the exact solution to an initial-value problem—it gives approximations. But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 11, 12, and 13.)

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, . . . , where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$, so Figure 14 shows that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly, $$y_2 = y_1 + hF(x_1, y_1)$$

In general, $$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

Euler's Method Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \ldots$$

EXAMPLE 3 Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

SOLUTION We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if $y(x)$ is the exact solution, then $y(0.3) \approx 1.362$.

Proceeding with similar calculations, we get the values in the table:

Computer software packages that produce numerical approximations to solutions of differential equations use methods that are refinements of Euler's method. Although Euler's method is simple and not as accurate, it is the basic idea on which the more accurate methods are based.

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

■

For a more accurate table of values in Example 3 we could decrease the step size. But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations. The following table shows the results of applying Euler's method with decreasing step size to the initial-value problem of Example 3.

TEC Module 9.2B shows how Euler's method works numerically and visually for a variety of differential equations and step sizes.

Notice that the Euler estimates in the table below seem to be approaching limits, namely, the true values of $y(0.5)$ and $y(1)$. Figure 15 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005. They are approaching the exact solution curve as the step size h approaches 0.

Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

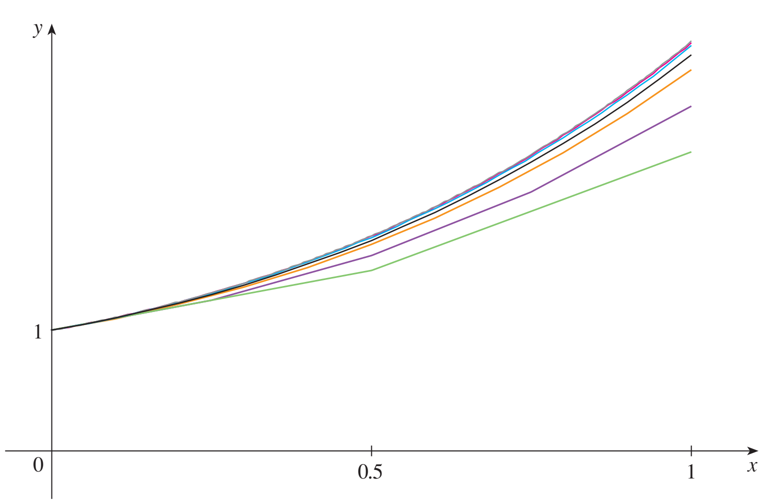


FIGURE 15 Euler approximation approaching the exact solution

Euler

Leonhard Euler (1707–1783) was the leading mathematician of the mid-18th century and the most prolific mathematician of all time. He was born in Switzerland but spent most of his career at the academies of science supported by Catherine the Great in St. Petersburg and Frederick the Great in Berlin. The collected works of Euler (pronounced *Oiler*) fill about 100 large volumes. As the French physicist Arago said, "Euler calculated without apparent effort, as men breathe or as eagles sustain themselves in the air." Euler's calculations and writings were not diminished by raising 13 children or being totally blind for the last 17 years of his life. In fact, when blind, he dictated his discoveries to his helpers from his prodigious memory and imagination. His treatises on calculus and most other mathematical subjects became the standard for mathematics instruction and the equation $e^{i\pi} + 1 = 0$ that he discovered brings together the five most famous numbers in all of mathematics.

EXAMPLE 4 In Example 2 we discussed a simple electric circuit with resistance 12 Ω, inductance 4 H, and a battery with voltage 60 V. If the switch is closed when $t = 0$, we modeled the current I at time t by the initial-value problem

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

Estimate the current in the circuit half a second after the switch is closed.

SOLUTION We use Euler's method with $F(t, I) = 15 - 3I$, $t_0 = 0$, $I_0 = 0$, and step size $h = 0.1$ second:

$$I_1 = 0 + 0.1(15 - 3 \cdot 0) = 1.5$$

$$I_2 = 1.5 + 0.1(15 - 3 \cdot 1.5) = 2.55$$

$$I_3 = 2.55 + 0.1(15 - 3 \cdot 2.55) = 3.285$$

$$I_4 = 3.285 + 0.1(15 - 3 \cdot 3.285) = 3.7995$$

$$I_5 = 3.7995 + 0.1(15 - 3 \cdot 3.7995) = 4.15965$$

So the current after 0.5 s is

$$I(0.5) \approx 4.16 \text{ A}$$

9.2 EXERCISES

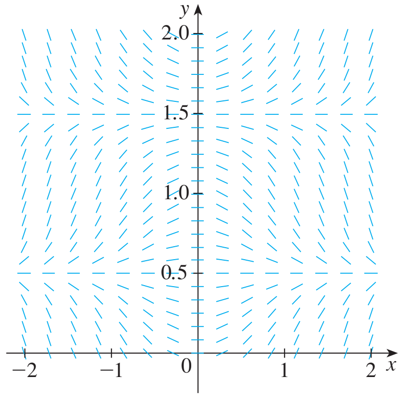
1. A direction field for the differential equation $y' = x \cos \pi y$ is shown.

(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = 0$ (ii) $y(0) = 0.5$

(iii) $y(0) = 1$ (iv) $y(0) = 1.6$

(b) Find all the equilibrium solutions.



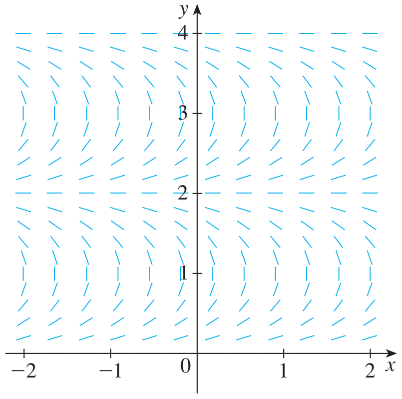
2. A direction field for the differential equation $y' = \tan\left(\frac{1}{2}\pi y\right)$ is shown.

(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = 1$ (ii) $y(0) = 0.2$

(iii) $y(0) = 2$ (iv) $y(1) = 3$

(b) Find all the equilibrium solutions.

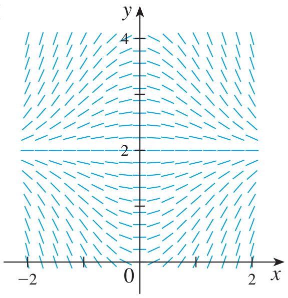


3–6 Match the differential equation with its direction field (labeled I–IV). Give reasons for your answer.

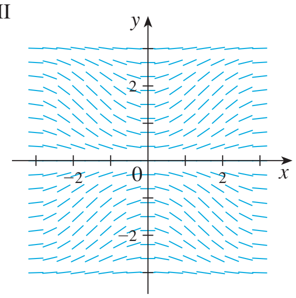
3. $y' = 2 - y$ **4.** $y' = x(2 - y)$

5. $y' = x + y - 1$ **6.** $y' = \sin x \sin y$

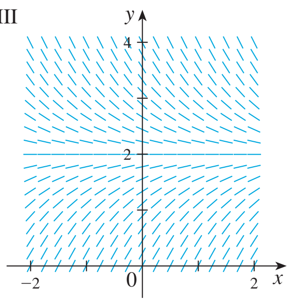
I



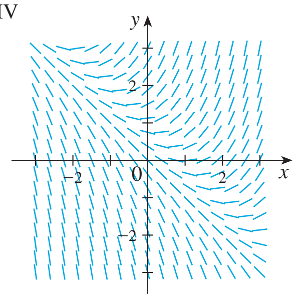
II



III



IV



7. Use the direction field labeled I (above) to sketch the graphs of the solutions that satisfy the given initial conditions.
(a) $y(0) = 1$ (b) $y(0) = 2.5$ (c) $y(0) = 3.5$

8. Use the direction field labeled III (above) to sketch the graphs of the solutions that satisfy the given initial conditions.
(a) $y(0) = 1$ (b) $y(0) = 2.5$ (c) $y(0) = 3.5$

9–10 Sketch a direction field for the differential equation. Then use it to sketch three solution curves.

9. $y' = \frac{1}{2}y$ **10.** $y' = x - y + 1$

11–14 Sketch the direction field of the differential equation. Then use it to sketch a solution curve that passes through the given point.

11. $y' = y - 2x$, $(1, 0)$ **12.** $y' = xy - x^2$, $(0, 1)$

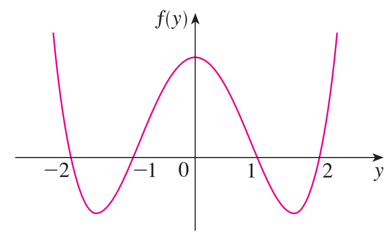
13. $y' = y + xy$, $(0, 1)$ **14.** $y' = x + y^2$, $(0, 0)$

CAS **15–16** Use a computer algebra system to draw a direction field for the given differential equation. Get a printout and sketch on it the solution curve that passes through (0, 1). Then use the CAS to draw the solution curve and compare it with your sketch.

15. $y' = x^2 y - \frac{1}{2}y^2$ **16.** $y' = \cos(x + y)$

CAS **17.** Use a computer algebra system to draw a direction field for the differential equation $y' = y^3 - 4y$. Get a printout and sketch on it solutions that satisfy the initial condition $y(0) = c$ for various values of c. For what values of c does $\lim_{t \to \infty} y(t)$ exist? What are the possible values for this limit?

18. Make a rough sketch of a direction field for the autonomous differential equation $y' = f(y)$, where the graph of f is as shown. How does the limiting behavior of solutions depend on the value of $y(0)$?

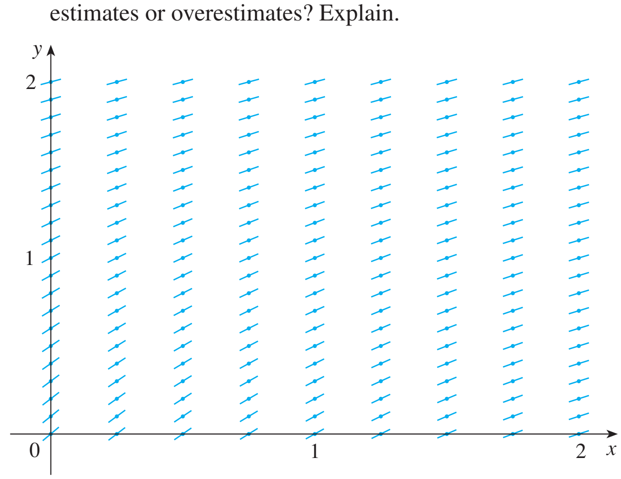


19. (a) Use Euler's method with each of the following step sizes to estimate the value of $y(0.4)$, where y is the solution of the initial-value problem $y' = y$, $y(0) = 1$.
(i) $h = 0.4$ (ii) $h = 0.2$ (iii) $h = 0.1$

(b) We know that the exact solution of the initial-value problem in part (a) is $y = e^x$. Draw, as accurately as you can, the graph of $y = e^x$, $0 \leqslant x \leqslant 0.4$, together with the Euler approximations using the step sizes in part (a). (Your sketches should resemble Figures 11, 12, and 13.) Use your sketches to decide whether your estimates in part (a) are underestimates or overestimates.

(c) The error in Euler's method is the difference between the exact value and the approximate value. Find the errors made in part (a) in using Euler's method to estimate the true value of $y(0.4)$, namely, $e^{0.4}$. What happens to the error each time the step size is halved?

20. A direction field for a differential equation is shown. Draw, with a ruler, the graphs of the Euler approximations to the solution curve that passes through the origin. Use step sizes $h = 1$ and $h = 0.5$. Will the Euler estimates be underestimates or overestimates? Explain.



21. Use Euler's method with step size 0.5 to compute the approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.

22. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = x^2y - \frac{1}{2}y^2$, $y(0) = 1$.

23. Use Euler's method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy$, $y(0) = 1$.

24. (a) Use Euler's method with step size 0.2 to estimate $y(0.6)$, where $y(x)$ is the solution of the initial-value problem $y' = \cos(x + y)$, $y(0) = 0$.
(b) Repeat part (a) with step size 0.1.

25. (a) Program a calculator or computer to use Euler's method to compute $y(1)$, where $y(x)$ is the solution of the initial-value problem

$$\frac{dy}{dx} + 3x^2y = 6x^2 \qquad y(0) = 3$$

(i) $h = 1$ (ii) $h = 0.1$
(iii) $h = 0.01$ (iv) $h = 0.001$

(b) Verify that $y = 2 + e^{-x^3}$ is the exact solution of the differential equation.
(c) Find the errors in using Euler's method to compute $y(1)$ with the step sizes in part (a). What happens to the error when the step size is divided by 10?

CAS **26.** (a) Program your computer algebra system, using Euler's method with step size 0.01, to calculate $y(2)$, where y is the solution of the initial-value problem

$$y' = x^3 - y^3 \qquad y(0) = 1$$

(b) Check your work by using the CAS to draw the solution curve.

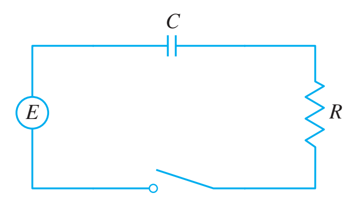
27. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs, C), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$, so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, and a battery gives a constant voltage of 60 V.
(a) Draw a direction field for this differential equation.
(b) What is the limiting value of the charge?
(c) Is there an equilibrium solution?
(d) If the initial charge is $Q(0) = 0$ C, use the direction field to sketch the solution curve.
(e) If the initial charge is $Q(0) = 0$ C, use Euler's method with step size 0.1 to estimate the charge after half a second.



28. In Exercise 9.1.14 we considered a 95°C cup of coffee in a 20°C room. Suppose it is known that the coffee cools at a rate of 1°C per minute when its temperature is 70°C.
(a) What does the differential equation become in this case?
(b) Sketch a direction field and use it to sketch the solution curve for the initial-value problem. What is the limiting value of the temperature?
(c) Use Euler's method with step size $h = 2$ minutes to estimate the temperature of the coffee after 10 minutes.

9.3 Separable Equations

We have looked at first-order differential equations from a geometric point of view (direction fields) and from a numerical point of view (Euler's method). What about the symbolic point of view? It would be nice to have an explicit formula for a solution of a differential equation. Unfortunately, that is not always possible. But in this section we examine a certain type of differential equation that *can* be solved explicitly.

A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "sep-

arated" into a function of x and a function of y. Equivalently, if $f(y) \neq 0$, we could write

1 $$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side. Then we integrate both sides of the equation:

2 $$\int h(y)\,dy = \int g(x)\,dx$$

Equation 2 defines y implicitly as a function of x. In some cases we may be able to solve for y in terms of x.

The technique for solving separable differential equations was first used by James Bernoulli (in 1690) in solving a problem about pendulums and by Leibniz (in a letter to Huygens in 1691). John Bernoulli explained the general method in a paper published in 1694.

We use the Chain Rule to justify this procedure: If h and g satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so $$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

and $$h(y)\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

EXAMPLE 1

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

SOLUTION

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

Figure 1 shows graphs of several members of the family of solutions of the differential equation in Example 1. The solution of the initial-value problem in part (b) is shown in red.

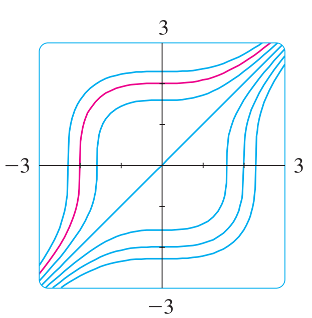


FIGURE 1

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$. Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

■

Some computer software can plot curves defined by implicit equations. Figure 2 shows the graphs of several members of the family of solutions of the differential equation in Example 2. As we look at the curves from left to right, the values of C are 3, 2, 1, 0, -1, -2, and -3.

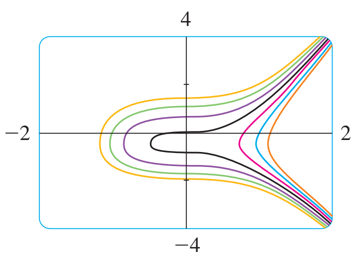


FIGURE 2

EXAMPLE 2 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

SOLUTION Writing the equation in differential form and integrating both sides, we have

$$(2y + \cos y)\,dy = 6x^2\,dx$$

$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$

3 $$y^2 + \sin y = 2x^3 + C$$

where C is a constant. Equation 3 gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x. ■

EXAMPLE 3 Solve the equation $y' = x^2 y$.

SOLUTION First we rewrite the equation using Leibniz notation:

$$\frac{dy}{dx} = x^2 y$$

If a solution y is a function that satisfies $y(x) \neq 0$ for some x, it follows from a uniqueness theorem for solutions of differential equations that $y(x) \neq 0$ for all x.

If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2\,dx \qquad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

This equation defines y implicitly as a function of x. But in this case we can solve explicitly for y as follows:

$$|y| = e^{\ln|y|} = e^{(x^3/3)+C} = e^C e^{x^3/3}$$

so $$y = \pm e^C e^{x^3/3}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form

$$y = Ae^{x^3/3}$$

where A is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$). ■

Figure 3 shows a direction field for the differential equation in Example 3. Compare it with Figure 4, in which we use the equation $y = Ae^{x^3/3}$ to graph solutions for several values of A. If you use the direction field to sketch solution curves with y-intercepts 5, 2, 1, -1, and -2, they will resemble the curves in Figure 4.

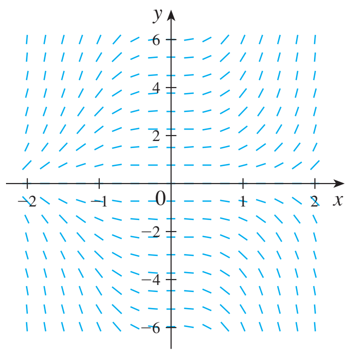


FIGURE 3

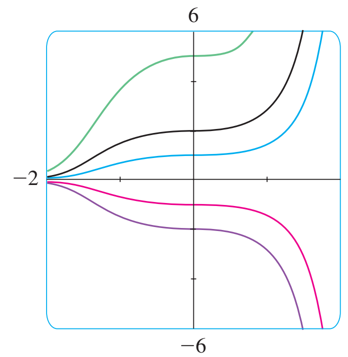


FIGURE 4

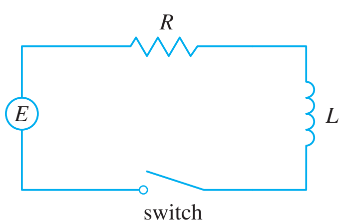


FIGURE 5

EXAMPLE 4 In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in Figure 5 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

Find an expression for the current in a circuit where the resistance is 12 Ω, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

SOLUTION With $L = 4$, $R = 12$, and $E(t) = 60$, the equation becomes

$$4\frac{dI}{dt} + 12I = 60 \qquad \text{or} \qquad \frac{dI}{dt} = 15 - 3I$$

and the initial-value problem is

$$\frac{dI}{dt} = 15 - 3I \qquad I(0) = 0$$

We recognize this equation as being separable, and we solve it as follows:

$$\int \frac{dI}{15 - 3I} = \int dt \qquad (15 - 3I \neq 0)$$

$$-\tfrac{1}{3}\ln|15 - 3I| = t + C$$

$$|15 - 3I| = e^{-3(t+C)}$$

$$15 - 3I = \pm e^{-3C}e^{-3t} = Ae^{-3t}$$

$$I = 5 - \tfrac{1}{3}Ae^{-3t}$$

Figure 6 shows how the solution in Example 4 (the current) approaches its limiting value. Comparison with Figure 9.2.10 shows that we were able to draw a fairly accurate solution curve from the direction field.

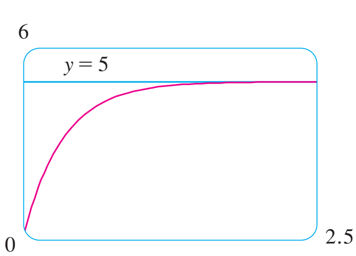


FIGURE 6

Since $I(0) = 0$, we have $5 - \frac{1}{3}A = 0$, so $A = 15$ and the solution is

$$I(t) = 5 - 5e^{-3t}$$

The limiting current, in amperes, is

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}(5 - 5e^{-3t}) = 5 - 5\lim_{t\to\infty} e^{-3t} = 5 - 0 = 5 \qquad \blacksquare$$

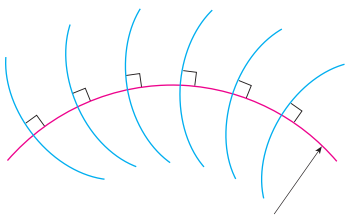


FIGURE 7

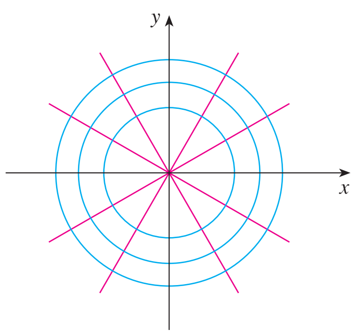


FIGURE 8

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 7). For instance, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin (see Figure 8). We say that the two families are orthogonal trajectories of each other.

EXAMPLE 5 Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

SOLUTION The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis. The first step is to find a single differential equation that is satisfied by all members of the family. If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously. To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\dfrac{x}{y^2}y} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{y}{2x}$$

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$. On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + C$$

4 $$x^2 + \frac{y^2}{2} = C$$

where C is an arbitrary positive constant. Thus the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 9. ■

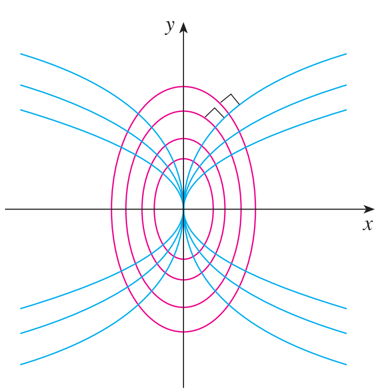


FIGURE 9

Orthogonal trajectories occur in various branches of physics. For example, in an electrostatic field the lines of force are orthogonal to the lines of constant potential. Also, the streamlines in aerodynamics are orthogonal trajectories of the velocity-equipotential curves.

Mixing Problems

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt. A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate. If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order separable differential equation. We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

EXAMPLE 6 A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

SOLUTION Let $y(t)$ be the amount of salt (in kilograms) after t minutes. We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$. Note that dy/dt is the rate of change of the amount of salt, so

5 $$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank. We have

$$\text{rate in} = \left(0.03\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = 0.75\,\frac{\text{kg}}{\text{min}}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $y(t)/5000$ (measured in kilograms per liter). Since the brine flows out at a rate of 25 L/min, we have

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}$$

Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Since $y(0) = 20$, we have $-\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Figure 10 shows the graph of the function $y(t)$ of Example 6. Notice that, as time goes by, the amount of salt approaches 150 kg.

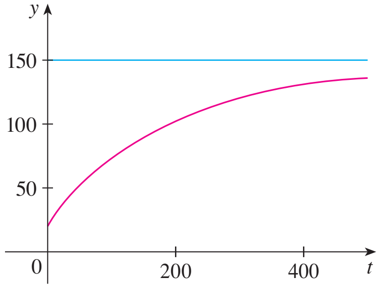


FIGURE 10

Therefore $$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$ and the right side is never 0, we deduce that $150 - y(t)$ is always positive. Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$ ■

9.3 EXERCISES

1–10 Solve the differential equation.

1. $\dfrac{dy}{dx} = 3x^2y^2$

2. $\dfrac{dy}{dx} = x\sqrt{y}$

3. $xyy' = x^2 + 1$

4. $y' + xe^y = 0$

5. $(e^y - 1)y' = 2 + \cos x$

6. $\dfrac{du}{dt} = \dfrac{1 + t^4}{ut^2 + u^4t^2}$

7. $\dfrac{d\theta}{dt} = \dfrac{t \sec\theta}{\theta e^{t^2}}$

8. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1 + R^2}}{\ln H}$

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1$

10. $\dfrac{dz}{dt} + e^{t+z} = 0$

11–18 Find the solution of the differential equation that satisfies the given initial condition.

11. $\dfrac{dy}{dx} = xe^y$, $y(0) = 0$

12. $\dfrac{dy}{dx} = \dfrac{x \sin x}{y}$, $y(0) = -1$

13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$

14. $x + 3y^2\sqrt{x^2 + 1}\,\dfrac{dy}{dx} = 0$, $y(0) = 1$

15. $x \ln x = y\left(1 + \sqrt{3 + y^2}\right) y'$, $y(1) = 1$

16. $\dfrac{dP}{dt} = \sqrt{Pt}$, $P(1) = 2$

17. $y' \tan x = a + y$, $y(\pi/3) = a$, $0 < x < \pi/2$

18. $\dfrac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$

19. Find an equation of the curve that passes through the point $(0, 2)$ and whose slope at (x, y) is x/y.

20. Find the function f such that $f'(x) = xf(x) - x$ and $f(0) = 2$.

21. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.

22. Solve the differential equation $xy' = y + xe^{y/x}$ by making the change of variable $v = y/x$.

23. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.
(b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.
(c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.

24. Solve the equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

CAS **25.** Solve the initial-value problem $y' = (\sin x)/\sin y$, $y(0) = \pi/2$, and graph the solution (if your CAS does implicit plots).

CAS **26.** Solve the equation $y' = x\sqrt{x^2 + 1}/(ye^y)$ and graph several members of the family of solutions (if your CAS does implicit plots). How does the solution curve change as the constant C varies?

CAS **27–28**
(a) Use a computer algebra system to draw a direction field for the differential equation. Get a printout and use it to sketch some solution curves without solving the differential equation.
(b) Solve the differential equation.
(c) Use the CAS to draw several members of the family of solutions obtained in part (b). Compare with the curves from part (a).

27. $y' = y^2$

28. $y' = xy$

29–32 Find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

29. $x^2 + 2y^2 = k^2$

30. $y^2 = kx^3$

31. $y = \dfrac{k}{x}$

32. $y = \dfrac{1}{x + k}$

33–35 An **integral equation** is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

33. $y(x) = 2 + \int_2^x [t - ty(t)]\,dt$

34. $y(x) = 2 + \int_1^x \dfrac{dt}{ty(t)}, \quad x > 0$

35. $y(x) = 4 + \int_0^x 2t\sqrt{y(t)}\,dt$

36. Find a function f such that $f(3) = 2$ and

$$(t^2 + 1)f'(t) + [f(t)]^2 + 1 = 0 \qquad t \neq 1$$

[*Hint:* Use the addition formula for $\tan(x + y)$ on Reference Page 2.]

37. Solve the initial-value problem in Exercise 9.2.27 to find an expression for the charge at time t. Find the limiting value of the charge.

38. In Exercise 9.2.28 we discussed a differential equation that models the temperature of a 95°C cup of coffee in a 20°C room. Solve the differential equation to find an expression for the temperature of the coffee at time t.

39. In Exercise 9.1.15 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

40. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: $A + B \rightarrow C$. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B]$$

(See Example 2.7.4.) Thus, if the initial concentrations are $[A] = a$ moles/L and $[B] = b$ moles/L and we write $x = [C]$, then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.

(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that $[C] = \frac{1}{2}a$ after 20 seconds?

41. In contrast to the situation of Exercise 40, experiments show that the reaction $H_2 + Br_2 \rightarrow 2HBr$ satisfies the rate law

$$\frac{d[HBr]}{dt} = k[H_2][Br_2]^{1/2}$$

and so for this reaction the differential equation becomes

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

where $x = [HBr]$ and a and b are the initial concentrations of hydrogen and bromine.

(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.

(b) If $a > b$, find t as a function of x. [*Hint:* In performing the integration, make the substitution $u = \sqrt{b - x}$.]

42. A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere with radius 2 m and temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

43. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < r/k$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

44. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.

(b) Solve the initial-value problem found in part (a).

(c) How long will it take for the new bills to account for 90% of the currency in circulation?

45. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution

is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

46. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

47. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

48. A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

49. When a raindrop falls, it increases in size and so its mass at time t is a function of t, namely, $m(t)$. The rate of growth of the mass is $km(t)$ for some positive constant k. When we apply Newton's Law of Motion to the raindrop, we get $(mv)' = gm$, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The *terminal velocity* of the raindrop is $\lim_{t\to\infty} v(t)$. Find an expression for the terminal velocity in terms of g and k.

50. An object of mass m is moving horizontally through a medium which resists the motion with a force that is a function of the velocity; that is,

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = f(v)$$

where $v = v(t)$ and $s = s(t)$ represent the velocity and position of the object at time t, respectively. For example, think of a boat moving through the water.

(a) Suppose that the resisting force is proportional to the velocity, that is, $f(v) = -kv$, k a positive constant. (This model is appropriate for small values of v.) Let $v(0) = v_0$ and $s(0) = s_0$ be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels from time $t = 0$?

(b) For larger values of v a better model is obtained by supposing that the resisting force is proportional to the square of the velocity, that is, $f(v) = -kv^2$, $k > 0$. (This model was first proposed by Newton.) Let v_0 and s_0 be the initial values of v and s. Determine v and s at any time t. What is the total distance that the object travels in this case?

51. *Allometric growth* in biology refers to relationships between sizes of parts of an organism (skull length and body length, for instance). If $L_1(t)$ and $L_2(t)$ are the sizes of two organs in an organism of age t, then L_1 and L_2 satisfy an allometric law if their specific growth rates are proportional:

$$\frac{1}{L_1}\frac{dL_1}{dt} = k\frac{1}{L_2}\frac{dL_2}{dt}$$

where k is a constant.

(a) Use the allometric law to write a differential equation relating L_1 and L_2 and solve it to express L_1 as a function of L_2.

(b) In a study of several species of unicellular algae, the proportionality constant in the allometric law relating B (cell biomass) and V (cell volume) was found to be $k = 0.0794$. Write B as a function of V.

52. A model for tumor growth is given by the Gompertz equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

where a and b are positive constants and V is the volume of the tumor measured in mm^3.

(a) Find a family of solutions for tumor volume as a function of time.

(b) Find the solution that has an initial tumor volume of $V(0) = 1$ mm^3.

53. Let $A(t)$ be the area of a tissue culture at time t and let M be the final area of the tissue when growth is complete. Most cell divisions occur on the periphery of the tissue and the number of cells on the periphery is proportional to $\sqrt{A(t)}$. So a reasonable model for the growth of tissue is obtained by assuming that the rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$.

(a) Formulate a differential equation and use it to show that the tissue grows fastest when $A(t) = \frac{1}{3}M$.

CAS (b) Solve the differential equation to find an expression for $A(t)$. Use a computer algebra system to perform the integration.

54. According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x = x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m(dv/dt)$ and so

$$m\frac{dv}{dt} = -\frac{mgR^2}{(x + R)^2}$$

(a) Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above

the surface reached by the object. Show that

$$v_0 = \sqrt{\frac{2gRh}{R + h}}$$

[*Hint:* By the Chain Rule, $m\,(dv/dt) = mv\,(dv/dx)$.]

(b) Calculate $v_e = \lim_{h\to\infty} v_0$. This limit is called the *escape velocity* for the earth.

(c) Use $R = 3960$ mi and $g = 32\ \text{ft/s}^2$ to calculate v_e in feet per second and in miles per second.

APPLIED PROJECT HOW FAST DOES A TANK DRAIN?

If water (or other liquid) drains from a tank, we expect that the flow will be greatest at first (when the water depth is greatest) and will gradually decrease as the water level decreases. But we need a more precise mathematical description of how the flow decreases in order to answer the kinds of questions that engineers ask: How long does it take for a tank to drain completely? How much water should a tank hold in order to guarantee a certain minimum water pressure for a sprinkler system?

Let $h(t)$ and $V(t)$ be the height and volume of water in a tank at time t. If water drains through a hole with area a at the bottom of the tank, then Torricelli's Law says that

$$\boxed{1} \qquad \frac{dV}{dt} = -a\sqrt{2gh}$$

where g is the acceleration due to gravity. So the rate at which water flows from the tank is proportional to the square root of the water height.

1. (a) Suppose the tank is cylindrical with height 6 ft and radius 2 ft and the hole is circular with radius 1 inch. If we take $g = 32\ \text{ft/s}^2$, show that h satisfies the differential equation

$$\frac{dh}{dt} = -\frac{1}{72}\sqrt{h}$$

(b) Solve this equation to find the height of the water at time t, assuming the tank is full at time $t = 0$.

(c) How long will it take for the water to drain completely?

Problem 2(b) is best done as a classroom demonstration or as a group project with three students in each group: a timekeeper to call out seconds, a bottle keeper to estimate the height every 10 seconds, and a record keeper to record these values.



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2. Because of the rotation and viscosity of the liquid, the theoretical model given by Equation 1 isn't quite accurate. Instead, the model

$$\boxed{2} \qquad \frac{dh}{dt} = k\sqrt{h}$$

is often used and the constant k (which depends on the physical properties of the liquid) is determined from data concerning the draining of the tank.

(a) Suppose that a hole is drilled in the side of a cylindrical bottle and the height h of the water (above the hole) decreases from 10 cm to 3 cm in 68 seconds. Use Equation 2 to find an expression for $h(t)$. Evaluate $h(t)$ for $t = 10, 20, 30, 40, 50, 60$.

(b) Drill a 4-mm hole near the bottom of the cylindrical part of a two-liter plastic soft-drink bottle. Attach a strip of masking tape marked in centimeters from 0 to 10, with 0 corresponding to the top of the hole. With one finger over the hole, fill the bottle with water to the 10-cm mark. Then take your finger off the hole and record the values of $h(t)$ for $t = 10, 20, 30, 40, 50, 60$ seconds. (You will probably find that it takes 68 seconds for the level to decrease to $h = 3$ cm.) Compare your data with the values of $h(t)$ from part (a). How well did the model predict the actual values?

3. In many parts of the world, the water for sprinkler systems in large hotels and hospitals is supplied by gravity from cylindrical tanks on or near the roofs of the buildings. Suppose such a tank has radius 10 ft and the diameter of the outlet is 2.5 inches. An engineer has to guaran-

tee that the water pressure will be at least 2160 lb/ft^2 for a period of 10 minutes. (When a fire happens, the electrical system might fail and it could take up to 10 minutes for the emergency generator and fire pump to be activated.) What height should the engineer specify for the tank in order to make such a guarantee? (Use the fact that the water pressure at a depth of d feet is $P = 62.5d$ lb/ft^2. See Section 8.3.)

4. Not all water tanks are shaped like cylinders. Suppose a tank has cross-sectional area $A(h)$ at height h. Then the volume of water up to height h is $V = \int_0^h A(u)\,du$ and so the Fundamental Theorem of Calculus gives $dV/dh = A(h)$. It follows that

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}$$

and so Torricelli's Law becomes

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we take $g = 10$ m/s^2, show that h satisfies the differential equation

$$(4h - h^2)\frac{dh}{dt} = -0.0001\sqrt{20h}$$

(b) How long will it take for the water to drain completely?

APPLIED PROJECT WHICH IS FASTER, GOING UP OR COMING DOWN?

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to earth from its maximum height? We will solve the problem in this project, but before getting started, think about that situation and make a guess based on your physical intuition.

In modeling force due to air resistance, various functions have been used, depending on the physical characteristics and speed of the ball. Here we use a linear model, $-pv$, but a quadratic model ($-pv^2$ on the way up and pv^2 on the way down) is another possibility for higher speeds (see Exercise 9.3.50). For a golf ball, experiments have shown that a good model is $-pv^{1.3}$ going up and $p|v|^{1.3}$ coming down. But no matter which force function $-f(v)$ is used [where $f(v) > 0$ for $v > 0$ and $f(v) < 0$ for $v < 0$], the answer to the question remains the same. See F. Brauer, "What Goes Up Must Come Down, Eventually," *American Mathematical Monthly* 108 (2001), pp. 437–440.

1. A ball with mass m is projected vertically upward from the earth's surface with a positive initial velocity v_0. We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t. In both the ascent and the descent, the total force acting on the ball is $-pv - mg$. [During ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negative and the resistance acts upward.] So, by Newton's Second Law, the equation of motion is

$$mv' = -pv - mg$$

Solve this differential equation to show that the velocity is

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}(1 - e^{-pt/m}) - \frac{mgt}{p}$$

3. Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p} \ln\left(\frac{mg + pv_0}{mg}\right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is $\frac{1}{10}$ of the speed.

4. Let t_2 be the time at which the ball falls back to earth. For the particular ball in Problem 3, estimate t_2 by using a graph of the height function $y(t)$. Which is faster, going up or coming down?

5. In general, it's not easy to find t_2 because it's impossible to solve the equation $y(t) = 0$ explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster: we determine whether $y(2t_1)$ is positive or negative. Show that

$$y(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2 \ln x\right)$$

where $x = e^{pt_1/m}$. Then show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2 \ln x$$

is increasing for $x > 1$. Use this result to decide whether $y(2t_1)$ is positive or negative. What can you conclude? Is ascent or descent faster?

9.4 Models for Population Growth

In this section we investigate differential equations that are used to model population growth: the law of natural growth, the logistic equation, and several others.

The Law of Natural Growth

One of the models for population growth that we considered in Section 9.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

1
$$\frac{dP}{dt} = kP$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

Because Equation 1 is a separable differential equation, we can solve it by the methods of Section 9.3:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C} = e^C e^{kt}$$

$$P = Ae^{kt}$$

where A ($= \pm e^C$ or 0) is an arbitrary constant. To see the significance of the constant A, we observe that

$$P(0) = Ae^{k\cdot 0} = A$$

Therefore A is the initial value of the function.

Examples and exercises on the use of (2) are given in Section 6.5.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Another way of writing Equation 1 is

$$\frac{1}{P}\frac{dP}{dt} = k$$

which says that the **relative growth rate** (the growth rate divided by the population size) is constant. Then (2) says that a population with constant relative growth rate must grow exponentially.

We can account for emigration (or "harvesting") from a population by modifying Equation 1: if the rate of emigration is a constant m, then the rate of change of the population is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \tag{3}$$

See Exercise 17 for the solution and consequences of Equation 3.

The Logistic Model

As we discussed in Section 9.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**:

$$\boxed{4} \qquad \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Notice from Equation 4 that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population P lies between 0 and M, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

EXAMPLE 1 Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after $t = 0$.

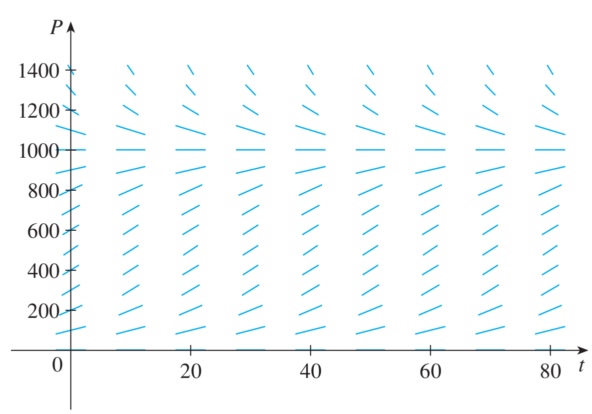


FIGURE 1
Direction field for the logistic equation in Example 1

The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$. Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing. The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$. In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500. (See Exercise 13.)

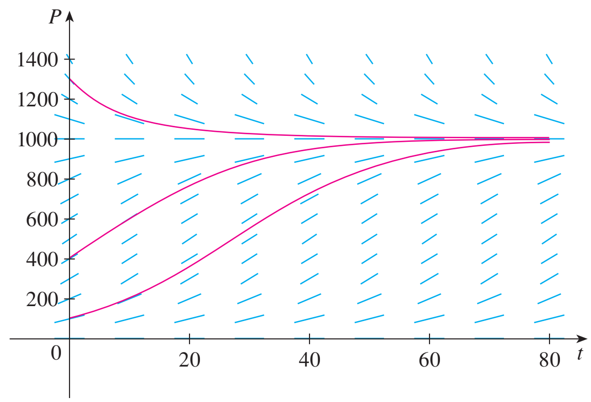


FIGURE 2
Solution curves for the logistic equation in Example 1

■

The logistic equation (4) is separable and so we can solve it explicitly using the method of Section 9.3. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

5 $$\int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions (see Section 7.4), we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k\,dt$$

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

6 $$\frac{M-P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$. Solving Equation 6 for P, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so
$$P = \frac{M}{1 + Ae^{-kt}}$$

We find the value of A by putting $t = 0$ in Equation 6. If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

7 $$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}$$

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

EXAMPLE 2 Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

SOLUTION The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$. So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \qquad \text{where } A = \frac{1000 - 100}{100} = 9$$

Thus
$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and 80 are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6 \qquad P(80) = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Solving this equation for t, we get

$$1 + 9e^{-0.08t} = \tfrac{10}{9}$$

$$e^{-0.08t} = \tfrac{1}{81}$$

$$-0.08t = \ln \tfrac{1}{81} = -\ln 81$$

$$t = \frac{\ln 81}{0.08} \approx 54.9$$

Compare the solution curve in Figure 3 with the lowest solution curve we drew from the direction field in Figure 2.

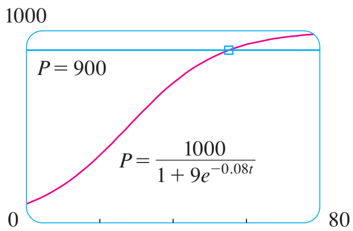


FIGURE 3

So the population reaches 900 when t is approximately 55. As a check on our work, we graph the population curve in Figure 3 and observe where it intersects the line $P = 900$. The cursor indicates that $t \approx 55$. ■

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

EXAMPLE 3 Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

SOLUTION Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt} = 2e^{0.7944t}$$

Gause used the same value of k for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$). The equation

$$\frac{1}{P_0}\frac{dP}{dt}\bigg|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

Then the solution of the logistic equation in Equation 7 gives

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}$$

where $$A = \frac{M - P_0}{P_0} = \frac{64 - 2}{2} = 31$$

So $$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

We notice from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model. For $t \geq 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.

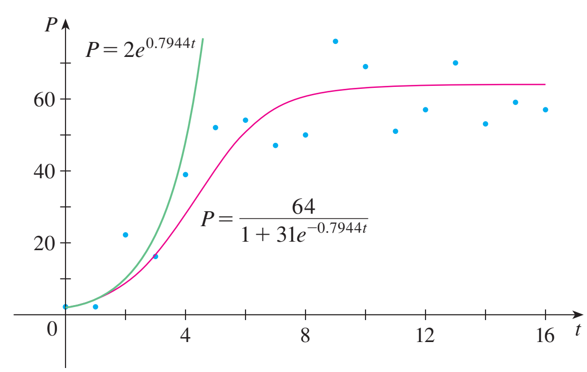


FIGURE 4
The exponential and logistic models for the *Paramecium* data

Many countries that formerly experienced exponential growth are now finding that their rates of population growth are declining and the logistic model provides a better model. The table in the margin shows midyear values of $B(t)$, the population of Belgium, in thousands, at time t, from 1980 to 2012. Figure 5 shows these data points together with a shifted logistic function obtained from a calculator with the ability to fit a logistic function to these points by regression. We see that the logistic model provides a very good fit.

t	$B(t)$	t	$B(t)$
1980	9,847	1998	10,217
1982	9,856	2000	10,264
1984	9,855	2002	10,312
1986	9,862	2004	10,348
1988	9,884	2006	10,379
1990	9,969	2008	10,404
1992	10,046	2010	10,423
1994	10,123	2012	10,438
1996	10,179		

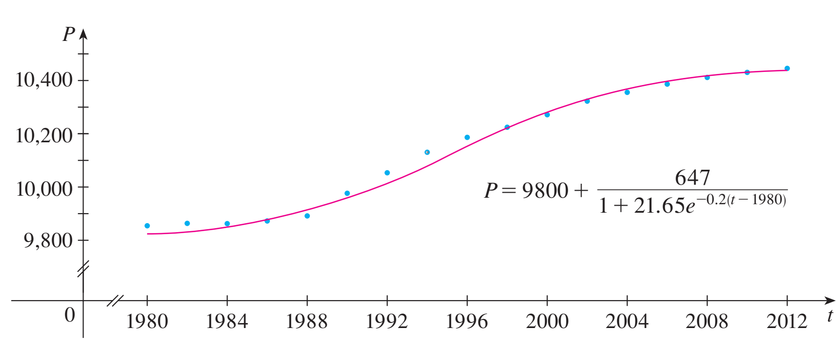


FIGURE 5
Logistic model for the population of Belgium

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. In Exercise 22 we look at the Gompertz growth function and in Exercises 23 and 24 we investigate seasonal-growth models.

Two of the other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.) This equation is explored in Exercises 19 and 20.

For some species there is a minimum population level m below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, $1 - m/P$, takes into account the consequences of a sparse population (see Exercise 21).

9.4 EXERCISES

1–2 A population grows according to the given logistic equation, where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?
(b) Write the solution of the equation.
(c) What is the population after 10 weeks?

1. $\dfrac{dP}{dt} = 0.04P\left(1 - \dfrac{P}{1200}\right), \quad P(0) = 60$

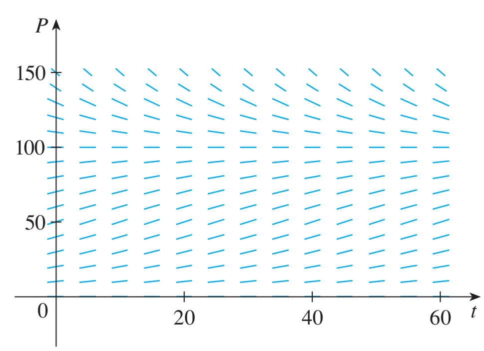
2. $\dfrac{dP}{dt} = 0.02P - 0.0004P^2, \quad P(0) = 40$

3. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?
(b) A direction field for this equation is shown. Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



(c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?
(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

4. Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.

(a) Write the logistic differential equation for these data.

(b) Draw a direction field (either by hand or with a computer algebra system). What does it tell you about the solution curves?
(c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?
(d) Program a calculator or computer to use Euler's method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.
(e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).
(f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).

5. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $k = 0.71$ per year.
(a) If $y(0) = 2 \times 10^7$ kg, find the biomass a year later.
(b) How long will it take for the biomass to reach 4×10^7 kg?

6. Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.
(a) What is the carrying capacity?
(b) What is $P'(0)$?
(c) When will the population reach 50% of the carrying capacity?

7. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

8. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.
(b) Use the data to estimate the initial relative growth rate.
(c) Find both an exponential model and a logistic model for these data.
(d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.
(e) Use your logistic model to estimate the number of yeast cells after 7 hours.

9. The population of the world was about 6.1 billion in 2000. Birth rates around that time ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 20 billion.
(a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)
(b) Use the logistic model to estimate the world population in the year 2010 and compare with the actual population of 6.9 billion.
(c) Use the logistic model to predict the world population in the years 2100 and 2500.

10. (a) Assume that the carrying capacity for the US population is 800 million. Use it and the fact that the population was 282 million in 2000 to formulate a logistic model for the US population.
(b) Determine the value of k in your model by using the fact that the population in 2010 was 309 million.
(c) Use your model to predict the US population in the years 2100 and 2200.
(d) Use your model to predict the year in which the US population will exceed 500 million.

11. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.
(a) Write a differential equation that is satisfied by y.
(b) Solve the differential equation.
(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

12. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.
(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.
(b) How long will it take for the population to increase to 5000?

13. (a) Show that if P satisfies the logistic equation (4), then

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) Deduce that a population grows fastest when it reaches half its carrying capacity.

14. For a fixed value of M (say $M = 10$), the family of logistic functions given by Equation 7 depends on the initial value P_0 and the proportionality constant k. Graph several members of this family. How does the graph change when P_0 varies? How does it change when k varies?

15. The table gives the midyear population of Japan, in thousands, from 1960 to 2010.

Year	Population	Year	Population
1960	94,092	1990	123,537
1965	98,883	1995	125,327
1970	104,345	2000	126,776
1975	111,573	2005	127,715
1980	116,807	2010	127,579
1985	120,754		

Use a calculator to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models. [*Hint:* Subtract 94,000 from each of the population figures. Then, after obtaining a model from your calculator, add 94,000 to get your final model. It might be helpful to choose $t = 0$ to correspond to 1960 or 1980.]

16. The table gives the midyear population of Norway, in thousands, from 1960 to 2010.

Year	Population	Year	Population
1960	3581	1990	4242
1965	3723	1995	4359
1970	3877	2000	4492
1975	4007	2005	4625
1980	4086	2010	4891
1985	4152		

Use a calculator to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models. [*Hint:* Subtract 3500 from each of the population figures. Then, after obtaining a model from your calculator, add 3500 to get your final model. It might be helpful to choose $t = 0$ to correspond to 1960.]

17. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.
(b) What condition on m will lead to an exponential expansion of the population?
(c) What condition on m will result in a constant population? A population decline?
(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

18. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.
(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.
(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.
(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

19. Let's modify the logistic differential equation of Example 1 as follows:

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

(a) Suppose $P(t)$ represents a fish population at time t, where t is measured in weeks. Explain the meaning of the final term in the equation (-15).
(b) Draw a direction field for this differential equation.
(c) What are the equilibrium solutions?
(d) Use the direction field to sketch several solution curves. Describe what happens to the fish population for various initial populations.
CAS (e) Solve this differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).

CAS **20.** Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.
(a) Use a CAS to draw direction fields for various values of c.

(b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?

(c) Use the differential equation to prove what you discovered graphically in part (b).

(d) What would you recommend for a limit to the weekly catch of this fish population?

21. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.

(b) For the case where $k = 0.08$, $M = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?

(c) Solve the differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial population P_0.

(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. [*Hint:* Show that the numerator in your expression for $P(t)$ is 0 for some value of t.]

22. Another model for a growth function for a limited population is given by the **Gompertz function**, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.

(a) Solve this differential equation.

(b) Compute $\lim_{t\to\infty} P(t)$.

(c) Graph the Gompertz growth function for $M = 1000$, $P_0 = 100$, and $c = 0.05$, and compare it with the logistic function in Example 2. What are the similarities? What are the differences?

(d) We know from Exercise 13 that the logistic function grows fastest when $P = M/2$. Use the Gompertz differential equation to show that the Gompertz function grows fastest when $P = M/e$.

23. In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.

(a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and ϕ are positive constants.

(b) By graphing the solution for several values of k, r, and ϕ, explain how the values of k, r, and ϕ affect the solution. What can you say about $\lim_{t\to\infty} P(t)$?

24. Suppose we alter the differential equation in Exercise 23 as follows:

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

(a) Solve this differential equation with the help of a table of integrals or a CAS.

(b) Graph the solution for several values of k, r, and ϕ. How do the values of k, r, and ϕ affect the solution? What can you say about $\lim_{t\to\infty} P(t)$ in this case?

25. Graphs of logistic functions (Figures 2 and 3) look suspiciously similar to the graph of the hyperbolic tangent function (Figure 6.7.3). Explain the similarity by showing that the logistic function given by Equation 7 can be written as

$$P(t) = \tfrac{1}{2}M\left[1 + \tanh\left(\tfrac{1}{2}k(t - c)\right)\right]$$

where $c = (\ln A)/k$. Thus the logistic function is really just a shifted hyperbolic tangent.

9.5 Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

2 $$y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \qquad \text{or} \qquad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor*. We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

3 $$I(x)\big(y' + P(x)y\big) = \big(I(x)y\big)'$$

If we can find such a function I, then Equation 1 becomes

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)\,Q(x)\,dx + C$$

so the solution would be

4 $$y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)\,P(x)y = \big(I(x)y\big)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln|I| = \int P(x)\,dx$$

$$I = Ae^{\int P(x)\,dx}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

5 $$I(x) = e^{\int P(x)\,dx}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

EXAMPLE 1 Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

SOLUTION The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or $$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$ ■

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

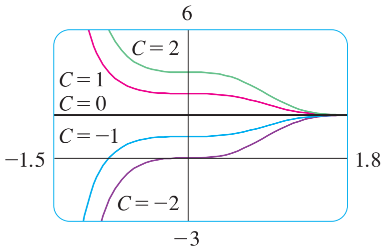


FIGURE 1

EXAMPLE 2 Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

SOLUTION We must first divide both sides by the coefficient of y' to put the differential equation into standard form:

6 $$y' + \frac{1}{x}y = \frac{1}{x^2} \qquad x > 0$$

The integrating factor is

$$I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Multiplication of Equation 6 by x gives

$$xy' + y = \frac{1}{x} \qquad \text{or} \qquad (xy)' = \frac{1}{x}$$

Then $$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so $$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$ ■

The solution of the initial-value problem in Example 2 is shown in Figure 2.

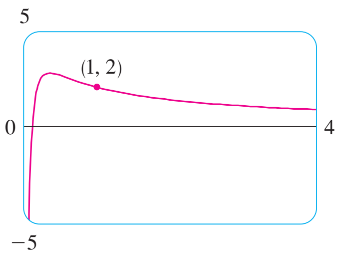


FIGURE 2

EXAMPLE 3 Solve $y' + 2xy = 1$.

SOLUTION The given equation is in the standard form for a linear equation. Multiplying by the integrating factor

$$e^{\int 2x\,dx} = e^{x^2}$$

we get $$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$$

or $$\left(e^{x^2}y\right)' = e^{x^2}$$

Therefore $$e^{x^2}y = \int e^{x^2}\,dx + C$$

Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer algebra system (Figure 3).

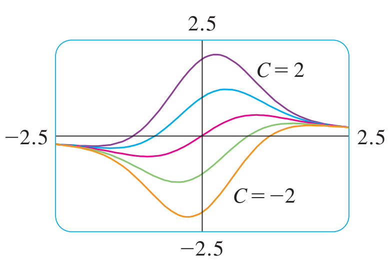


FIGURE 3

Recall from Section 7.5 that $\int e^{x^2}dx$ can't be expressed in terms of elementary functions. Nonetheless, it's a perfectly good function and we can leave the answer as

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

Another way of writing the solution is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + Ce^{-x^2}$$

(Any number can be chosen for the lower limit of integration.) ■

Application to Electric Circuits

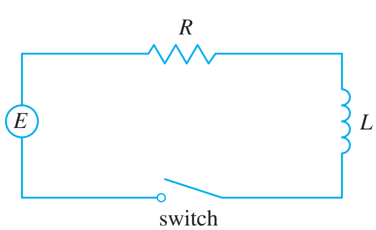


FIGURE 4

In Section 9.2 we considered the simple electric circuit shown in Figure 4: An electromotive force (usually a battery or generator) produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

7 $$L\frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation. The solution gives the current I at time t.

EXAMPLE 4 Suppose that in the simple circuit of Figure 4 the resistance is 12 Ω and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find (a) $I(t)$, (b) the current after 1 second, and (c) the limiting value of the current.

The differential equation in Example 4 is both linear and separable, so an alternative method is to solve it as a separable equation (Example 9.3.4). If we replace the battery by a generator, however, we get an equation that is linear but not separable (Example 5).

SOLUTION
(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 7, we obtain the initial-value problem

$$4\frac{dI}{dt} + 12I = 60 \qquad I(0) = 0$$

or

$$\frac{dI}{dt} + 3I = 15 \qquad I(0) = 0$$

Multiplying by the integrating factor $e^{\int 3\,dt} = e^{3t}$, we get

$$e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t}$$

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}$$

$$e^{3t}I = \int 15e^{3t}\,dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Figure 5 shows how the current in Example 4 approaches its limiting value.

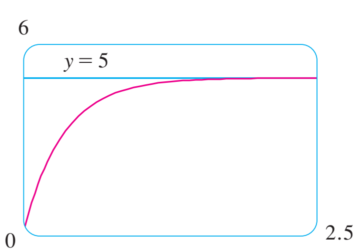


FIGURE 5

Since $I(0) = 0$, we have $5 + C = 0$, so $C = -5$ and

$$I(t) = 5(1 - e^{-3t})$$

(b) After 1 second the current is

$$I(1) = 5(1 - e^{-3}) \approx 4.75 \text{ A}$$

(c) The limiting value of the current is given by

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} 5(1 - e^{-3t}) = 5 - 5\lim_{t\to\infty} e^{-3t} = 5 - 0 = 5$$ ■

Figure 6 shows the graph of the current when the battery is replaced by a generator.

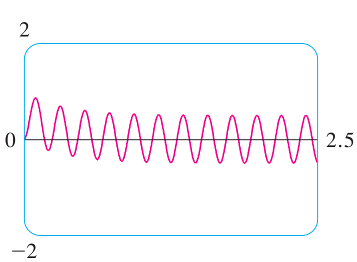


FIGURE 6

EXAMPLE 5 Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of $E(t) = 60 \sin 30t$ volts. Find $I(t)$.

SOLUTION This time the differential equation becomes

$$4\frac{dI}{dt} + 12I = 60 \sin 30t \qquad \text{or} \qquad \frac{dI}{dt} + 3I = 15 \sin 30t$$

The same integrating factor e^{3t} gives

$$\frac{d}{dt}(e^{3t}I) = e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t} \sin 30t$$

Using Formula 98 in the Table of Integrals, we have

$$e^{3t}I = \int 15e^{3t} \sin 30t\, dt = 15 \frac{e^{3t}}{909}(3 \sin 30t - 30 \cos 30t) + C$$

$$I = \tfrac{5}{101}(\sin 30t - 10 \cos 30t) + Ce^{-3t}$$

Since $I(0) = 0$, we get

$$-\tfrac{50}{101} + C = 0$$

so

$$I(t) = \tfrac{5}{101}(\sin 30t - 10 \cos 30t) + \tfrac{50}{101}e^{-3t}$$

9.5 EXERCISES

1–4 Determine whether the differential equation is linear.

1. $y' + x\sqrt{y} = x^2$

2. $y' - x = y \tan x$

3. $ue^t = t + \sqrt{t}\,\dfrac{du}{dt}$

4. $\dfrac{dR}{dt} + t \cos R = e^{-t}$

5–14 Solve the differential equation.

5. $y' + y = 1$

6. $y' - y = e^x$

7. $y' = x - y$

8. $4x^3y + x^4y' = \sin^3x$

9. $xy' + y = \sqrt{x}$

10. $2xy' + y = 2\sqrt{x}$

11. $xy' - 2y = x^2, \quad x > 0$

12. $y' + 2xy = 1$

13. $t^2 \dfrac{dy}{dt} + 3ty = \sqrt{1 + t^2}, \quad t > 0$

14. $t \ln t \dfrac{dr}{dt} + r = te^t$

15–20 Solve the initial-value problem.

15. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

16. $t^3 \dfrac{dy}{dt} + 3t^2y = \cos t, \quad y(\pi) = 0$

17. $t \dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

18. $xy' + y = x \ln x, \quad y(1) = 0$

19. $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

20. $(x^2 + 1)\dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$

21–22 Solve the differential equation and use a calculator to graph several members of the family of solutions. How does the solution curve change as C varies?

21. $xy' + 2y = e^x$

22. $xy' = x^2 + 2y$

23. A **Bernoulli differential equation** (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

24–25 Use the method of Exercise 23 to solve the differential equation.

24. $xy' + y = -xy^2$

25. $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$

26. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

27. In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.

28. In the circuit shown in Figure 4, a generator supplies a voltage of $E(t) = 40 \sin 60t$ volts, the inductance is 1 H, the resistance is 20 Ω, and $I(0) = 1$ A.
(a) Find $I(t)$.
(b) Find the current after 0.1 seconds.
(c) Use a graphing device to draw the graph of the current function.

29. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage

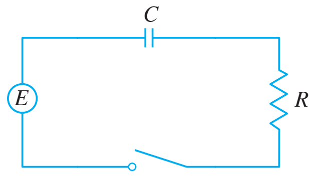
drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$ (see Example 2.7.3), so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, a battery gives a constant voltage of 60 V, and the initial charge is $Q(0) = 0$ C. Find the charge and the current at time t.



30. In the circuit of Exercise 29, $R = 2\ \Omega$, $C = 0.01$ F, $Q(0) = 0$, and $E(t) = 10 \sin 60t$. Find the charge and the current at time t.

31. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a *learning curve*. In Exercise 9.1.15 we proposed the differential equation

$$\frac{dP}{dt} = k[M - P(t)]$$

as a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

32. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 31 and assuming that $P(0) = 0$, estimate the maximum number of units per hour that each worker is capable of processing.

33. In Section 9.3 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable differentiable equations. (See Example 6 in that section.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

34. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

35. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv$$

(a) Solve this as a linear equation to show that

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

(b) What is the limiting velocity?
(c) Find the distance the object has fallen after t seconds.

36. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 35(a) to find dv/dm and show that heavier objects *do* fall faster than lighter ones.

37. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

38. To account for seasonal variation in the logistic differential equation, we could allow k and M to be functions of t:

$$\frac{dP}{dt} = k(t)P\left(1 - \frac{P}{M(t)}\right)$$

(a) Verify that the substitution $z = 1/P$ transforms this equation into the linear equation

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)}$$

(b) Write an expression for the solution of the linear equation in part (a) and use it to show that if the carrying capacity M

is constant, then

$$P(t) = \frac{M}{1 + CMe^{-\int k(t)\,dt}}$$

Deduce that if $\int_0^\infty k(t)\,dt = \infty$, then $\lim_{t\to\infty} P(t) = M$. [This will be true if $k(t) = k_0 + a\cos bt$ with $k_0 > 0$, which describes a positive intrinsic growth rate with a periodic seasonal variation.]

(c) If k is constant but M varies, show that

$$z(t) = e^{-kt}\int_0^t \frac{ke^{ks}}{M(s)}\,ds + Ce^{-kt}$$

and use l'Hospital's Rule to deduce that if $M(t)$ has a limit as $t \to \infty$, then $P(t)$ has the same limit.

9.6 Predator-Prey Systems

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section we consider more realistic models that take into account the interaction of two species in the same habitat. We will see that these models take the form of a pair of linked differential equations.

We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey. Examples of prey and predators include rabbits and wolves in an isolated forest, food-fish and sharks, aphids and ladybugs, and bacteria and amoebas. Our model will have two dependent variables and both are functions of time. We let $R(t)$ be the number of prey (using R for rabbits) and $W(t)$ be the number of predators (with W for wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR \qquad \text{where } k \text{ is a positive constant}$$

In the absence of prey, we assume that the predator population would decline through mortality at a rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

With both species present, however, we assume that the principal cause of death among the prey is being eaten by a predator, and the birth and survival rates of the predators depend on their available food supply, namely, the prey. We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product RW. (The more there are of either population, the more encounters there are likely to be.) A system of two differential equations that incorporates these assumptions is as follows:

W represents the predators.
R represents the prey.

$$\boxed{1} \qquad \frac{dR}{dt} = kR - aRW \qquad\qquad \frac{dW}{dt} = -rW + bRW$$

where k, r, a, and b are positive constants. Notice that the term $-aRW$ decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

The equations in (1) are known as the **predator-prey equations**, or the **Lotka-Volterra equations**. A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predators as functions of time. Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).


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EXAMPLE 1 Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations (1) with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the **equilibrium solutions**) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

SOLUTION

(a) With the given values of k, a, r, and b, the Lotka-Volterra equations become

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both R and W will be constant if both derivatives are 0, that is,

$$R' = R(0.08 - 0.001W) = 0$$

$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by $R = 0$ and $W = 0$. (This makes sense: If there are no rabbits or wolves, the populations are certainly not going to increase.) The other constant solution is

$$W = \frac{0.08}{0.001} = 80$$

$$R = \frac{0.02}{0.00002} = 1000$$

So the equilibrium populations consist of 80 wolves and 1000 rabbits. This means that 1000 rabbits are just enough to support a constant wolf population of 80. There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits).

(b) We use the Chain Rule to eliminate t:

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$$

so

$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(c) If we think of W as a function of R, we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We draw the direction field for this differential equation in Figure 1 and we use it to sketch several solution curves in Figure 2. If we move along a solution curve, we observe how the relationship between R and W changes as time passes. Notice that the curves appear to be closed in the sense that if we travel along a curve, we always return to the same point. Notice also that the point (1000, 80) is inside all the solution curves. That point is called an *equilibrium point* because it corresponds to the equilibrium solution $R = 1000$, $W = 80$.

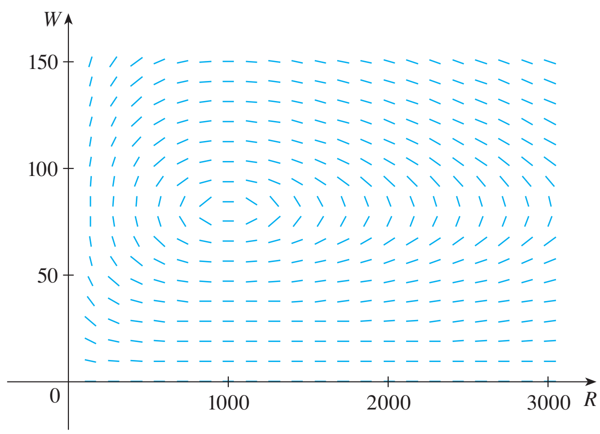


FIGURE 1
Direction field for the predator-prey system

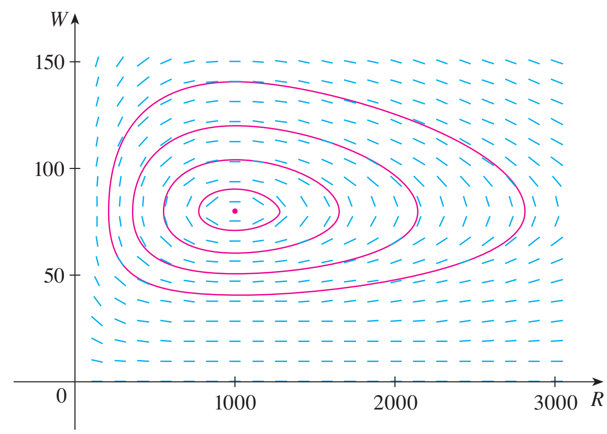


FIGURE 2
Phase portrait of the system

When we represent solutions of a system of differential equations as in Figure 2, we refer to the RW-plane as the **phase plane**, and we call the solution curves **phase trajectories**. So a phase trajectory is a path traced out by solutions (R, W) as time goes by. A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 2.

(d) Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point $P_0(1000, 40)$. Figure 3 shows this phase trajectory with the direction

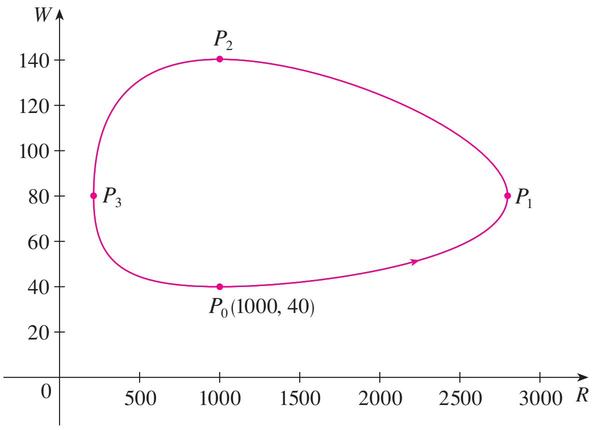


FIGURE 3
Phase trajectory through (1000, 40)

field removed. Starting at the point P_0 at time $t = 0$ and letting t increase, do we move clockwise or counterclockwise around the phase trajectory? If we put $R = 1000$ and $W = 40$ in the first differential equation, we get

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40) = 80 - 40 = 40$$

Since $dR/dt > 0$, we conclude that R is increasing at P_0 and so we move counterclockwise around the phase trajectory.

We see that at P_0 there aren't enough wolves to maintain a balance between the populations, so the rabbit population increases. That results in more wolves and eventually there are so many wolves that the rabbits have a hard time avoiding them. So the number of rabbits begins to decline (at P_1, where we estimate that R reaches its maximum population of about 2800). This means that at some later time the wolf population starts to fall (at P_2, where $R = 1000$ and $W \approx 140$). But this benefits the rabbits, so their population later starts to increase (at P_3, where $W = 80$ and $R \approx 210$). As a consequence, the wolf population eventually starts to increase as well. This happens when the populations return to their initial values of $R = 1000$ and $W = 40$, and the entire cycle begins again.

(e) From the description in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$. Suppose the points P_1, P_2, and P_3 in Figure 3 are reached at times t_1, t_2, and t_3. Then we can sketch graphs of R and W as in Figure 4.

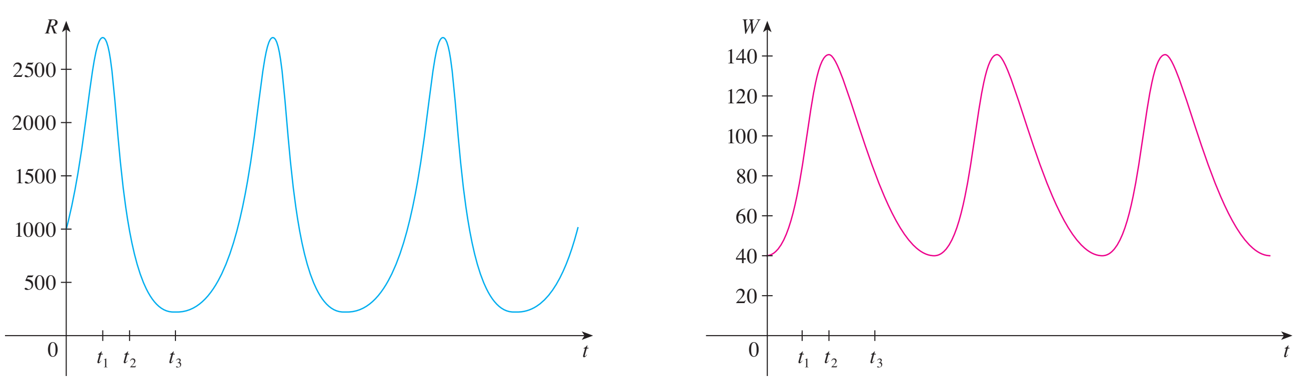


FIGURE 4 Graphs of the rabbit and wolf populations as functions of time

To make the graphs easier to compare, we draw the graphs on the same axes but with different scales for R and W, as in Figure 5 on page 671. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves. ■

An important part of the modeling process, as we discussed in Section 1.2, is to interpret our mathematical conclusions as real-world predictions and to test the predictions against real data. The Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s. Figure 6 shows graphs of the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded by the company over a 90-year period. You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur and the period of these cycles is roughly 10 years.

TEC In Module 9.6 you can change the coefficients in the Lotka-Volterra equations and observe the resulting changes in the phase trajectory and graphs of the rabbit and wolf populations.

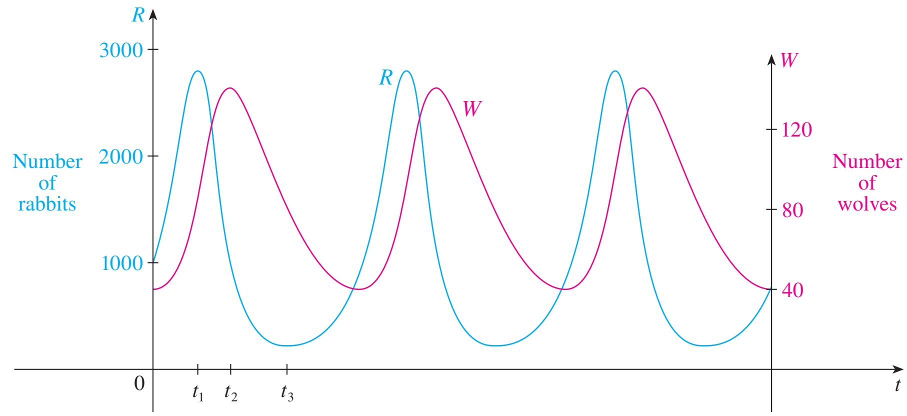


FIGURE 5
Comparison of the rabbit and wolf populations

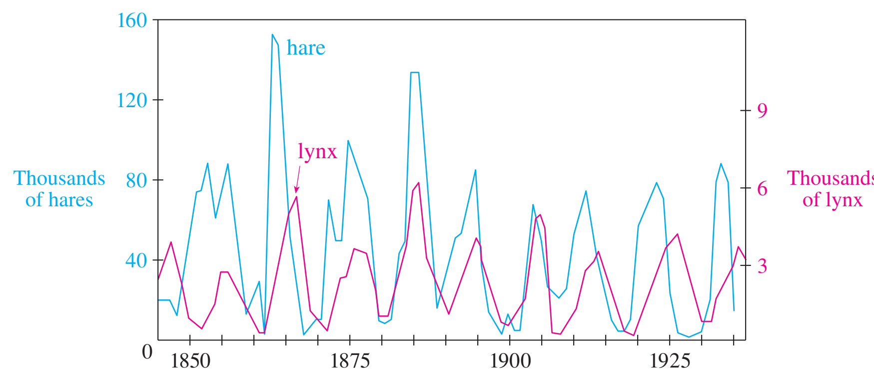


FIGURE 6
Relative abundance of hare and lynx from Hudson's Bay Company records



Jeffrey Lepore / Science Source

Although the relatively simple Lotka-Volterra model has had some success in explaining and predicting coupled populations, more sophisticated models have also been proposed. One way to modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity M. Then the Lotka-Volterra equations (1) are replaced by the system of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{M}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

This model is investigated in Exercises 11 and 12.

Models have also been proposed to describe and predict population levels of two or more species that compete for the same resources or cooperate for mutual benefit. Such models are explored in Exercises 2–4.

9.6 EXERCISES

1. For each predator-prey system, determine which of the variables, x or y, represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Do the predators feed only on the prey or do they have additional food sources? Explain.

(a) $\dfrac{dx}{dt} = -0.05x + 0.0001xy$

$\dfrac{dy}{dt} = 0.1y - 0.005xy$

(b) $\dfrac{dx}{dt} = 0.2x - 0.0002x^2 - 0.006xy$

$\dfrac{dy}{dt} = -0.015y + 0.00008xy$

2. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for instance). Decide whether each system describes competition or cooperation and explain why it is a reasonable

model. (Ask yourself what effect an increase in one species has on the growth rate of the other.)

(a) $\dfrac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$

$\dfrac{dy}{dt} = 0.08x + 0.00004xy$

(b) $\dfrac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$

$\dfrac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$

3. The system of differential equations

$$\frac{dx}{dt} = 0.5x - 0.004x^2 - 0.001xy$$

$$\frac{dy}{dt} = 0.4y - 0.001y^2 - 0.002xy$$

is a model for the populations of two species.

(a) Does the model describe cooperation, or competition, or a predator-prey relationship?

(b) Find the equilibrium solutions and explain their significance.

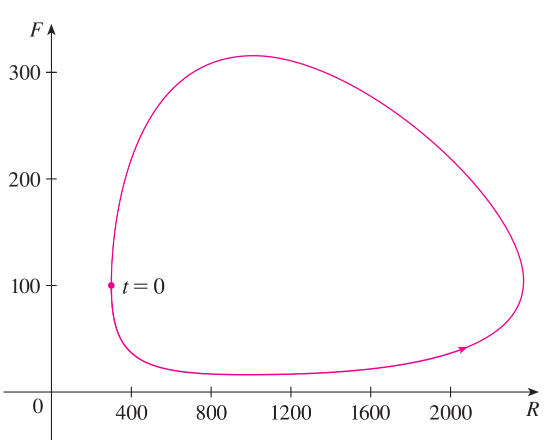
4. Lynx eat snowshoe hares and snowshoe hares eat woody plants like willows. Suppose that, in the absence of hares, the willow population will grow exponentially and the lynx population will decay exponentially. In the absence of lynx and willow, the hare population will decay exponentially. If $L(t)$, $H(t)$, and $W(t)$ represent the populations of these three species at time t, write a system of differential equations as a model for their dynamics. If the constants in your equation are all positive, explain why you have used plus or minus signs.

5–6 A phase trajectory is shown for populations of rabbits (R) and foxes (F).

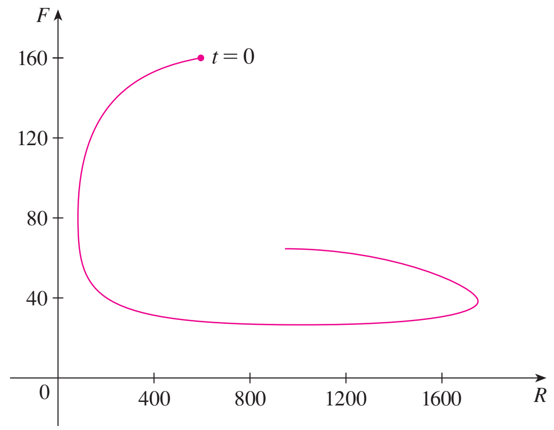
(a) Describe how each population changes as time goes by.

(b) Use your description to make a rough sketch of the graphs of R and F as functions of time.

5.

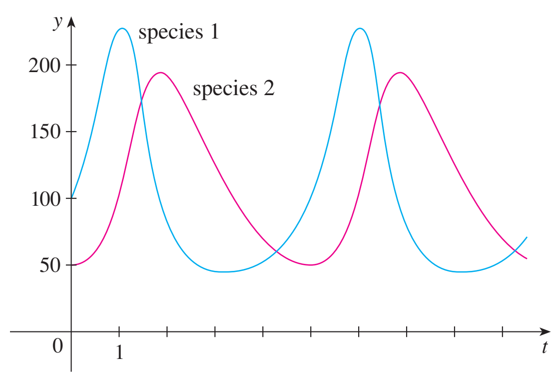


6.

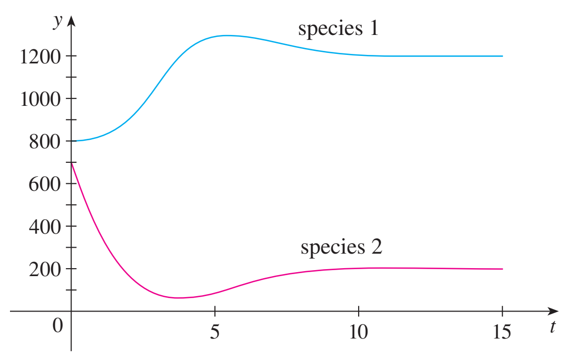


7–8 Graphs of populations of two species are shown. Use them to sketch the corresponding phase trajectory.

7.



8.



9. In Example 1(b) we showed that the rabbit and wolf populations satisfy the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

By solving this separable differential equation, show that

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

where C is a constant.

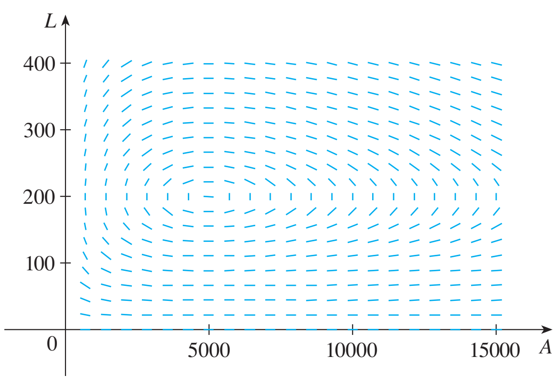
It is impossible to solve this equation for W as an explicit function of R (or vice versa). If you have a computer algebra system that graphs implicitly defined curves, use this equation and your CAS to draw the solution curve that passes through the point (1000, 40) and compare with Figure 3.

10. Populations of aphids and ladybugs are modeled by the equations

$$\frac{dA}{dt} = 2A - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.
(b) Find an expression for dL/dA.
(c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?



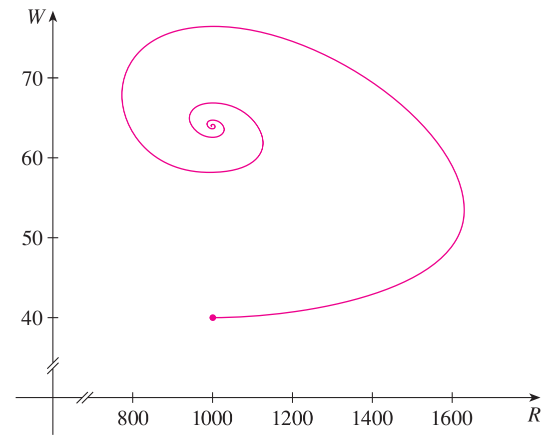
(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

11. In Example 1 we used Lotka-Volterra equations to model populations of rabbits and wolves. Let's modify those equations as follows:

$$\frac{dR}{dt} = 0.08R(1 - 0.0002R) - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

(a) According to these equations, what happens to the rabbit population in the absence of wolves?
(b) Find all the equilibrium solutions and explain their significance.
(c) The figure shows the phase trajectory that starts at the point (1000, 40). Describe what eventually happens to the rabbit and wolf populations.



(d) Sketch graphs of the rabbit and wolf populations as functions of time.

CAS **12.** In Exercise 10 we modeled populations of aphids and ladybugs with a Lotka-Volterra system. Suppose we modify those equations as follows:

$$\frac{dA}{dt} = 2A(1 - 0.0001A) - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) In the absence of ladybugs, what does the model predict about the aphids?
(b) Find the equilibrium solutions.
(c) Find an expression for dL/dA.
(d) Use a computer algebra system to draw a direction field for the differential equation in part (c). Then use the direction field to sketch a phase portrait. What do the phase trajectories have in common?
(e) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
(f) Use part (e) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

9 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a differential equation?
(b) What is the order of a differential equation?
(c) What is an initial condition?

2. What can you say about the solutions of the equation $y' = x^2 + y^2$ just by looking at the differential equation?

3. What is a direction field for the differential equation $y' = F(x, y)$?

4. Explain how Euler's method works.

5. What is a separable differential equation? How do you solve it?

6. What is a first-order linear differential equation? How do you solve it?

7. (a) Write a differential equation that expresses the law of natural growth. What does it say in terms of relative growth rate?
(b) Under what circumstances is this an appropriate model for population growth?
(c) What are the solutions of this equation?

8. (a) Write the logistic differential equation.
(b) Under what circumstances is this an appropriate model for population growth?

9. (a) Write Lotka-Volterra equations to model populations of food-fish (F) and sharks (S).
(b) What do these equations say about each population in the absence of the other?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. All solutions of the differential equation $y' = -1 - y^4$ are decreasing functions.

2. The function $f(x) = (\ln x)/x$ is a solution of the differential equation $x^2y' + xy = 1$.

3. The equation $y' = x + y$ is separable.

4. The equation $y' = 3y - 2x + 6xy - 1$ is separable.

5. The equation $e^x y' = y$ is linear.

6. The equation $y' + xy = e^y$ is linear.

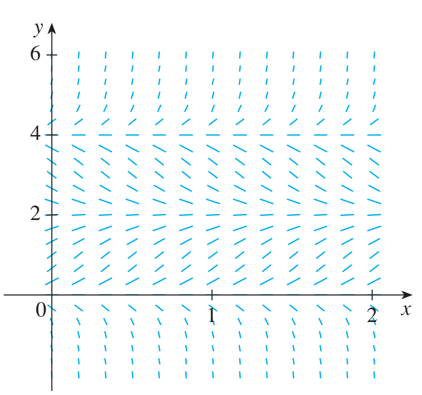
7. If y is the solution of the initial-value problem

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right) \qquad y(0) = 1$$

then $\lim_{t\to\infty} y = 5$.

EXERCISES

1. (a) A direction field for the differential equation $y' = y(y - 2)(y - 4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.
(i) $y(0) = -0.3$ (ii) $y(0) = 1$
(iii) $y(0) = 3$ (iv) $y(0) = 4.3$
(b) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite? What are the equilibrium solutions?



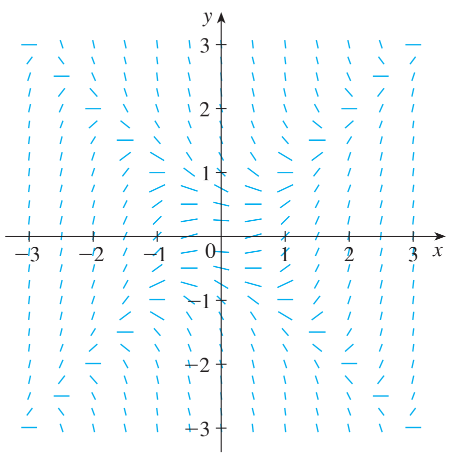
2. (a) Sketch a direction field for the differential equation $y' = x/y$. Then use it to sketch the four solutions that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$, $y(2) = 1$, and $y(-2) = 1$.

(b) Check your work in part (a) by solving the differential equation explicitly. What type of curve is each solution curve?

3. (a) A direction field for the differential equation $y' = x^2 - y^2$ is shown. Sketch the solution of the initial-value problem

$$y' = x^2 - y^2 \qquad y(0) = 1$$

Use your graph to estimate the value of $y(0.3)$.



(b) Use Euler's method with step size 0.1 to estimate $y(0.3)$, where $y(x)$ is the solution of the initial-value problem in part (a). Compare with your estimate from part (a).

(c) On what lines are the centers of the horizontal line segments of the direction field in part (a) located? What happens when a solution curve crosses these lines?

4. (a) Use Euler's method with step size 0.2 to estimate $y(0.4)$, where $y(x)$ is the solution of the initial-value problem

$$y' = 2xy^2 \qquad y(0) = 1$$

(b) Repeat part (a) with step size 0.1.

(c) Find the exact solution of the differential equation and compare the value at 0.4 with the approximations in parts (a) and (b).

5–8 Solve the differential equation.

5. $y' = xe^{-\sin x} - y\cos x$

6. $\dfrac{dx}{dt} = 1 - t + x - tx$

7. $2ye^{y^2}y' = 2x + 3\sqrt{x}$

8. $x^2y' - y = 2x^3e^{-1/x}$

9–11 Solve the initial-value problem.

9. $\dfrac{dr}{dt} + 2tr = r, \quad r(0) = 5$

10. $(1 + \cos x)y' = (1 + e^{-y})\sin x, \quad y(0) = 0$

11. $xy' - y = x\ln x, \quad y(1) = 2$

12. Solve the initial-value problem $y' = 3x^2e^y$, $y(0) = 1$, and graph the solution.

13–14 Find the orthogonal trajectories of the family of curves.

13. $y = ke^x$

14. $y = e^{kx}$

15. (a) Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right) \qquad P(0) = 100$$

and use it to find the population when $t = 20$.

(b) When does the population reach 1200?

16. (a) The population of the world was 6.1 billion in 2000 and 6.9 billion in 2010. Find an exponential model for these data and use the model to predict the world population in the year 2020.

(b) According to the model in part (a), when will the world population exceed 10 billion?

(c) Use the data in part (a) to find a logistic model for the population. Assume a carrying capacity of 20 billion. Then use the logistic model to predict the population in 2020. Compare with your prediction from the exponential model.

(d) According to the logistic model, when will the world population exceed 10 billion? Compare with your prediction in part (b).

17. The von Bertalanffy growth model is used to predict the length $L(t)$ of a fish over a period of time. If L_∞ is the largest length for a species, then the hypothesis is that the rate of growth in length is proportional to $L_\infty - L$, the length yet to be achieved.

(a) Formulate and solve a differential equation to find an expression for $L(t)$.

(b) For the North Sea haddock it has been determined that $L_\infty = 53$ cm, $L(0) = 10$ cm, and the constant of proportionality is 0.2. What does the expression for $L(t)$ become with these data?

18. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

19. One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people

and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How long does it take for 80% of the population to become infected?

20. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

where k is a positive constant. Find R as a function of S.

21. The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k+h}\right)$$

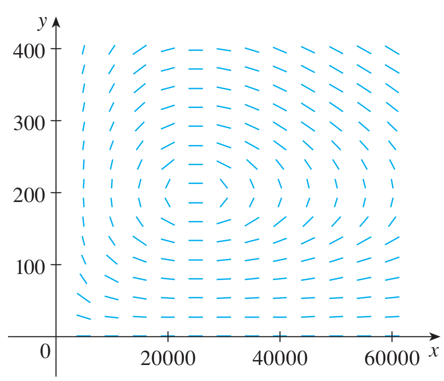
where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation to find a relationship between h and t.

22. Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = 0.4x - 0.002xy$$

$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain.
(b) Find the equilibrium solutions and explain their significance.
(c) Find an expression for dy/dx.
(d) The direction field for the differential equation in part (c) is shown. Use it to sketch the phase trajectory corresponding to initial populations of 100 birds and 40,000 insects. Then use the phase trajectory to describe how both populations change.



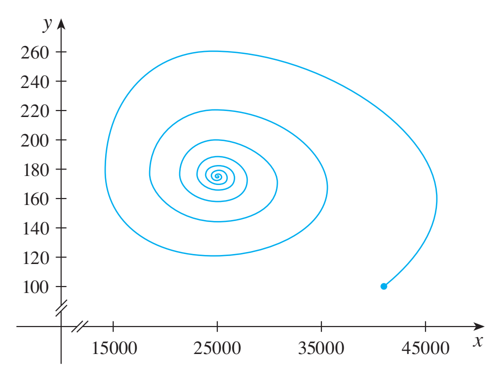
(e) Use part (d) to make rough sketches of the bird and insect populations as functions of time. How are these graphs related to each other?

23. Suppose the model of Exercise 22 is replaced by the equations

$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy$$

$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) According to these equations, what happens to the insect population in the absence of birds?
(b) Find the equilibrium solutions and explain their significance.
(c) The figure shows the phase trajectory that starts with 100 birds and 40,000 insects. Describe what eventually happens to the bird and insect populations.



(d) Sketch graphs of the bird and insect populations as functions of time.

24. Barbara weighs 60 kg and is on a diet of 1600 calories per day, of which 850 are used automatically by basal metabolism. She spends about 15 cal/kg/day times her weight doing exercise. If 1 kg of fat contains 10,000 cal and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

Problems Plus

1. Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\,dt \qquad \text{for all real } x$$

2. A student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, he was lucky and got the correct answer. The function f that he used was $f(x) = e^{x^2}$ and the domain of his problem was the interval $\left(\frac{1}{2}, \infty\right)$. What was the function g?

3. Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.

4. Find all functions f that satisfy the equation

$$\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right) = -1$$

5. Find the curve $y = f(x)$ such that $f(x) \geqslant 0$, $f(0) = 0$, $f(1) = 1$, and the area under the graph of f from 0 to x is proportional to the $(n + 1)$st power of $f(x)$.

6. A *subtangent* is a portion of the x-axis that lies directly beneath the segment of a tangent line from the point of contact to the x-axis. Find the curves that pass through the point $(c, 1)$ and whose subtangents all have length c.

7. A peach pie is taken out of the oven at 5:00 PM. At that time it is piping hot, 100°C. At 5:10 PM its temperature is 80°C; at 5:20 PM it is 65°C. What is the temperature of the room?

8. Snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 PM but only 3 km from 1 PM to 2 PM. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.]

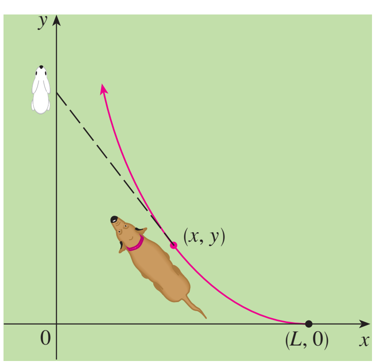


FIGURE FOR PROBLEM 9

9. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.

(iii) The dog runs at the same speed as the rabbit.

(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$. [*Hint:* Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.]

(c) Does the dog ever catch the rabbit?

10. (a) Suppose that the dog in Problem 9 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

11. A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries ore at a rate of $60{,}000\pi$ ft^3/h and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?

(b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of $20{,}000\pi$ ft^3/h when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

12. Find the curve that passes through the point (3, 2) and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

13. Recall that the normal line to a curve at a point P on the curve is the line that passes through P and is perpendicular to the tangent line at P. Find the curve that passes through the point (3, 2) and has the property that if the normal line is drawn at any point on the curve, then the y-intercept of the normal line is always 6.

14. Find all curves with the property that if the normal line is drawn at any point P on the curve, then the part of the normal line between P and the x-axis is bisected by the y-axis.

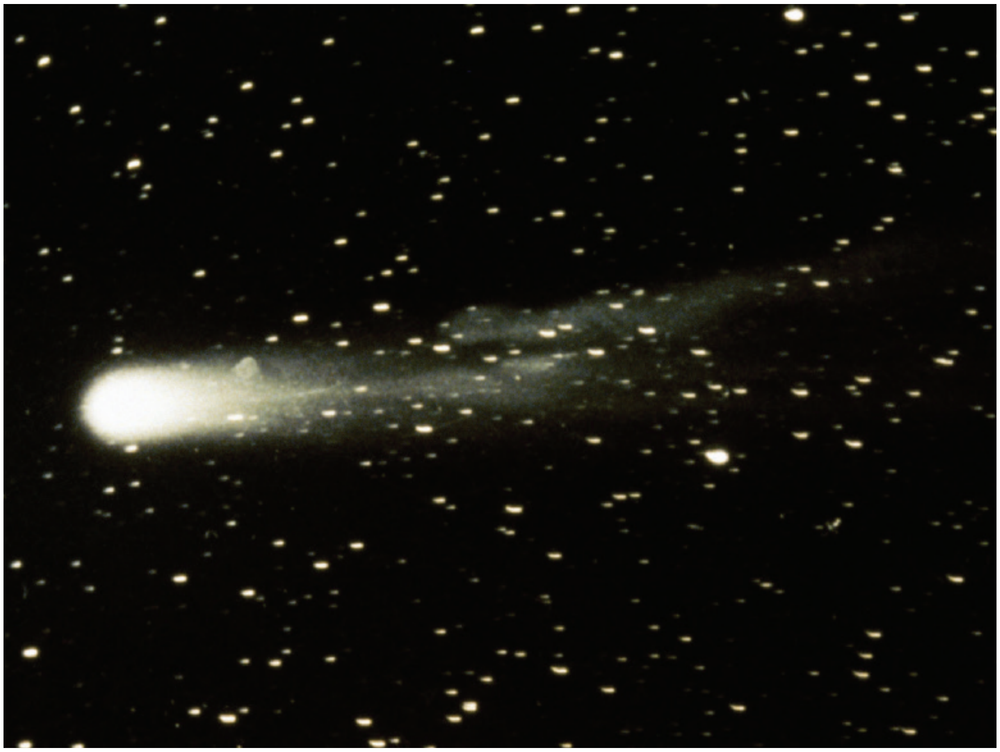
15. Find all curves with the property that if a line is drawn from the origin to any point (x, y) on the curve, and then a tangent is drawn to the curve at that point and extended to meet the x-axis, the result is an isosceles triangle with equal sides meeting at (x, y).

16. (a) An outfielder fields a baseball 280 ft away from home plate and throws it directly to the catcher with an initial velocity of 100 ft/s. Assume that the velocity $v(t)$ of the ball after t seconds satisfies the differential equation $dv/dt = -\frac{1}{10}v$ because of air resistance. How long does it take for the ball to reach home plate? (Ignore any vertical motion of the ball.)

(b) The manager of the team wonders whether the ball will reach home plate sooner if it is relayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 105 ft/s. The manager clocks the relay time of the shortstop (catching, turning, throwing) at half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach home plate? Should the manager encourage a direct throw or a relayed throw? What if the shortstop can throw at 115 ft/s?

(c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?

10 Parametric Equations and Polar Coordinates



The photo shows Halley's comet as it passed Earth in 1986. Due to return in 2061, it was named after Edmond Halley (1656–1742), the English scientist who first recognized its periodicity. In Section 10.6 you will see how polar coordinates provide a convenient equation for the elliptical path of its orbit.

Stocktrek / Stockbyte / Getty Images

SO FAR WE HAVE DESCRIBED plane curves by giving y as a function of x $[y = f(x)]$ or x as a function of y $[x = g(y)]$ or by giving a relation between x and y that defines y implicitly as a function of x $[f(x, y) = 0]$. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both x and y are given in terms of a third variable t called a parameter $[x = f(t), y = g(t)]$. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.

10.1 Curves Defined by Parametric Equations

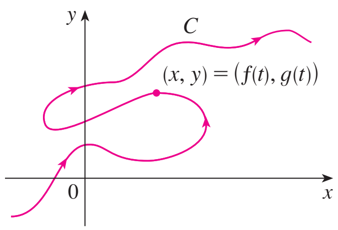


FIGURE 1

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

SOLUTION Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 0$, then $x = 0$, $y = 1$ and so the corresponding point is $(0, 1)$. In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

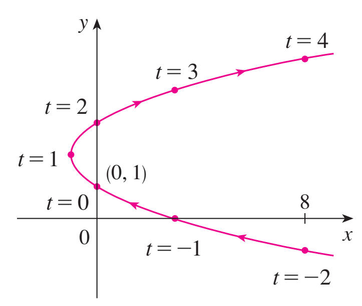


FIGURE 2

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter t as follows. We obtain $t = y - 1$ from the second equation and substitute into the first equation. This gives

$$x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$$

and so the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$. ■

This equation in x and y describes *where* the particle has been, but it doesn't tell us *when* the particle was at a particular point. The parametric equations have an advantage—they tell us *when* the particle was at a point. They also indicate the *direction* of the motion.

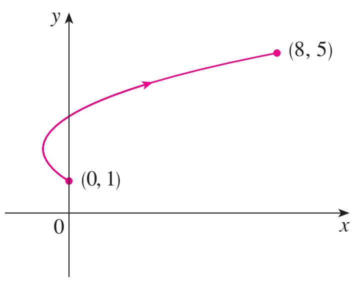


FIGURE 3

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number. But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \leqslant t \leqslant 4$$

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

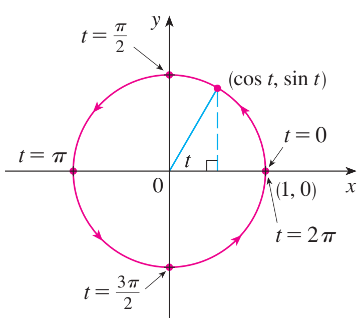


FIGURE 4

EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 4. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point (1, 0). ■

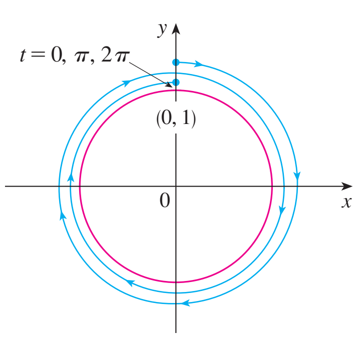


FIGURE 5

EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

SOLUTION Again we have

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at (0, 1) and moves *twice* around the circle in the clockwise direction as indicated in Figure 5. ■

Examples 2 and 3 show that different sets of parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular way.

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

SOLUTION If we take the equations of the unit circle in Example 2 and multiply the expressions for x and y by r, we get $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtain para-

metric equations of the circle (Figure 6) with center (h, k) and radius r:

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$

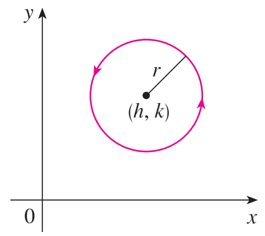


FIGURE 6
$x = h + r\cos t,\ y = k + r\sin t$

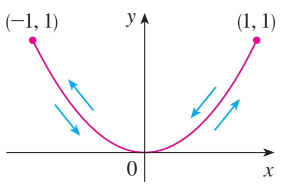


FIGURE 7

EXAMPLE 5 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 7.)

TEC Module 10.1A gives an animation of the relationship between motion along a parametric curve $x = f(t)$, $y = g(t)$ and motion along the graphs of f and g as functions of t. Clicking on TRIG gives you the family of parametric curves

$$x = a\cos bt \qquad y = c\sin dt$$

If you choose $a = b = c = d = 1$ and click on **animate**, you will see how the graphs of $x = \cos t$ and $y = \sin t$ relate to the circle in Example 2. If you choose $a = b = c = 1$, $d = 2$, you will see graphs as in Figure 8. By clicking on **animate** or moving the t-slider to the right, you can see from the color coding how motion along the graphs of $x = \cos t$ and $y = \sin 2t$ corresponds to motion along the parametric curve, which is called a **Lissajous figure**.

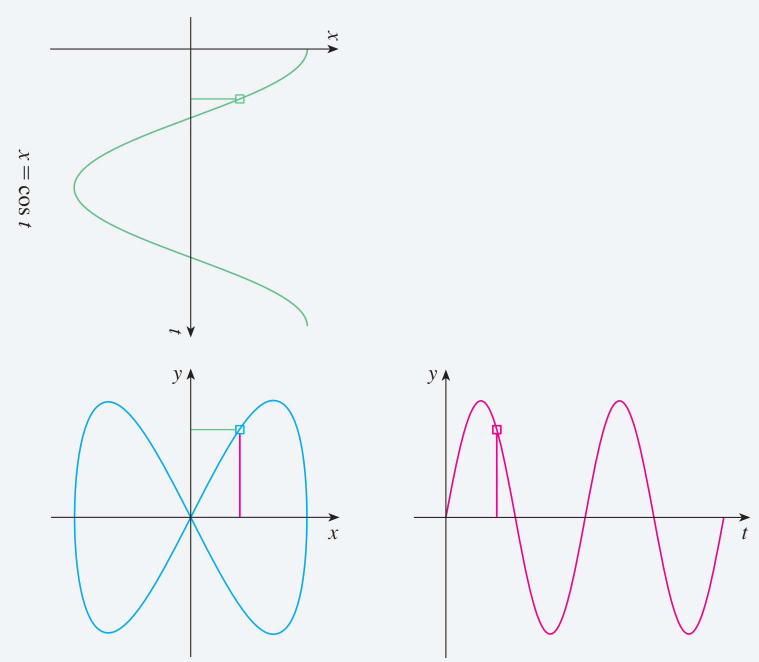


FIGURE 8 $x = \cos t \quad y = \sin 2t$ $\qquad$ $y = \sin 2t$

Graphing Devices

Most graphing calculators and other graphing devices can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

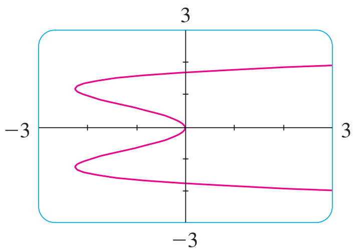


FIGURE 9

EXAMPLE 6 Use a graphing device to graph the curve $x = y^4 - 3y^2$.

SOLUTION If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

Using these parametric equations to graph the curve, we obtain Figure 9. It would be possible to solve the given equation ($x = y^4 - 3y^2$) for y as four functions of x and graph them individually, but the parametric equations provide a much easier method. ■

In general, if we need to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

Notice also that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t \qquad y = f(t)$$

Graphing devices are particularly useful for sketching complicated parametric curves. For instance, the curves shown in Figures 10, 11, and 12 would be virtually impossible to produce by hand.

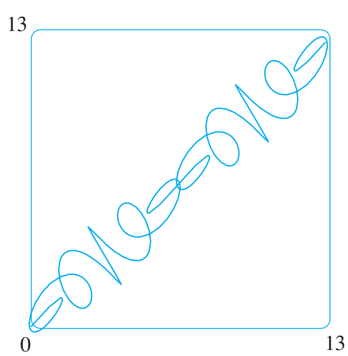


FIGURE 10
$x = t + \sin 5t$
$y = t + \sin 6t$

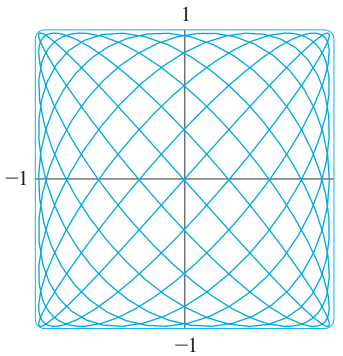


FIGURE 11
$x = \sin 9t$
$y = \sin 10t$

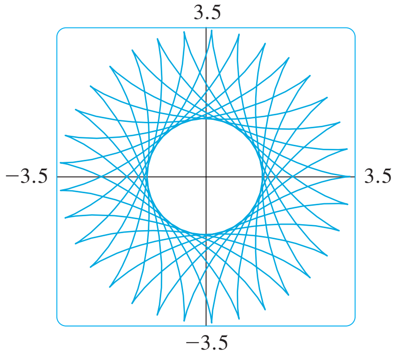


FIGURE 12
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t$

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Laboratory Project after Section 10.2 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in laser printers and in documents viewed electronically.

■ The Cycloid

TEC An animation in Module 10.1B shows how the cycloid is formed as the circle moves.

EXAMPLE 7 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

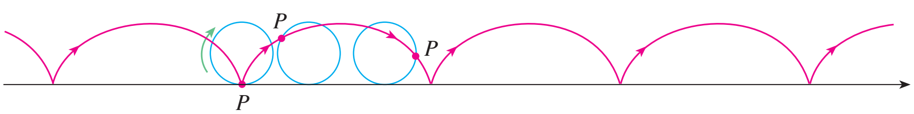


FIGURE 13

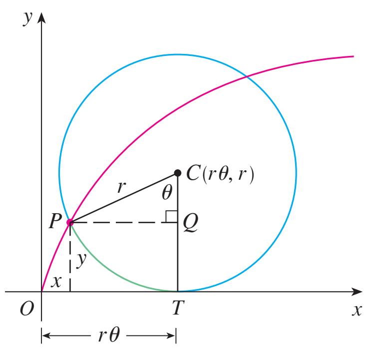


FIGURE 14

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 14 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 14 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

1 $$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$. Although Equations 1 were derived from Figure 14, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ (see Exercise 39).

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated and not as convenient to work with as the parametric equations. ■

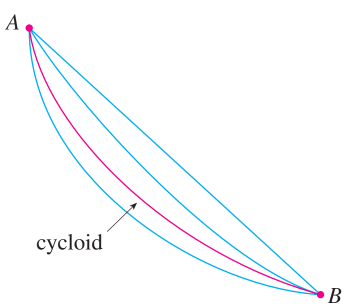


FIGURE 15

One of the first people to study the cycloid was Galileo, who proposed that bridges be built in the shape of cycloids and who tried to find the area under one arch of a cycloid. Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 15, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 16). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

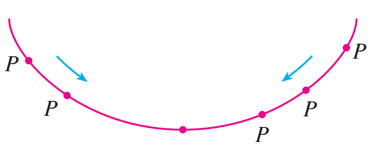


FIGURE 16

Families of Parametric Curves

EXAMPLE 8 Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

SOLUTION We use a graphing device to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 17. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

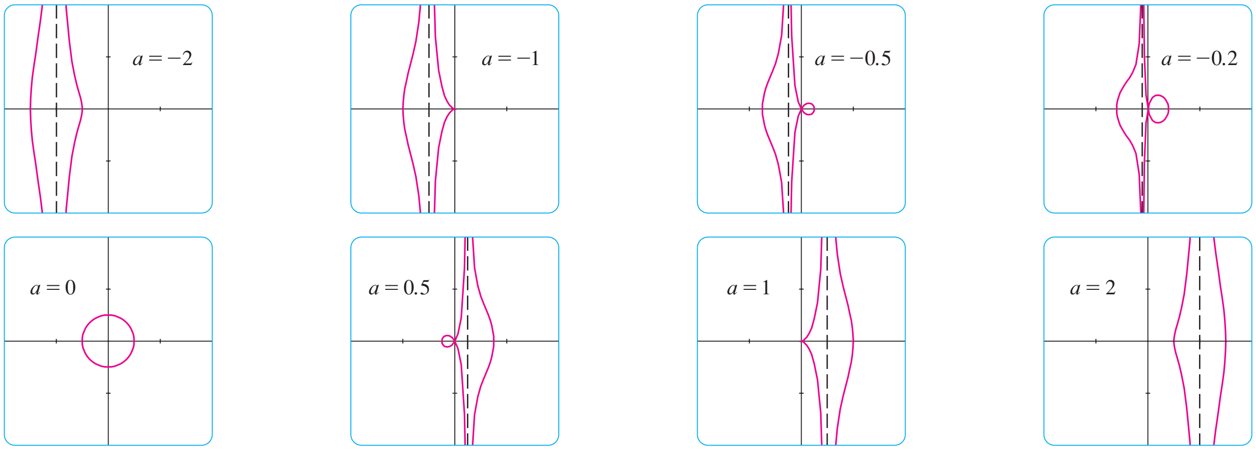


FIGURE 17
Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell. ■

10.1 EXERCISES

1–4 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

1. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \leqslant t \leqslant 2$

2. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \leqslant t \leqslant 2$

3. $x = t + \sin t, \quad y = \cos t, \quad -\pi \leqslant t \leqslant \pi$

4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \leqslant t \leqslant 2$

5–10
(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

5. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

6. $x = 3t + 2, \quad y = 2t + 3$

7. $x = t^2 - 3, \quad y = t + 2, \quad -3 \leqslant t \leqslant 3$

8. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \leqslant t \leqslant 2\pi$

9. $x = \sqrt{t}, \quad y = 1 - t$

10. $x = t^2, \quad y = t^3$

11–18
(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

11. $x = \sin \frac{1}{2}\theta, \quad y = \cos \frac{1}{2}\theta, \quad -\pi \leqslant \theta \leqslant \pi$

12. $x = \frac{1}{2}\cos\theta, \quad y = 2\sin\theta, \quad 0 \leqslant \theta \leqslant \pi$

13. $x = \sin t, \quad y = \csc t, \quad 0 < t < \pi/2$

14. $x = e^t, \quad y = e^{-2t}$

15. $x = t^2, \quad y = \ln t$

16. $x = \sqrt{t + 1}, \quad y = \sqrt{t - 1}$

17. $x = \sinh t, \quad y = \cosh t$

18. $x = \tan^2\theta, \quad y = \sec\theta, \quad -\pi/2 < \theta < \pi/2$

19–22 Describe the motion of a particle with position (x, y) as t varies in the given interval.

19. $x = 5 + 2\cos \pi t, \quad y = 3 + 2\sin \pi t, \quad 1 \leqslant t \leqslant 2$

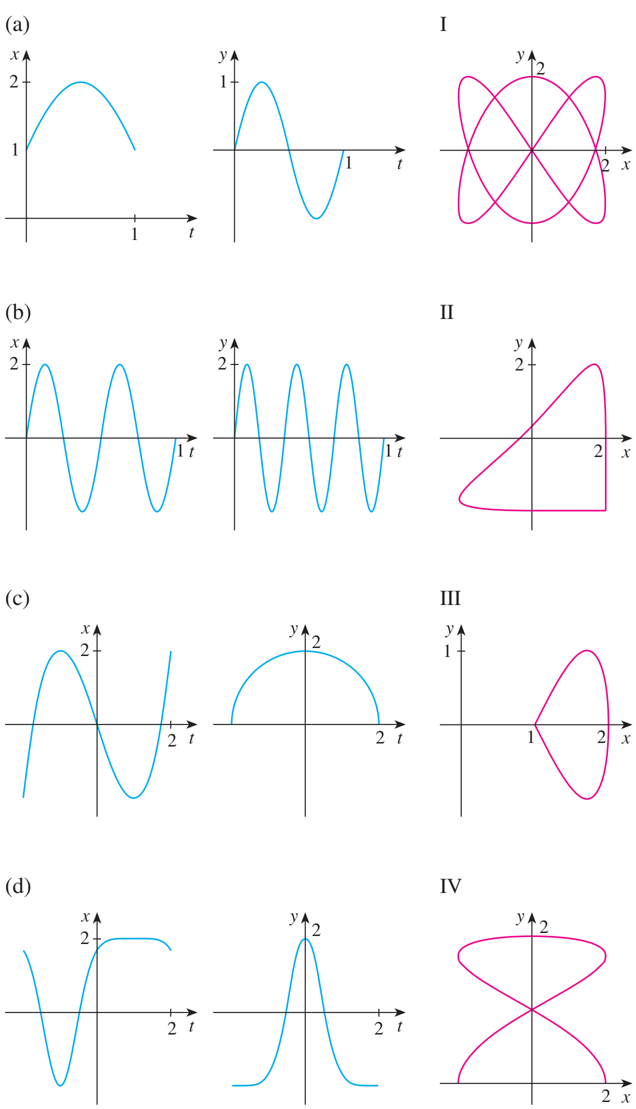
20. $x = 2 + \sin t, \quad y = 1 + 3\cos t, \quad \pi/2 \leqslant t \leqslant 2\pi$

21. $x = 5\sin t, \quad y = 2\cos t, \quad -\pi \leqslant t \leqslant 5\pi$

22. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leqslant t \leqslant 2\pi$

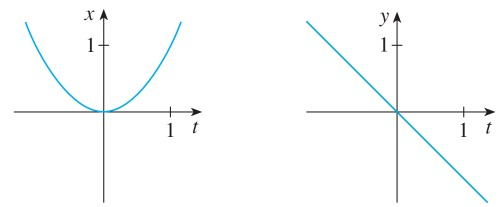
23. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

24. Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the parametric curves labeled I–IV. Give reasons for your choices.

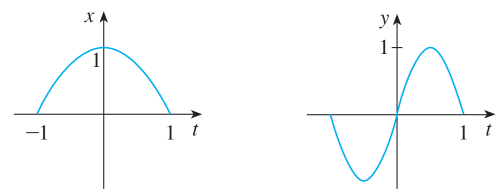


25–27 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

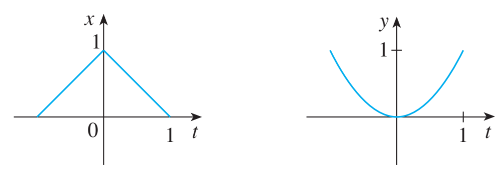
25.



26.



27.



28. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1, \quad y = t^2$

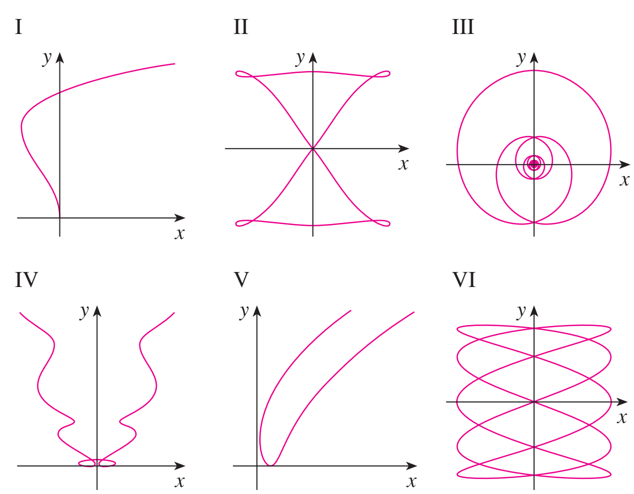
(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$

(d) $x = \cos 5t, \quad y = \sin 2t$

(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$



29. Graph the curve $x = y - 2\sin \pi y$.

30. Graph the curves $y = x^3 - 4x$ and $x = y^3 - 4y$ and find their points of intersection correct to one decimal place.

31. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

where $0 \leqslant t \leqslant 1$, describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

32. Use a graphing device and the result of Exercise 31(a) to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

33. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y - 1)^2 = 4$ in the manner described.

(a) Once around clockwise, starting at $(2, 1)$

(b) Three times around counterclockwise, starting at $(2, 1)$

(c) Halfway around counterclockwise, starting at $(0, 3)$

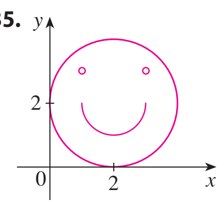
34. (a) Find parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$. [*Hint:* Modify the equations of the circle in Example 2.]

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4,$ and 8.

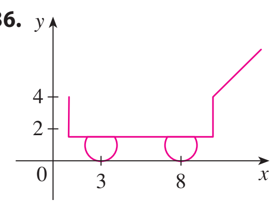
(c) How does the shape of the ellipse change as b varies?

35–36 Use a graphing calculator or computer to reproduce the picture.

35.



36.



37–38 Compare the curves represented by the parametric equations. How do they differ?

37. (a) $x = t^3, \quad y = t^2$ (b) $x = t^6, \quad y = t^4$

(c) $x = e^{-3t}, \quad y = e^{-2t}$

38. (a) $x = t, \quad y = t^{-2}$ (b) $x = \cos t, \quad y = \sec^2 t$

(c) $x = e^t, \quad y = e^{-2t}$

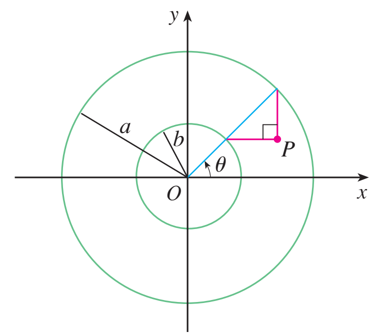
39. Derive Equations 1 for the case $\pi/2 < \theta < \pi$.

40. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid, and assuming the line is the x-axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

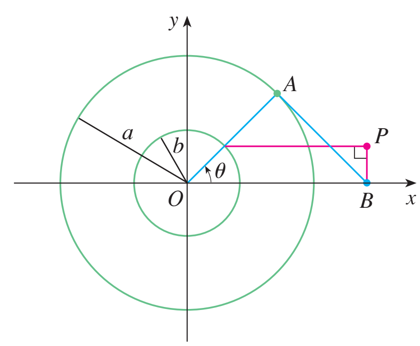
$$x = r\theta - d\sin\theta \qquad y = r - d\cos\theta$$

Sketch the trochoid for the cases $d < r$ and $d > r$.

41. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



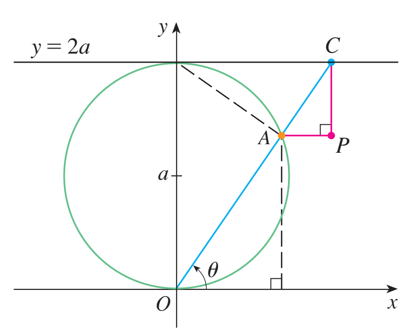
42. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.



43. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that parametric equations for this curve can be written as

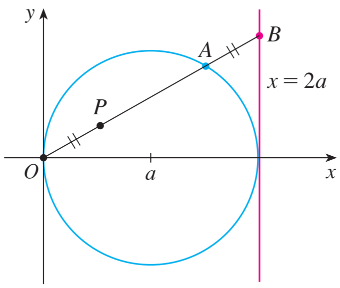
$$x = 2a\cot\theta \qquad y = 2a\sin^2\theta$$

Sketch the curve.



44. (a) Find parametric equations for the set of all points P as shown in the figure such that $|OP| = |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)

(b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



45. Suppose that the position of one particle at time t is given by

$$x_1 = 3 \sin t \qquad y_1 = 2 \cos t \qquad 0 \leqslant t \leqslant 2\pi$$

and the position of a second particle is given by

$$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(a) Graph the paths of both particles. How many points of intersection are there?

(b) Are any of these points of intersection *collision points*? In other words, are the particles ever at the same place at the same time? If so, find the collision points.

(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

46. If a projectile is fired with an initial velocity of v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s^2).

(a) If a gun is fired with $\alpha = 30°$ and $v_0 = 500$ m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?

(b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.

(c) Show that the path is parabolic by eliminating the parameter.

47. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

48. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

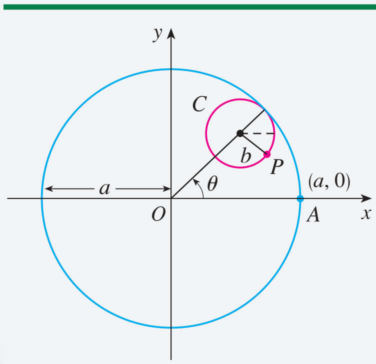
49. Graph several members of the family of curves with parametric equations $x = t + a \cos t$, $y = t + a \sin t$, where $a > 0$. How does the shape change as a increases? For what values of a does the curve have a loop?

50. Graph several members of the family of curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. What features do the curves have in common? What happens as n increases?

51. The curves with equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when a, b, and n vary. (Take n to be a positive integer.)

52. Investigate the family of curves defined by the parametric equations $x = \cos t$, $y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

LABORATORY PROJECT RUNNING CIRCLES AROUND CIRCLES



In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

$$x = (a - b) \cos \theta + b \cos\left(\frac{a - b}{b}\theta\right) \qquad y = (a - b) \sin \theta - b \sin\left(\frac{a - b}{b}\theta\right)$$

2. Use a graphing device (or the interactive graphic in TEC Module 10.1B) to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the

TEC Look at Module 10.1B to see how hypocycloids and epicycloids are formed by the motion of rolling circles.

graph? Show that if we take $a = 4$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

3. Now try $b = 1$ and $a = n/d$, a fraction where n and d have no common factor. First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when $n = d + 1$?

4. What happens if $b = 1$ and a is irrational? Experiment with an irrational number like $\sqrt{2}$ or $e - 2$. Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

10.2 Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, areas, arc length, and surface area.

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

If we think of the curve as being traced out by a moving particle, then dy/dt and dx/dt are the vertical and horizontal velocities of the particle and Formula 1 says that the slope of the tangent is the ratio of these velocities.

1
$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0$$

Equation 1 (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t. We see from (1) that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$). This information is useful for sketching parametric curves.

As we know from Chapter 4, it is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

Note that $\dfrac{d^2y}{dx^2} \neq \dfrac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.
(a) Show that C has two tangents at the point (3, 0) and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

SOLUTION
(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at (3, 0). Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

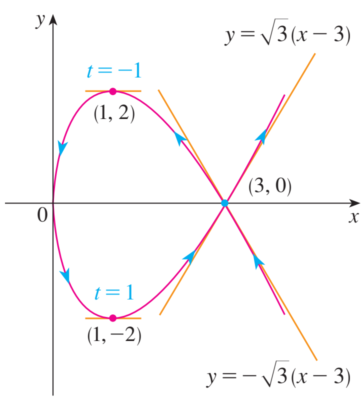


FIGURE 1

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$. C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on C is (0, 0).

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C in Figure 1. ■

EXAMPLE 2
(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$. (See Example 10.1.7.)
(b) At what points is the tangent horizontal? When is it vertical?

SOLUTION
(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \qquad \text{or} \qquad \sqrt{3}\,x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.

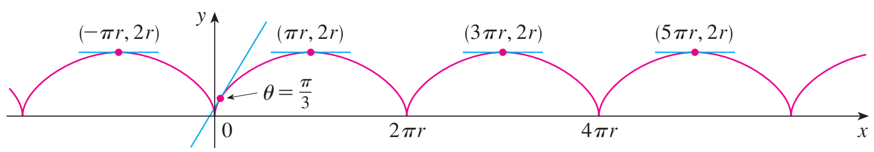


FIGURE 2

(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n - 1)\pi$, n an integer. The corresponding point on the cycloid is $((2n - 1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^+} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta \to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$. ■

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

The limits of integration for t are found as usual with the Substitution Rule. When $x = a$, t is either α or β. When $x = b$, t is the remaining value.

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t) f'(t)\,dt\right]$$

EXAMPLE 3 Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

(See Figure 3.)

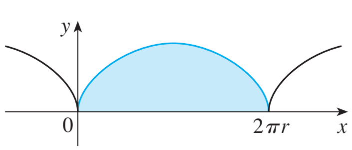


FIGURE 3

SOLUTION One arch of the cycloid is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= r^2 \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= r^2\left(\tfrac{3}{2} \cdot 2\pi\right) = 3\pi r^2 \end{aligned}$$ ■

The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 10.1.7). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

2
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

3
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

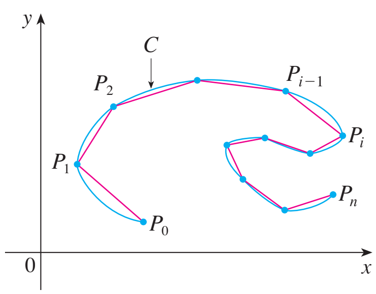


FIGURE 4

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C. (See Figure 4.)

As in Section 8.1, we define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\, \Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\, \Delta t$$

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**})\Delta t]^2}$$

$$= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t$$

and so

4
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t$$

The sum in (4) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in (4) is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

5 Theorem If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$ of Section 8.1.

EXAMPLE 4 If we use the representation of the unit circle given in Example 10.1.2,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 5 gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation given in Example 10.1.3,

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in Theorem 5 gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} 2\, dt = 4\pi$$

Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β. ■

EXAMPLE 5 Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

SOLUTION From Example 3 we see that one arch is described by the parameter interval $0 \leqslant \theta \leqslant 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 5). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

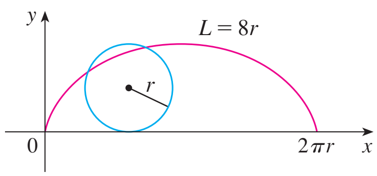


FIGURE 5

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

$$= 2r[2 + 2] = 8r$$ ■

Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. Suppose the curve c given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f', g' are continuous, $g(t) \geqslant 0$, is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by

6 $$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 6 Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leqslant t \leqslant \pi$$

about the x-axis. Therefore, from Formula 6, we get

$$S = \int_0^\pi 2\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= 2\pi\int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi\int_0^\pi r\sin t \cdot r\, dt$$

$$= 2\pi r^2\int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2$$ ■

10.2 EXERCISES

1–2 Find dy/dx.

1. $x = \dfrac{t}{1+t}, \quad y = \sqrt{1+t}$

2. $x = te^t, \quad y = t + \sin t$

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = t^3 + 1, \quad y = t^4 + t; \quad t = -1$

4. $x = \sqrt{t}, \quad y = t^2 - 2t; \quad t = 4$

5. $x = t\cos t, \quad y = t\sin t; \quad t = \pi$

6. $x = e^t \sin \pi t, \quad y = e^{2t}; \quad t = 0$

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. $x = 1 + \ln t, \quad y = t^2 + 2; \quad (1, 3)$

8. $x = 1 + \sqrt{t}, \quad y = e^{t^2}; \quad (2, e)$

9–10 Find an equation of the tangent to the curve at the given point. Then graph the curve and the tangent.

9. $x = t^2 - t, \quad y = t^2 + t + 1; \quad (0, 3)$

10. $x = \sin \pi t, \quad y = t^2 + t; \quad (0, 2)$

11–16 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

11. $x = t^2 + 1, \quad y = t^2 + t$

12. $x = t^3 + 1, \quad y = t^2 - t$

13. $x = e^t, \quad y = te^{-t}$

14. $x = t^2 + 1, \quad y = e^t - 1$

15. $x = t - \ln t, \quad y = t + \ln t$

16. $x = \cos t, \quad y = \sin 2t, \quad 0 < t < \pi$

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = t^3 - 3t, \quad y = t^2 - 3$

18. $x = t^3 - 3t, \quad y = t^3 - 3t^2$

19. $x = \cos \theta, \quad y = \cos 3\theta$

20. $x = e^{\sin\theta}, \quad y = e^{\cos\theta}$

21. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

23–24 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

23. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

24. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

25. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

26. Graph the curve $x = -2\cos t$, $y = \sin t + \sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of θ. (See Exercise 10.1.40.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ in terms of θ. (Astroids are explored in the Laboratory Project on page 689.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

29. At what point(s) on the curve $x = 3t^2 + 1$, $y = t^3 - 1$ does the tangent line have slope $\frac{1}{2}$?

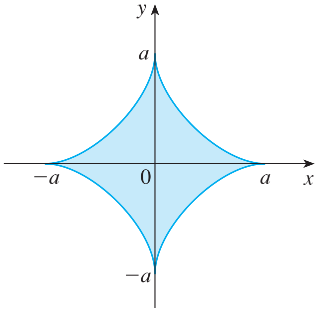
30. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

33. Find the area enclosed by the x-axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.

34. Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Laboratory Project on page 689.)



35. Find the area under one arch of the trochoid of Exercise 10.1.40 for the case $d < r$.

36. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t + e^{-t}, \quad y = t - e^{-t}, \quad 0 \leqslant t \leqslant 2$

38. $x = t^2 - t, \quad y = t^4, \quad 1 \leqslant t \leqslant 4$

39. $x = t - 2 \sin t, \quad y = 1 - 2 \cos t, \quad 0 \leqslant t \leqslant 4\pi$

40. $x = t + \sqrt{t}, \quad y = t - \sqrt{t}, \quad 0 \leqslant t \leqslant 1$

41–44 Find the exact length of the curve.

41. $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

42. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 2$

43. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant 1$

44. $x = 3 \cos t - \cos 3t, \quad y = 3 \sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

45–46 Graph the curve and find its exact length.

45. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leqslant t \leqslant \pi$

46. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

47. Graph the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ and find its length correct to four decimal places.

48. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

49. Use Simpson's Rule with $n = 6$ to estimate the length of the curve $x = t - e^t$, $y = t + e^t$, $-6 \leqslant t \leqslant 6$.

50. In Exercise 10.1.43 you were asked to derive the parametric equations $x = 2a \cot \theta$, $y = 2a \sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

51–52 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

51. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

52. $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

53. Show that the total length of the ellipse $x = a \sin \theta$, $y = b \cos \theta$, $a > b > 0$, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta} \, d\theta$$

where e is the eccentricity of the ellipse $\left(e = c/a, \text{ where } c = \sqrt{a^2 - b^2}\right)$.

54. Find the total length of the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$, where $a > 0$.

CAS **55.** (a) Graph the **epitrochoid** with equations

$$x = 11 \cos t - 4 \cos(11t/2)$$
$$y = 11 \sin t - 4 \sin(11t/2)$$

What parameter interval gives the complete curve?
(b) Use your CAS to find the approximate length of this curve.

CAS **56.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2) \, du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2) \, du$$

where C and S are the Fresnel functions that were introduced in Chapter 4.
(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?
(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

57–60 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

57. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

58. $x = \sin t, \quad y = \sin 2t, \quad 0 \leqslant t \leqslant \pi/2$

59. $x = t + e^t, \quad y = e^{-t}, \quad 0 \leqslant t \leqslant 1$

60. $x = t^2 - t^3, \quad y = t + t^4, \quad 0 \leqslant t \leqslant 1$

61–63 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

61. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

62. $x = 2t^2 + 1/t, \quad y = 8\sqrt{t}, \quad 1 \leqslant t \leqslant 3$

63. $x = a \cos^3\theta, \quad y = a \sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

64. Graph the curve

$$x = 2 \cos \theta - \cos 2\theta \qquad y = 2 \sin \theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the exact area of the resulting surface. (Use your graph to help find the correct parameter interval.)

65–66 Find the surface area generated by rotating the given curve about the y-axis.

65. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

66. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

---

67. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

68. Use Formula 1 to derive Formula 6 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

69. The **curvature** at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.

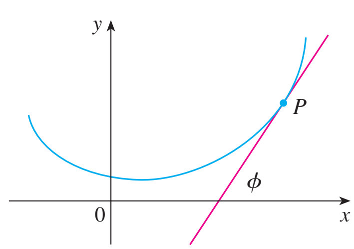
(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$



70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.

(b) At what point does this parabola have maximum curvature?

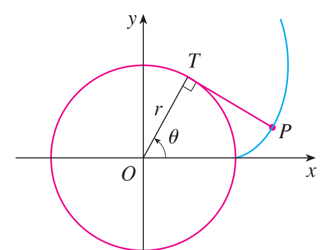
71. Use the formula in Exercise 69(a) to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

72. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.

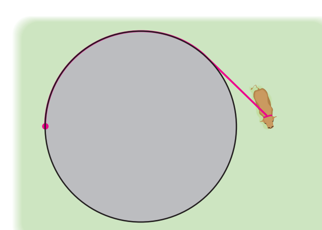
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$



74. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the grazing area available for the cow.



LABORATORY PROJECT BÉZIER CURVES

Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910–1999), who worked in the automotive industry. A cubic Bézier curve is determined by four *control points*, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1 - t)^3 + 3x_1t(1 - t)^2 + 3x_2t^2(1 - t) + x_3t^3$$

$$y = y_0(1 - t)^3 + 3y_1t(1 - t)^2 + 3y_2t^2(1 - t) + y_3t^3$$

where $0 \leqslant t \leqslant 1$. Notice that when $t = 0$ we have $(x, y) = (x_0, y_0)$ and when $t = 1$ we have $(x, y) = (x_3, y_3)$, so the curve starts at P_0 and ends at P_3.

1. Graph the Bézier curve with control points $P_0(4, 1)$, $P_1(28, 48)$, $P_2(50, 42)$, and $P_3(40, 5)$. Then, on the same screen, graph the line segments P_0P_1, P_1P_2, and P_2P_3. (Exercise 10.1.31 shows how to do this.) Notice that the middle control points P_1 and P_2 don't lie on the curve; the curve starts at P_0, heads toward P_1 and P_2 without reaching them, and ends at P_3.

2. From the graph in Problem 1, it appears that the tangent at P_0 passes through P_1 and the tangent at P_3 passes through P_2. Prove it.

3. Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.

4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that gives a reasonable representation of the letter C.

5. More complicated shapes can be represented by piecing together two or more Bézier curves. Suppose the first Bézier curve has control points P_0, P_1, P_2, P_3 and the second one has control points P_3, P_4, P_5, P_6. If we want these two pieces to join together smoothly, then the tangents at P_3 should match and so the points P_2, P_3, and P_4 all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.

10.3 Polar Coordinates

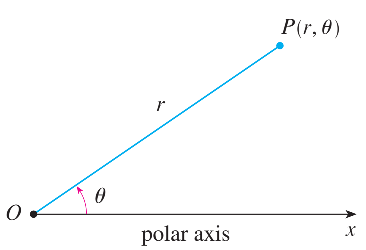


FIGURE 1

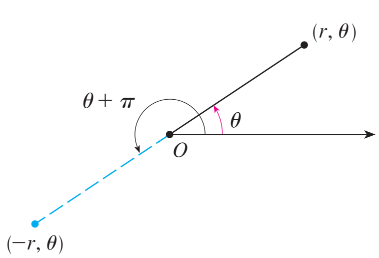


FIGURE 2

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1 Plot the points whose polar coordinates are given.

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

SOLUTION The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.

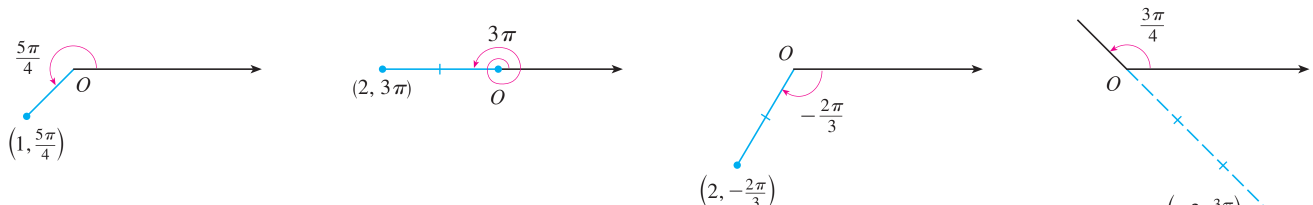


FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

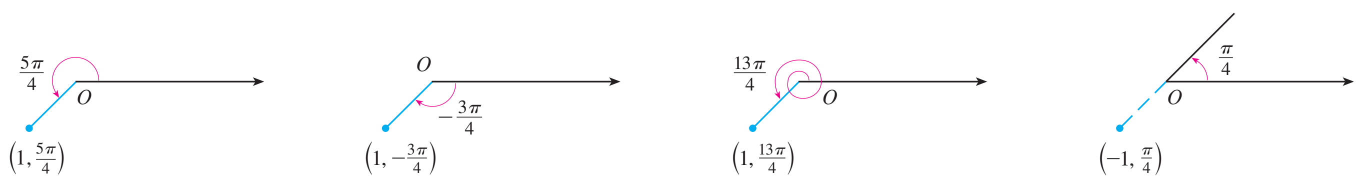


FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \qquad \text{and} \qquad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

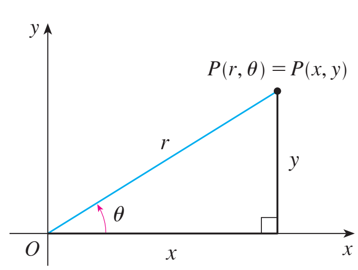


FIGURE 5

$$\cos \theta = \frac{x}{r} \qquad \sin \theta = \frac{y}{r}$$

and so

1
$$x = r \cos \theta \qquad y = r \sin \theta$$

Although Equations 1 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ. (See the general definition of $\sin \theta$ and $\cos \theta$ in Appendix D.)

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

2
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

which can be deduced from Equations 1 or simply read from Figure 5.

EXAMPLE 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

SOLUTION Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the point is $(1, \sqrt{3})$ in Cartesian coordinates. ■

EXAMPLE 3 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

SOLUTION If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$. ■

NOTE Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leqslant \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2. As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

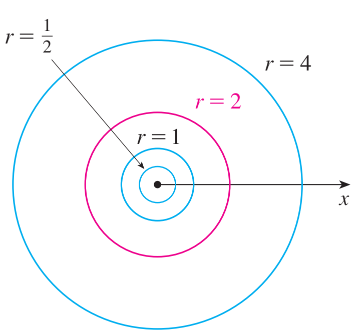


FIGURE 6

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

EXAMPLE 4 What curve is represented by the polar equation $r = 2$?

SOLUTION The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.) ■

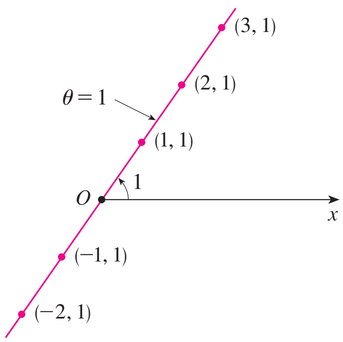


FIGURE 7

EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant. ■

EXAMPLE 6

(a) Sketch the curve with polar equation $r = 2\cos\theta$.

(b) Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, since if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

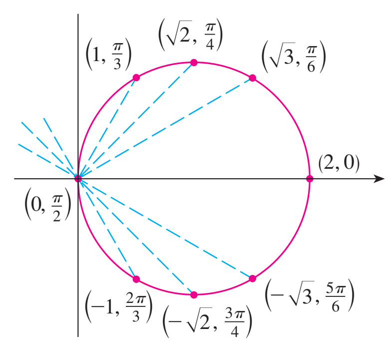


FIGURE 8
Table of values and graph of $r = 2\cos\theta$

(b) To convert the given equation to a Cartesian equation we use Equations 1 and 2. From $x = r\cos\theta$ we have $\cos\theta = x/r$, so the equation $r = 2\cos\theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \qquad \text{or} \qquad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1, 0)$ and radius 1. ■

Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation $r = 2\cos\theta$. The angle OPQ is a right angle (Why?) and so $r/2 = \cos\theta$.

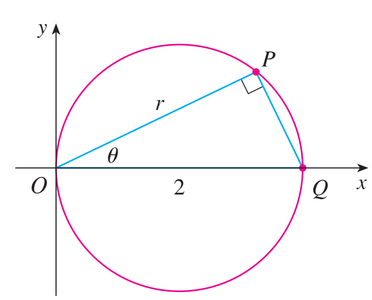


FIGURE 9

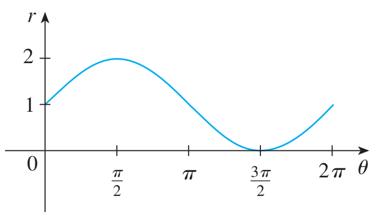


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

EXAMPLE 7 Sketch the curve $r = 1 + \sin\theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

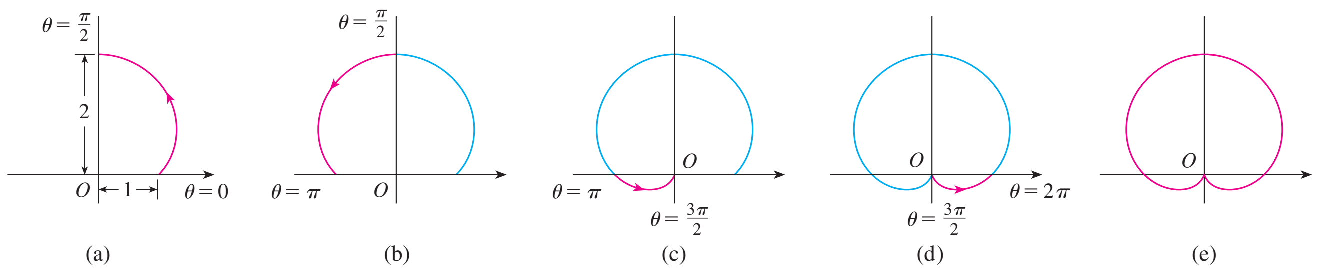


FIGURE 11 Stages in sketching the cardioid $r = 1 + \sin\theta$ ■

EXAMPLE 8 Sketch the curve $r = \cos 2\theta$.

TEC Module 10.3 helps you see how polar curves are traced out by showing animations similar to Figures 10–13.

SOLUTION As in Example 7, we first sketch $r = \cos 2\theta$, $0 \leqslant \theta \leqslant 2\pi$, in Cartesian coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

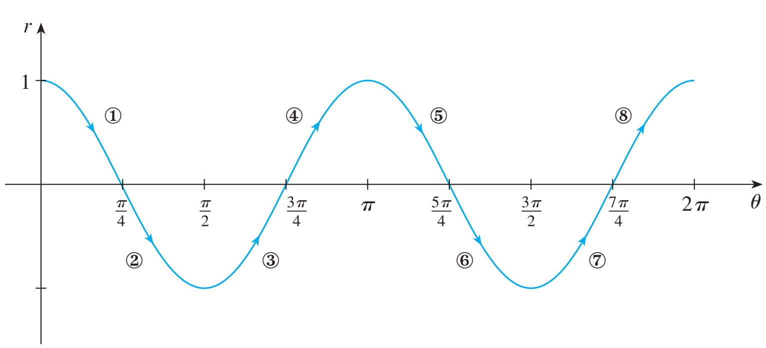


FIGURE 12
$r = \cos 2\theta$ in Cartesian coordinates

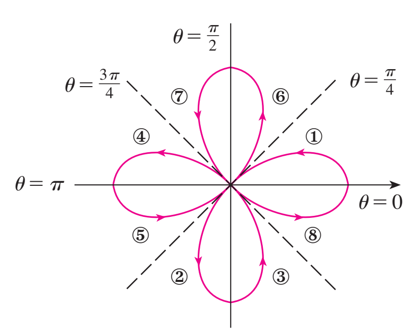


FIGURE 13
Four-leaved rose $r = \cos 2\theta$ ■

Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

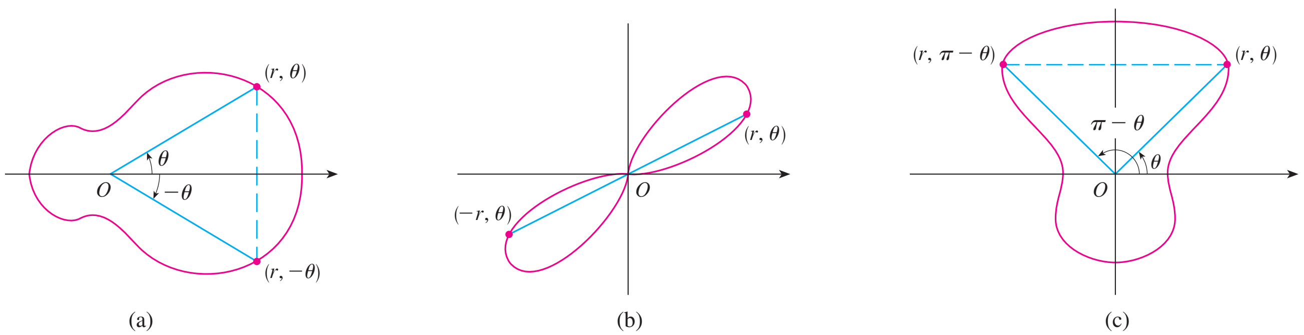


FIGURE 14

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \leqslant \theta \leqslant \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves (Equation 10.2.1) and the Product Rule, we have

3
$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if } \frac{dr}{d\theta} \neq 0$$

For instance, in Example 8 we found that $r = \cos 2\theta = 0$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos 2\theta$ at the origin.

EXAMPLE 9
(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.
(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

SOLUTION Using Equation 3 with $r = 1 + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \pi/2)$, $(\frac{1}{2}, 7\pi/6)$, $(\frac{1}{2}, 11\pi/6)$ and vertical tangents at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry,

$$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

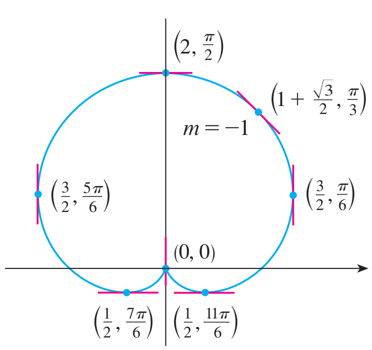


FIGURE 15
Tangent lines for $r = 1 + \sin\theta$

Thus there is a vertical tangent line at the pole (see Figure 15). ■

NOTE Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\,\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

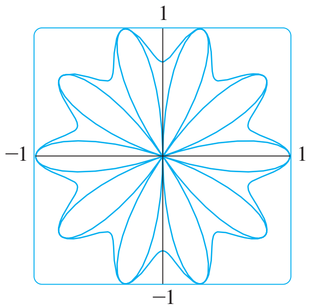


FIGURE 16
$r = \sin^3(2.5\theta) + \cos^3(2.5\theta)$

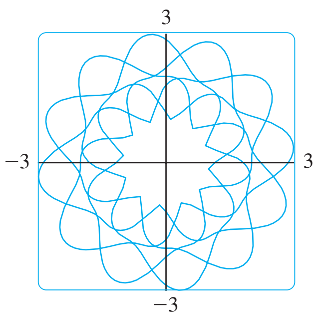


FIGURE 17
$r = 2 + \sin^3(2.4\theta)$

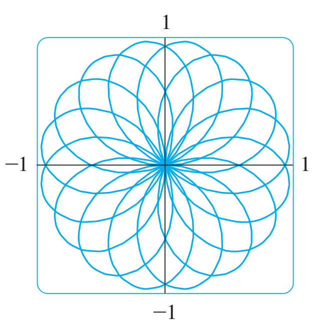


FIGURE 18
$r = \sin(8\theta/5)$

Graphing Polar Curves with Graphing Devices

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Some machines require that the parameter be called t rather than θ.

EXAMPLE 10 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin(8\theta/5)\cos\theta \qquad y = r\sin\theta = \sin(8\theta/5)\sin\theta$$

In any case we need to determine the domain for θ. So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops. ■

EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

SOLUTION Figure 19 on page 706 shows computer-drawn graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c

In Exercise 53 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

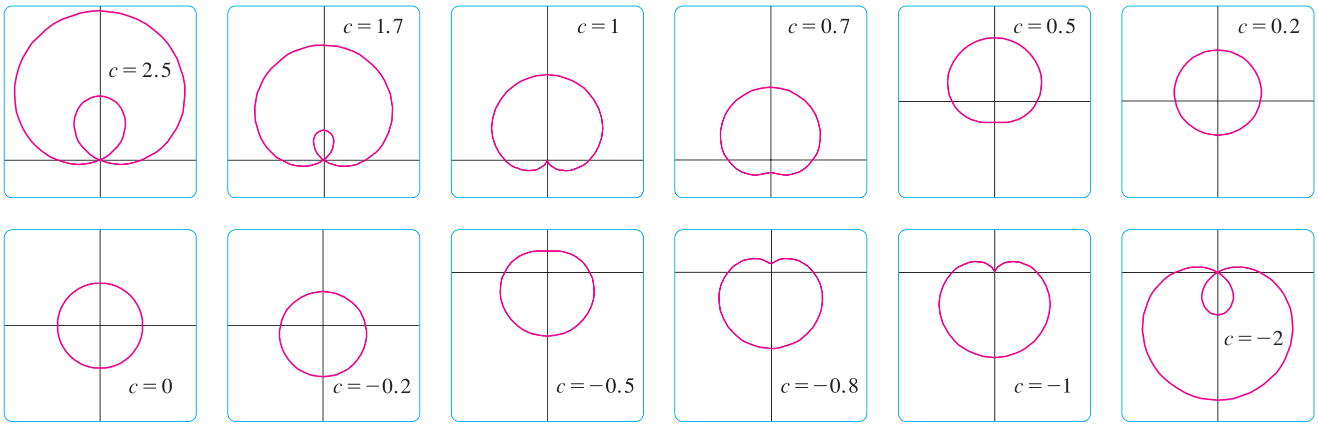


FIGURE 19
Members of the family of limaçons $r = 1 + c \sin\theta$

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. ■

Limaçons arise in the study of planetary motion. In particular, the trajectory of Mars, as viewed from the planet Earth, has been modeled by a limaçon with a loop, as in the parts of Figure 19 with $|c| > 1$.

10.3 EXERCISES

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(1, \pi/4)$ (b) $(-2, 3\pi/2)$ (c) $(3, -\pi/3)$

2. (a) $(2, 5\pi/6)$ (b) $(1, -2\pi/3)$ (c) $(-1, 5\pi/4)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(2, 3\pi/2)$ (b) $(\sqrt{2}, \pi/4)$ (c) $(-1, -\pi/6)$

4. (a) $(4, 4\pi/3)$ (b) $(-2, 3\pi/4)$ (c) $(-3, -\pi/3)$

5–6 The Cartesian coordinates of a point are given.
(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.
(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(-4, 4)$ (b) $(3, 3\sqrt{3})$

6. (a) $(\sqrt{3}, -1)$ (b) $(-6, 0)$

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $r \geqslant 1$

8. $0 \leqslant r < 2, \quad \pi \leqslant \theta \leqslant 3\pi/2$

9. $r \geqslant 0, \quad \pi/4 \leqslant \theta \leqslant 3\pi/4$

10. $1 \leqslant r \leqslant 3, \quad \pi/6 < \theta < 5\pi/6$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $r \geqslant 1, \quad \pi \leqslant \theta \leqslant 2\pi$

13. Find the distance between the points with polar coordinates $(4, 4\pi/3)$ and $(6, 5\pi/3)$.

14. Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

15–20 Identify the curve by finding a Cartesian equation for the curve.

15. $r^2 = 5$

16. $r = 4 \sec\theta$

17. $r = 5 \cos\theta$

18. $\theta = \pi/3$

19. $r^2 \cos 2\theta = 1$

20. $r^2 \sin 2\theta = 1$

21–26 Find a polar equation for the curve represented by the given Cartesian equation.

21. $y = 2$

22. $y = x$

23. $y = 1 + 3x$

24. $4y^2 = x$

25. $x^2 + y^2 = 2cx$

26. $x^2 - y^2 = 4$

27–28 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

27. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point (3, 3)

28. (a) A circle with radius 5 and center (2, 3)
(b) A circle centered at the origin with radius 4

29–46 Sketch the curve with the given polar equation by first sketching the graph of r as a function of θ in Cartesian coordinates.

29. $r = -2\sin\theta$

30. $r = 1 - \cos\theta$

31. $r = 2(1 + \cos\theta)$

32. $r = 1 + 2\cos\theta$

33. $r = \theta,\ \theta \geqslant 0$

34. $r = \theta^2,\ -2\pi \leqslant \theta \leqslant 2\pi$

35. $r = 3\cos 3\theta$

36. $r = -\sin 5\theta$

37. $r = 2\cos 4\theta$

38. $r = 2\sin 6\theta$

39. $r = 1 + 3\cos\theta$

40. $r = 1 + 5\sin\theta$

41. $r^2 = 9\sin 2\theta$

42. $r^2 = \cos 4\theta$

43. $r = 2 + \sin 3\theta$

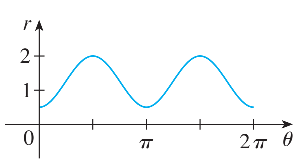
44. $r^2\theta = 1$

45. $r = \sin(\theta/2)$

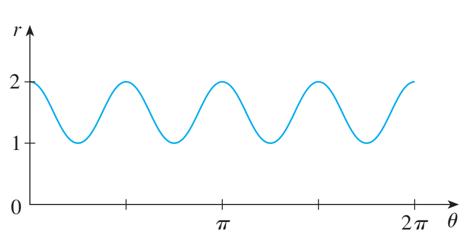
46. $r = \cos(\theta/3)$

47–48 The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

47.



48.



---

49. Show that the polar curve $r = 4 + 2\sec\theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r\to\pm\infty} x = 2$. Use this fact to help sketch the conchoid.

50. Show that the curve $r = 2 - \csc\theta$ (also a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r\to\pm\infty} y = -1$. Use this fact to help sketch the conchoid.

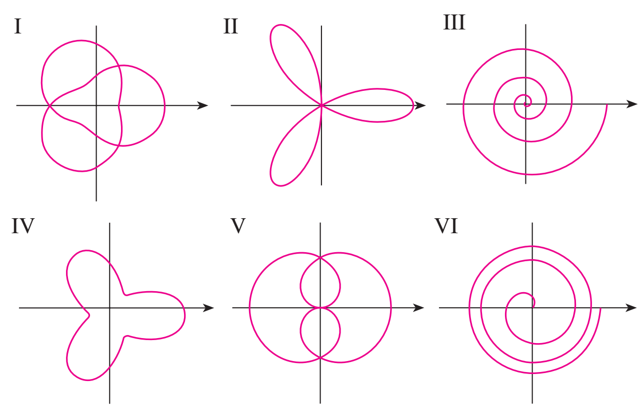
51. Show that the curve $r = \sin\theta\ \tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

52. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

53. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 19 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

54. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \ln\theta,\quad 1 \leqslant \theta \leqslant 6\pi$ (b) $r = \theta^2,\quad 0 \leqslant \theta \leqslant 8\pi$
(c) $r = \cos 3\theta$ (d) $r = 2 + \cos 3\theta$
(e) $r = \cos(\theta/2)$ (f) $r = 2 + \cos(3\theta/2)$



55–60 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

55. $r = 2\cos\theta,\quad \theta = \pi/3$

56. $r = 2 + \sin 3\theta,\quad \theta = \pi/4$

57. $r = 1/\theta,\quad \theta = \pi$

58. $r = \cos(\theta/3),\quad \theta = \pi$

59. $r = \cos 2\theta,\quad \theta = \pi/4$

60. $r = 1 + 2\cos\theta,\quad \theta = \pi/3$

61–64 Find the points on the given curve where the tangent line is horizontal or vertical.

61. $r = 3\cos\theta$

62. $r = 1 - \sin\theta$

63. $r = 1 + \cos\theta$

64. $r = e^\theta$

65. Show that the polar equation $r = a \sin\theta + b \cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

66. Show that the curves $r = a \sin\theta$ and $r = a \cos\theta$ intersect at right angles.

67–72 Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

67. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

68. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

69. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

70. $r = |\tan\theta|^{|\cot\theta|}$ (valentine curve)

71. $r = 1 + \cos^{999}\theta$ (Pac-Man curve)

72. $r = 2 + \cos(9\theta/4)$

73. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

74. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

75. Investigate the family of curves with polar equations $r = 1 + c\cos\theta$, where c is a real number. How does the shape change as c changes?

76. Investigate the family of polar curves

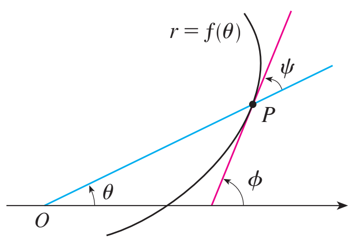
$$r = 1 + \cos^n\theta$$

where n is a positive integer. How does the shape change as n increases? What happens as n becomes large? Explain the shape for large n by considering the graph of r as a function of θ in Cartesian coordinates.

77. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



78. (a) Use Exercise 77 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^\theta$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

LABORATORY PROJECT FAMILIES OF POLAR CURVES

In this project you will discover the interesting and beautiful shapes that members of families of polar curves can take. You will also see how the shape of the curve changes when you vary the constants.

1. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?

(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

2. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

3. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

4. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

$$r^4 - 2c^2r^2 \cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

10.4 Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

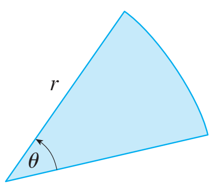


FIGURE 1

1 $$A = \tfrac{1}{2}r^2\theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$. (See also Exercise 7.3.35.)

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leqslant 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

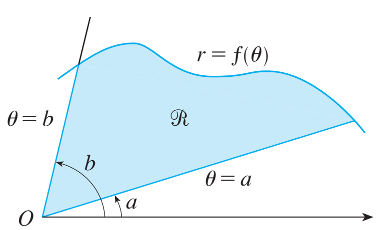


FIGURE 2

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2 \,\Delta\theta$$

and so an approximation to the total area A of $\mathcal{R}$ is

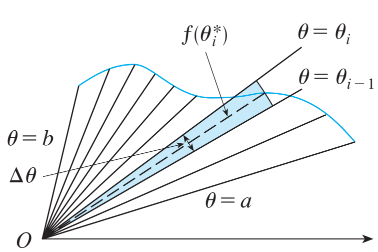


FIGURE 3

2 $$A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \,\Delta\theta$$

It appears from Figure 3 that the approximation in (2) improves as $n \to \infty$. But the sums in (2) are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2 \,\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2 \,d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathcal{R}$ is

3 $$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

Formula 3 is often written as

4 $$A = \int_a^b \tfrac{1}{2}r^2\,d\theta$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4, it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

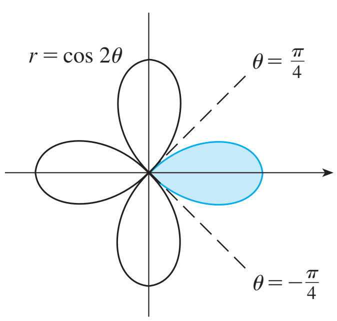


FIGURE 4

EXAMPLE 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION The curve $r = \cos 2\theta$ was sketched in Example 10.3.8. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\,d\theta = \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$ ■

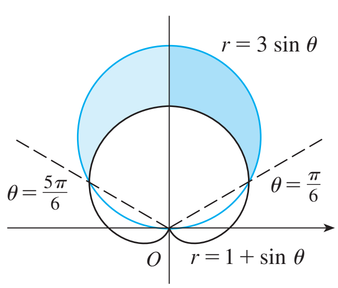


FIGURE 5

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid (see Example 10.3.7) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$, which gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\,d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2\,d\theta$$

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$A = 2\left[\tfrac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta - \tfrac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\,d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right]$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} = \pi$$ ■

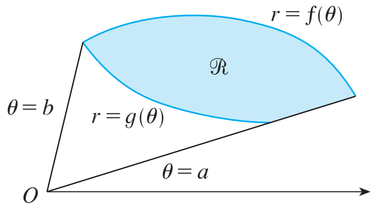


FIGURE 6

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let $\mathcal{R}$ be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ and $0 < b - a \leqslant 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \tfrac{1}{2}[g(\theta)]^2\,d\theta$$

$$= \tfrac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

CAUTION The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$ and found only two such points, $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3\sin\theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin\theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

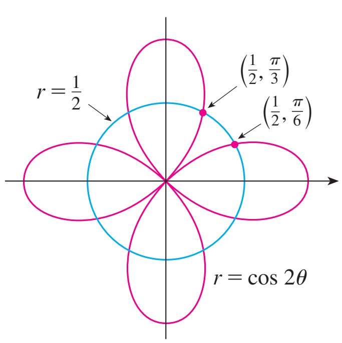


FIGURE 7

EXAMPLE 3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

SOLUTION If we solve the equations $r = \cos 2\theta$ and $r = \frac{1}{2}$, we get $\cos 2\theta = \frac{1}{2}$ and, therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. Thus the values of θ between 0 and 2π that satisfy both equations are $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$. We have found four points of intersection: $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$, $\left(\frac{1}{2}, 7\pi/6\right)$, and $\left(\frac{1}{2}, 11\pi/6\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\frac{1}{2}, \pi/3\right)$, $\left(\frac{1}{2}, 2\pi/3\right)$, $\left(\frac{1}{2}, 4\pi/3\right)$, and $\left(\frac{1}{2}, 5\pi/3\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\frac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\frac{1}{2}$. ■

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, we can use Theorem 10.2.5 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

5
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

EXAMPLE 4 Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 10.3.7.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so Formula 5 gives

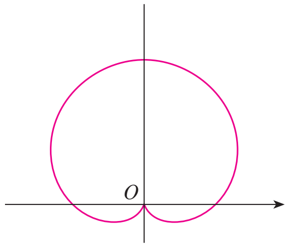


FIGURE 8
$r = 1 + \sin\theta$

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta}\,d\theta = \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\,d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$. ■

10.4 EXERCISES

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = e^{-\theta/4}$, $\pi/2 \leqslant \theta \leqslant \pi$

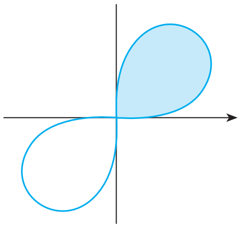
2. $r = \cos\theta$, $0 \leqslant \theta \leqslant \pi/6$

3. $r = \sin\theta + \cos\theta$, $0 \leqslant \theta \leqslant \pi$

4. $r = 1/\theta$, $\pi/2 \leqslant \theta \leqslant 2\pi$

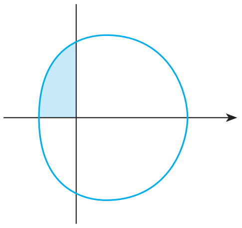
5–8 Find the area of the shaded region.

5.



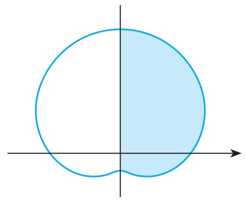
$r^2 = \sin 2\theta$

6.



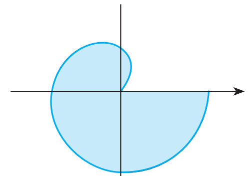
$r = 2 + \cos\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sqrt{\ln\theta},\ 1 \leqslant \theta \leqslant 2\pi$

9–12 Sketch the curve and find the area that it encloses.

9. $r = 2\sin\theta$

10. $r = 1 - \sin\theta$

11. $r = 3 + 2\cos\theta$

12. $r = 2 - \cos\theta$

13–16 Graph the curve and find the area that it encloses.

13. $r = 2 + \sin 4\theta$

14. $r = 3 - 2\cos 4\theta$

15. $r = \sqrt{1 + \cos^2(5\theta)}$

16. $r = 1 + 5\sin 6\theta$

17–21 Find the area of the region enclosed by one loop of the curve.

17. $r = 4\cos 3\theta$

18. $r^2 = 4\cos 2\theta$

19. $r = \sin 4\theta$

20. $r = 2\sin 5\theta$

21. $r = 1 + 2\sin\theta$ (inner loop)

22. Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 4\sin\theta, \quad r = 2$

24. $r = 1 - \sin\theta, \quad r = 1$

25. $r^2 = 8\cos 2\theta, \quad r = 2$

26. $r = 1 + \cos\theta, \quad r = 2 - \cos\theta$

27. $r = 3\cos\theta, \quad r = 1 + \cos\theta$

28. $r = 3\sin\theta, \quad r = 2 - \sin\theta$

29–34 Find the area of the region that lies inside both curves.

29. $r = 3\sin\theta, \quad r = 3\cos\theta$

30. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

31. $r = \sin 2\theta, \quad r = \cos 2\theta$

32. $r = 3 + 2\cos\theta, \quad r = 3 + 2\sin\theta$

33. $r^2 = 2\sin 2\theta, \quad r = 1$

34. $r = a\sin\theta, \quad r = b\cos\theta, \quad a > 0,\ b > 0$

35. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

37. $r = \sin\theta, \quad r = 1 - \sin\theta$

38. $r = 1 + \cos\theta, \quad r = 1 - \sin\theta$

39. $r = 2\sin 2\theta, \quad r = 1$

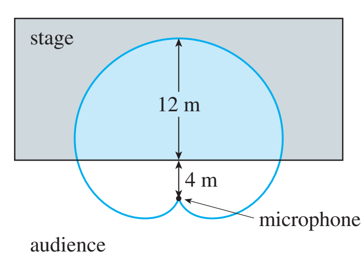
40. $r = \cos 3\theta, \quad r = \sin 3\theta$

41. $r = \sin\theta, \quad r = \sin 2\theta$

42. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

43. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of θ at which they intersect. Then use these values to estimate the area that lies inside both curves.

44. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



45–48 Find the exact length of the polar curve.

45. $r = 2\cos\theta, \quad 0 \leqslant \theta \leqslant \pi$

46. $r = 5^\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

47. $r = \theta^2, \quad 0 \leqslant \theta \leqslant 2\pi$

48. $r = 2(1 + \cos\theta)$

49–50 Find the exact length of the curve. Use a graph to determine the parameter interval.

49. $r = \cos^4(\theta/4)$

50. $r = \cos^2(\theta/2)$

51–54 Use a calculator to find the length of the curve correct to four decimal places. If necessary, graph the curve to determine the parameter interval.

51. One loop of the curve $r = \cos 2\theta$

52. $r = \tan\theta, \quad \pi/6 \leqslant \theta \leqslant \pi/3$

53. $r = \sin(6 \sin\theta)$

54. $r = \sin(\theta/4)$

55. (a) Use Formula 10.2.6 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where f' is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

56. (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where f' is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.

(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

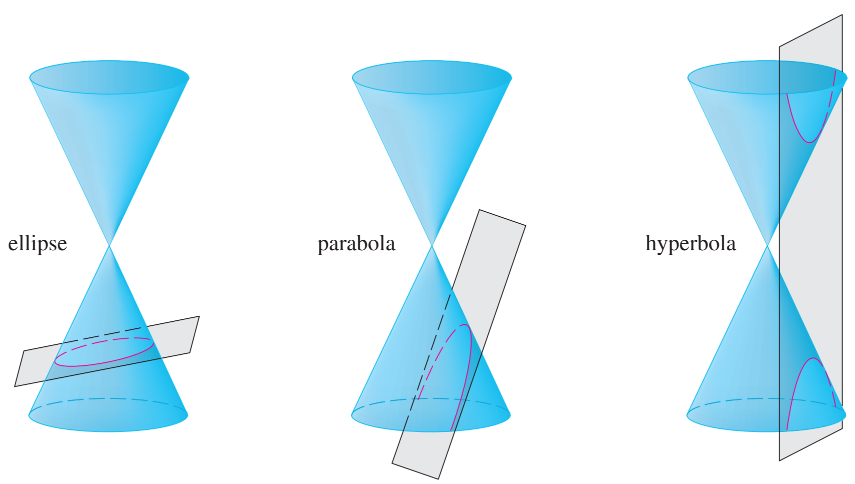


FIGURE 1
Conics

Parabolas

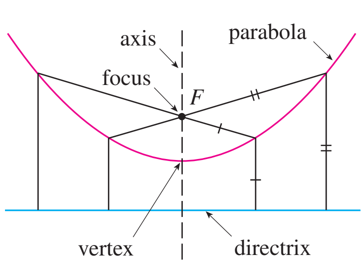


FIGURE 2

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 18 on page 202 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola,

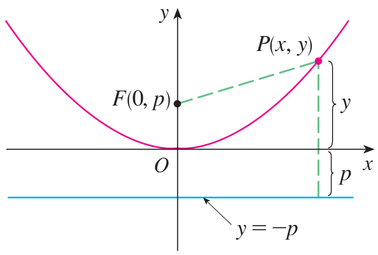


FIGURE 3

then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

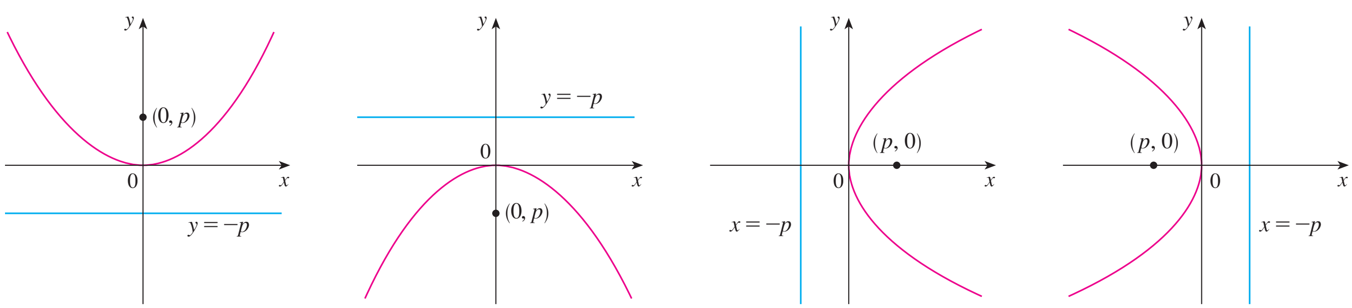
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

1 An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If we write $a = 1/(4p)$, then the standard equation of a parabola (1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because (1) is unchanged when x is replaced by $-x$.



(a) $x^2 = 4py,\ p > 0$ (b) $x^2 = 4py,\ p < 0$ (c) $y^2 = 4px,\ p > 0$ (d) $y^2 = 4px,\ p < 0$

FIGURE 4

If we interchange x and y in (1), we obtain

2 $$y^2 = 4px$$

which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the x-axis, which is the axis of the parabola.

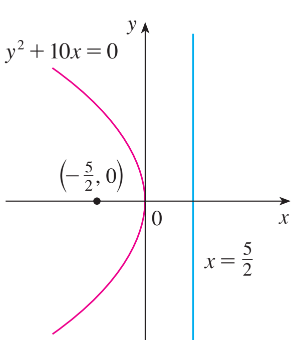


FIGURE 5

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

SOLUTION If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = \left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5. ■

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

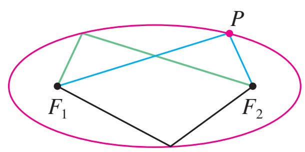


FIGURE 6

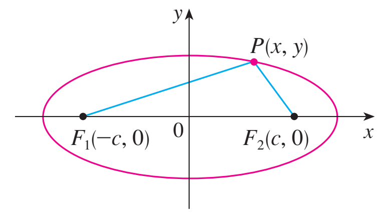


FIGURE 7

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is,
$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or
$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to
$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From triangle F_1F_2P in Figure 7 we can see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

3
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

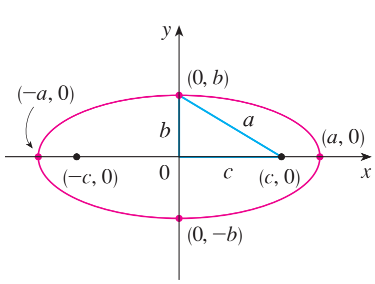


FIGURE 8
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a \geqslant b$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. The line segment joining $(0, b)$ and $(0, -b)$ is the **minor axis**. Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

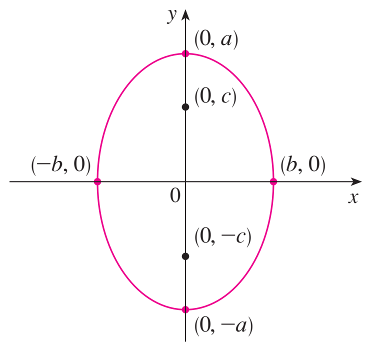


FIGURE 9
$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1,\ a \geqslant b$

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (4). (See Figure 9.)

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

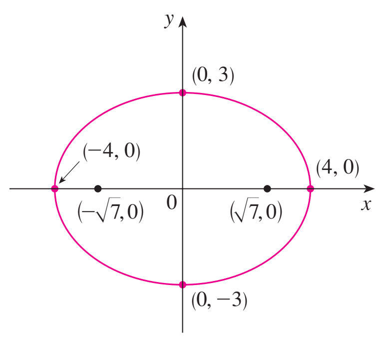


FIGURE 10
$9x^2 + 16y^2 = 144$

EXAMPLE 2 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $\left(\pm\sqrt{7}, 0\right)$. The graph is sketched in Figure 10. ■

EXAMPLE 3 Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

SOLUTION Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$. ■

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 65). This principle is used in *lithotripsy,* a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

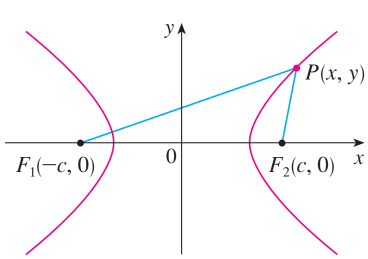


FIGURE 11
P is on the hyperbola when $|PF_1| - |PF_2| = \pm 2a$.

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the **foci**) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle's Law, Ohm's Law, supply and demand curves). A particularly

significant application of hyperbolas was found in the navigation systems developed in World Wars I and II (see Exercise 51).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 52 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

6
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

This shows that $x^2 \geqslant a^2$, so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*.

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. (See Exercise 3.5.57, where these lines are shown to be slant asymptotes.)

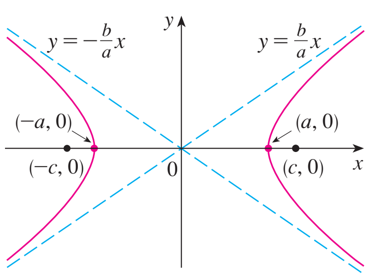


FIGURE 12
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

7 The hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information, which is illustrated in Figure 13.

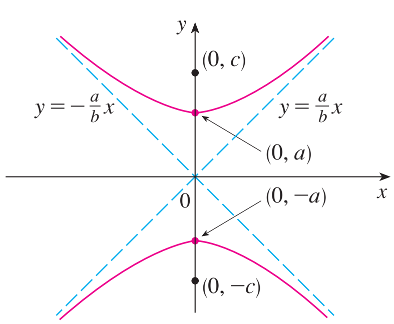


FIGURE 13
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

8 The hyperbola
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$
has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

EXAMPLE 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

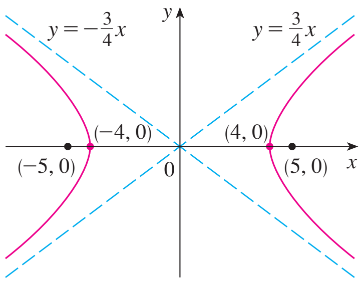


FIGURE 14
$9x^2 - 16y^2 = 144$

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in (7) with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14. ■

EXAMPLE 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

SOLUTION From (8) and the given information, we see that $a = 1$ and $a/b = 2$. Thus $b = a/2 = \frac{1}{2}$ and $c^2 = a^2 + b^2 = \frac{5}{4}$. The foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is

$$y^2 - 4x^2 = 1$$ ■

Shifted Conics

As discussed in Appendix C, we shift conics by taking the standard equations (1), (2), (4), (5), (7), and (8) and replacing x and y by $x - h$ and $y - k$.

EXAMPLE 6 Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$ and vertices $(1, -2)$, $(5, -2)$.

SOLUTION The major axis is the line segment that joins the vertices $(1, -2)$, $(5, -2)$ and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is $(3, -2)$, we replace x and y in (4) by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse. ■

EXAMPLE 7 Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

SOLUTION We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$

$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$

$$4(y-1)^2 - 9(x-4)^2 = 36$$

$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

This is in the form (8) except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $\left(4, 1 + \sqrt{13}\right)$ and $\left(4, 1 - \sqrt{13}\right)$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15. ■

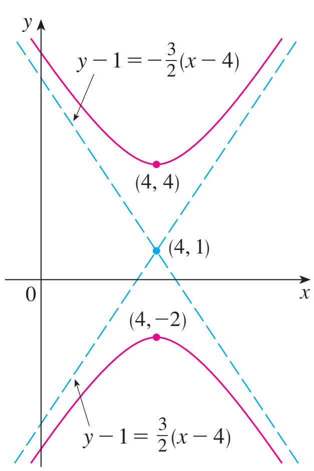


FIGURE 15
$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

10.5 EXERCISES

1–8 Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x^2 = 6y$

2. $2y^2 = 5x$

3. $2x = -y^2$

4. $3x^2 + 8y = 0$

5. $(x + 2)^2 = 8(y - 3)$

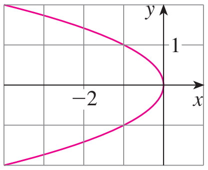
6. $(y - 2)^2 = 2x + 1$

7. $y^2 + 6y + 2x + 1 = 0$

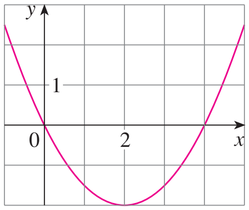
8. $2x^2 - 16x - 3y + 38 = 0$

9–10 Find an equation of the parabola. Then find the focus and directrix.

9.



10.



11–16 Find the vertices and foci of the ellipse and sketch its graph.

11. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$

12. $\dfrac{x^2}{36} + \dfrac{y^2}{8} = 1$

13. $x^2 + 9y^2 = 9$

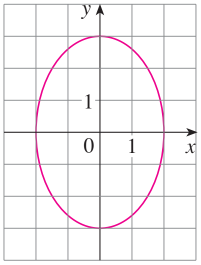
14. $100x^2 + 36y^2 = 225$

15. $9x^2 - 18x + 4y^2 = 27$

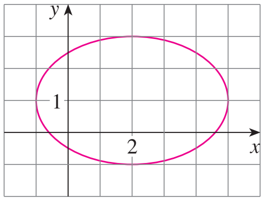
16. $x^2 + 3y^2 + 2x - 12y + 10 = 0$

17–18 Find an equation of the ellipse. Then find its foci.

17.



18.



19–24 Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$

20. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$

21. $x^2 - y^2 = 100$

22. $y^2 - 16x^2 = 16$

23. $x^2 - y^2 + 2y = 2$

24. $9y^2 - 4x^2 - 36y - 8x = 4$

25–30 Identify the type of conic section whose equation is given and find the vertices and foci.

25. $4x^2 = y^2 + 4$

26. $4x^2 = y + 4$

27. $x^2 = 4y - 2y^2$

28. $y^2 - 2 = x^2 - 2x$

29. $3x^2 - 6x - 2y = 1$

30. $x^2 - 2x + 2y^2 - 8y + 7 = 0$

31–48 Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex $(0, 0)$, focus $(1, 0)$

32. Parabola, focus $(0, 0)$, directrix $y = 6$

33. Parabola, focus $(-4, 0)$, directrix $x = 2$

34. Parabola, focus $(2, -1)$, vertex $(2, 3)$

35. Parabola, vertex $(3, -1)$, horizontal axis, passing through $(-15, 2)$

36. Parabola, vertical axis, passing through $(0, 4)$, $(1, 3)$, and $(-2, -6)$

37. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$

38. Ellipse, foci $(0, \pm\sqrt{2})$, vertices $(0, \pm 2)$

39. Ellipse, foci $(0, 2)$, $(0, 6)$, vertices $(0, 0)$, $(0, 8)$

40. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$

41. Ellipse, center $(-1, 4)$, vertex $(-1, 0)$, focus $(-1, 6)$

42. Ellipse, foci $(\pm 4, 0)$, passing through $(-4, 1.8)$

43. Hyperbola, vertices $(\pm 3, 0)$, foci $(\pm 5, 0)$

44. Hyperbola, vertices $(0, \pm 2)$, foci $(0, \pm 5)$

45. Hyperbola, vertices $(-3, -4)$, $(-3, 6)$, foci $(-3, -7)$, $(-3, 9)$

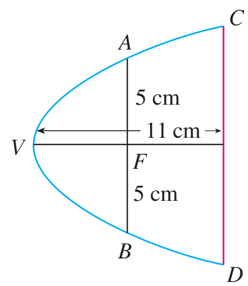
46. Hyperbola, vertices $(-1, 2)$, $(7, 2)$, foci $(-2, 2)$, $(8, 2)$

47. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$

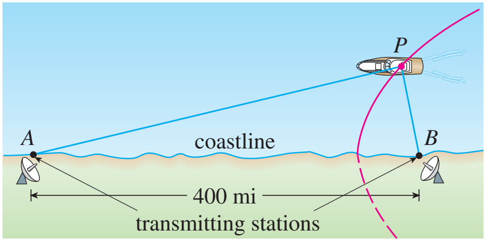
48. Hyperbola, foci $(2, 0)$, $(2, 8)$, asymptotes $y = 3 + \frac{1}{2}x$ and $y = 5 - \frac{1}{2}x$

49. The point in a lunar orbit nearest the surface of the moon is called *perilune* and the point farthest from the surface is called *apolune*. The *Apollo 11* spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

50. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.
(a) Find an equation of the parabola.
(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.



51. The LORAN (LOng RAnge Navigation) radio navigation system was widely used until the 1990s when it was superseded by the GPS system. In the LORAN system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 400 mi due east of station A on a coastline. A ship received the signal from B 1200 microseconds (μs) before it received the signal from A.
(a) Assuming that radio signals travel at a speed of 980 ft/μs, find an equation of the hyperbola on which the ship lies.
(b) If the ship is due north of B, how far off the coastline is the ship?



52. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

53. Show that the function defined by the upper branch of the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is concave upward.

54. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

55. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k - 16} = 1$$

in each of the following cases:
(a) $k > 16$ (b) $0 < k < 16$ (c) $k < 0$
(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of k is.

56. (a) Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (x_0, y_0) can be written as

$$y_0 y = 2p(x + x_0)$$

(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

57. Show that the tangent lines to the parabola $x^2 = 4py$ drawn from any point on the directrix are perpendicular.

58. Show that if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

59. Use parametric equations and Simpson's Rule with $n = 8$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$.

60. The dwarf planet Pluto travels in an elliptical orbit around the sun (at one focus). The length of the major axis is 1.18×10^{10} km and the length of the minor axis is 1.14×10^{10} km. Use Simpson's Rule with $n = 10$ to estimate the distance traveled by the planet during one complete orbit around the sun.

61. Find the area of the region enclosed by the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the vertical line through a focus.

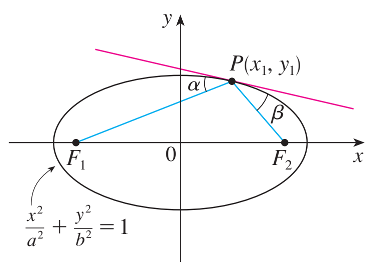
62. (a) If an ellipse is rotated about its major axis, find the volume of the resulting solid.
(b) If it is rotated about its minor axis, find the resulting volume.

63. Find the centroid of the region enclosed by the x-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

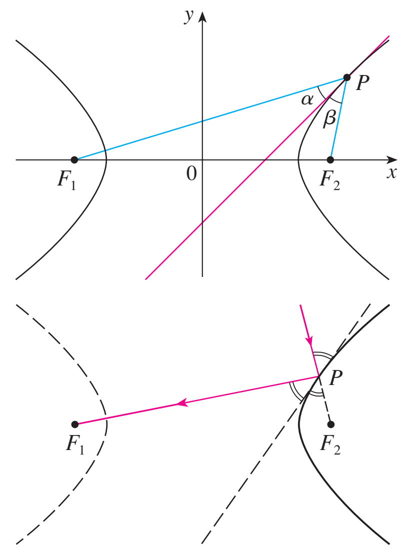
64. (a) Calculate the surface area of the ellipsoid that is generated by rotating an ellipse about its major axis.
(b) What is the surface area if the ellipse is rotated about its minor axis?

65. Let $P(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines

PF_1, PF_2 and the ellipse as shown in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 17 on page 201 to show that $\tan \alpha = \tan \beta$.]



66. Let $P(x_1, y_1)$ be a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the hyperbola as shown in the figure. Prove that $\alpha = \beta$. (This is the reflection property of the hyperbola. It shows that light aimed at a focus F_2 of a hyperbolic mirror is reflected toward the other focus F_1.)



10.6 Conic Sections in Polar Coordinates

In the preceding section we defined the parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conic sections in terms of a focus and directrix. Furthermore, if we place the focus at the origin, then a conic section has a simple polar equation, which provides a convenient description of the motion of planets, satellites, and comets.

1 Theorem Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

PROOF Notice that if the eccentricity is $e = 1$, then $|PF| = |Pl|$ and so the given condition simply becomes the definition of a parabola as given in Section 10.5.

Let us place the focus F at the origin and the directrix parallel to the y-axis and d units to the right. Thus the directrix has equation $x = d$ and is perpendicular to the

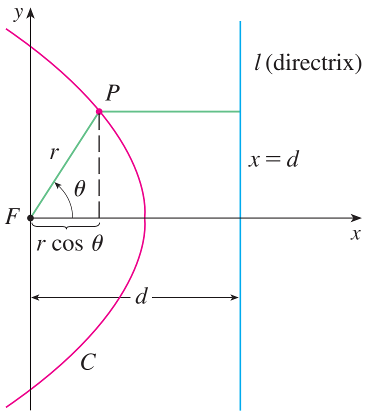


FIGURE 1

polar axis. If the point P has polar coordinates (r, θ), we see from Figure 1 that

$$|PF| = r \qquad |Pl| = d - r\cos\theta$$

Thus the condition $|PF|/|Pl| = e$, or $|PF| = e|Pl|$, becomes

2 $$r = e(d - r\cos\theta)$$

If we square both sides of this polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2)$$

or $$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2$$

After completing the square, we have

3 $$\left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2}$$

If $e < 1$, we recognize Equation 3 as the equation of an ellipse. In fact, it is of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

4 $$h = -\frac{e^2d}{1 - e^2} \qquad a^2 = \frac{e^2d^2}{(1 - e^2)^2} \qquad b^2 = \frac{e^2d^2}{1 - e^2}$$

In Section 10.5 we found that the foci of an ellipse are at a distance c from the center, where

5 $$c^2 = a^2 - b^2 = \frac{e^4d^2}{(1 - e^2)^2}$$

This shows that $$c = \frac{e^2d}{1 - e^2} = -h$$

and confirms that the focus as defined in Theorem 1 means the same as the focus defined in Section 10.5. It also follows from Equations 4 and 5 that the eccentricity is given by

$$e = \frac{c}{a}$$

If $e > 1$, then $1 - e^2 < 0$ and we see that Equation 3 represents a hyperbola. Just as we did before, we could rewrite Equation 3 in the form

$$\frac{(x - h)^2}{a^2} - \frac{y^2}{b^2} = 1$$

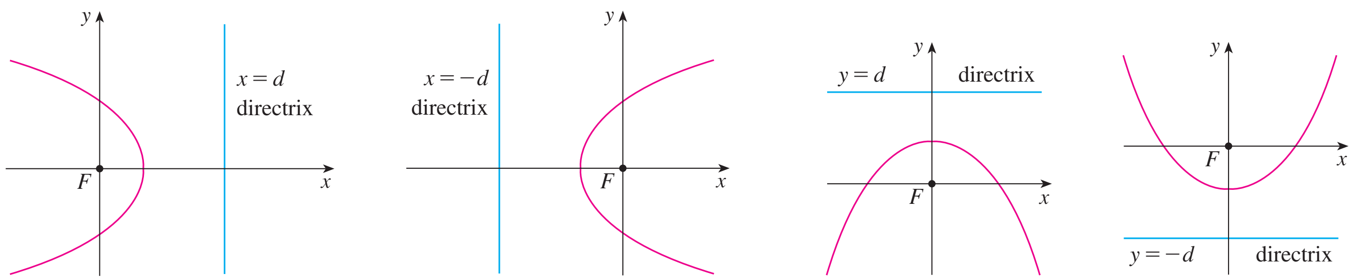
and see that

$$e = \frac{c}{a} \qquad \text{where} \quad c^2 = a^2 + b^2$$ ■

By solving Equation 2 for r, we see that the polar equation of the conic shown in Figure 1 can be written as

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the left of the focus as $x = -d$, or if the directrix is chosen to be parallel to the polar axis as $y = \pm d$, then the polar equation of the conic is given by the following theorem, which is illustrated by Figure 2. (See Exercises 21–23.)



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

FIGURE 2
Polar equations of conics

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

EXAMPLE 1 Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

SOLUTION Using Theorem 6 with $e = 1$ and $d = 6$, and using part (d) of Figure 2, we see that the equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

■

EXAMPLE 2 A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

SOLUTION Dividing numerator and denominator by 3, we write the equation as

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

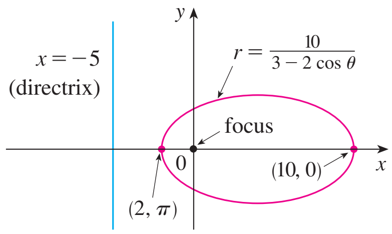


FIGURE 3

From Theorem 6 we see that this represents an ellipse with $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$, we have

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation $x = -5$. When $\theta = 0$, $r = 10$; when $\theta = \pi$, $r = 2$. So the vertices have polar coordinates $(10, 0)$ and $(2, \pi)$. The ellipse is sketched in Figure 3. ■

EXAMPLE 3 Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

SOLUTION Writing the equation in the form

$$r = \frac{6}{1 + 2\sin\theta}$$

we see that the eccentricity is $e = 2$ and the equation therefore represents a hyperbola. Since $ed = 6$, $d = 3$ and the directrix has equation $y = 3$. The vertices occur when $\theta = \pi/2$ and $3\pi/2$, so they are $(2, \pi/2)$ and $(-6, 3\pi/2) = (6, \pi/2)$. It is also useful to plot the x-intercepts. These occur when $\theta = 0, \pi$; in both cases $r = 6$. For additional accuracy we could draw the asymptotes. Note that $r \to \pm\infty$ when $1 + 2\sin\theta \to 0^+$ or 0^- and $1 + 2\sin\theta = 0$ when $\sin\theta = -\frac{1}{2}$. Thus the asymptotes are parallel to the rays $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The hyperbola is sketched in Figure 4.

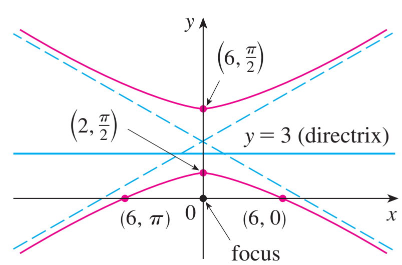


FIGURE 4
$r = \dfrac{12}{2 + 4\sin\theta}$ ■

When rotating conic sections, we find it much more convenient to use polar equations than Cartesian equations. We just use the fact (see Exercise 10.3.73) that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α.

EXAMPLE 4 If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

SOLUTION We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

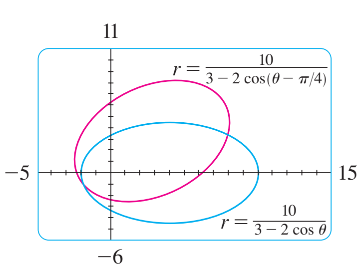


FIGURE 5

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about its left focus. ■

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity e. Notice that when e is close to 0 the ellipse is nearly circular, whereas it becomes more elongated as $e \to 1^-$. When $e = 1$, of course, the conic is a parabola.

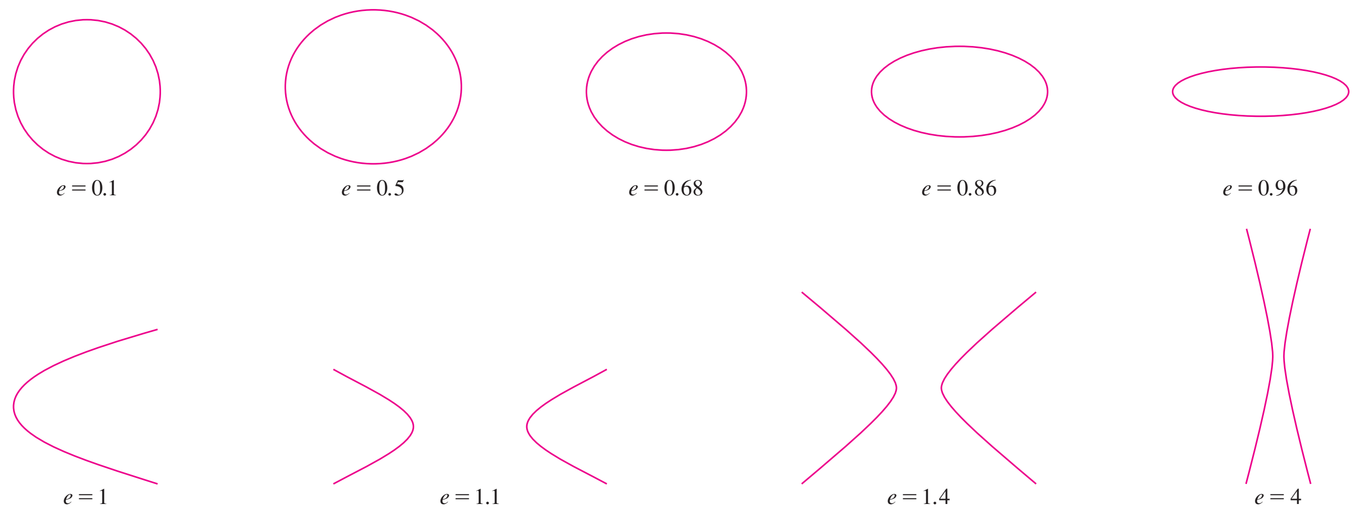


FIGURE 6

Kepler's Laws

In 1609 the German mathematician and astronomer Johannes Kepler, on the basis of huge amounts of astronomical data, published the following three laws of planetary motion.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Although Kepler formulated his laws in terms of the motion of planets around the sun, they apply equally well to the motion of moons, comets, satellites, and other bodies that orbit subject to a single gravitational force. In Section 13.4 we will show how to deduce Kepler's Laws from Newton's Laws. Here we use Kepler's First Law, together with the polar equation of an ellipse, to calculate quantities of interest in astronomy.

For purposes of astronomical calculations, it's useful to express the equation of an ellipse in terms of its eccentricity e and its semimajor axis a. We can write the distance d from the focus to the directrix in terms of a if we use (4):

$$a^2 = \frac{e^2 d^2}{(1 - e^2)^2} \quad \Rightarrow \quad d^2 = \frac{a^2(1 - e^2)^2}{e^2} \quad \Rightarrow \quad d = \frac{a(1 - e^2)}{e}$$

So $ed = a(1 - e^2)$. If the directrix is $x = d$, then the polar equation is

$$r = \frac{ed}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

7 The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

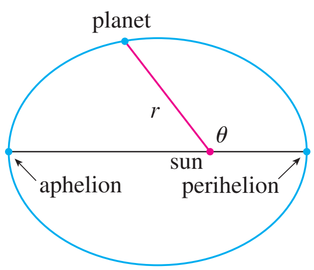


FIGURE 7

The positions of a planet that are closest to and farthest from the sun are called its **perihelion** and **aphelion**, respectively, and correspond to the vertices of the ellipse (see Figure 7). The distances from the sun to the perihelion and aphelion are called the **perihelion distance** and **aphelion distance**, respectively. In Figure 1 on page 723 the sun is at the focus F, so at perihelion we have $\theta = 0$ and, from Equation 7,

$$r = \frac{a(1 - e^2)}{1 + e\cos 0} = \frac{a(1 - e)(1 + e)}{1 + e} = a(1 - e)$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1 + e)$.

8 The perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

EXAMPLE 5

(a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.
(b) Find the distance from the earth to the sun at perihelion and at aphelion.

SOLUTION

(a) The length of the major axis is $2a = 2.99 \times 10^8$, so $a = 1.495 \times 10^8$. We are given that $e = 0.017$ and so, from Equation 7, an equation of the earth's orbit around the sun is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{(1.495 \times 10^8)[1 - (0.017)^2]}{1 + 0.017\cos\theta}$$

or, approximately,

$$r = \frac{1.49 \times 10^8}{1 + 0.017\cos\theta}$$

(b) From (8), the perihelion distance from the earth to the sun is

$$a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017) \approx 1.47 \times 10^8 \text{ km}$$

and the aphelion distance is

$$a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017) \approx 1.52 \times 10^8 \text{ km}$$ ■

10.6 EXERCISES

1–8 Write a polar equation of a conic with the focus at the origin and the given data.

1. Ellipse, eccentricity $\frac{1}{2}$, directrix $x = 4$

2. Parabola, directrix $x = -3$

3. Hyperbola, eccentricity 1.5, directrix $y = 2$

4. Hyperbola, eccentricity 3, directrix $x = 3$

5. Ellipse, eccentricity $\frac{2}{3}$, vertex $(2, \pi)$

6. Ellipse, eccentricity 0.6, directrix $r = 4 \csc\theta$

7. Parabola, vertex $(3, \pi/2)$

8. Hyperbola, eccentricity 2, directrix $r = -2 \sec\theta$

9–16 (a) Find the eccentricity, (b) identify the conic, (c) give an equation of the directrix, and (d) sketch the conic.

9. $r = \dfrac{4}{5 - 4\sin\theta}$

10. $r = \dfrac{1}{2 + \sin\theta}$

11. $r = \dfrac{2}{3 + 3\sin\theta}$

12. $r = \dfrac{5}{2 - 4\cos\theta}$

13. $r = \dfrac{9}{6 + 2\cos\theta}$

14. $r = \dfrac{1}{3 - 3\sin\theta}$

15. $r = \dfrac{3}{4 - 8\cos\theta}$

16. $r = \dfrac{4}{2 + 3\cos\theta}$

17. (a) Find the eccentricity and directrix of the conic $r = 1/(1 - 2\sin\theta)$ and graph the conic and its directrix.
(b) If this conic is rotated counterclockwise about the origin through an angle $3\pi/4$, write the resulting equation and graph its curve.

18. Graph the conic $r = 4/(5 + 6\cos\theta)$ and its directrix. Also graph the conic obtained by rotating this curve about the origin through an angle $\pi/3$.

19. Graph the conics $r = e/(1 - e\cos\theta)$ with $e = 0.4, 0.6, 0.8$, and 1.0 on a common screen. How does the value of e affect the shape of the curve?

20. (a) Graph the conics $r = ed/(1 + e\sin\theta)$ for $e = 1$ and various values of d. How does the value of d affect the shape of the conic?
(b) Graph these conics for $d = 1$ and various values of e. How does the value of e affect the shape of the conic?

21. Show that a conic with focus at the origin, eccentricity e, and directrix $x = -d$ has polar equation

$$r = \frac{ed}{1 - e\cos\theta}$$

22. Show that a conic with focus at the origin, eccentricity e, and directrix $y = d$ has polar equation

$$r = \frac{ed}{1 + e\sin\theta}$$

23. Show that a conic with focus at the origin, eccentricity e, and directrix $y = -d$ has polar equation

$$r = \frac{ed}{1 - e\sin\theta}$$

24. Show that the parabolas $r = c/(1 + \cos\theta)$ and $r = d/(1 - \cos\theta)$ intersect at right angles.

25. The orbit of Mars around the sun is an ellipse with eccentricity 0.093 and semimajor axis 2.28×10^8 km. Find a polar equation for the orbit.

26. Jupiter's orbit has eccentricity 0.048 and the length of the major axis is 1.56×10^9 km. Find a polar equation for the orbit.

27. The orbit of Halley's comet, last seen in 1986 and due to return in 2061, is an ellipse with eccentricity 0.97 and one focus at the sun. The length of its major axis is 36.18 AU. [An astronomical unit (AU) is the mean distance between the earth and the sun, about 93 million miles.] Find a polar equation for the orbit of Halley's comet. What is the maximum distance from the comet to the sun?

28. Comet Hale-Bopp, discovered in 1995, has an elliptical orbit with eccentricity 0.9951. The length of the orbit's major axis is 356.5 AU. Find a polar equation for the orbit of this comet. How close to the sun does it come?



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29. The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the sun is 4.6×10^7 km. Find its maximum distance from the sun.

30. The distance from the dwarf planet Pluto to the sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion. Find the eccentricity of Pluto's orbit.

31. Using the data from Exercise 29, find the distance traveled by the planet Mercury during one complete orbit around the sun. (If your calculator or computer algebra system evaluates definite integrals, use it. Otherwise, use Simpson's Rule.)

10 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a parametric curve?
(b) How do you sketch a parametric curve?

2. (a) How do you find the slope of a tangent to a parametric curve?
(b) How do you find the area under a parametric curve?

3. Write an expression for each of the following:
(a) The length of a parametric curve
(b) The area of the surface obtained by rotating a parametric curve about the x-axis

4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.
(b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.
(c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?

5. (a) How do you find the slope of a tangent line to a polar curve?
(b) How do you find the area of a region bounded by a polar curve?
(c) How do you find the length of a polar curve?

6. (a) Give a geometric definition of a parabola.
(b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?

7. (a) Give a definition of an ellipse in terms of foci.
(b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

8. (a) Give a definition of a hyperbola in terms of foci.
(b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
(c) Write equations for the asymptotes of the hyperbola in part (b).

9. (a) What is the eccentricity of a conic section?
(b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?
(c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent when $t = 1$.

2. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then

$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$

3. The length of the curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$.

4. If a point is represented by (x, y) in Cartesian coordinates (where $x \neq 0$) and (r, θ) in polar coordinates, then $\theta = \tan^{-1}(y/x)$.

5. The polar curves

$$r = 1 - \sin 2\theta \qquad r = \sin 2\theta - 1$$

have the same graph.

6. The equations $r = 2$, $x^2 + y^2 = 4$, and $x = 2 \sin 3t$, $y = 2 \cos 3t$ $(0 \leqslant t \leqslant 2\pi)$ all have the same graph.

7. The parametric equations $x = t^2$, $y = t^4$ have the same graph as $x = t^3$, $y = t^6$.

8. The graph of $y^2 = 2y + 3x$ is a parabola.

9. A tangent line to a parabola intersects the parabola only once.

10. A hyperbola never intersects its directrix.

EXERCISES

1–4 Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leqslant t \leqslant 1$

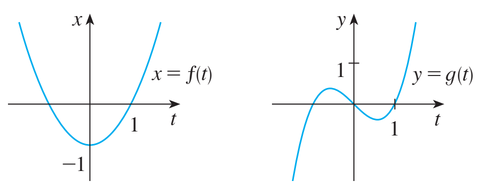
2. $x = 1 + e^{2t}, \quad y = e^t$

3. $x = \cos\theta, \quad y = \sec\theta, \quad 0 \leqslant \theta < \pi/2$

4. $x = 2\cos\theta, \quad y = 1 + \sin\theta$

5. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.

6. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



7. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.
(b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.

8. Sketch the region consisting of points whose polar coordinates satisfy $1 \leqslant r < 2$ and $\pi/6 \leqslant \theta \leqslant 5\pi/6$.

9–16 Sketch the polar curve.

9. $r = 1 + \sin\theta$

10. $r = \sin 4\theta$

11. $r = \cos 3\theta$

12. $r = 3 + \cos 3\theta$

13. $r = 1 + \cos 2\theta$

14. $r = 2\cos(\theta/2)$

15. $r = \dfrac{3}{1 + 2\sin\theta}$

16. $r = \dfrac{3}{2 - 2\cos\theta}$

17–18 Find a polar equation for the curve represented by the given Cartesian equation.

17. $x + y = 2$

18. $x^2 + y^2 = 2$

19. The curve with polar equation $r = (\sin\theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a machine to check your sketch.

20. Graph the ellipse $r = 2/(4 - 3\cos\theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

21–24 Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

21. $x = \ln t, \quad y = 1 + t^2; \quad t = 1$

22. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$

23. $r = e^{-\theta}; \quad \theta = \pi$

24. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

25–26 Find dy/dx and d^2y/dx^2.

25. $x = t + \sin t, \quad y = t - \cos t$

26. $x = 1 + t^2, \quad y = t - t^3$

27. Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t$, $y = t^2 + t + 1$. Then use calculus to find the exact coordinates.

28. Find the area enclosed by the loop of the curve in Exercise 27.

29. At what points does the curve

$$x = 2a\cos t - a\cos 2t \qquad y = 2a\sin t - a\sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

30. Find the area enclosed by the curve in Exercise 29.

31. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.

32. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.

33. Find the points of intersection of the curves $r = 2$ and $r = 4\cos\theta$.

34. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.

35. Find the area of the region that lies inside both of the circles $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$.

36. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

37–40 Find the length of the curve.

37. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 2$

38. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

39. $r = 1/\theta, \quad \pi \leqslant \theta \leqslant 2\pi$

40. $r = \sin^3(\theta/3), \quad 0 \leqslant \theta \leqslant \pi$

41–42 Find the area of the surface obtained by rotating the given curve about the x-axis.

41. $x = 4\sqrt{t}, \quad y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}, \quad 1 \leqslant t \leqslant 4$

42. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

43. The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \qquad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning "to turn or twist"). Investigate how these curves vary as c varies.

44. A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

45–48 Find the foci and vertices and sketch the graph.

45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

46. $4x^2 - y^2 = 16$

47. $6y^2 + x - 36y + 55 = 0$

48. $25x^2 + 4y^2 + 50x - 16y = 59$

49. Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

50. Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

51. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

52. Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

53. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

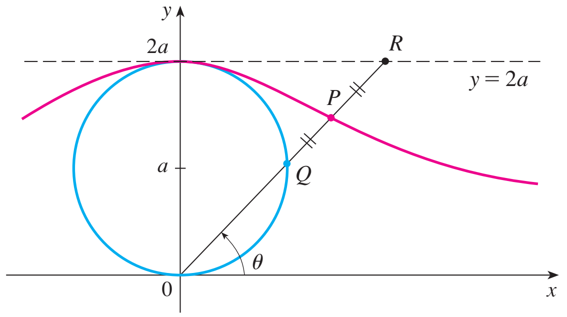
54. Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

55. Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4 \sec\theta$.

56. Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e\cos\theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

57. In the figure the circle of radius a is stationary, and for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$ is called the **long-bow curve**. Find parametric equations for this curve.



58. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?

(b) Find the points on the curve where the tangent lines are horizontal or vertical.

(c) Show that the line $y = -x - 1$ is a slant asymptote.

(d) Sketch the curve.

(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.

(f) Show that the polar equation can be written in the form

$$r = \frac{3\sec\theta\,\tan\theta}{1 + \tan^3\theta}$$

(g) Find the area enclosed by the loop of this curve.

CAS (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

Problems Plus

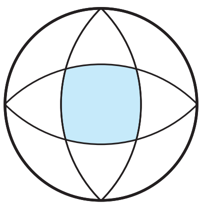


FIGURE FOR PROBLEM 1

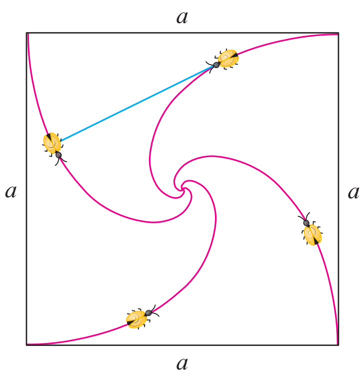


FIGURE FOR PROBLEM 4

1. The outer circle in the figure has radius 1 and the centers of the interior circular arcs lie on the outer circle. Find the area of the shaded region.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.
(b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geqslant x \geqslant 0$ initially.)
(c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin\theta$, where $0 \leqslant c \leqslant 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

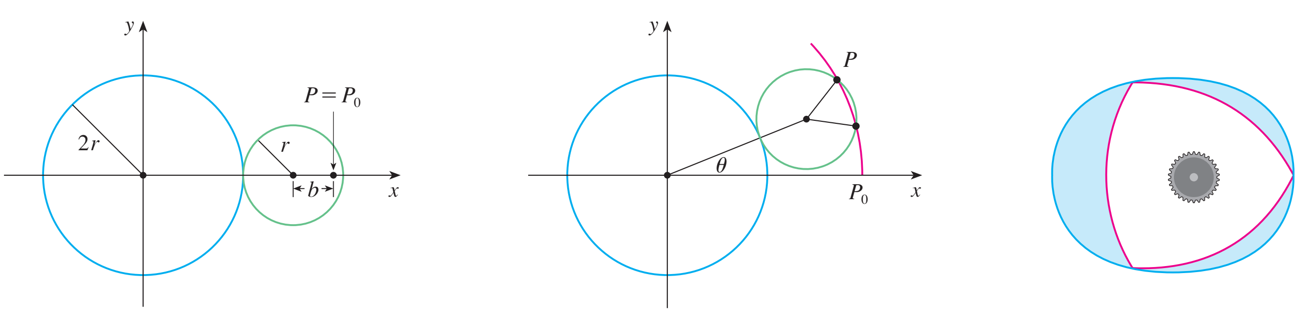
4. Four bugs are placed at the four corners of a square with side length a. The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.
(a) Find the polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)
(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.

5. Show that any tangent line to a hyperbola touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.

6. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C. A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x-axis.
(a) Using θ as a parameter, show that parametric equations of the path traced out by P are

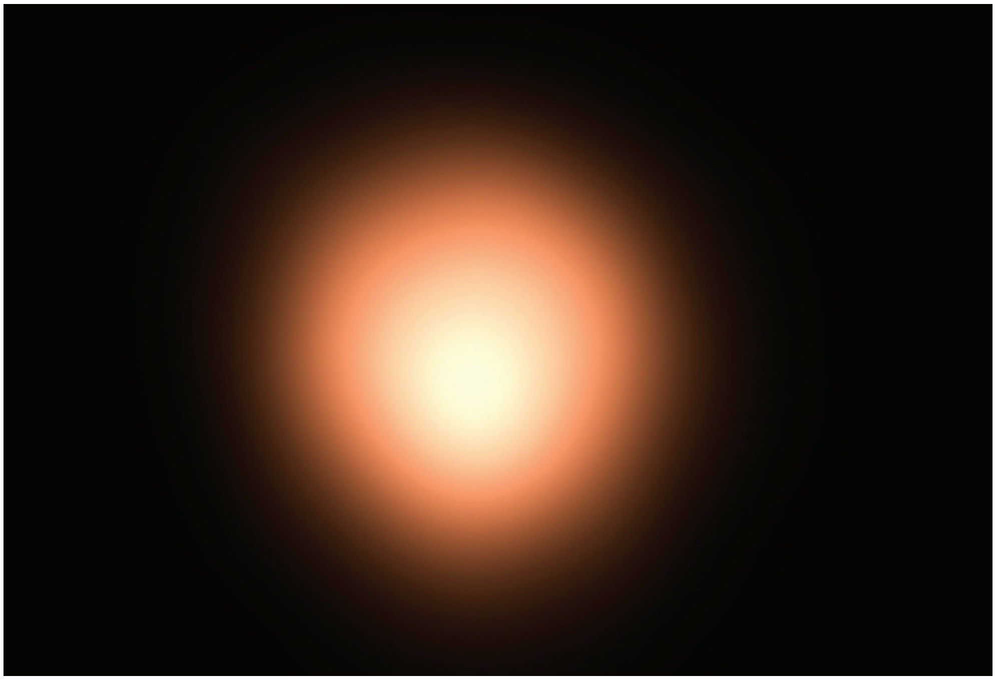
$$x = b\cos 3\theta + 3r\cos\theta \qquad y = b\sin 3\theta + 3r\sin\theta$$

Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid*. The path traced out by P for $0 < b < r$ is called an *epitrochoid*.
(b) Graph the curve for various values of b between 0 and r.
(c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.
Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.
(d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leqslant \frac{3}{2}(2 - \sqrt{3})r$.



(i) (ii) (iii)

11 Infinite Sequences and Series



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Betelgeuse is a red supergiant star, one of the largest and brightest of the observable stars. In the project on page 823 you are asked to compare the radiation emitted by Betelgeuse with that of other stars.

INFINITE SEQUENCES AND SERIES WERE introduced briefly in *A Preview of Calculus* in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2}. (Recall that we have previously been unable to do this.) Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1 Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \dfrac{n}{n+1}$ $\qquad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\qquad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\qquad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$ $\qquad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos\dfrac{n\pi}{6},\ n \geqslant 0$ $\qquad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$ ■

EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5} \qquad a_2 = -\frac{4}{25} \qquad a_3 = \frac{5}{125} \qquad a_4 = -\frac{6}{625} \qquad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n + 2$. The denominators are the

powers of 5, so a_n has denominator 5^n. The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1. In Example 1(b) the factor $(-1)^n$ meant we started with a negative term. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$ ■

EXAMPLE 3 Here are some sequences that don't have a simple defining equation.
(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$$

(c) **The Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 83). ■

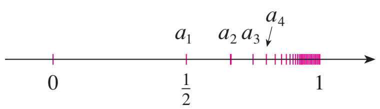


FIGURE 1

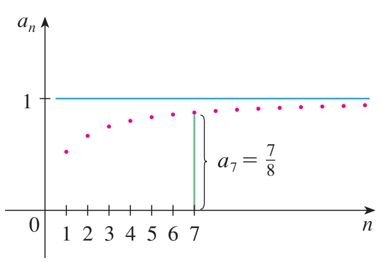


FIGURE 2

A sequence such as the one in Example 1(a), $a_n = n/(n+1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 3.4.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L.

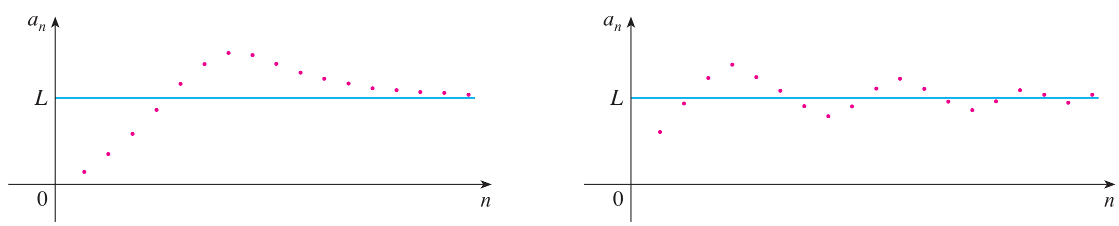


FIGURE 3
Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Compare this definition with Definition 3.4.5.

Definition 2 is illustrated by Figure 4, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

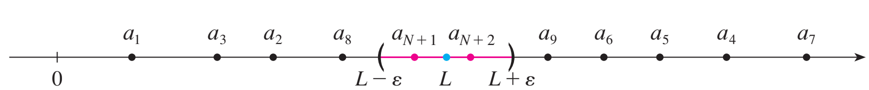


FIGURE 4

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

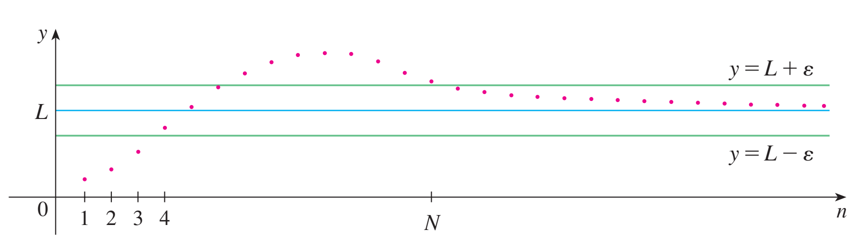


FIGURE 5

If you compare Definition 2 with Definition 3.4.5, you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

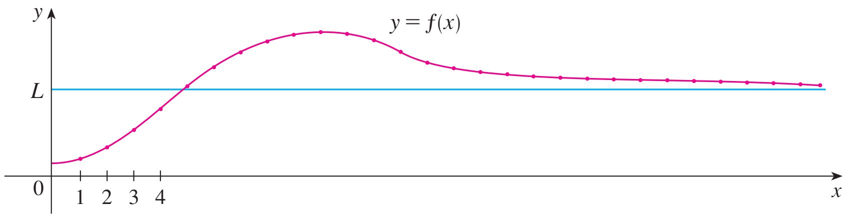


FIGURE 6

In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$ (Theorem 3.4.4), we have

4
$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. The following precise definition is similar to Definition 3.4.7.

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

The Limit Laws given in Section 1.6 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

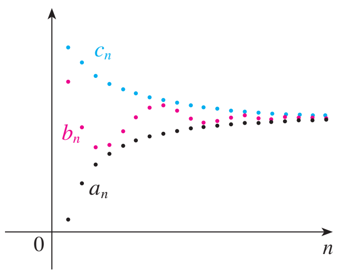


FIGURE 7
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as Exercise 87.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 4 Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

SOLUTION The method is similar to the one we used in Section 3.4: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \dfrac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

Here we used Equation 4 with $r = 1$. ■

This shows that the guess we made earlier from Figures 1 and 2 was correct.

EXAMPLE 5 Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n:

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\dfrac{10}{n^2} + \dfrac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent. ■

EXAMPLE 6 Calculate $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$ ■

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

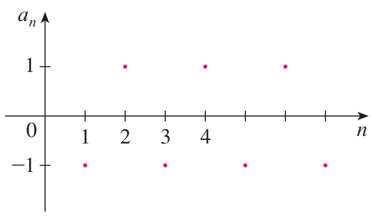


FIGURE 8

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent. ■

The graph of the sequence in Example 8 is shown in Figure 9 and supports our answer.

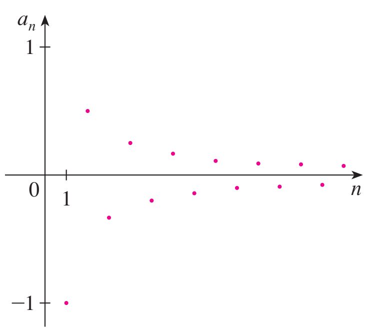


FIGURE 9

EXAMPLE 8 Evaluate $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by Theorem 6,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$ ■

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is left as Exercise 88.

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

EXAMPLE 9 Find $\lim_{n\to\infty} \sin(\pi/n)$.

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n\to\infty} \sin(\pi/n) = \sin\left(\lim_{n\to\infty} (\pi/n)\right) = \sin 0 = 0$$ ■

Creating Graphs of Sequences

Some computer algebra systems have special commands that enable us to create sequences and graph them directly. With most graphing calculators, however, sequences can be graphed by using parametric equations. For instance, the sequence in Example 10 can be graphed by entering the parametric equations

$$x = t \qquad y = t!/t^t$$

and graphing in dot mode, starting with $t = 1$ and setting the t-step equal to 1. The result is shown in Figure 10.

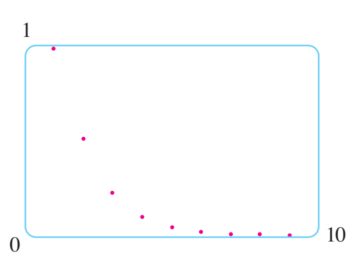


FIGURE 10

EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \to \infty$ but here we have no corresponding function for use with l'Hospital's Rule ($x!$ is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

8 $$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

It appears from these expressions and the graph in Figure 10 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leqslant \frac{1}{n}$$

We know that $1/n \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem. ■

EXAMPLE 11 For what values of r is the sequence $\{r^n\}$ convergent?

SOLUTION We know from Section 3.4 and the graphs of the exponential functions in Section 6.2 (or Section 6.4*) that $\lim_{x\to\infty} a^x = \infty$ for $a > 1$ and $\lim_{x\to\infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n\to\infty} 1^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n = 0$ by Theorem 6. If $r \leqslant -1$, then $\{r^n\}$ diverges as in Example 7. Figure 11 shows the graphs for various values of r. (The case $r = -1$ is shown in Figure 8.)

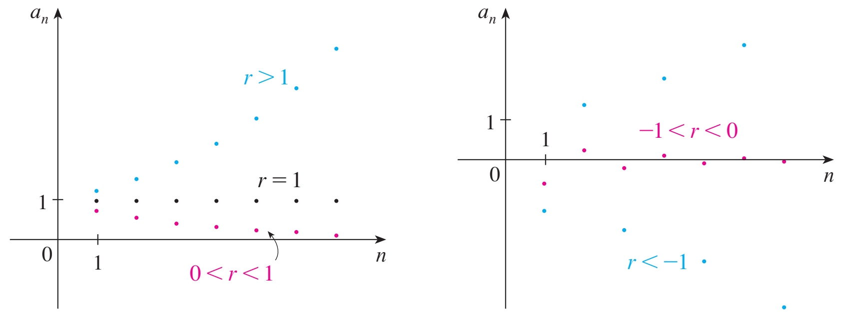


FIGURE 11
The sequence $a_n = r^n$

■

The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{ \dfrac{3}{n+5} \right\}$ is decreasing because

The right side is smaller because it has a larger denominator.

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geqslant 1$. ■

EXAMPLE 13 Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

SOLUTION 1 We must show that $a_{n+1} < a_n$, that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \quad &\iff \quad (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff \quad n^3+n^2+n+1 < n^3+2n^2+2n \\ &\iff \quad 1 < n^2+n \end{aligned}$$

Since $n \geqslant 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

SOLUTION 2 Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{whenever } x^2 > 1$$

Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing. ■

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$ but is divergent from Example 7] and not every

monotonic sequence is convergent ($a_n = n \to \infty$). But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \leqslant M$ for all n, then the terms are forced to crowd together and approach some number L.

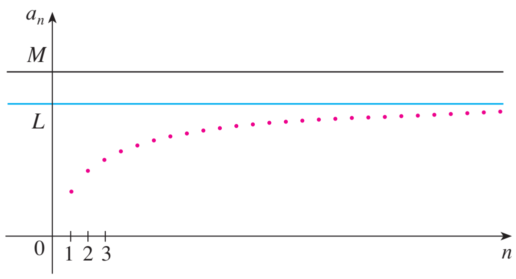


FIGURE 12

The proof of Theorem 12 is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leqslant M$ for all x in S), then S has a **least upper bound** b. (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leqslant M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

PROOF Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geqslant 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geqslant a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leqslant L - a_n < \varepsilon$$

since $a_n \leqslant L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

so $\lim_{n \to \infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing. ■

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

EXAMPLE 14 Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \qquad a_{n+1} = \tfrac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

SOLUTION We begin by computing the first several terms:

$$a_1 = 2 \qquad a_2 = \tfrac{1}{2}(2 + 6) = 4 \qquad a_3 = \tfrac{1}{2}(4 + 6) = 5$$

$$a_4 = \tfrac{1}{2}(5 + 6) = 5.5 \qquad a_5 = 5.75 \qquad a_6 = 5.875$$

$$a_7 = 5.9375 \qquad a_8 = 5.96875 \qquad a_9 = 5.984375$$

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geqslant 1$. This is true for $n = 1$ because $a_2 = 4 > a_1$. If we assume that it is true for $n = k$, then we have

Mathematical induction is often used in dealing with recursive sequences. See page 99 for a discussion of the Principle of Mathematical Induction.

$$a_{k+1} > a_k$$

so $$a_{k+1} + 6 > a_k + 6$$

and $$\tfrac{1}{2}(a_{k+1} + 6) > \tfrac{1}{2}(a_k + 6)$$

Thus $$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$. Therefore the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n. (Since the sequence is increasing, we already know that it has a lower bound: $a_n \geqslant a_1 = 2$ for all n.) We know that $a_1 < 6$, so the assertion is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_k < 6$$

so $$a_k + 6 < 12$$

and $$\tfrac{1}{2}(a_k + 6) < \tfrac{1}{2}(12) = 6$$

Thus $$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n\to\infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \tfrac{1}{2}(a_n + 6) = \tfrac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) = \tfrac{1}{2}(L + 6)$$

Since $a_n \to L$, it follows that $a_{n+1} \to L$ too (as $n \to \infty$, $n + 1 \to \infty$ also). So we have

A proof of this fact is requested in Exercise 70.

$$L = \tfrac{1}{2}(L + 6)$$

Solving this equation for L, we get $L = 6$, as we predicted. ■

11.1 EXERCISES

1. (a) What is a sequence?
   (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
   (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
   (b) What is a divergent sequence? Give two examples.

3–12 List the first five terms of the sequence.

3. $a_n = \dfrac{2^n}{2n+1}$

4. $a_n = \dfrac{n^2-1}{n^2+1}$

5. $a_n = \dfrac{(-1)^{n-1}}{5^n}$

6. $a_n = \cos\dfrac{n\pi}{2}$

7. $a_n = \dfrac{1}{(n+1)!}$

8. $a_n = \dfrac{(-1)^n n}{n!+1}$

9. $a_1 = 1, \quad a_{n+1} = 5a_n - 3$

10. $a_1 = 6, \quad a_{n+1} = \dfrac{a_n}{n}$

11. $a_1 = 2, \quad a_{n+1} = \dfrac{a_n}{1+a_n}$

12. $a_1 = 2, \quad a_2 = 1, \quad a_{n+1} = a_n - a_{n-1}$

13–18 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

13. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots\right\}$

14. $\left\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \dots\right\}$

15. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \dots\right\}$

16. $\{5, 8, 11, 14, 17, \dots\}$

17. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots\right\}$

18. $\{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

19. $a_n = \dfrac{3n}{1+6n}$

20. $a_n = 2 + \dfrac{(-1)^n}{n}$

21. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

22. $a_n = 1 + \dfrac{10^n}{9^n}$

23–56 Determine whether the sequence converges or diverges. If it converges, find the limit.

23. $a_n = \dfrac{3+5n^2}{n+n^2}$

24. $a_n = \dfrac{3+5n^2}{1+n}$

25. $a_n = \dfrac{n^4}{n^3-2n}$

26. $a_n = 2 + (0.86)^n$

27. $a_n = 3^n 7^{-n}$

28. $a_n = \dfrac{3\sqrt{n}}{\sqrt{n}+2}$

29. $a_n = e^{-1/\sqrt{n}}$

30. $a_n = \dfrac{4^n}{1+9^n}$

31. $a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}}$

32. $a_n = \cos\left(\dfrac{n\pi}{n+1}\right)$

33. $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$

34. $a_n = e^{2n/(n+2)}$

35. $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$

36. $a_n = \dfrac{(-1)^{n+1} n}{n+\sqrt{n}}$

37. $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}$

38. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$

39. $\{\sin n\}$

40. $a_n = \dfrac{\tan^{-1} n}{n}$

41. $\{n^2 e^{-n}\}$

42. $a_n = \ln(n+1) - \ln n$

43. $a_n = \dfrac{\cos^2 n}{2^n}$

44. $a_n = \sqrt[n]{2^{1+3n}}$

45. $a_n = n\sin(1/n)$

46. $a_n = 2^{-n}\cos n\pi$

47. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

48. $a_n = \sqrt[n]{n}$

49. $a_n = \ln(2n^2+1) - \ln(n^2+1)$

50. $a_n = \dfrac{(\ln n)^2}{n}$

51. $a_n = \arctan(\ln n)$

52. $a_n = n - \sqrt{n+1}\sqrt{n+3}$

53. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$

54. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

55. $a_n = \dfrac{n!}{2^n}$

56. $a_n = \dfrac{(-3)^n}{n!}$

57–63 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 739 for advice on graphing sequences.)

57. $a_n = (-1)^n \dfrac{n}{n+1}$

58. $a_n = \dfrac{\sin n}{n}$

59. $a_n = \arctan\left(\dfrac{n^2}{n^2+4}\right)$

60. $a_n = \sqrt[n]{3^n + 5^n}$

61. $a_n = \dfrac{n^2 \cos n}{1+n^2}$

62. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$

63. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}$

64. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n \geqslant 1$$

(b) What happens if the first term is $a_1 = 2$?

65. If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars.
(a) Find the first five terms of the sequence $\{a_n\}$.
(b) Is the sequence convergent or divergent? Explain.

66. If you deposit \$100 at the end of every month into an account that pays 3% interest per year compounded monthly, the amount of interest accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.0025^n - 1}{0.0025} - n\right)$$

(a) Find the first six terms of the sequence.
(b) How much interest will you have earned after two years?

67. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
(a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000$$

(b) How many catfish are in the pond after six months?

68. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

69. For what values of r is the sequence $\{nr^n\}$ convergent?

70. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geqslant 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

71. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

72–78 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

72. $a_n = \cos n$

73. $a_n = \dfrac{1}{2n+3}$

74. $a_n = \dfrac{1-n}{2+n}$

75. $a_n = n(-1)^n$

76. $a_n = 2 + \dfrac{(-1)^n}{n}$

77. $a_n = 3 - 2ne^{-n}$

78. $a_n = n^3 - 3n + 3$

79. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

80. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.
(a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.
(b) Find $\lim_{n\to\infty} a_n$.

81. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

82. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leqslant 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

83. (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is f_n, where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).
(b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

84. (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a)), \ldots,$ $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.
(b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L to five decimal places.

85. (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 2.

86. Use Definition 2 directly to prove that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

87. Prove Theorem 6.
[*Hint:* Use either Definition 2 or the Squeeze Theorem.]

88. Prove Theorem 7.

89. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

90. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$.

(a) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.
(c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.
(d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.
(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
(f) Use Theorem 12 to show that $\lim_{n\to\infty} (1 + 1/n)^n$ exists. (The limit is e. See Equation 6.4.9 or 6.4*.9.)

91. Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.
(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.

92. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.
(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$$

93. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.
(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.
(b) Show that $p_{n+1} < (b/a)p_n$.
(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; in other words, the population dies out.
(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.

LABORATORY PROJECT CAS LOGISTIC SEQUENCES

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the nth generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \leqslant p_n \leqslant 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 9.4. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first n terms of this sequence starting with an initial population p_0, where $0 < p_0 < 1$. Use this program to do the following.

1. Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that $1 < k < 3$. Graph each sequence. Do the sequences appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

2. Calculate terms of the sequence for a value of k between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?

3. Experiment with values of k between 3.4 and 3.5. What happens to the terms?

4. For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

11.2 Series

The current record for computing a decimal approximation for π was obtained by Shigeru Kondo and Alexander Yee in 2011 and contains more than 10 trillion decimal places.

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ \ldots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$\boxed{1} \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Does it make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the nth term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots$. The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1. (See also Figure 11 in *A Preview of Calculus*, page 6.) In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1. So it seems reasonable to say that the sum of this infinite series is 1 and to write

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

2 Definition Given a series $\Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Compare with the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_1^t f(x)\,dx$$

To find this integral we integrate from 1 to t and then let $t \to \infty$. For a series, we sum from 1 to n and then let $n \to \infty$.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{2n}{3n+5} = \lim_{n \to \infty} \frac{2}{3 + \frac{5}{n}} = \frac{2}{3}$$ ■

In Example 1 we were *given* an expression for the sum of the first n terms, but it's usually not easy to *find* such an expression. In Example 2, however, we look at a famous series for which we *can* find an explicit formula for s_n.

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (We have already considered the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$ on page 748.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n \to \infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3
$$s_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n \to \infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leqslant -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n \to \infty} s_n$ does not exist. Therefore the geometric series diverges in those cases. ■

Figure 1 provides a geometric demonstration of the result in Example 2. If the triangles are constructed as shown and s is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar} \quad \text{so} \quad s = \frac{a}{1 - r}$$

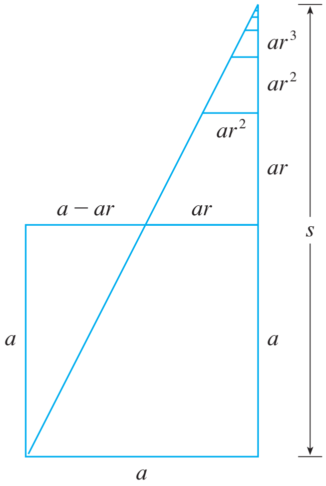


FIGURE 1

We summarize the results of Example 2 as follows.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

In words: The sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

EXAMPLE 3 Find the sum of the geometric series

$$5 - \tfrac{10}{3} + \tfrac{20}{9} - \tfrac{40}{27} + \cdots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by (4) and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$ ■

What do we really mean when we say that the sum of the series in Example 3 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first ten partial sums s_n and the graph in Figure 2 shows how the sequence of partial sums approaches 3.

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975

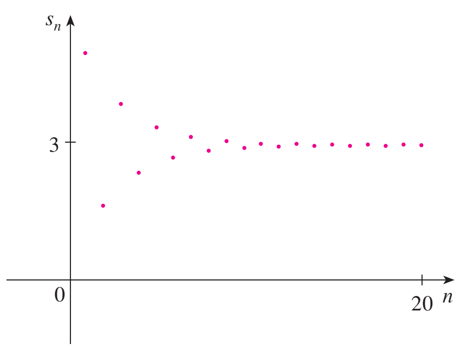


FIGURE 2

EXAMPLE 4 Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

SOLUTION Let's rewrite the nth term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\tfrac{4}{3}\right)^{n-1}$$

Another way to identify a and r is to write out the first few terms:

$$4 + \tfrac{16}{3} + \tfrac{64}{9} + \cdots$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges by (4). ■

EXAMPLE 5 A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL.

(a) Find the concentration after three days.

(b) What is the concentration after the nth dose?
(c) What is the limiting concentration?

SOLUTION
(a) Just before the daily dose of medication is administered, the concentration is reduced to 30% of the preceding day's concentration, that is, $0.3C_n$. With the new dose, the concentration is increased by 0.2 mg/mL and so

$$C_{n+1} = 0.2 + 0.3C_n$$

Starting with $C_0 = 0$ and putting $n = 0, 1, 2$ into this equation, we get

$$C_1 = 0.2 + 0.3C_0 = 0.2$$

$$C_2 = 0.2 + 0.3C_1 = 0.2 + 0.2(0.3) = 0.26$$

$$C_3 = 0.2 + 0.3C_2 = 0.2 + 0.2(0.3) + 0.2(0.3)^2 = 0.278$$

The concentration after three days is 0.278 mg/mL.
(b) After the nth dose the concentration is

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + 0.2(0.3)^{n-1}$$

This is a finite geometric series with $a = 0.2$ and $r = 0.3$, so by Formula 3 we have

$$C_n = \frac{0.2[1 - (0.3)^n]}{1 - 0.3} = \frac{2}{7}[1 - (0.3)^n] \text{ mg/mL}$$

(c) Because $0.3 < 1$, we know that $\lim_{n\to\infty} (0.3)^n = 0$. So the limiting concentration is

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{2}{7}[1 - (0.3)^n] = \frac{2}{7}(1 - 0) = \frac{2}{7} \text{ mg/mL}$$ ■

EXAMPLE 6 Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

SOLUTION

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$. Therefore

$$2.3\overline{17} = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$ ■

EXAMPLE 7 Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

SOLUTION Notice that this series starts with $n = 0$ and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.)

TEC Module 11.2 explores a series that depends on an angle θ in a triangle and enables you to see how rapidly the series converges when θ varies.

Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and (4) gives

$$\boxed{5} \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$ ■

EXAMPLE 8 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

and so
$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$ ■

Figure 3 illustrates Example 8 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \to 0$ and $s_n \to 1$. See Exercises 78 and 79 for two geometric interpretations of Example 8.

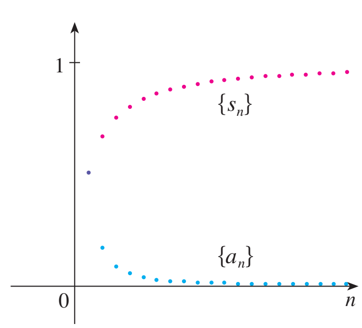


FIGURE 3

EXAMPLE 9 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums s_2, s_4, s_8, s_{16}, s_{32}, . . . and show that they become large.

$$\begin{aligned}
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}
\end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

The method used in Example 9 for showing that the harmonic series diverges is due to the French scholar Nicole Oresme (1323–1382).

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges. ■

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\Sigma\, a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = s - s = 0$$ ■

NOTE 1 With any *series* $\Sigma\, a_n$ we associate two *sequences:* the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\Sigma\, a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse of Theorem 6 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent. Observe that for the harmonic series $\Sigma\, 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but we showed in Example 9 that $\Sigma\, 1/n$ is divergent.

7 Test for Divergence If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 10 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

SOLUTION

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence. ■

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\Sigma\, a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence of $\Sigma\, a_n$. Remember the warning in Note 2: if $\lim_{n\to\infty} a_n = 0$, the series $\Sigma\, a_n$ might converge or it might diverge.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\,(a_n + b_n)$, and $\Sigma\,(a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1. For instance, here is how part (ii) of Theorem 8 is proved: Let

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

The nth partial sum for the series $\Sigma\,(a_n + b_n)$ is

$$u_n = \sum_{i=1}^{n} (a_i + b_i)$$

and, using Equation 4.2.10, we have

$$\begin{aligned} \lim_{n\to\infty} u_n &= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i \\ &= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t \end{aligned}$$

Therefore $\Sigma\,(a_n + b_n)$ is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$ ■

EXAMPLE 11 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$.

SOLUTION The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 8 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3 \cdot 1 + 1 = 4$$

■

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series $\sum_{n=1}^{\infty} n/(n^3 + 1)$ is convergent. Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.2 EXERCISES

1. (a) What is the difference between a sequence and a series?
   (b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

3–4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$

4. $s_n = \dfrac{n^2 - 1}{4n^2 + 1}$

5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \sin n$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

9–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \cos n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

15. Let $a_n = \dfrac{2n}{3n+1}$.
(a) Determine whether $\{a_n\}$ is convergent.
(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

(b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

17. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$

18. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$

19. $10 - 2 + 0.4 - 0.08 + \cdots$

20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

21. $\displaystyle\sum_{n=1}^{\infty} 12(0.73)^{n-1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} + \dfrac{1}{12} + \dfrac{1}{15} + \cdots$

28. $\dfrac{1}{3} + \dfrac{2}{9} + \dfrac{1}{27} + \dfrac{2}{81} + \dfrac{1}{243} + \dfrac{2}{729} + \cdots$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$

31. $\displaystyle\sum_{n=1}^{\infty} 3^{n+1}4^{-n}$

32. $\displaystyle\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$

35. $\displaystyle\sum_{k=1}^{\infty} (\sin 100)^k$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

37. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

38. $\displaystyle\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

39. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

40. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$

41. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 8). If it is convergent, find its sum.

43. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

44. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

46. $\displaystyle\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

47. $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

48. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

49. Let $x = 0.99999\ldots$.
(a) Do you think that $x < 1$ or $x = 1$?
(b) Sum a geometric series to find the value of x.
(c) How many decimal representations does the number 1 have?
(d) Which numbers have more than one decimal representation?

50. A sequence of terms is defined by

$$a_1 = 1 \qquad a_n = (5-n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

51–56 Express the number as a ratio of integers.

51. $0.\overline{8} = 0.8888\ldots$

52. $0.\overline{46} = 0.46464646\ldots$

53. $2.\overline{516} = 2.516516516\ldots$

54. $10.1\overline{35} = 10.135353535\ldots$

55. $1.234\overline{567}$

56. $5.\overline{71358}$

57–63 Find the values of x for which the series converges. Find the sum of the series for those values of x.

57. $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$

58. $\displaystyle\sum_{n=1}^{\infty} (x+2)^n$

59. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

60. $\displaystyle\sum_{n=0}^{\infty} (-4)^n(x-5)^n$

61. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{x^n}$

62. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

63. $\displaystyle\sum_{n=0}^{\infty} e^{nx}$

64. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

CAS **65–66** Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

65. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$

66. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

---

67. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n - 1}{n + 1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

68. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

69. A doctor prescribes a 100-mg antibiotic tablet to be taken every eight hours. Just before each tablet is taken, 20% of the drug remains in the body.

(a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?

(b) If Q_n is the quantity of the antibiotic in the body just after the nth tablet is taken, find an equation that expresses Q_{n+1} in terms of Q_n.

(c) What quantity of the antibiotic remains in the body in the long run?

70. A patient is injected with a drug every 12 hours. Immediately before each injection the concentration of the drug has been reduced by 90% and the new dose increases the concentration by 1.5 mg/L.

(a) What is the concentration after three doses?

(b) If C_n is the concentration after the nth dose, find a formula for C_n as a function of n.

(c) What is the limiting value of the concentration?

71. A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug remains in the body.

(a) What quantity of the drug is in the body after the third tablet? After the nth tablet?

(b) What quantity of the drug remains in the body in the long run?

72. After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents time in hours and a is a positive constant.

(a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n + 1)$st injection.

(b) Determine the limiting pre-injection concentration.

(c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.

73. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.

(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.

(b) Show that $\lim_{n\to\infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

74. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.

(b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)

(c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

75. Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

76. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

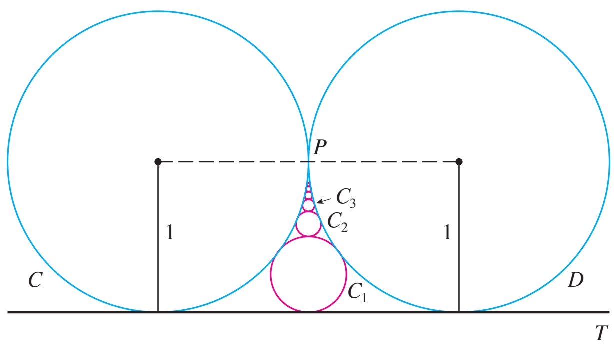
77. In Example 9 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See Exercise 6.2.109.)

If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

78. Graph the curves $y = x^n$, $0 \leqslant x \leqslant 1$, for $n = 0, 1, 2, 3, 4, \ldots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 8, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = 1$$

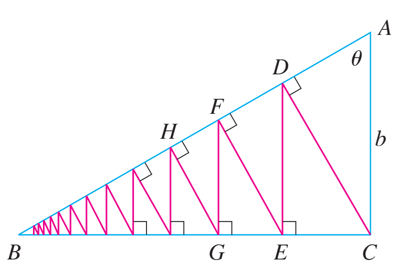
79. The figure shows two circles C and D of radius 1 that touch at P. The line T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2. This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 8.



80. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.



81. What is wrong with the following calculation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

82. Suppose that $\sum_{n=1}^{\infty} a_n$ ($a_n \neq 0$) is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

83. Prove part (i) of Theorem 8.

84. If $\sum a_n$ is divergent and $c \neq 0$, show that $\sum ca_n$ is divergent.

85. If $\sum a_n$ is convergent and $\sum b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]

86. If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum (a_n + b_n)$ necessarily divergent?

87. Suppose that a series $\sum a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \leqslant 1000$ for all n. Explain why $\sum a_n$ must be convergent.

88. The Fibonacci sequence was defined in Section 11.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Show that each of the following statements is true.

(a) $\dfrac{1}{f_{n-1} f_{n+1}} = \dfrac{1}{f_{n-1} f_n} - \dfrac{1}{f_n f_{n+1}}$

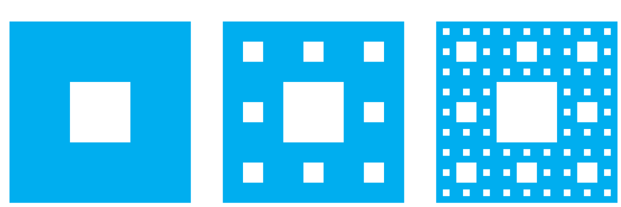
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = 1$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = 2$

89. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval [0, 1] and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in [0, 1] after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

(b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.

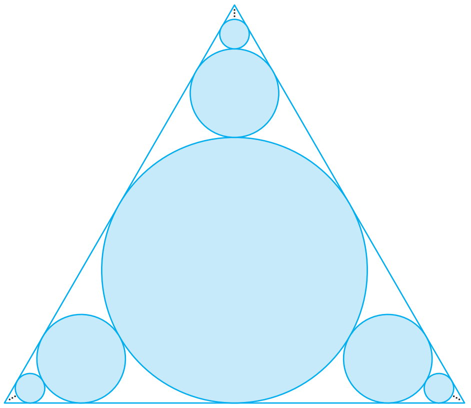


90. (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geqslant 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.

(b) Find $\lim_{n\to\infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

91. Consider the series $\sum_{n=1}^{\infty} n/(n+1)!$.
(a) Find the partial sums s_1, s_2, s_3, and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.
(b) Use mathematical induction to prove your guess.
(c) Show that the given infinite series is convergent, and find its sum.

92. In the figure at the right there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum 1/[n(n+1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

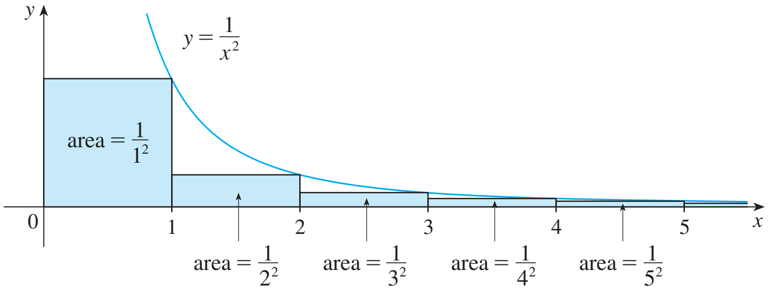


FIGURE 1

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geqslant 1$, which is the value of the integral $\int_1^\infty (1/x^2)\,dx$. In Section 7.8 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\,dx = 2$$

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$, but the proof of this fact is quite difficult. (See Problem 6 in the Problems Plus following Chapter 15.)]

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

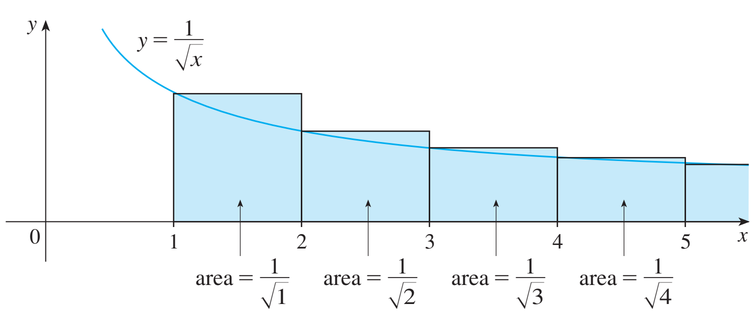


FIGURE 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \geqslant 1$, which is

equal to the integral $\int_1^\infty (1/\sqrt{x})\,dx$. But we know from Section 7.8 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite; that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test. (The proof is given at the end of this section.)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^\infty \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent by Note 4 of Section 11.2.

EXAMPLE 1 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_1^\infty 1/(x^2 + 1)\,dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2 + 1)$ is convergent. ■

EXAMPLE 2 For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION If $p < 0$, then $\lim_{n\to\infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

In order to use the Integral Test we need to be able to evaluate $\int_1^\infty f(x)\,dx$ and therefore we have to be able to find an antiderivative of f. Frequently this is difficult or impossible, so we need other tests for convergence too.

It follows from the Integral Test that the series $\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. (For $p = 1$, this series is the harmonic series discussed in Example 11.2.9.) ■

The series in Example 2 is called the ***p*-series**. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

1 The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

EXAMPLE 3
(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.
(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$. ■

NOTE We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{whereas} \qquad \int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\, dx$$

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, when $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \frac{(\ln x)^2}{2} \Bigg]_1^t$$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\Sigma\, (\ln n)/n$ is also divergent by the Integral Test. ■

Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\Sigma\, a_n$ is convergent and we now want to find an approximation to the sum s of the series. Of course, any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that

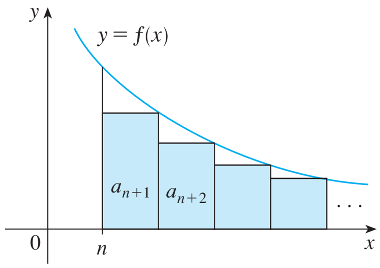


FIGURE 3

$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^\infty f(x)\,dx$$

Similarly, we see from Figure 4 that

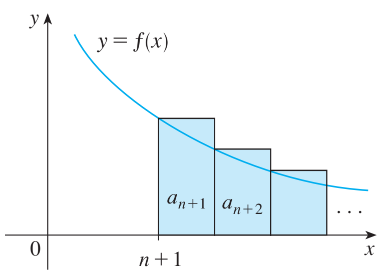


FIGURE 4

$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^\infty f(x)\,dx$$

So we have proved the following error estimate.

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\,dx \leqslant R_n \leqslant \int_n^\infty f(x)\,dx$$

EXAMPLE 5

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

SOLUTION In both parts (a) and (b) we need to know $\int_n^\infty f(x)\,dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a) Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate in (2), we have

$$R_{10} \leqslant \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leqslant 0.0005$. Since

$$R_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005. ■

If we add s_n to each side of the inequalities in (2), we get

3

$$s_n + \int_{n+1}^\infty f(x)\,dx \leqslant s \leqslant s_n + \int_n^\infty f(x)\,dx$$

because $s_n + R_n = s$. The inequalities in (3) give a lower bound and an upper bound for s. They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Although Euler was able to calculate the exact sum of the p-series for $p = 2$, nobody has been able to find the exact sum for $p = 3$. In Example 6, however, we show how to *estimate* this sum.

EXAMPLE 6 Use (3) with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$.

SOLUTION The inequalities in (3) become

$$s_{10} + \int_{11}^\infty \frac{1}{x^3}\,dx \leqslant s \leqslant s_{10} + \int_{10}^\infty \frac{1}{x^3}\,dx$$

From Example 5 we know that

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \leqslant s \leqslant s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leqslant s \leqslant 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx 1.2021 \qquad \text{with error} < 0.0005$$ ■

If we compare Example 6 with Example 5, we see that the improved estimate in (3) can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 5 but only 10 terms in Example 6.

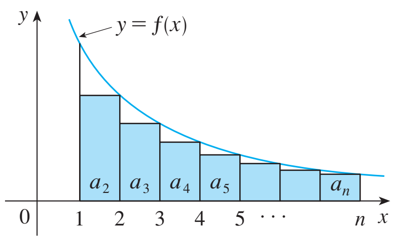


FIGURE 5

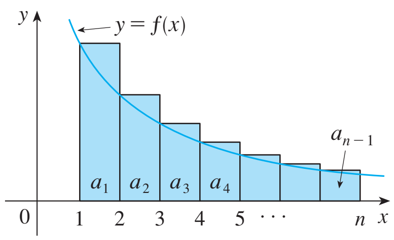


FIGURE 6

Proof of the Integral Test

We have already seen the basic idea behind the proof of the Integral Test in Figures 1 and 2 for the series $\Sigma\, 1/n^2$ and $\Sigma\, 1/\sqrt{n}$. For the general series $\Sigma\, a_n$, look at Figures 5 and 6. The area of the first shaded rectangle in Figure 5 is the value of f at the right endpoint of $[1, 2]$, that is, $f(2) = a_2$. So, comparing the areas of the shaded rectangles with the area under $y = f(x)$ from 1 to n, we see that

$$\boxed{4} \qquad a_2 + a_3 + \cdots + a_n \leqslant \int_1^n f(x)\, dx$$

(Notice that this inequality depends on the fact that f is decreasing.) Likewise, Figure 6 shows that

$$\boxed{5} \qquad \int_1^n f(x)\, dx \leqslant a_1 + a_2 + \cdots + a_{n-1}$$

(i) If $\int_1^\infty f(x)\, dx$ is convergent, then (4) gives

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\, dx \leqslant \int_1^\infty f(x)\, dx$$

since $f(x) \geqslant 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^{n} a_i \leqslant a_1 + \int_1^\infty f(x)\, dx = M, \text{ say}$$

Since $s_n \leqslant M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n + 1) \geqslant 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\Sigma\, a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\, dx$ is divergent, then $\int_1^n f(x)\, dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$. But (5) gives

$$\int_1^n f(x)\, dx \leqslant \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so $\Sigma\, a_n$ diverges. ■

11.3 EXERCISES

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^\infty \frac{1}{x^{1.3}}\, dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\, dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty} n^{-3}$

4. $\displaystyle\sum_{n=1}^{\infty} n^{-0.3}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n - 1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n - 1)^4}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

8. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

9–26 Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

10. $\displaystyle\sum_{n=3}^{\infty} n^{-0.9999}$

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots$

12. $\dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \dfrac{1}{11} + \dfrac{1}{13} + \cdots$

13. $\dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{11} + \dfrac{1}{15} + \dfrac{1}{19} + \cdots$

14. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^{3/2}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4}$

20. $\displaystyle\sum_{n=3}^{\infty} \frac{3n - 4}{n^2 - 2n}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

23. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

24. $\displaystyle\sum_{k=1}^{\infty} ke^{-k^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

29–32 Find the values of p for which the series is convergent.

29. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

30. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\,[\ln(\ln n)]^p}$

31. $\displaystyle\sum_{n=1}^{\infty} n(1 + n^2)^p$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

34. Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 760.) Use this fact to find the sum of each series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n + 1)^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

35. Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$ (b) $\displaystyle\sum_{k=5}^{\infty} \frac{1}{(k - 2)^4}$

36. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use (3) with $n = 10$ to give an improved estimate of the sum.
(c) Compare your estimate in part (b) with the exact value given in Exercise 35.
(d) Find a value of n so that s_n is within 0.00001 of the sum.

37. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using (3) with $n = 10$.
(c) Compare your estimate in part (b) with the exact value given in Exercise 34.
(d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

38. Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.

39. Estimate $\sum_{n=1}^{\infty} (2n + 1)^{-6}$ correct to five decimal places.

40. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

41. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

CAS **42.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.
(b) Find an upper bound for the error in the approximation $s \approx s_n$.
(c) What is the smallest value of n such that this upper bound is less than 0.05?
(d) Find s_n for this value of n.

43. (a) Use (4) to show that if s_n is the nth partial sum of the harmonic series, then

$$s_n \leqslant 1 + \ln n$$

(b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

44. Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)

(a) Draw a picture like Figure 6 with $f(x) = 1/x$ and interpret t_n as an area [or use (5)] to show that $t_n > 0$ for all n.

(b) Interpret

$$t_n - t_{n+1} = [\ln(n + 1) - \ln n] - \frac{1}{n + 1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore $\{t_n\}$ is a decreasing sequence.

(c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.

45. Find all positive values of b for which the series $\Sigma_{n=1}^{\infty} b^{\ln n}$ converges.

46. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n + 1} \right)$$

11.4 The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

1
$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\Sigma_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series (1) is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series (1) has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

It is important to keep in mind the distinction between a sequence and a series. A sequence is a list of numbers, whereas a series is a sum. With every series Σa_n there are associated two sequences: the sequence $\{a_n\}$ of terms and the sequence $\{s_n\}$ of partial sums.

PROOF

(i) Let $\quad s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing $(s_{n+1} = s_n + a_{n+1} \geqslant s_n)$. Also $t_n \to t$, so $t_n \leqslant t$ for all n. Since $a_i \leqslant b_i$, we have $s_n \leqslant t_n$. Thus $s_n \leqslant t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus Σa_n converges.

(ii) If Σb_n is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geqslant b_i$ so $s_n \geqslant t_n$. Thus $s_n \to \infty$. Therefore Σa_n diverges. ■

Standard Series for Use with the Comparison Test

In using the Comparison Test we must, of course, have some known series Σb_n for the purpose of comparison. Most of the time we use one of these series:

- A p-series $\left[\Sigma 1/n^p \text{ converges if } p > 1 \text{ and diverges if } p \leqslant 1; \text{ see (11.3.1)}\right]$
- A geometric series $\left[\Sigma ar^{n-1} \text{ converges if } |r| < 1 \text{ and diverges if } |r| \geqslant 1; \text{ see (11.2.4)}\right]$

EXAMPLE 1 Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\Sigma\, 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test. ■

NOTE 1 Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 2 Test the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 11.3.4, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geqslant 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geqslant 3$$

We know that $\Sigma\, 1/k$ is divergent (p-series with $p = 1$). Thus the given series is divergent by the Comparison Test. ■

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\Sigma\, b_n = \Sigma\left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\Sigma\, 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma\left(\frac{1}{2}\right)^n$. In such cases the following test can be used.

Exercises 40 and 41 deal with the cases $c = 0$ and $c = \infty$.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N$$

If $\Sigma\, b_n$ converges, so does $\Sigma\, Mb_n$. Thus $\Sigma\, a_n$ converges by part (i) of the Comparison Test. If $\Sigma\, b_n$ diverges, so does $\Sigma\, mb_n$ and part (ii) of the Comparison Test shows that $\Sigma\, a_n$ diverges. ■

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\Sigma\, 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test. ■

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \dfrac{3}{n}}{2\sqrt{\dfrac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\Sigma\, b_n = 2\,\Sigma\, 1/n^{1/2}$ is divergent (p-series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test. ■

Notice that in testing many series we find a suitable comparison series $\Sigma\, b_n$ by keeping only the highest powers in the numerator and denominator.

Estimating Sums

If we have used the Comparison Test to show that a series $\Sigma\, a_n$ converges by comparison with a series $\Sigma\, b_n$, then we may be able to estimate the sum $\Sigma\, a_n$ by comparing remainders. As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

For the comparison series $\Sigma\, b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leqslant b_n$ for all n, we have $R_n \leqslant T_n$. If $\Sigma\, b_n$ is a p-series, we can estimate its remainder T_n as in Section 11.3. If $\Sigma\, b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (see Exercises 35 and 36). In either case we know that R_n is smaller than T_n.

EXAMPLE 5 Use the sum of the first 100 terms to approximate the sum of the series $\Sigma\, 1/(n^3 + 1)$. Estimate the error involved in this approximation.

SOLUTION Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. The remainder T_n for the comparison series $\Sigma\, 1/n^3$ was estimated in Example 11.3.5 using the Remainder Estimate for the Integral Test. There we found that

$$T_n \leqslant \int_n^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \leqslant T_n \leqslant \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leqslant \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005. ■

11.4 EXERCISES

1. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be convergent.
(a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

2. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be divergent.
(a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
(b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

3–32 Determine whether the series converges or diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 8}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n - 1}{n^3 + 1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1 + k^3}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3 + 4k + 3}}$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k - 1)(k^2 - 1)}{(k + 1)(k^2 + 4)^2}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \cos n}{e^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} + 2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{n^3 + n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + n + 1}{n^4 + n^2}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1 + n}}{2 + n}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{n + 2}{(n + 1)^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{5 + 2n}{(1 + n^2)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 3^n}{n + 2^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + 1}{ne^n + 1}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5 + n^5}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4}$

35. $\displaystyle\sum_{n=1}^{\infty} 5^{-n} \cos^2 n$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 4^n}$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers 0, 1, 2, . . . , 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

38. For what values of p does the series $\Sigma_{n=2}^{\infty} 1/(n^p \ln n)$ converge?

39. Prove that if $a_n \geqslant 0$ and $\Sigma\, a_n$ converges, then $\Sigma\, a_n^2$ also converges.

40. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is convergent. Prove that if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0$$

then $\Sigma\, a_n$ is also convergent.

(b) Use part (a) to show that the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$

41. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is divergent. Prove that if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$$

then $\Sigma\, a_n$ is also divergent.

(b) Use part (a) to show that the series diverges.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

42. Give an example of a pair of series $\Sigma\, a_n$ and $\Sigma\, b_n$ with positive terms where $\lim_{n \to \infty} (a_n/b_n) = 0$ and $\Sigma\, b_n$ diverges, but $\Sigma\, a_n$ converges. (Compare with Exercise 40.)

43. Show that if $a_n > 0$ and $\lim_{n \to \infty} n a_n \neq 0$, then $\Sigma\, a_n$ is divergent.

44. Show that if $a_n > 0$ and $\Sigma\, a_n$ is convergent, then $\Sigma \ln(1 + a_n)$ is convergent.

45. If $\Sigma\, a_n$ is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent?

46. If $\Sigma\, a_n$ and $\Sigma\, b_n$ are both convergent series with positive terms, is it true that $\Sigma\, a_n b_n$ is also convergent?

11.5 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leqslant b_n$ for all n

(ii) $\displaystyle\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Before giving the proof let's look at Figure 1, which gives a picture of the idea behind the proof. We first plot $s_1 = b_1$ on a number line. To find s_2 we subtract b_2, so s_2 is to the left of s_1. Then to find s_3 we add b_3, so s_3 is to the right of s_2. But, since $b_3 < b_2$, s_3 is to the left of s_1. Continuing in this manner, we see that the partial sums oscillate back and forth. Since $b_n \to 0$, the successive steps are becoming smaller and smaller. The even partial sums $s_2, s_4, s_6, \ldots$ are increasing and the odd partial sums $s_1, s_3, s_5, \ldots$ are decreasing. Thus it seems plausible that both are converging to some number s, which is the sum of the series. Therefore we consider the even and odd partial sums separately in the following proof.

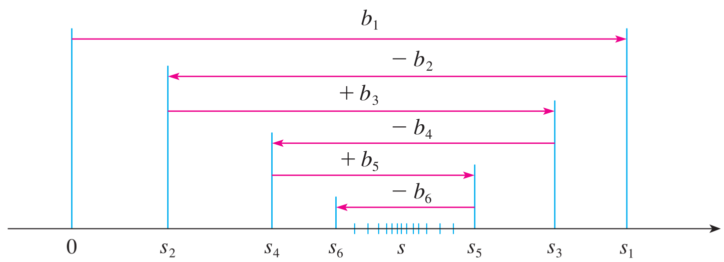


FIGURE 1

PROOF OF THE ALTERNATING SERIES TEST We first consider the even partial sums:

$$s_2 = b_1 - b_2 \geqslant 0 \qquad \text{since } b_2 \leqslant b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geqslant s_2 \qquad \text{since } b_4 \leqslant b_3$$

In general $$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geqslant s_{2n-2} \qquad \text{since } b_{2n} \leqslant b_{2n-1}$$

Thus $$0 \leqslant s_2 \leqslant s_4 \leqslant s_6 \leqslant \cdots \leqslant s_{2n} \leqslant \cdots$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in parentheses is positive, so $s_{2n} \leqslant b_1$ for all n. Therefore the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \qquad \text{[by condition (ii)]} \\ &= s \end{aligned}$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ [see Exercise 11.1.92(a)] and so the series is convergent. ■

Figure 2 illustrates Example 1 by showing the graphs of the terms $a_n = (-1)^{n-1}/n$ and the partial sums s_n. Notice how the values of s_n zigzag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (see Exercise 36).

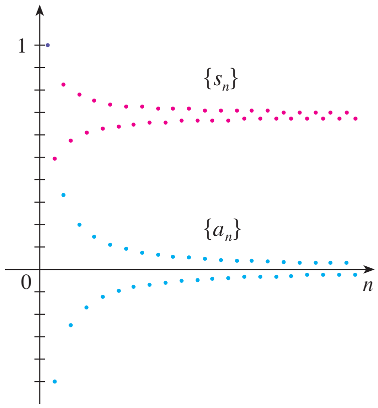


FIGURE 2

EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

$$\text{(i)}\quad b_{n+1} < b_n \qquad \text{because} \qquad \frac{1}{n+1} < \frac{1}{n}$$

$$\text{(ii)}\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test. ■

EXAMPLE 2 The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \dfrac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence. ■

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3+1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3+1)$, we find that

$$f'(x) = \frac{x(2 - x^3)}{(x^3+1)^2}$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geqslant 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Instead of verifying condition (i) of the Alternating Series Test by computing a derivative, we could verify that $b_{n+1} < b_n$ directly by using the technique of Solution 1 of Example 11.1.13.

Condition (ii) is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\dfrac{1}{n}}{1 + \dfrac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test. ■

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

You can see geometrically why the Alternating Series Estimation Theorem is true by looking at Figure 1 (on page 773). Notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on. Notice also that s lies between any two consecutive partial sums.

Alternating Series Estimation Theorem If $s = \Sigma\,(-1)^{n-1}b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then
$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. (There we showed that s is larger than all the even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n| = b_{n+1}$$ ■

By definition, $0! = 1$.

EXAMPLE 4 Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)}\quad \frac{1}{(n+1)!} = \frac{1}{n!\,(n+1)} < \frac{1}{n!}$$

$$\text{(ii)}\quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \quad \frac{1}{n!} \to 0 \text{ as } n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$$= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots$$

Notice that
$$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

and
$$s_6 = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.368056$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leqslant b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places. ■

In Section 11.10 we will prove that $e^x = \Sigma_{n=0}^{\infty}\, x^n/n!$ for all x, so what we have obtained in Example 4 is actually an approximation to the number e^{-1}.

NOTE The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

11.5 EXERCISES

1. (a) What is an alternating series?
(b) Under what conditions does an alternating series converge?
(c) If these conditions are satisfied, what can you say about the remainder after n terms?

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

4. $\frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3 + 5n}$

6. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{3n - 1}{2n + 1}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2 + n + 1}$

9. $\sum_{n=1}^{\infty} (-1)^n e^{-n}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n + 3}$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 4}$

12. $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

13. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

15. $\sum_{n=0}^{\infty} \frac{\sin\left(n + \frac{1}{2}\right)\pi}{1 + \sqrt{n}}$

16. $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

17. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

20. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

21–22 Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.

21. $\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

22. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

23–26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $\quad (|\text{error}| < 0.00005)$

24. $\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{3}\right)^n}{n}$ $\quad (|\text{error}| < 0.0005)$

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n}$ $\quad (|\text{error}| < 0.0005)$

26. $\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)^n$ $\quad (|\text{error}| < 0.00005)$

27–30 Approximate the sum of the series correct to four decimal places.

27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$

29. $\sum_{n=1}^{\infty} (-1)^n n e^{-2n}$

30. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 4^n}$

31. Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ an overestimate or an underestimate of the total sum? Explain.

32–34 For what values of p is each series convergent?

32. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + p}$

34. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

35. Show that the series $\Sigma\,(-1)^{n-1}b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

36. Use the following steps to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$.

(b) From Exercise 11.3.44 we have

$$h_n - \ln n \to \gamma \qquad \text{as } n \to \infty$$

and therefore

$$h_{2n} - \ln(2n) \to \gamma \qquad \text{as } n \to \infty$$

Use these facts together with part (a) to show that $s_{2n} \to \ln 2$ as $n \to \infty$.

11.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 3 that the idea of absolute convergence sometimes helps in such cases.

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Notice that if $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

EXAMPLE 1 The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$). ■

EXAMPLE 2 We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (see Example 11.5.1), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent. ■

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

PROOF Observe that the inequality

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\Sigma\, a_n$ is absolutely convergent, then $\Sigma\, |a_n|$ is convergent, so $\Sigma\, 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\Sigma\, (a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent. ■

EXAMPLE 3 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Figure 1 shows the graphs of the terms a_n and partial sums s_n of the series in Example 3. Notice that the series is not alternating but has positive and negative terms.

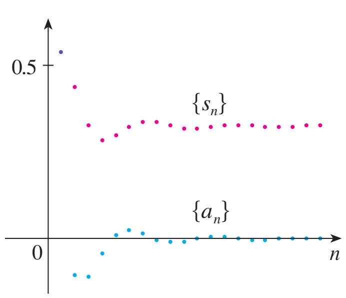


FIGURE 1

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.) We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leqslant 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$$

We know that $\Sigma\, 1/n^2$ is convergent (p-series with $p = 2$) and therefore $\Sigma\, |\cos n|/n^2$ is convergent by the Comparison Test. Thus the given series $\Sigma\, (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3. ■

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

PROOF

(i) The idea is to compare the given series with a convergent geometric series. Since $L < 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \qquad \text{and} \qquad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| < r \qquad \text{whenever } n \geqslant N$$

or, equivalently,

4 $$|a_{n+1}| < |a_n|\, r \qquad \text{whenever } n \geqslant N$$

Putting n successively equal to $N, N+1, N+2, \ldots$ in (4), we obtain

$$|a_{N+1}| < |a_N|\, r$$
$$|a_{N+2}| < |a_{N+1}|\, r < |a_N|\, r^2$$
$$|a_{N+3}| < |a_{N+2}|\, r < |a_N|\, r^3$$

and, in general,

5 $$|a_{N+k}| < |a_N|\, r^k \qquad \text{for all } k \geqslant 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|\, r^k = |a_N|\, r + |a_N|\, r^2 + |a_N|\, r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the inequality (5), together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (Recall that a finite number of terms doesn't affect convergence.) Therefore $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \text{whenever } n \geqslant N$$

This means that $|a_{n+1}| > |a_n|$ whenever $n \geqslant N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence. ■

NOTE Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \qquad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}} \to 1 \qquad \text{as } n \to \infty$$

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as we will see in Examples 4 and 5.

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

EXAMPLE 4 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent. ■

Estimating Sums

In the last three sections we used various methods for estimating the sum of a series—the method depended on which test was used to prove convergence. What about series for which the Ratio Test works? There are two possibilities: If the series happens to be an alternating series, as in Example 4, then it is best to use the methods of Section 11.5. If the terms are all positive, then use the special methods explained in Exercise 46.

EXAMPLE 5 Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \qquad \text{as } n \to \infty$$

(see Equation 6.4.9 or 6.4*.9). Since $e > 1$, the given series is divergent by the Ratio Test. ■

NOTE Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geqslant n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

The following test is convenient to apply when nth powers occur. Its proof is similar to the proof of the Ratio Test and is left as Exercise 49.

The Root Test

(i) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

EXAMPLE 6 Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \dfrac{3}{n}}{3 + \dfrac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test. ■

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

if $\Sigma\, a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\Sigma\, a_n$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\boxed{6} \qquad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2$$

(See Exercise 11.5.36.) If we multiply this series by $\frac{1}{2}$, we get

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Inserting zeros between the terms of this series, we have

Adding these zeros does not affect the sum of the series; each term in the sequence of partial sums is repeated, but the limit is the same.

$$\boxed{7} \qquad 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Now we add the series in Equations 6 and 7 using Theorem 11.2.8:

$$\boxed{8} \qquad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2}\ln 2$$

Notice that the series in (8) contains the same terms as in (6) but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to r.

A proof of this fact is outlined in Exercise 52.

11.6 EXERCISES

1. What can you say about the series $\Sigma\, a_n$ in each of the following cases?

(a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8$ (b) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8$

(c) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

2–6 Determine whether the series is absolutely convergent or conditionally convergent.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

3. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{5n+1}$ **4.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3+1}$

5. $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

6. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 4}$

7–24 Use the Ratio Test to determine whether the series is convergent or divergent.

7. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

8. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

10. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

11. $\sum_{k=1}^{\infty} \frac{1}{k!}$

12. $\sum_{k=1}^{\infty} ke^{-k}$

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

14. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

15. $\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$

16. $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$

17. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

18. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

19. $\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$

20. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

21. $1 - \frac{2!}{1 \cdot 3} + \frac{3!}{1 \cdot 3 \cdot 5} - \frac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots + (-1)^{n-1} \frac{n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} + \cdots$

22. $\frac{2}{3} + \frac{2 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots$

23. $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$

25–30 Use the Root Test to determine whether the series is convergent or divergent.

25. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

26. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

27. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

28. $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

29. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

30. $\sum_{n=0}^{\infty} (\arctan n)^n$

31–38 Use any test to determine whether the series is absolutely convergent, conditionally convergent, or divergent.

31. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

32. $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$

33. $\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$

34. $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$

35. $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n$

36. $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1 + n\sqrt{n}}$

37. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

38. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

39. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

40. A series $\Sigma\, a_n$ is defined by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

41–42 Let $\{b_n\}$ be a sequence of positive numbers that converges to $\frac{1}{2}$. Determine whether the given series is absolutely convergent.

41. $\sum_{n=1}^{\infty} \frac{b_n^n \cos n\pi}{n}$

42. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}$

43. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

44. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

45. (a) Show that $\Sigma_{n=0}^{\infty}\, x^n/n!$ converges for all x.

(b) Deduce that $\lim_{n \to \infty} x^n/n! = 0$ for all x.

46. Let $\Sigma\, a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n \to \infty} r_n = L < 1$, so $\Sigma\, a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leqslant \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leqslant \frac{a_{n+1}}{1 - L}$$

47. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/(n2^n)$. Use Exercise 46 to estimate the error in using s_5 as an approximation to the sum of the series.
(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

48. Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 46 to estimate the error.

49. Prove the Root Test. [*Hint for part (i):* Take any number r such that $L < r < 1$ and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geqslant N$.]

50. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.
(a) Verify that the series is convergent.
(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

51. Given any series Σa_n, we define a series Σa_n^+ whose terms are all the positive terms of Σa_n and a series Σa_n^- whose terms are all the negative terms of Σa_n. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad a_n^- = \frac{a_n - |a_n|}{2}$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.
(a) If Σa_n is absolutely convergent, show that both of the series Σa_n^+ and Σa_n^- are convergent.
(b) If Σa_n is conditionally convergent, show that both of the series Σa_n^+ and Σa_n^- are divergent.

52. Prove that if Σa_n is a conditionally convergent series and r is any real number, then there is a rearrangement of Σa_n whose sum is r. [*Hints:* Use the notation of Exercise 51. Take just enough positive terms a_n^+ so that their sum is greater than r. Then add just enough negative terms a_n^- so that the cumulative sum is less than r. Continue in this manner and use Theorem 11.2.6.]

53. Suppose the series Σa_n is conditionally convergent.
(a) Prove that the series $\Sigma n^2 a_n$ is divergent.
(b) Conditional convergence of Σa_n is not enough to determine whether Σna_n is convergent. Show this by giving an example of a conditionally convergent series such that Σna_n converges and an example where Σna_n diverges.

11.7 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form Σar^{n-1} or Σar^n, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if Σa_n has some negative terms, then we can apply the Comparison Test to $\Sigma |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n \to \infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. ■

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$ ■

EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\, dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. ■

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test. ■

EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test. ■

EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test. ■

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 1}$

2. $\sum_{n=1}^{\infty} \frac{n - 1}{n^3 + 1}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^3 + 1}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^2 + 1}$

5. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + n)^{3n}}$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

9. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n} \right)$

12. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2 + 1}}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1 + 2^n}$

15. $\sum_{k=1}^{\infty} \frac{2^{k-1} 3^{k+1}}{k^k}$

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 + n}$

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n - 1)}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)}$

21. $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

22. $\sum_{k=1}^{\infty} \frac{1}{2 + \sin k}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k + 1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j + 5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$

32. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\sum_{n=1}^{\infty} \left(\frac{n}{n + 1} \right)^{n^2}$

34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

35. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

11.8 Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\boxed{2} \qquad \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geqslant 1$. (See Equation 11.2.5.)

Trigonometric Series
A power series is a series in which each term is a power function. A **trigonometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trigonometric functions. This type of series is discussed on the website

www.stewartcalculus.com

Click on *Additional Topics* and then on *Fourier Series*.

In fact if we put $x = \frac{1}{2}$ in the geometric series (2) we get the convergent series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

but if we put $x = 2$ in (2) we get the divergent series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \cdots$$

More generally, a series of the form

3 $$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a**. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 3 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$. Notice also that when $x = a$, all of the terms are 0 for $n \geqslant 1$ and so the power series (3) always converges when $x = a$.

EXAMPLE 1 For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

SOLUTION We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

Notice that

$$(n+1)! = (n+1)n(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1$$
$$= (n+1)n!$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$. ■

EXAMPLE 2 For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION Let $a_n = (x-3)^n/n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \qquad \text{as } n\to\infty$$

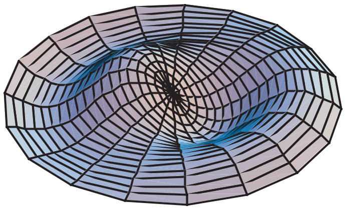
By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x - 3| < 1$ and divergent when $|x - 3| > 1$. Now

$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x - 3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\Sigma\, 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\Sigma\, (-1)^n/n$, which converges by the Alternating Series Test. Thus the given power series converges for $2 \leqslant x < 4$. ■


Membrane courtesy of National Film Board of Canada



Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a **Bessel function**, after the German astronomer Friedrich Bessel (1784–1846), and the function given in Exercise 35 is another example of a Bessel function. In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$. ■

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

$$s_0(x) = 1$$

$$s_1(x) = 1 - \frac{x^2}{4}$$

$$s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

$$s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

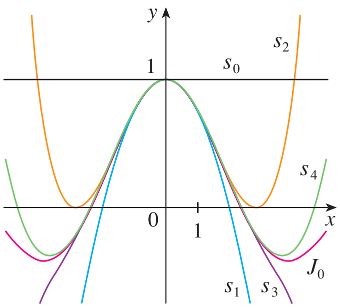


FIGURE 1
Partial sums of the Bessel function J_0

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included. Figure 2 shows a more complete graph of the Bessel function.

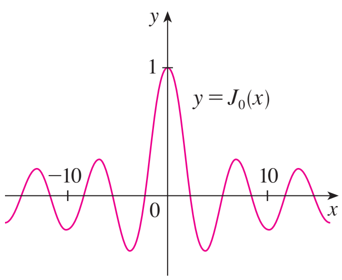


FIGURE 2

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 2, the infinite interval $(-\infty, \infty)$ in Example 3, and a collapsed interval $[0, 0] = \{0\}$ in Example 1]. The following theorem, proved in Appendix F, says that this is true in general.

4 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \qquad (a - R, a + R] \qquad [a - R, a + R) \qquad [a - R, a + R]$$

The situation is illustrated in Figure 3.

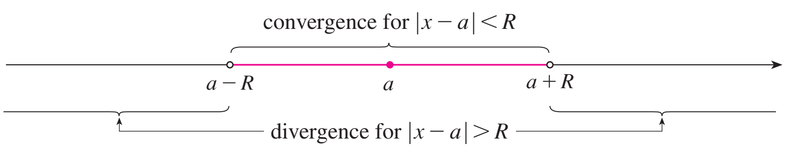


FIGURE 3

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n / \sqrt{n+1}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \left| -3x \sqrt{\frac{n+1}{n+2}} \right|$$

$$= 3 \sqrt{\frac{1 + (1/n)}{1 + (2/n)}} \, |x| \to 3|x| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (Use the Integral Test or simply observe that it is a p-series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore the given power series converges when $-\frac{1}{3} < x \leqslant \frac{1}{3}$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$. ■

EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n / 3^{n+1}$, then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right|$$

$$= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2|<3$ can be written as $-5<x<1$, so we test the series at the endpoints -5 and 1. When $x=-5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x=1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5<x<1$, so the interval of convergence is $(-5, 1)$. ■

11.8 EXERCISES

1. What is a power series?

2. (a) What is the radius of convergence of a power series? How do you find it?
(b) What is the interval of convergence of a power series? How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

5. $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\sum_{n=1}^{\infty} n^n x^n$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$

10. $\sum_{n=1}^{\infty} 2^n n^2 x^n$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n$

13. $\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$

14. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

15. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

17. $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

18. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n} (x+6)^n$

19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

20. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

21. $\sum_{n=1}^{\infty} \frac{n}{b^n} (x-a)^n, \quad b>0$

22. $\sum_{n=2}^{\infty} \frac{b^n}{\ln n} (x-a)^n, \quad b>0$

23. $\sum_{n=1}^{\infty} n!(2x-1)^n$

24. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

25. $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

26. $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

29. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, can we conclude that each of the following series is convergent?

(a) $\sum_{n=0}^{\infty} c_n(-2)^n$ (b) $\sum_{n=0}^{\infty} c_n(-4)^n$

30. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x=-4$ and diverges when $x=6$. What can be said about the convergence or divergence of the following series?

(a) $\sum_{n=0}^{\infty} c_n$ (b) $\sum_{n=0}^{\infty} c_n 8^n$

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$ (d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

32. Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is
(a) (p, q) (b) $(p, q]$ (c) $[p, q)$ (d) $[p, q]$

33. Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

34. Graph the first several partial sums $s_n(x)$ of the series $\sum_{n=0}^{\infty} x^n$, together with the sum function $f(x) = 1/(1 - x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

35. The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$$

is called the *Bessel function of order 1*.
(a) Find its domain.
(b) Graph the first several partial sums on a common screen.
CAS (c) If your CAS has built-in Bessel functions, graph J_1 on the same screen as the partial sums in part (b) and observe how the partial sums approximate J_1.

36. The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).
(a) Find the domain of the Airy function.
(b) Graph the first several partial sums on a common screen.
CAS (c) If your CAS has built-in Airy functions, graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A.

37. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

38. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geqslant 0$, find the interval of convergence of the series and a formula for $f(x)$.

39. Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

40. Suppose that the power series $\sum c_n(x - a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence of the power series.

41. Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

42. Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.)

We start with an equation that we have seen before:

1
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We first encountered this equation in Example 11.2.7, where we obtained it by observing that the series is a geometric series with $a = 1$ and $r = x$. But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

A geometric illustration of Equation 1 is shown in Figure 1. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x)$$

where

$$s_n(x) = 1 + x + x^2 + \cdots + x^n$$

is the nth partial sum. Notice that as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

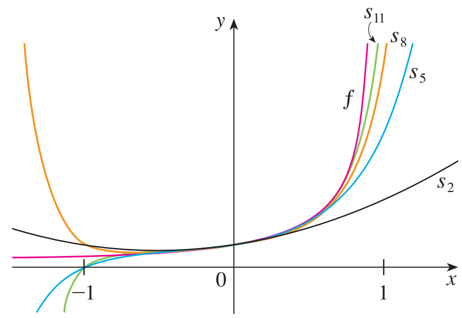


FIGURE 1 $f(x) = \dfrac{1}{1-x}$ and some partial sums

EXAMPLE 1 Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

SOLUTION Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the interval of convergence is $(-1, 1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.) ■

EXAMPLE 2 Find a power series representation for $1/(x + 2)$.

SOLUTION In order to put this function in the form of the left side of Equation 1, we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\dfrac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2, 2)$. ■

EXAMPLE 3 Find a power series representation of $x^3/(x + 2)$.

SOLUTION Since this function is just x^3 times the function in Example 2, all we have to do is to multiply that series by x^3:

It's legitimate to move x^3 across the sigma sign because it doesn't depend on n. [Use Theorem 11.2.8(i) with $c = x^3$.]

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

$$= \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{8}x^5 - \tfrac{1}{16}x^6 + \cdots$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

As in Example 2, the interval of convergence is $(-2, 2)$. ■

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

In part (ii), $\int c_0\,dx = c_0x + C_1$ is written as $c_0(x-a) + C$, where $C = C_1 + ac_0$, so all the terms of the series have the same form.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

(iv) $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 38.)

NOTE 2 Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 39.)

NOTE 3 The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We will discuss this method in Chapter 17.

EXAMPLE 4 In Example 11.8.3 we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all x. Thus, by Theorem 2, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$ ■

EXAMPLE 5 Express $1/(1-x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

If we wish, we can replace n by $n + 1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$. ■

EXAMPLE 6 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Integrating both sides of this equation, we get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \qquad |x| < 1$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1 + 0) = C$.

Thus $C = 0$ and

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$. ■

EXAMPLE 7 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\tan^{-1}x = \int \frac{1}{1 + x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1. ■

The power series for $\tan^{-1}x$ obtained in Example 7 is called *Gregory's series* after the Scottish mathematician James Gregory (1638–1675), who had anticipated some of Newton's discoveries. We have shown that Gregory's series is valid when $-1 < x < 1$, but it turns out (although it isn't easy to prove) that it is also valid when $x = \pm 1$. Notice that when $x = 1$ the series becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This beautiful result is known as the Leibniz formula for π.

EXAMPLE 8

(a) Evaluate $\int [1/(1 + x^7)]\,dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} [1/(1 + x^7)]\,dx$ correct to within 10^{-7}.

SOLUTION

(a) The first step is to express the integrand, $1/(1 + x^7)$, as the sum of a power series. As in Example 1, we start with Equation 1 and replace x by $-x^7$:

$$\frac{1}{1 + x^7} = \frac{1}{1 - (-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots$$

Now we integrate term by term:

$$\int \frac{1}{1 + x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n + 1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

This example demonstrates one way in which power series representations are useful. Integrating $1/(1 + x^7)$ by hand is incredibly difficult. Different computer algebra systems return different forms of the answer, but they are all extremely complicated. (If you have a CAS, try it yourself.) The infinite series answer that we obtain in Example 8(a) is actually much easier to deal with than the finite answer provided by a CAS.

(b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$ ■

11.9 EXERCISES

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–10 Find a power series representation for the function and determine the interval of convergence.

3. $f(x) = \dfrac{1}{1+x}$

4. $f(x) = \dfrac{5}{1-4x^2}$

5. $f(x) = \dfrac{2}{3-x}$

6. $f(x) = \dfrac{4}{2x+3}$

7. $f(x) = \dfrac{x^2}{x^4+16}$

8. $f(x) = \dfrac{x}{2x^2+1}$

9. $f(x) = \dfrac{x-1}{x+2}$

10. $f(x) = \dfrac{x+a}{x^2+a^2}, \quad a > 0$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

11. $f(x) = \dfrac{2x-4}{x^2-4x+3}$

12. $f(x) = \dfrac{2x+3}{x^2+3x+2}$

13. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

14. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?

(b) Use part (a) to find a power series for $f(x) = x \ln(1-x)$.

(c) By putting $x = \frac{1}{2}$ in your result from part (a), express ln 2 as the sum of an infinite series.

15–20 Find a power series representation for the function and determine the radius of convergence.

15. $f(x) = \ln(5-x)$

16. $f(x) = x^2 \tan^{-1}(x^3)$

17. $f(x) = \dfrac{x}{(1+4x)^2}$

18. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

19. $f(x) = \dfrac{1+x}{(1-x)^2}$

20. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

21–24 Find a power series representation for f, and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

21. $f(x) = \dfrac{x^2}{x^2 + 1}$

22. $f(x) = \ln(1 + x^4)$

23. $f(x) = \ln\left(\dfrac{1 + x}{1 - x}\right)$

24. $f(x) = \tan^{-1}(2x)$

25–28 Evaluate the indefinite integral as a power series. What is the radius of convergence?

25. $\displaystyle\int \frac{t}{1 - t^8}\,dt$

26. $\displaystyle\int \frac{t}{1 + t^3}\,dt$

27. $\displaystyle\int x^2 \ln(1 + x)\,dx$

28. $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

29–32 Use a power series to approximate the definite integral to six decimal places.

29. $\displaystyle\int_0^{0.3} \frac{x}{1 + x^3}\,dx$

30. $\displaystyle\int_0^{1/2} \arctan(x/2)\,dx$

31. $\displaystyle\int_0^{0.2} x\ln(1 + x^2)\,dx$

32. $\displaystyle\int_0^{0.3} \frac{x^2}{1 + x^4}\,dx$

33. Use the result of Example 7 to compute arctan 0.2 correct to five decimal places.

34. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

35. (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

36. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n + 1)!2^{2n+1}}$$

(a) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

37. (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$

(b) Show that $f(x) = e^x$.

38. Let $f_n(x) = (\sin nx)/n^2$. Show that the series $\Sigma f_n(x)$ converges for all values of x but the series of derivatives $\Sigma f_n'(x)$ diverges when $x = 2n\pi$, n an integer. For what values of x does the series $\Sigma f_n''(x)$ converge?

39. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

40. (a) Starting with the geometric series $\Sigma_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} nx^{n-1} \qquad |x| < 1$$

(b) Find the sum of each of the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} nx^n, \quad |x| < 1$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) Find the sum of each of the following series.

(i) $\displaystyle\sum_{n=2}^{\infty} n(n - 1)x^n, \quad |x| < 1$

(ii) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

41. Use the power series for $\tan^{-1}x$ to prove the following expression for π as the sum of an infinite series:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)3^n}$$

42. (a) By completing the square, show that

$$\int_0^{1/2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$$

(b) By factoring $x^3 + 1$ as a sum of cubes, rewrite the integral in part (a). Then express $1/(x^3 + 1)$ as the sum of a power series and use it to prove the following formula for π:

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n}\left(\frac{2}{3n + 1} + \frac{1}{3n + 2}\right)$$

11.10 Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems: Which functions have power series representations? How can we find such representations?

We start by supposing that f is any function that can be represented by a power series

1 $$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \qquad |x - a| < R$$

Let's try to determine what the coefficients c_n must be in terms of f. To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

By Theorem 11.9.2, we can differentiate the series in Equation 1 term by term:

2 $$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots \qquad |x - a| < R$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

3 $$f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots \qquad |x - a| < R$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

4 $$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots \qquad |x - a| < R$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Solving this equation for the nth coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$

Taylor and Maclaurin

The Taylor series is named after the English mathematician Brook Taylor (1685–1731) and the Maclaurin series is named in honor of the Scottish mathematician Colin Maclaurin (1698–1746) despite the fact that the Maclaurin series is really just a special case of the Taylor series. But the idea of representing particular functions as sums of power series goes back to Newton, and the general Taylor series was known to the Scottish mathematician James Gregory in 1668 and to the Swiss mathematician John Bernoulli in the 1690s. Taylor was apparently unaware of the work of Gregory and Bernoulli when he published his discoveries on series in 1715 in his book *Methodus incrementorum directa et inversa*. Maclaurin series are named after Colin Maclaurin because he popularized them in his calculus textbook *Treatise of Fluxions* published in 1742.

The series in Equation 6 is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$ the Taylor series becomes

$$\boxed{7} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

NOTE We have shown that *if* f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series. An example of such a function is given in Exercise 84.

EXAMPLE 1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

SOLUTION If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence we let $a_n = x^n/n!$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n + 1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n + 1} \to 0 < 1$$

so, by the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$. ■

The conclusion we can draw from Theorem 5 and Example 1 is that *if* e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x *does* have a power series representation?

Let's investigate the more general question: under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

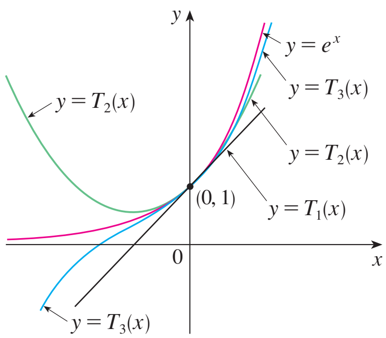


FIGURE 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

Notice that T_n is a polynomial of degree n called the ***n*th-degree Taylor polynomial of *f* at *a***. For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2,$ and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

We have therefore proved the following theorem.

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following theorem.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leq d$$

To see why this is true for $n = 1$, we assume that $|f''(x)| \leq M$. In particular, we have $f''(x) \leq M$, so for $a \leq x \leq a + d$ we have

$$\int_a^x f''(t)\,dt \leq \int_a^x M\,dt$$

An antiderivative of f'' is f', so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leq M(x - a) \qquad \text{or} \qquad f'(x) \leq f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t)\,dt \leq \int_a^x [f'(a) + M(t - a)]\,dt$$

$$f(x) - f(a) \leq f'(a)(x - a) + M\frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leq \frac{M}{2}(x - a)^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leq \frac{M}{2}(x - a)^2$$

A similar argument, using $f''(x) \geq -M$, shows that

$$R_1(x) \geq -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leq \frac{M}{2}|x - a|^2$$

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$.

This proves Taylor's Inequality for the case where $n = 1$. The result for any n is proved in a similar way by integrating $n + 1$ times. (See Exercise 83 for the case $n = 2$.)

NOTE In Section 11.11 we will explore the use of Taylor's Inequality in approximating functions. Our immediate use of it is in conjunction with Theorem 8.

In applying Theorems 8 and 9 it is often helpful to make use of the following fact.

10

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

Formulas for the Taylor Remainder Term

As alternatives to Taylor's Inequality, we have the following formulas for the remainder term. If $f^{(n+1)}$ is continuous on an interval I and $x \in I$, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x - t)^n f^{(n+1)}(t)\,dt$$

This is called the *integral form of the remainder term*. Another formula, called *Lagrange's form of the remainder term,* states that there is a number z between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - a)^{n+1}$$

This version is an extension of the Mean Value Theorem (which is the case $n = 0$).

Proofs of these formulas, together with discussions of how to use them to solve the examples of Sections 11.10 and 11.11, are given on the website

www.stewartcalculus.com

Click on *Additional Topics* and then on *Formulas for the Remainder Term in Taylor series*.

This is true because we know from Example 1 that the series $\Sigma\, x^n/n!$ converges for all x and so its nth term approaches 0.

EXAMPLE 2 Prove that e^x is equal to the sum of its Maclaurin series.

SOLUTION If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \leqslant d$, then $|f^{(n+1)}(x)| = e^x \leqslant e^d$. So Taylor's Inequality, with $a = 0$ and $M = e^d$, says that

$$|R_n(x)| \leqslant \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leqslant d$$

Notice that the same constant $M = e^d$ works for every value of n. But, from Equation 10, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x. By Theorem 8, e^x is equal to the sum of its Maclaurin series, that is,

11 $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$ ■

In particular, if we put $x = 1$ in Equation 11, we obtain the following expression for the number e as a sum of an infinite series:

In 1748 Leonhard Euler used Equation 12 to find the value of e correct to 23 digits. In 2010 Shigeru Kondo, again using the series in (12), computed e to more than one trillion decimal places. The special techniques employed to speed up the computation are explained on the website

numbers.computation.free.fr

12 $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

EXAMPLE 3 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

SOLUTION We have $f^{(n)}(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series (6), we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Again it can be verified, as in Example 1, that the radius of convergence is $R = \infty$. As in Example 2 we can verify that $\lim_{n\to\infty} R_n(x) = 0$, so

13 $$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x$$ ■

We have two power series expansions for e^x, the Maclaurin series in Equation 11 and the Taylor series in Equation 13. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

EXAMPLE 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

SOLUTION We arrange our computation in two columns as follows:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Figure 2 shows the graph of $\sin x$ together with its Taylor (or Maclaurin) polynomials

$$T_1(x) = x$$
$$T_3(x) = x - \frac{x^3}{3!}$$
$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Notice that, as n increases, $T_n(x)$ becomes a better approximation to $\sin x$.

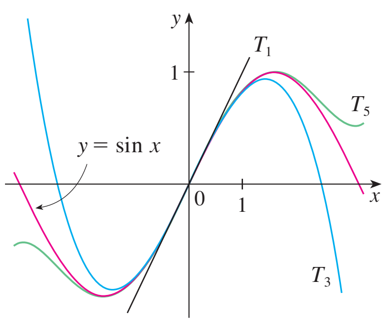


FIGURE 2

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we know that $|f^{(n+1)}(x)| \leqslant 1$ for all x. So we can take $M = 1$ in Taylor's Inequality:

14
$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 10 the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series by Theorem 8. ■

We state the result of Example 4 for future reference.

15
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

EXAMPLE 5 Find the Maclaurin series for $\cos x$.

SOLUTION We could proceed directly as in Example 4, but it's easier to differentiate the Maclaurin series for $\sin x$ given by Equation 15:

$$\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Since the Maclaurin series for sin x converges for all x, Theorem 11.9.2 tells us that the differentiated series for cos x also converges for all x. Thus

16
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

The Maclaurin series for e^x, sin x, and cos x that we found in Examples 2, 4, and 5 were discovered, using different methods, by Newton. These equations are remarkable because they say we know everything about each of these functions if we know all its derivatives at the single number 0.

EXAMPLE 6 Find the Maclaurin series for the function $f(x) = x \cos x$.

SOLUTION Instead of computing derivatives and substituting in Equation 7, it's easier to multiply the series for cos x (Equation 16) by x:

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

EXAMPLE 7 Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

SOLUTION Arranging our work in columns, we have

$$f(x) = \sin x \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

and this pattern repeats indefinitely. Therefore the Taylor series at $\pi/3$ is

$$f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

We have obtained two different series representations for sin x, the Maclaurin series in Example 4 and the Taylor series in Example 7. It is best to use the Maclaurin series for values of x near 0 and the Taylor series for x near $\pi/3$. Notice that the third Taylor polynomial T_3 in Figure 3 is a good approximation to sin x near $\pi/3$ but not as good near 0. Compare it with the third Maclaurin polynomial T_3 in Figure 2, where the opposite is true.

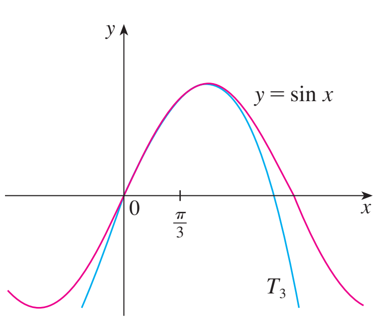


FIGURE 3

The proof that this series represents sin x for all x is very similar to that in Example 4. [Just replace x by $x - \pi/3$ in (14).] We can write the series in sigma notation if we separate the terms that contain $\sqrt{3}$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

The power series that we obtained by indirect methods in Examples 5 and 6 and in Section 11.9 are indeed the Taylor or Maclaurin series of the given functions because Theorem 5 asserts that, no matter how a power series representation $f(x) = \Sigma\, c_n(x - a)^n$ is obtained, it is always true that $c_n = f^{(n)}(a)/n!$. In other words, the coefficients are uniquely determined.

EXAMPLE 8 Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

SOLUTION Arranging our work in columns, we have

$$\begin{aligned}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k - 1)(1 + x)^{k-2} & f''(0) &= k(k - 1) \\
f'''(x) &= k(k - 1)(k - 2)(1 + x)^{k-3} & f'''(0) &= k(k - 1)(k - 2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k - 1) \cdots (k - n + 1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k - 1) \cdots (k - n + 1)
\end{aligned}$$

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k - 1) \cdots (k - n + 1)}{n!} x^n$$

This series is called the **binomial series**. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test. If the nth term is a_n, then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{k(k - 1) \cdots (k - n + 1)(k - n)x^{n+1}}{(n + 1)!} \cdot \frac{n!}{k(k - 1) \cdots (k - n + 1)x^n} \right|$$

$$= \frac{|k - n|}{n + 1} |x| = \frac{\left| 1 - \dfrac{k}{n} \right|}{1 + \dfrac{1}{n}} |x| \to |x| \qquad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. ■

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k - 1)(k - 2) \cdots (k - n + 1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series. It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult. The proof outlined in Exercise 85 is much easier.

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. It turns out that the series converges at 1 if $-1 < k \leqslant 0$ and at both endpoints if $k \geqslant 0$. Notice that if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$, so $\binom{k}{n} = 0$ for $n > k$. This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer. (See Reference Page 1.)

EXAMPLE 9 Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{4 - x}}$ and its radius of convergence.

SOLUTION We rewrite $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4 - x}} = \frac{1}{\sqrt{4\left(1 - \dfrac{x}{4}\right)}} = \frac{1}{2\sqrt{1 - \dfrac{x}{4}}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}$$

Using the binomial series with $k = -\frac{1}{2}$ and with x replaced by $-x/4$, we have

$$\frac{1}{\sqrt{4 - x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3\right.$$

$$\left. + \cdots + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2} - n + 1)}{n!}\left(-\frac{x}{4}\right)^n + \cdots\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1 \cdot 3}{2!8^2}x^2 + \frac{1 \cdot 3 \cdot 5}{3!8^3}x^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!8^n}x^n + \cdots\right]$$

We know from (17) that this series converges when $|-x/4| < 1$, that is, $|x| < 4$, so the radius of convergence is $R = 4$. ■

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

Table 1

Important Maclaurin Series and Their Radii of Convergence

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1}x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

EXAMPLE 10 Find the sum of the series $\dfrac{1}{1 \cdot 2} - \dfrac{1}{2 \cdot 2^2} + \dfrac{1}{3 \cdot 2^3} - \dfrac{1}{4 \cdot 2^4} + \cdots$.

SOLUTION With sigma notation we can write the given series as

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}$$

Then from Table 1 we see that this series matches the entry for $\ln(1 + x)$ with $x = \frac{1}{2}$. So

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} = \ln\left(1 + \tfrac{1}{2}\right) = \ln \tfrac{3}{2}$$ ■

TEC Module 11.10/11.11 enables you to see how successive Taylor polynomials approach the original function.

One reason that Taylor series are important is that they enable us to integrate functions that we couldn't previously handle. In fact, in the introduction to this chapter we mentioned that Newton often integrated functions by first expressing them as power series and then integrating the series term by term. The function $f(x) = e^{-x^2}$ can't be integrated by techniques discussed so far because its antiderivative is not an elementary function (see Section 7.5). In the following example we use Newton's idea to integrate this function.

EXAMPLE 11

(a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

SOLUTION

(a) First we find the Maclaurin series for $f(x) = e^{-x^2}$. Although it's possible to use the direct method, let's find it simply by replacing x with $-x^2$ in the series for e^x given in

Table 1. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Now we integrate term by term:

$$\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx$$

$$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This series converges for all x because the original series for e^{-x^2} converges for all x.

(b) The Fundamental Theorem of Calculus gives

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

We can take $C = 0$ in the antiderivative in part (a).

$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots$$

$$\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} \approx 0.7475$$

The Alternating Series Estimation Theorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$ ■

Another use of Taylor series is illustrated in the next example. The limit could be found with l'Hospital's Rule, but instead we use a series.

EXAMPLE 12 Evaluate $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

SOLUTION Using the Maclaurin series for e^x, we have

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - x}{x^2}$$

Some computer algebra systems compute limits in this way.

$$= \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right) = \frac{1}{2}$$

because power series are continuous functions. ■

Multiplication and Division of Power Series

If power series are added or subtracted, they behave like polynomials (Theorem 11.2.8 shows this). In fact, as the following example illustrates, they can also be multiplied and divided like polynomials. We find only the first few terms because the calculations for the later terms become tedious and the initial terms are the most important ones.

EXAMPLE 13 Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

SOLUTION

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

We multiply these expressions, collecting like terms just as for polynomials:

$$\begin{array}{rl} & 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ \times & x \qquad\quad - \frac{1}{6}x^3 + \cdots \\ \hline & x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots \\ + & - \frac{1}{6}x^3 - \frac{1}{6}x^4 - \cdots \\ \hline & x + x^2 + \frac{1}{3}x^3 + \cdots \end{array}$$

Thus
$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

We use a procedure like long division:

$$\begin{array}{r|l} & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\ & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\ \hline & \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\ & \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\ \hline & \frac{2}{15}x^5 + \cdots \end{array}$$

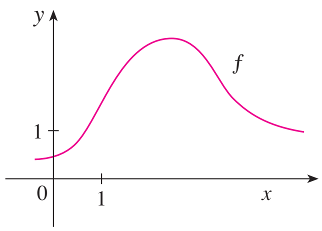
Thus
$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$ ■

Although we have not attempted to justify the formal manipulations used in Example 13, they are legitimate. There is a theorem which states that if both $f(x) = \Sigma\, c_n x^n$ and $g(x) = \Sigma\, b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

11.10 EXERCISES

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. The graph of f is shown.



(a) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

5–10 Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

5. $f(x) = xe^x, \quad a = 0$

6. $f(x) = \dfrac{1}{1+x}, \quad a = 2$

7. $f(x) = \sqrt[3]{x}, \quad a = 8$

8. $f(x) = \ln x, \quad a = 1$

9. $f(x) = \sin x, \quad a = \pi/6$

10. $f(x) = \cos^2 x, \quad a = 0$

11–18 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

11. $f(x) = (1-x)^{-2}$

12. $f(x) = \ln(1+x)$

13. $f(x) = \cos x$

14. $f(x) = e^{-2x}$

15. $f(x) = 2^x$

16. $f(x) = x\cos x$

17. $f(x) = \sinh x$

18. $f(x) = \cosh x$

19–26 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

19. $f(x) = x^5 + 2x^3 + x, \quad a = 2$

20. $f(x) = x^6 - x^4 + 2, \quad a = -2$

21. $f(x) = \ln x, \quad a = 2$

22. $f(x) = 1/x, \quad a = -3$

23. $f(x) = e^{2x}, \quad a = 3$

24. $f(x) = \cos x, \quad a = \pi/2$

25. $f(x) = \sin x, \quad a = \pi$

26. $f(x) = \sqrt{x}, \quad a = 16$

27. Prove that the series obtained in Exercise 13 represents $\cos x$ for all x.

28. Prove that the series obtained in Exercise 25 represents $\sin x$ for all x.

29. Prove that the series obtained in Exercise 17 represents $\sinh x$ for all x.

30. Prove that the series obtained in Exercise 18 represents $\cosh x$ for all x.

31–34 Use the binomial series to expand the function as a power series. State the radius of convergence.

31. $\sqrt[4]{1-x}$

32. $\sqrt[3]{8+x}$

33. $\dfrac{1}{(2+x)^3}$

34. $(1-x)^{3/4}$

35–44 Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

35. $f(x) = \arctan(x^2)$

36. $f(x) = \sin(\pi x/4)$

37. $f(x) = x\cos 2x$

38. $f(x) = e^{3x} - e^{2x}$

39. $f(x) = x\cos\left(\frac{1}{2}x^2\right)$

40. $f(x) = x^2\ln(1+x^3)$

41. $f(x) = \dfrac{x}{\sqrt{4+x^2}}$

42. $f(x) = \dfrac{x^2}{\sqrt{2+x}}$

43. $f(x) = \sin^2 x \quad \left[\textit{Hint:} \text{ Use } \sin^2 x = \frac{1}{2}(1-\cos 2x).\right]$

44. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases}$

45–48 Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

45. $f(x) = \cos(x^2)$

46. $f(x) = \ln(1+x^2)$

47. $f(x) = xe^{-x}$

48. $f(x) = \tan^{-1}(x^3)$

49. Use the Maclaurin series for $\cos x$ to compute $\cos 5°$ correct to five decimal places.

50. Use the Maclaurin series for e^x to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

51. (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.
(b) Use part (a) to find the Maclaurin series for $\sin^{-1}x$.

52. (a) Expand $1/\sqrt[4]{1+x}$ as a power series.
(b) Use part (a) to estimate $1/\sqrt[4]{1.1}$ correct to three decimal places.

53–56 Evaluate the indefinite integral as an infinite series.

53. $\int \sqrt{1+x^3}\,dx$

54. $\int x^2 \sin(x^2)\,dx$

55. $\int \frac{\cos x - 1}{x}\,dx$

56. $\int \arctan(x^2)\,dx$

57–60 Use series to approximate the definite integral to within the indicated accuracy.

57. $\int_0^{1/2} x^3 \arctan x\,dx$ (four decimal places)

58. $\int_0^1 \sin(x^4)\,dx$ (four decimal places)

59. $\int_0^{0.4} \sqrt{1+x^4}\,dx$ $(|\text{error}| < 5 \times 10^{-6})$

60. $\int_0^{0.5} x^2 e^{-x^2}\,dx$ $(|\text{error}| < 0.001)$

61–65 Use series to evaluate the limit.

61. $\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$

62. $\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$

63. $\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

64. $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$

65. $\lim_{x\to 0} \frac{x^3 - 3x + 3\tan^{-1}x}{x^5}$

66. Use the series in Example 13(b) to evaluate

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

We found this limit in Example 6.8.4 using l'Hospital's Rule three times. Which method do you prefer?

67–72 Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

67. $y = e^{-x^2}\cos x$

68. $y = \sec x$

69. $y = \frac{x}{\sin x}$

70. $y = e^x \ln(1+x)$

71. $y = (\arctan x)^2$

72. $y = e^x \sin^2 x$

73–80 Find the sum of the series.

73. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

74. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

75. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\,5^n}$

76. $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

77. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

78. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

79. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

80. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots$

81. Show that if p is an nth-degree polynomial, then

$$p(x+1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}$$

82. If $f(x) = (1+x^3)^{30}$, what is $f^{(58)}(0)$?

83. Prove Taylor's Inequality for $n = 2$, that is, prove that if $|f'''(x)| \leqslant M$ for $|x-a| \leqslant d$, then

$$|R_2(x)| \leqslant \frac{M}{6}|x-a|^3 \qquad \text{for } |x-a| \leqslant d$$

84. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.
(b) Graph the function in part (a) and comment on its behavior near the origin.

85. Use the following steps to prove (17).
(a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1+x} \qquad -1 < x < 1$$

(b) Let $h(x) = (1+x)^{-k}g(x)$ and show that $h'(x) = 0$.
(c) Deduce that $g(x) = (1+x)^k$.

86. In Exercise 10.2.53 it was shown that the length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, where $a > b > 0$, is

$$L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where $e = \sqrt{a^2-b^2}/a$ is the eccentricity of the ellipse.
Expand the integrand as a binomial series and use the result of Exercise 7.1.50 to express L as a series in powers of the eccentricity up to the term in e^6.

LABORATORY PROJECT CAS AN ELUSIVE LIMIT

This project deals with the function

$$f(x) = \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

1. Use your computer algebra system to evaluate $f(x)$ for $x = 1, 0.1, 0.01, 0.001$, and 0.0001. Does it appear that f has a limit as $x \to 0$?
2. Use the CAS to graph f near $x = 0$. Does it appear that f has a limit as $x \to 0$?
3. Try to evaluate $\lim_{x \to 0} f(x)$ with l'Hospital's Rule, using the CAS to find derivatives of the numerator and denominator. What do you discover? How many applications of l'Hospital's Rule are required?
4. Evaluate $\lim_{x \to 0} f(x)$ by using the CAS to find sufficiently many terms in the Taylor series of the numerator and denominator. (Use the command `taylor` in Maple or `Series` in Mathematica.)
5. Use the limit command on your CAS to find $\lim_{x \to 0} f(x)$ directly. (Most computer algebra systems use the method of Problem 4 to compute limits.)
6. In view of the answers to Problems 4 and 5, how do you explain the results of Problems 1 and 2?

WRITING PROJECT HOW NEWTON DISCOVERED THE BINOMIAL SERIES

The Binomial Theorem, which gives the expansion of $(a + b)^k$, was known to Chinese mathematicians many centuries before the time of Newton for the case where the exponent k is a positive integer. In 1665, when he was 22, Newton was the first to discover the infinite series expansion of $(a + b)^k$ when k is a fractional exponent (positive or negative). He didn't publish his discovery, but he stated it and gave examples of how to use it in a letter (now called the *epistola prior*) dated June 13, 1676, that he sent to Henry Oldenburg, secretary of the Royal Society of London, to transmit to Leibniz. When Leibniz replied, he asked how Newton had discovered the binomial series. Newton wrote a second letter, the *epistola posterior* of October 24, 1676, in which he explained in great detail how he arrived at his discovery by a very indirect route. He was investigating the areas under the curves $y = (1 - x^2)^{n/2}$ from 0 to x for $n = 0, 1, 2, 3, 4, \ldots$. These are easy to calculate if n is even. By observing patterns and interpolating, Newton was able to guess the answers for odd values of n. Then he realized he could get the same answers by expressing $(1 - x^2)^{n/2}$ as an infinite series.

Write a report on Newton's discovery of the binomial series. Start by giving the statement of the binomial series in Newton's notation (see the *epistola prior* on page 285 of [4] or page 402 of [2]). Explain why Newton's version is equivalent to Theorem 17 on page 807. Then read Newton's *epistola posterior* (page 287 in [4] or page 404 in [2]) and explain the patterns that Newton discovered in the areas under the curves $y = (1 - x^2)^{n/2}$. Show how he was able to guess the areas under the remaining curves and how he verified his answers. Finally, explain how these discoveries led to the binomial series. The books by Edwards [1] and Katz [3] contain commentaries on Newton's letters.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 178–187.
2. John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader* (London: MacMillan Press, 1987).
3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 463–466.
4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton University Press, 1969).

11.11 Applications of Taylor Polynomials

In this section we explore two types of applications of Taylor polynomials. First we look at how they are used to approximate functions—computer scientists like them because polynomials are the simplest of functions. Then we investigate how physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, the velocity of water waves, and building highways across a desert.

Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In Section 11.10 we introduced the notation $T_n(x)$ for the nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a. Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since f is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so T_n can be used as an approximation to f: $f(x) \approx T_n(x)$.

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x-a)$$

is the same as the linearization of f at a that we discussed in Section 2.9. Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

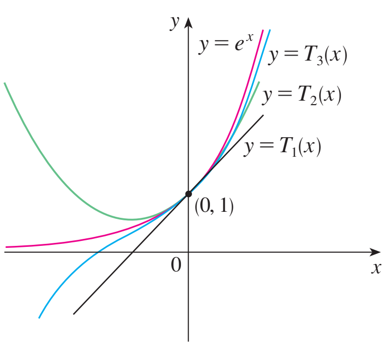


FIGURE 1

To illustrate these ideas let's take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1. The graph of T_1 is the tangent line to $y = e^x$ at $(0, 1)$; this tangent line is the best linear approximation to e^x near $(0, 1)$. The graph of T_2 is the parabola $y = 1 + x + x^2/2$, and the graph of T_3 is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than T_2. The next Taylor polynomial T_4 would be an even better approximation, and so on.

	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower. In fact, the farther x is from 0, the more slowly $T_n(x)$ converges to e^x.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality (Theorem 11.10.9), which says that if $|f^{(n+1)}(x)| \leqslant M$, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1}$$

EXAMPLE 1

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leqslant x \leqslant 9$?

SOLUTION

(a)

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad f(8) = 2$$

$$f'(x) = \tfrac{1}{3}x^{-2/3} \qquad f'(8) = \tfrac{1}{12}$$

$$f''(x) = -\tfrac{2}{9}x^{-5/3} \qquad f''(8) = -\tfrac{1}{144}$$

$$f'''(x) = \tfrac{10}{27}x^{-8/3}$$

Thus the second-degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example. But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leqslant \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \leqslant M$. Because $x \geqslant 7$, we have $x^{8/3} \geqslant 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leqslant \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore we can take $M = 0.0021$. Also $7 \leqslant x \leqslant 9$, so $-1 \leqslant x - 8 \leqslant 1$ and $|x - 8| \leqslant 1$. Then Taylor's Inequality gives

$$|R_2(x)| \leqslant \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leqslant x \leqslant 9$, the approximation in part (a) is accurate to within 0.0004. ■

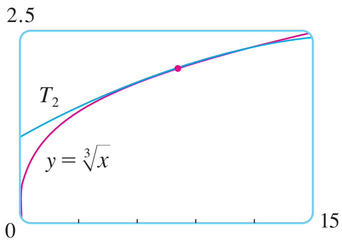


FIGURE 2

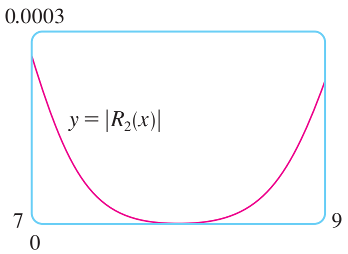


FIGURE 3

Let's use a graphing device to check the calculation in Example 1. Figure 2 shows that the graphs of $y = \sqrt[3]{x}$ and $y = T_2(x)$ are very close to each other when x is near 8. Figure 3 shows the graph of $|R_2(x)|$ computed from the expression

$$|R_2(x)| = |\sqrt[3]{x} - T_2(x)|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \leqslant x \leqslant 9$. Thus the error estimate from graphical methods is slightly better than the error estimate from Taylor's Inequality in this case.

EXAMPLE 2

(a) What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leqslant x \leqslant 0.3$? Use this approximation to find $\sin 12°$ correct to six decimal places.

(b) For what values of x is this approximation accurate to within 0.00005?

SOLUTION

(a) Notice that the Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating for all nonzero values of x, and the successive terms decrease in size because $|x| < 1$, so we can use the Alternating Series Estimation Theorem. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040}$$

If $-0.3 \leqslant x \leqslant 0.3$, then $|x| \leqslant 0.3$, so the error is smaller than

$$\frac{(0.3)^7}{5040} \approx 4.3 \times 10^{-8}$$

To find $\sin 12°$ we first convert to radian measure:

$$\sin 12° = \sin\left(\frac{12\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$

$$\approx \frac{\pi}{15} - \left(\frac{\pi}{15}\right)^3 \frac{1}{3!} + \left(\frac{\pi}{15}\right)^5 \frac{1}{5!} \approx 0.20791169$$

Thus, correct to six decimal places, $\sin 12° \approx 0.207912$.

(b) The error will be smaller than 0.00005 if

$$\frac{|x|^7}{5040} < 0.00005$$

Solving this inequality for x, we get

$$|x|^7 < 0.252 \qquad \text{or} \qquad |x| < (0.252)^{1/7} \approx 0.821$$

So the given approximation is accurate to within 0.00005 when $|x| < 0.82$. ■

TEC Module 11.10/11.11 graphically shows the remainders in Taylor polynomial approximations.

What if we use Taylor's Inequality to solve Example 2? Since $f^{(7)}(x) = -\cos x$, we have $|f^{(7)}(x)| \leqslant 1$ and so

$$|R_6(x)| \leqslant \frac{1}{7!}|x|^7$$

So we get the same estimates as with the Alternating Series Estimation Theorem.

What about graphical methods? Figure 4 shows the graph of

$$|R_6(x)| = \left|\sin x - \left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5\right)\right|$$

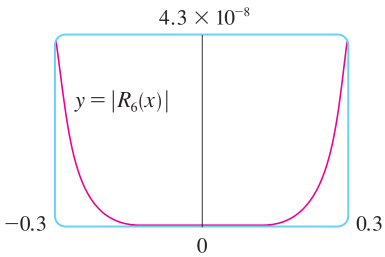


FIGURE 4

and we see from it that $|R_6(x)| < 4.3 \times 10^{-8}$ when $|x| \leqslant 0.3$. This is the same estimate that we obtained in Example 2. For part (b) we want $|R_6(x)| < 0.00005$, so we graph both $y = |R_6(x)|$ and $y = 0.00005$ in Figure 5. By placing the cursor on the right intersection point we find that the inequality is satisfied when $|x| < 0.82$. Again this is the same estimate that we obtained in the solution to Example 2.

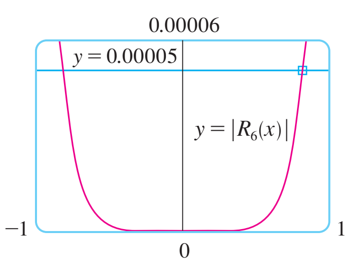


FIGURE 5

If we had been asked to approximate sin 72° instead of sin 12° in Example 2, it would have been wise to use the Taylor polynomials at $a = \pi/3$ (instead of $a = 0$) because they are better approximations to $\sin x$ for values of x close to $\pi/3$. Notice that 72° is close to 60° (or $\pi/3$ radians) and the derivatives of $\sin x$ are easy to compute at $\pi/3$.

Figure 6 shows the graphs of the Maclaurin polynomial approximations

$$T_1(x) = x \qquad\qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad\qquad T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

to the sine curve. You can see that as n increases, $T_n(x)$ is a good approximation to $\sin x$ on a larger and larger interval.

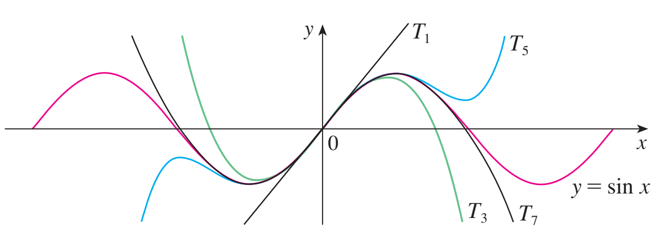


FIGURE 6

One use of the type of calculation done in Examples 1 and 2 occurs in calculators and computers. For instance, when you press the sin or e^x key on your calculator, or when a computer programmer uses a subroutine for a trigonometric or exponential or Bessel function, in many machines a polynomial approximation is calculated. The polynomial is often a Taylor polynomial that has been modified so that the error is spread more evenly throughout an interval.

Applications to Physics

Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series. In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor's Inequality can then be used to gauge the accuracy of the approximation. The following example shows one way in which this idea is used in special relativity.

EXAMPLE 3 In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.
(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leqslant 100$ m/s.

SOLUTION
(a) Using the expressions given for K and m, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

With $x = -v^2/c^2$, the Maclaurin series for $(1 + x)^{-1/2}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$.) Therefore we have

$$(1 + x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

The upper curve in Figure 7 is the graph of the expression for the kinetic energy K of an object with velocity v in special relativity. The lower curve shows the function used for K in classical Newtonian physics. When v is much smaller than the speed of light, the curves are practically identical.

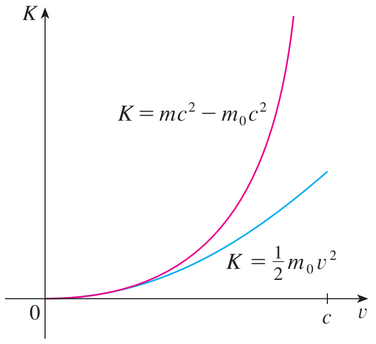


FIGURE 7

and

$$K = m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right]$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K \approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \tfrac{1}{2}m_0v^2$$

(b) If $x = -v^2/c^2$, $f(x) = m_0c^2[(1+x)^{-1/2} - 1]$, and M is a number such that $|f''(x)| \leqslant M$, then we can use Taylor's Inequality to write

$$|R_1(x)| \leqslant \frac{M}{2!}x^2$$

We have $f''(x) = \frac{3}{4}m_0c^2(1+x)^{-5/2}$ and we are given that $|v| \leqslant 100$ m/s, so

$$|f''(x)| = \frac{3m_0c^2}{4(1 - v^2/c^2)^{5/2}} \leqslant \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (=M)$$

Thus, with $c = 3 \times 10^8$ m/s,

$$|R_1(x)| \leqslant \frac{1}{2} \cdot \frac{3m_0c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

So when $|v| \leqslant 100$ m/s, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.2 \times 10^{-10})m_0$. ■

Another application to physics occurs in optics. Figure 8 is adapted from *Optics*, 4th ed., by Eugene Hecht (San Francisco, 2002), page 153. It depicts a wave from the point source S meeting a spherical interface of radius R centered at C. The ray SA is refracted toward P.

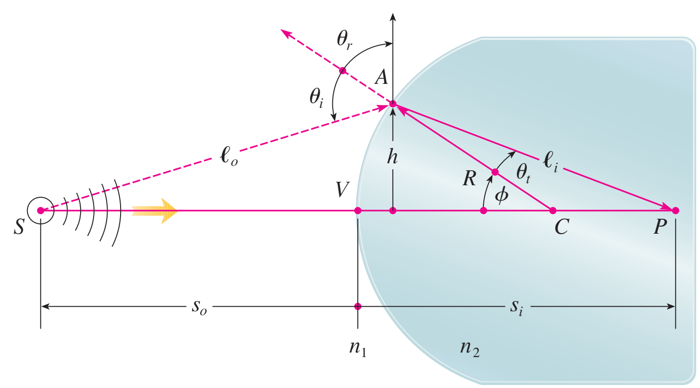


FIGURE 8
Refraction at a spherical interface
Source: Adapted from E. Hecht, *Optics,* 4e (Upper Saddle River, NJ: Pearson Education, 2002).

Using Fermat's principle that light travels so as to minimize the time taken, Hecht derives the equation

1 $$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right)$$

where n_1 and n_2 are indexes of refraction and ℓ_o, ℓ_i, s_o, and s_i are the distances indicated in Figure 8. By the Law of Cosines, applied to triangles ACS and ACP, we have

2 $$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi}$$

$$\ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Here we use the identity
$$\cos(\pi - \phi) = -\cos\phi$$

Because Equation 1 is cumbersome to work with, Gauss, in 1841, simplified it by using the linear approximation $\cos \phi \approx 1$ for small values of ϕ. (This amounts to using the Taylor polynomial of degree 1.) Then Equation 1 becomes the following simpler equation [as you are asked to show in Exercise 34(a)]:

$$\boxed{3} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

The resulting optical theory is known as *Gaussian optics*, or *first-order optics*, and has become the basic theoretical tool used to design lenses.

A more accurate theory is obtained by approximating $\cos \phi$ by its Taylor polynomial of degree 3 (which is the same as the Taylor polynomial of degree 2). This takes into account rays for which ϕ is not so small, that is, rays that strike the surface at greater distances h above the axis. In Exercise 34(b) you are asked to use this approximation to derive the more accurate equation

$$\boxed{4} \qquad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R} + h^2\left[\frac{n_1}{2s_o}\left(\frac{1}{s_o} + \frac{1}{R}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

The resulting optical theory is known as *third-order optics*.

Other applications of Taylor polynomials to physics and engineering are explored in Exercises 32, 33, 35, 36, 37, and 38, and in the Applied Project on page 823.

11.11 EXERCISES

1. (a) Find the Taylor polynomials up to degree 5 for $f(x) = \sin x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
(b) Evaluate f and these polynomials at $x = \pi/4$, $\pi/2$, and π.
(c) Comment on how the Taylor polynomials converge to $f(x)$.

2. (a) Find the Taylor polynomials up to degree 3 for $f(x) = \tan x$ centered at $a = 0$. Graph f and these polynomials on a common screen.
(b) Evaluate f and these polynomials at $x = \pi/6$, $\pi/4$, and $\pi/3$.
(c) Comment on how the Taylor polynomials converge to $f(x)$.

3–10 Find the Taylor polynomial $T_3(x)$ for the function f centered at the number a. Graph f and T_3 on the same screen.

3. $f(x) = e^x, \quad a = 1$

4. $f(x) = \sin x, \quad a = \pi/6$

5. $f(x) = \cos x, \quad a = \pi/2$

6. $f(x) = e^{-x} \sin x, \quad a = 0$

7. $f(x) = \ln x, \quad a = 1$

8. $f(x) = x \cos x, \quad a = 0$

9. $f(x) = xe^{-2x}, \quad a = 0$

10. $f(x) = \tan^{-1}x, \quad a = 1$

CAS **11–12** Use a computer algebra system to find the Taylor polynomials T_n centered at a for $n = 2, 3, 4, 5$. Then graph these polynomials and f on the same screen.

11. $f(x) = \cot x, \quad a = \pi/4$

12. $f(x) = \sqrt[3]{1 + x^2}, \quad a = 0$

13–22
(a) Approximate f by a Taylor polynomial with degree n at the number a.
(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
(c) Check your result in part (b) by graphing $|R_n(x)|$.

13. $f(x) = 1/x, \quad a = 1, \quad n = 2, \quad 0.7 \leqslant x \leqslant 1.3$

14. $f(x) = x^{-1/2}, \quad a = 4, \quad n = 2, \quad 3.5 \leqslant x \leqslant 4.5$

15. $f(x) = x^{2/3}, \quad a = 1, \quad n = 3, \quad 0.8 \leqslant x \leqslant 1.2$

16. $f(x) = \sin x, \quad a = \pi/6, \quad n = 4, \quad 0 \leqslant x \leqslant \pi/3$

17. $f(x) = \sec x, \quad a = 0, \quad n = 2, \quad -0.2 \leqslant x \leqslant 0.2$

18. $f(x) = \ln(1 + 2x), \quad a = 1, \quad n = 3, \quad 0.5 \leqslant x \leqslant 1.5$

19. $f(x) = e^{x^2}, \quad a = 0, \quad n = 3, \quad 0 \leqslant x \leqslant 0.1$

20. $f(x) = x \ln x, \quad a = 1, \quad n = 3, \quad 0.5 \leqslant x \leqslant 1.5$

21. $f(x) = x \sin x, \quad a = 0, \quad n = 4, \quad -1 \leqslant x \leqslant 1$

22. $f(x) = \sinh 2x, \quad a = 0, \quad n = 5, \quad -1 \leqslant x \leqslant 1$

23. Use the information from Exercise 5 to estimate cos 80° correct to five decimal places.

24. Use the information from Exercise 16 to estimate sin 38° correct to five decimal places.

25. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

26. How many terms of the Maclaurin series for $\ln(1 + x)$ do you need to use to estimate ln 1.4 to within 0.001?

27–29 Use the Alternating Series Estimation Theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.

27. $\sin x \approx x - \dfrac{x^3}{6} \quad (|\text{error}| < 0.01)$

28. $\cos x \approx 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} \quad (|\text{error}| < 0.005)$

29. $\arctan x \approx x - \dfrac{x^3}{3} + \dfrac{x^5}{5} \quad (|\text{error}| < 0.05)$

30. Suppose you know that

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n + 1)}$$

and the Taylor series of f centered at 4 converges to $f(x)$ for all x in the interval of convergence. Show that the fifth-degree Taylor polynomial approximates $f(5)$ with error less than 0.0002.

31. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?

32. The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in units of ohm-meters (Ω-m). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20}\, e^{\alpha(t-20)}$$

where t is the temperature in °C. There are tables that list the values of α (called the temperature coefficient) and ρ_{20} (the resistivity at 20°C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by its first- or second-degree Taylor polynomial at $t = 20$.

(a) Find expressions for these linear and quadratic approximations.

(b) For copper, the tables give $\alpha = 0.0039/°\text{C}$ and $\rho_{20} = 1.7 \times 10^{-8}\ \Omega\text{-m}$. Graph the resistivity of copper and the linear and quadratic approximations for $-250°\text{C} \leqslant t \leqslant 1000°\text{C}$.

(c) For what values of t does the linear approximation agree with the exponential expression to within one percent?

33. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D + d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



34. (a) Derive Equation 3 for Gaussian optics from Equation 1 by approximating $\cos \phi$ in Equation 2 by its first-degree Taylor polynomial.

(b) Show that if $\cos \phi$ is replaced by its third-degree Taylor polynomial in Equation 2, then Equation 1 becomes Equation 4 for third-order optics. [*Hint:* Use the first two terms in the binomial series for ℓ_o^{-1} and ℓ_i^{-1}. Also, use $\phi \approx \sin \phi$.]

35. If a water wave with length L moves with velocity v across a body of water with depth d, as in the figure on page 822, then

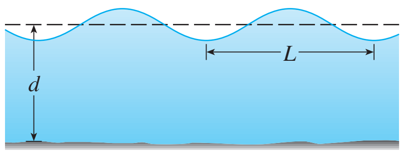
$$v^2 = \frac{gL}{2\pi} \tanh \frac{2\pi d}{L}$$

(a) If the water is deep, show that $v \approx \sqrt{gL/(2\pi)}$.

(b) If the water is shallow, use the Maclaurin series for tanh to show that $v \approx \sqrt{gd}$. (Thus in shallow water the

velocity of a wave tends to be independent of the length of the wave.)

(c) Use the Alternating Series Estimation Theorem to show that if $L > 10d$, then the estimate $v^2 \approx gd$ is accurate to within $0.014gL$.

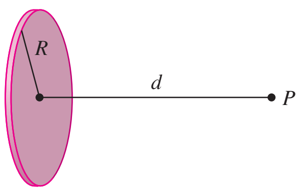


36. A uniformly charged disk has radius R and surface charge density σ as in the figure. The electric potential V at a point P at a distance d along the perpendicular central axis of the disk is

$$V = 2\pi k_e \sigma \left(\sqrt{d^2 + R^2} - d\right)$$

where k_e is a constant (called Coulomb's constant). Show that

$$V \approx \frac{\pi k_e R^2 \sigma}{d} \quad \text{for large } d$$



37. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth.

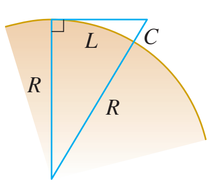
(a) If R is the radius of the earth and L is the length of the highway, show that the correction is

$$C = R \sec(L/R) - R$$

(b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}$$

(c) Compare the corrections given by the formulas in parts (a) and (b) for a highway that is 100 km long. (Take the radius of the earth to be 6370 km.)



38. The period of a pendulum with length L that makes a maximum angle θ_0 with the vertical is

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. (In Exercise 7.7.42 we approximated this integral using Simpson's Rule.)

(a) Expand the integrand as a binomial series and use the result of Exercise 7.1.50 to show that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 3^2}{2^2 4^2}k^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k^6 + \cdots\right]$$

If θ_0 is not too large, the approximation $T \approx 2\pi\sqrt{L/g}$, obtained by using only the first term in the series, is often used. A better approximation is obtained by using two terms:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \tfrac{1}{4}k^2\right)$$

(b) Notice that all the terms in the series after the first one have coefficients that are at most $\frac{1}{4}$. Use this fact to compare this series with a geometric series and show that

$$2\pi\sqrt{\frac{L}{g}}\left(1 + \tfrac{1}{4}k^2\right) \leqslant T \leqslant 2\pi\sqrt{\frac{L}{g}}\,\frac{4 - 3k^2}{4 - 4k^2}$$

(c) Use the inequalities in part (b) to estimate the period of a pendulum with $L = 1$ meter and $\theta_0 = 10°$. How does it compare with the estimate $T \approx 2\pi\sqrt{L/g}$? What if $\theta_0 = 42°$?

39. In Section 3.8 we considered Newton's method for approximating a root r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations $x_2, x_3, \ldots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Taylor's Inequality with $n = 1$, $a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n, and x_{n+1}, and $|f''(x)| \leqslant M$, $|f'(x)| \geqslant K$ for all $x \in I$, then

$$|x_{n+1} - r| \leqslant \frac{M}{2K}|x_n - r|^2$$

[This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n + 1$ is at most $(M/2K)10^{-2m}$.]

APPLIED PROJECT RADIATION FROM THE STARS



Luke Dodd / Science Source

Any object emits radiation when heated. A *blackbody* is a system that absorbs all the radiation that falls on it. For instance, a matte black surface or a large cavity with a small hole in its wall (like a blast furnace) is a blackbody and emits blackbody radiation. Even the radiation from the sun is close to being blackbody radiation.

Proposed in the late 19th century, the Rayleigh-Jeans Law expresses the energy density of blackbody radiation of wavelength λ as

$$f(\lambda) = \frac{8\pi kT}{\lambda^4}$$

where λ is measured in meters, T is the temperature in kelvins (K), and k is Boltzmann's constant. The Rayleigh-Jeans Law agrees with experimental measurements for long wavelengths but disagrees drastically for short wavelengths. [The law predicts that $f(\lambda) \to \infty$ as $\lambda \to 0^+$ but experiments have shown that $f(\lambda) \to 0$.] This fact is known as the *ultraviolet catastrophe.*

In 1900 Max Planck found a better model (known now as Planck's Law) for blackbody radiation:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1}$$

where λ is measured in meters, T is the temperature (in kelvins), and

$$h = \text{Planck's constant} = 6.6262 \times 10^{-34}\ \text{J}\cdot\text{s}$$

$$c = \text{speed of light} = 2.997925 \times 10^{8}\ \text{m/s}$$

$$k = \text{Boltzmann's constant} = 1.3807 \times 10^{-23}\ \text{J/K}$$

1. Use l'Hospital's Rule to show that

$$\lim_{\lambda \to 0^+} f(\lambda) = 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} f(\lambda) = 0$$

for Planck's Law. So this law models blackbody radiation better than the Rayleigh-Jeans Law for short wavelengths.

2. Use a Taylor polynomial to show that, for large wavelengths, Planck's Law gives approximately the same values as the Rayleigh-Jeans Law.

3. Graph f as given by both laws on the same screen and comment on the similarities and differences. Use $T = 5700$ K (the temperature of the sun). (You may want to change from meters to the more convenient unit of micrometers: 1 mm $= 10^{-6}$ m.)

4. Use your graph in Problem 3 to estimate the value of λ for which $f(\lambda)$ is a maximum under Planck's Law.

5. Investigate how the graph of f changes as T varies. (Use Planck's Law.) In particular, graph f for the stars Betelgeuse ($T = 3400$ K), Procyon ($T = 6400$ K), and Sirius ($T = 9200$ K), as well as the sun. How does the total radiation emitted (the area under the curve) vary with T? Use the graph to comment on why Sirius is known as a blue star and Betelgeuse as a red star.

11 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a convergent sequence?
   (b) What is a convergent series?
   (c) What does $\lim_{n\to\infty} a_n = 3$ mean?
   (d) What does $\sum_{n=1}^{\infty} a_n = 3$ mean?
2. (a) What is a bounded sequence?
   (b) What is a monotonic sequence?
   (c) What can you say about a bounded monotonic sequence?
3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
   (b) What is a p-series? Under what circumstances is it convergent?
4. Suppose $\sum a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?
5. State the following.
   (a) The Test for Divergence
   (b) The Integral Test
   (c) The Comparison Test
   (d) The Limit Comparison Test
   (e) The Alternating Series Test
   (f) The Ratio Test
   (g) The Root Test
6. (a) What is an absolutely convergent series?
   (b) What can you say about such a series?
   (c) What is a conditionally convergent series?
7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?
   (b) If a series is convergent by the Comparison Test, how do you estimate its sum?
   (c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?
8. (a) Write the general form of a power series.
   (b) What is the radius of convergence of a power series?
   (c) What is the interval of convergence of a power series?
9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.
   (a) How do you differentiate f? What is the radius of convergence of the series for f'?
   (b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?
10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.
    (b) Write an expression for the Taylor series of f centered at a.
    (c) Write an expression for the Maclaurin series of f.
    (d) How do you show that $f(x)$ is equal to the sum of its Taylor series?
    (e) State Taylor's Inequality.
11. Write the Maclaurin series and the interval of convergence for each of the following functions.
    (a) $1/(1-x)$ (b) e^x (c) $\sin x$
    (d) $\cos x$ (e) $\tan^{-1}x$ (f) $\ln(1+x)$
12. Write the binomial series expansion of $(1+x)^k$. What is the radius of convergence of this series?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.
2. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.
3. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{2n+1} = L$.
4. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-2)^n$ is convergent.
5. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-6)^n$ is convergent.
6. If $\sum c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$.
7. The Ratio Test can be used to determine whether $\sum 1/n^3$ converges.
8. The Ratio Test can be used to determine whether $\sum 1/n!$ converges.
9. If $0 \leqslant a_n \leqslant b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$
11. If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$.
12. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.
13. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots$ converges for all x, then $f'''(0) = 2$.
14. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent.
15. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.
16. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.
17. If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

18. If $a_n > 0$ and $\lim_{n\to\infty} (a_{n+1}/a_n) < 1$, then $\lim_{n\to\infty} a_n = 0$.

19. $0.99999\ldots = 1$

20. If $\lim_{n\to\infty} a_n = 2$, then $\lim_{n\to\infty} (a_{n+3} - a_n) = 0$.

21. If a finite number of terms are added to a convergent series, then the new series is still convergent.

22. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then $\sum_{n=1}^{\infty} a_n b_n = AB$.

EXERCISES

1–8 Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \dfrac{2 + n^3}{1 + 2n^3}$

2. $a_n = \dfrac{9^{n+1}}{10^n}$

3. $a_n = \dfrac{n^3}{1 + n^2}$

4. $a_n = \cos(n\pi/2)$

5. $a_n = \dfrac{n \sin n}{n^2 + 1}$

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$

7. $\{(1 + 3/n)^{4n}\}$

8. $\{(-10)^n/n!\}$

9. A sequence is defined recursively by the equations $a_1 = 1$, $a_{n+1} = \frac{1}{3}(a_n + 4)$. Show that $\{a_n\}$ is increasing and $a_n < 2$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

10. Show that $\lim_{n\to\infty} n^4 e^{-n} = 0$ and use a graph to find the smallest value of N that corresponds to $\varepsilon = 0.1$ in the precise definition of a limit.

11–22 Determine whether the series is convergent or divergent.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n + 1}}$

15. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n + 1}\right)$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 3n}{1 + (1.2)^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + 2n^2)^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{5^n n!}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

21. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n + 1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n + 1} - \sqrt{n - 1}}{n}$

23–26 Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

23. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n + 1) 3^n}{2^{2n+1}}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$

27–31 Find the sum of the series.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n + 3)}$

29. $\displaystyle\sum_{n=1}^{\infty} [\tan^{-1}(n + 1) - \tan^{-1} n]$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$

31. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots$

32. Express the repeating decimal $4.17326326326\ldots$ as a fraction.

33. Show that $\cosh x \geqslant 1 + \frac{1}{2}x^2$ for all x.

34. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

35. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ correct to four decimal places.

36. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n^6$ and estimate the error in using it as an approximation to the sum of the series.

(b) Find the sum of this series correct to five decimal places.

37. Use the sum of the first eight terms to approximate the sum of the series $\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$. Estimate the error involved in this approximation.

38. (a) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ is convergent.

(b) Deduce that $\displaystyle\lim_{n\to\infty} \frac{n^n}{(2n)!} = 0$.

39. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series

$$\sum_{n=1}^{\infty} \left(\frac{n + 1}{n}\right) a_n$$

is also absolutely convergent.

40–43 Find the radius of convergence and interval of convergence of the series.

40. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\,4^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

44. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

45. Find the Taylor series of $f(x) = \sin x$ at $a = \pi/6$.

46. Find the Taylor series of $f(x) = \cos x$ at $a = \pi/3$.

47–54 Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin x$, $\tan^{-1}x$, and $\ln(1+x)$.

47. $f(x) = \dfrac{x^2}{1+x}$

48. $f(x) = \tan^{-1}(x^2)$

49. $f(x) = \ln(4-x)$

50. $f(x) = xe^{2x}$

51. $f(x) = \sin(x^4)$

52. $f(x) = 10^x$

53. $f(x) = 1/\sqrt[4]{16-x}$

54. $f(x) = (1-3x)^{-5}$

55. Evaluate $\displaystyle\int \frac{e^x}{x}\,dx$ as an infinite series.

56. Use series to approximate $\int_0^1 \sqrt{1+x^4}\,dx$ correct to two decimal places.

57–58

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Graph f and T_n on a common screen.

(c) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(d) Check your result in part (c) by graphing $|R_n(x)|$.

57. $f(x) = \sqrt{x}$, $\quad a = 1$, $\quad n = 3$, $\quad 0.9 \leqslant x \leqslant 1.1$

58. $f(x) = \sec x$, $\quad a = 0$, $\quad n = 2$, $\quad 0 \leqslant x \leqslant \pi/6$

59. Use series to evaluate the following limit.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

60. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity for an object on the surface of the earth.

(a) Express F as a series in powers of h/R.

(b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R. Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within one percent. (Use $R = 6400$ km.)

61. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x.

(a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \cdots = 0$$

(b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \cdots = 0$$

62. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

Problems Plus

Before you look at the solution of the example, cover it up and first try to solve the problem yourself.

EXAMPLE Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$.

SOLUTION The problem-solving principle that is relevant here is *recognizing something familiar*. Does the given series look anything like a series that we already know? Well, it does have some ingredients in common with the Maclaurin series for the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We can make this series look more like our given series by replacing x by $x+2$:

$$e^{x+2} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \cdots$$

But here the exponent in the numerator matches the number in the denominator whose factorial is taken. To make that happen in the given series, let's multiply and divide by $(x+2)^3$:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$

$$= (x+2)^{-3}\left[\frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \cdots\right]$$

We see that the series between brackets is just the series for e^{x+2} with the first three terms missing. So

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = (x+2)^{-3}\left[e^{x+2} - 1 - (x+2) - \frac{(x+2)^2}{2!}\right]$$ ■

Problems

1. If $f(x) = \sin(x^3)$, find $f^{(15)}(0)$.

2. A function f is defined by

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is f continuous?

3. (a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x$.
(b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

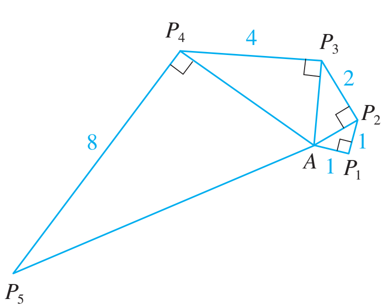


FIGURE FOR PROBLEM 4

4. Let $\{P_n\}$ be a sequence of points determined as in the figure. Thus $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$, and angle AP_nP_{n+1} is a right angle. Find $\lim_{n\to\infty} \angle P_nAP_{n+1}$.

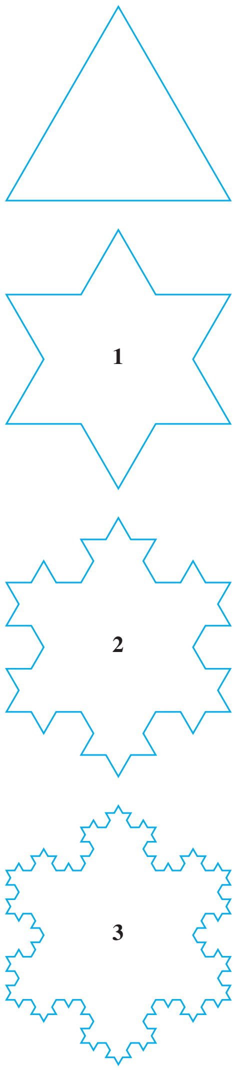


FIGURE FOR PROBLEM 5

5. To construct the **snowflake curve**, start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part (see the figure). Step 2 is to repeat step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely.

(a) Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve (the curve obtained after step n of the construction), respectively. Find formulas for s_n, l_n, and p_n.

(b) Show that $p_n \to \infty$ as $n \to \infty$.

(c) Sum an infinite series to find the area enclosed by the snowflake curve.

Note: Parts (b) and (c) show that the snowflake curve is infinitely long but encloses only a finite area.

6. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

7. (a) Show that for $xy \neq -1$,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

(b) Show that $\arctan \frac{120}{119} - \arctan \frac{1}{239} = \pi/4$.

(c) Deduce the following formula of John Machin (1680–1751):

$$4 \arctan \tfrac{1}{5} - \arctan \tfrac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

$$0.1973955597 < \arctan \tfrac{1}{5} < 0.1973955616$$

(e) Show that

$$0.004184075 < \arctan \tfrac{1}{239} < 0.004184077$$

(f) Deduce that, correct to seven decimal places, $\pi \approx 3.1415927$.

Machin used this method in 1706 to find π correct to 100 decimal places. Recently, with the aid of computers, the value of π has been computed to increasingly greater accuracy. In 2013 Shigeru Kondo and Alexander Yee computed the value of π to more than 12 trillion decimal places!

8. (a) Prove a formula similar to the one in Problem 7(a) but involving arccot instead of arctan.

(b) Find the sum of the series $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$.

9. Use the result of Problem 7(a) to find the sum of the series $\sum_{n=1}^{\infty} \arctan(2/n^2)$.

10. If $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n \to \infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = 0$$

If you don't see how to prove this, try the problem-solving strategy of *using analogy* (see page 98). Try the special cases $k = 1$ and $k = 2$ first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

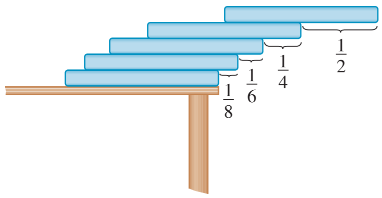


FIGURE FOR PROBLEM 12

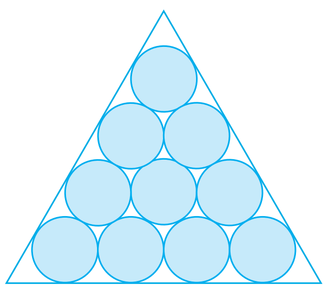


FIGURE FOR PROBLEM 15

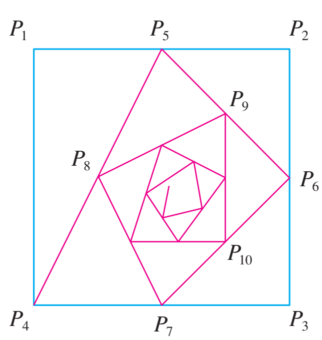


FIGURE FOR PROBLEM 18

11. Find the interval of convergence of $\Sigma_{n=1}^{\infty} n^3x^n$ and find its sum.

12. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Use the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (Try it yourself with a deck of cards.) Consider centers of mass.

13. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$.

14. If $p > 1$, evaluate the expression

$$\frac{1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots}{1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdots}$$

15. Suppose that circles of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case $n = 4$.) If A is the area of the triangle and A_n is the total area occupied by the n rows of circles, show that

$$\lim_{n\to\infty} \frac{A_n}{A} = \frac{\pi}{2\sqrt{3}}$$

16. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

Find the sum of the series $\Sigma_{n=0}^{\infty} a_n$.

17. If the curve $y = e^{-x/10}\sin x$, $x \geqslant 0$, is rotated about the x-axis, the resulting solid looks like an infinite decreasing string of beads.

(a) Find the exact volume of the nth bead. (Use either a table of integrals or a computer algebra system.)

(b) Find the total volume of the beads.

18. Starting with the vertices $P_1(0, 1)$, $P_2(1, 1)$, $P_3(1, 0)$, $P_4(0, 0)$ of a square, we construct further points as shown in the figure: P_5 is the midpoint of P_1P_2, P_6 is the midpoint of P_2P_3, P_7 is the midpoint of P_3P_4, and so on. The polygonal spiral path $P_1P_2P_3P_4P_5P_6P_7\ldots$ approaches a point P inside the square.

(a) If the coordinates of P_n are (x_n, y_n), show that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ and find a similar equation for the y-coordinates.

(b) Find the coordinates of P.

19. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

20. Carry out the following steps to show that

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots = \ln 2$$

(a) Use the formula for the sum of a finite geometric series (11.2.3) to get an expression for

$$1 - x + x^2 - x^3 + \cdots + x^{2n-2} - x^{2n-1}$$

(b) Integrate the result of part (a) from 0 to 1 to get an expression for

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

as an integral.

(c) Deduce from part (b) that

$$\left| \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1)(2n)} - \int_0^1 \frac{dx}{1+x} \right| < \int_0^1 x^{2n}\, dx$$

(d) Use part (c) to show that the sum of the given series is ln 2.

21. Find all the solutions of the equation

$$1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$$

[*Hint:* Consider the cases $x \geqslant 0$ and $x < 0$ separately.]

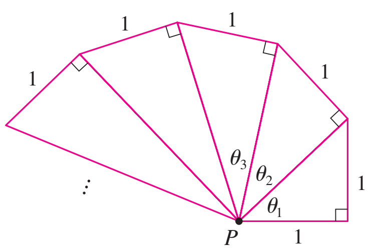


FIGURE FOR PROBLEM 22

22. Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around P by showing that $\Sigma\, \theta_n$ is a divergent series.

23. Consider the series whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

24. (a) Show that the Maclaurin series of the function

$$f(x) = \frac{x}{1 - x - x^2} \qquad \text{is} \qquad \sum_{n=1}^{\infty} f_n x^n$$

where f_n is the nth Fibonacci number, that is, $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. [*Hint:* Write $x/(1 - x - x^2) = c_0 + c_1 x + c_2 x^2 + \cdots$ and multiply both sides of this equation by $1 - x - x^2$.]

(b) By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Maclaurin series in a different way, find an explicit formula for the nth Fibonacci number.

25. Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

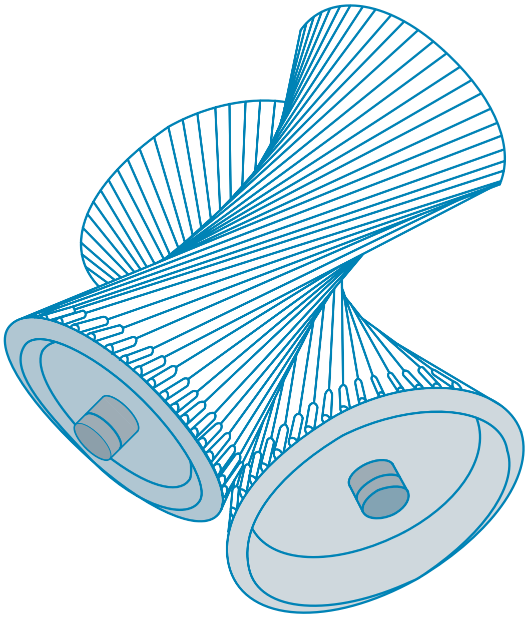
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$.

26. Prove that if $n > 1$, the nth partial sum of the harmonic series is not an integer.

Hint: Let 2^k be the largest power of 2 that is less than or equal to n and let M be the product of all odd integers that are less than or equal to n. Suppose that $s_n = m$, an integer. Then $M2^k s_n = M2^k m$. The right side of this equation is even. Prove that the left side is odd by showing that each of its terms is an even integer, except for the last one.

12 Vectors and the Geometry of Space



Each of these gears has the shape of a hyperboloid, a type of surface we will study in Section 12.6. The shape allows the gears to transmit motion between skew (neither parallel nor intersecting) axes.

IN THIS CHAPTER WE INTRODUCE vectors and coordinate systems for three-dimensional space. This will be the setting for our study of the calculus of functions of two variables in Chapter 14 because the graph of such a function is a surface in space. In this chapter we will see that vectors provide particularly simple descriptions of lines and planes in space.

12.1 Three-Dimensional Coordinate Systems

3D Space

To locate a point in a plane, we need two numbers. We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers.

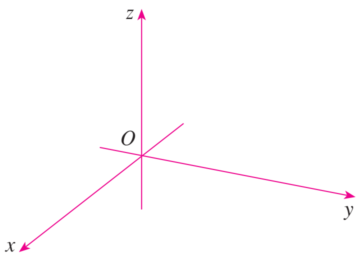


FIGURE 1
Coordinate axes

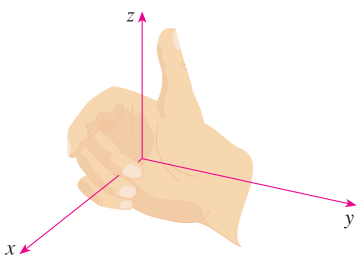


FIGURE 2
Right-hand rule

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

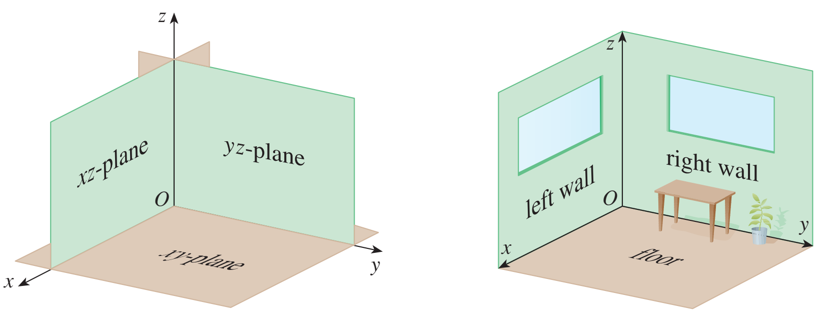


FIGURE 3 (a) Coordinate planes (b)

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.

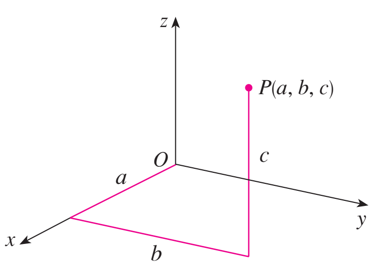


FIGURE 4

The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.

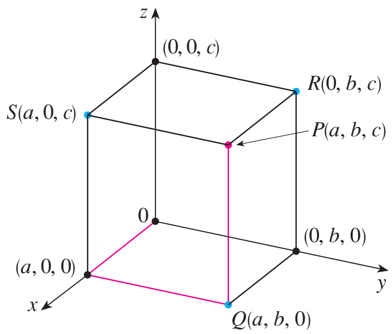


FIGURE 5

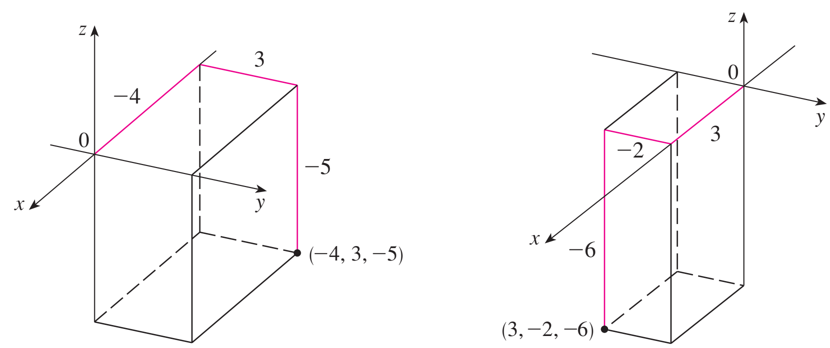


FIGURE 6

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**. Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Surfaces

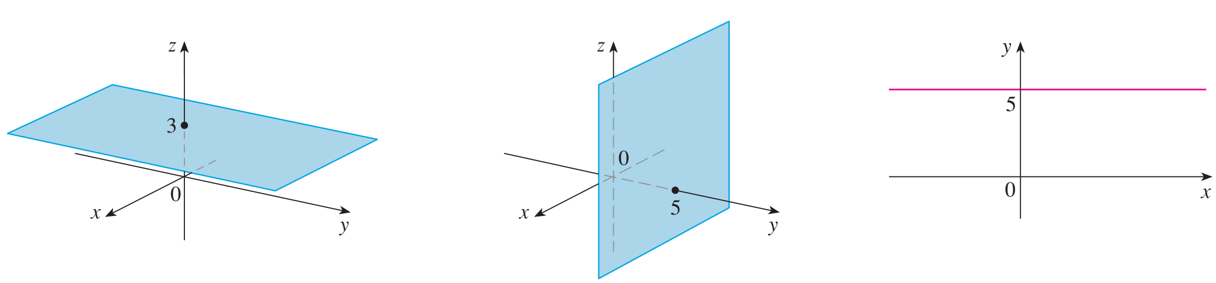
In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in $\mathbb{R}^3$.

EXAMPLE 1 What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ (b) $y = 5$

SOLUTION

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3 (x and y can each be any value). This is the horizontal plane that is parallel to the xy-plane and three units above it as in Figure 7(a).



(a) $z = 3$, a plane in $\mathbb{R}^3$ (b) $y = 5$, a plane in $\mathbb{R}^3$ (c) $y = 5$, a line in $\mathbb{R}^2$

FIGURE 7

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5. This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b). ■

NOTE When an equation is given, we must understand from the context whether it represents a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$. In Example 1, $y = 5$ represents a plane in $\mathbb{R}^3$, but of course $y = 5$ can also represent a line in $\mathbb{R}^2$ if we are dealing with two-dimensional analytic geometry. See Figure 7(b) and (c).

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane. In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.

EXAMPLE 2
(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

SOLUTION
(a) Because $z = 3$, the points lie in the horizontal plane $z = 3$ from Example 1(a). Because $x^2 + y^2 = 1$, the points lie on the circle with radius 1 and center on the z-axis. See Figure 8.

(b) Given that $x^2 + y^2 = 1$, with no restrictions on z, we see that the point (x, y, z) could lie on a circle in any horizontal plane $z = k$. So the surface $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consists of all possible horizontal circles $x^2 + y^2 = 1$, $z = k$, and is therefore the circular cylinder with radius 1 whose axis is the z-axis. See Figure 9.

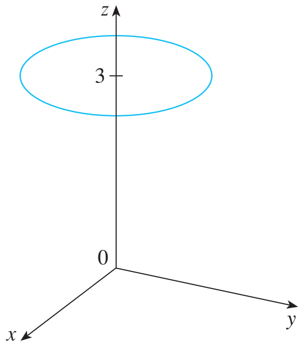


FIGURE 8
The circle $x^2 + y^2 = 1$, $z = 3$

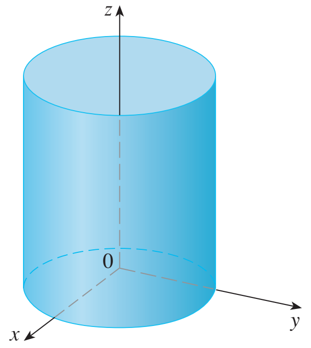


FIGURE 9
The cylinder $x^2 + y^2 = 1$ ■

EXAMPLE 3 Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

SOLUTION The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$. This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$. The portion of this plane that lies in the first octant is sketched in Figure 10. ■

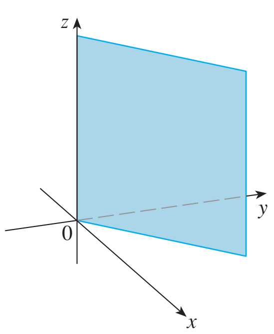


FIGURE 10
The plane $y = x$

Distance and Spheres

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

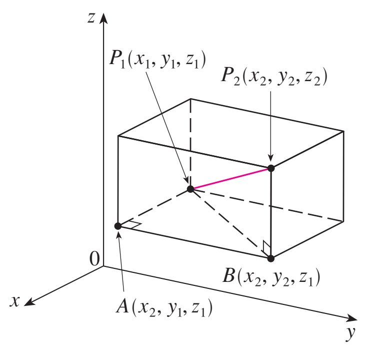


FIGURE 11

To see why this formula is true, we construct a rectangular box as in Figure 11, where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$

and

$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EXAMPLE 4 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = 3$$ ■

EXAMPLE 5 Find an equation of a sphere with radius r and center $C(h, k, l)$.

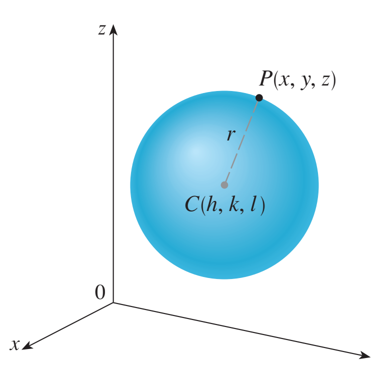


FIGURE 12

SOLUTION By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. (See Figure 12.) Thus P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$ ■

The result of Example 5 is worth remembering.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

EXAMPLE 7 What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4 \qquad z \leqslant 0$$

SOLUTION The inequalities

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4$$

can be rewritten as

$$1 \leqslant \sqrt{x^2 + y^2 + z^2} \leqslant 2$$

so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z \leqslant 0$, so the points lie on or below the xy-plane. Thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. It is sketched in Figure 13. ■

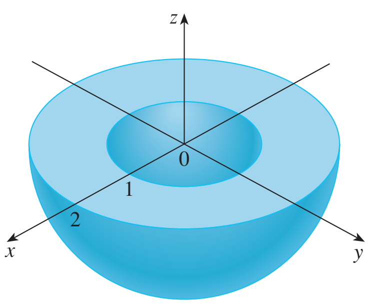


FIGURE 13

12.1 EXERCISES

1. Suppose you start at the origin, move along the x-axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?

2. Sketch the points $(1, 5, 3)$, $(0, 2, -3)$, $(-3, 0, 2)$, and $(2, -2, -1)$ on a single set of coordinate axes.

3. Which of the points $A(-4, 0, -1)$, $B(3, 1, -5)$, and $C(2, 4, 6)$ is closest to the yz-plane? Which point lies in the xz-plane?

4. What are the projections of the point $(2, 3, 5)$ on the xy-, yz-, and xz-planes? Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

5. What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with sketches.

6. What does the equation $y = 3$ represent in $\mathbb{R}^3$? What does $z = 5$ represent? What does the pair of equations $y = 3$, $z = 5$ represent? In other words, describe the set of points (x, y, z) such that $y = 3$ and $z = 5$. Illustrate with a sketch.

7. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.

8. Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x^2 + z^2 = 9$.

9–10 Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

9. $P(3, -2, -3)$, $Q(7, 0, 1)$, $R(1, 2, 1)$

10. $P(2, -1, 0)$, $Q(4, 1, 1)$, $R(4, -5, 4)$

11. Determine whether the points lie on a straight line.
(a) $A(2, 4, 2)$, $B(3, 7, -2)$, $C(1, 3, 3)$
(b) $D(0, -5, 5)$, $E(1, -2, 4)$, $F(3, 4, 2)$

12. Find the distance from $(4, -2, 6)$ to each of the following.
(a) The xy-plane
(b) The yz-plane
(c) The xz-plane
(d) The x-axis
(e) The y-axis
(f) The z-axis

13. Find an equation of the sphere with center $(-3, 2, 5)$ and radius 4. What is the intersection of this sphere with the yz-plane?

14. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

15. Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has center $(3, 8, 1)$.

16. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

17–20 Show that the equation represents a sphere, and find its center and radius.

17. $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$

18. $x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$

19. $2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$

20. $3x^2 + 3y^2 + 3z^2 = 10 + 6y + 12z$

---

21. (a) Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(b) Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$. (A *median* of a triangle is a line segment that joins a vertex to the midpoint of the opposite side.)

22. Find an equation of a sphere if one of its diameters has endpoints $(5, 4, 3)$ and $(1, 6, -9)$.

23. Find equations of the spheres with center $(2, -3, 6)$ that touch (a) the xy-plane, (b) the yz-plane, (c) the xz-plane.

24. Find an equation of the largest sphere with center $(5, 4, 9)$ that is contained in the first octant.

25–38 Describe in words the region of $\mathbb{R}^3$ represented by the equation(s) or inequality.

25. $x = 5$

26. $y = -2$

27. $y < 8$

28. $z \geqslant -1$

29. $0 \leqslant z \leqslant 6$

30. $y^2 = 4$

31. $x^2 + y^2 = 4, \quad z = -1$

32. $x^2 + y^2 = 4$

33. $x^2 + y^2 + z^2 = 4$

34. $x^2 + y^2 + z^2 \leqslant 4$

35. $1 \leqslant x^2 + y^2 + z^2 \leqslant 5$

36. $x = z$

37. $x^2 + z^2 \leqslant 9$

38. $x^2 + y^2 + z^2 > 2z$

39–42 Write inequalities to describe the region.

39. The region between the yz-plane and the vertical plane $x = 5$

40. The solid cylinder that lies on or below the plane $z = 8$ and on or above the disk in the xy-plane with center the origin and radius 2

41. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

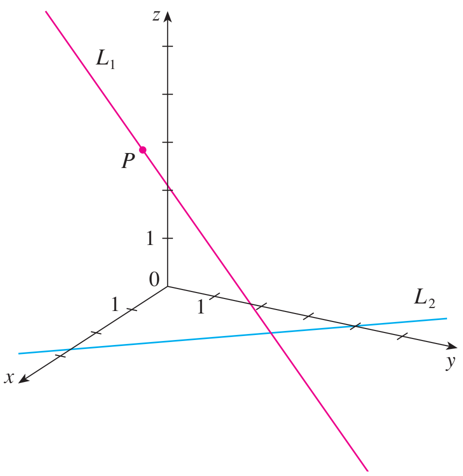
42. The solid upper hemisphere of the sphere of radius 2 centered at the origin

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43. The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 onto the xy-plane. (In other words, the points on L_2 are directly beneath, or above, the points on L_1.)

(a) Find the coordinates of the point P on the line L_1.

(b) Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz-plane, respectively.



44. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

45. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

46. Find the volume of the solid that lies inside both of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$$

and $$x^2 + y^2 + z^2 = 4$$

47. Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

48. Describe and sketch a solid with the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle.

12.2 Vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter ($\vec{v}$).

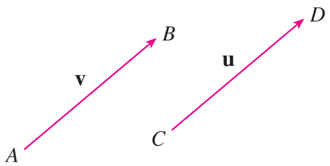


FIGURE 1
Equivalent vectors

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector v**, shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position. We say that **u** and **v** are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

Combining Vectors

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$ as in Figure 2. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

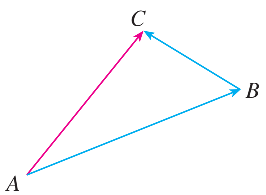


FIGURE 2

In general, if we start with vectors **u** and **v**, we first move **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

> **Definition of Vector Addition** If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

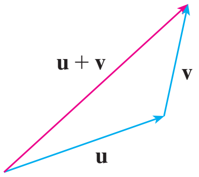


FIGURE 3
The Triangle Law

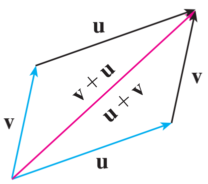


FIGURE 4
The Parallelogram Law

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**. Completing the parallelogram, we see that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This also gives another way to construct the sum: if we place **u** and **v** so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)



FIGURE 5

EXAMPLE 1 Draw the sum of the vectors **a** and **b** shown in Figure 5.

SOLUTION First we move **b** and place its tail at the tip of **a**, being careful to draw a copy of **b** that has the same length and direction. Then we draw the vector $\mathbf{a} + \mathbf{b}$ [see

Figure 6(a)] starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

Alternatively, we could place **b** so it starts where **a** starts and construct **a** + **b** by the Parallelogram Law as in Figure 6(b).

TEC Visual 12.2 shows how the Triangle and Parallelogram Laws work for various vectors **a** and **b**.

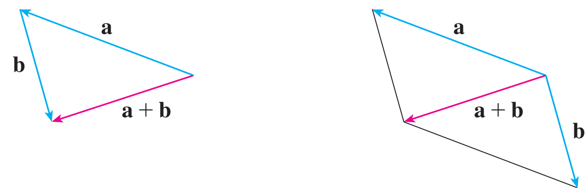


FIGURE 6 (a) (b)

It is possible to multiply a vector by a real number c. (In this context we call the real number c a **scalar** to distinguish it from a vector.) For instance, we want 2**v** to be the same vector as **v** + **v**, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

> **Definition of Scalar Multiplication** If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

This definition is illustrated in Figure 7. We see that real numbers work like scaling factors here; that's why we call them scalars. Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as **v** but points in the opposite direction. We call it the **negative** of **v**.

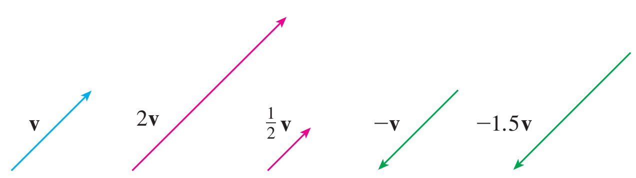


FIGURE 7
Scalar multiples of **v**

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of **v**, $-\mathbf{v}$, and then adding it to **u** by the Parallelogram Law as in Figure 8(a). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$, the vector $\mathbf{u} - \mathbf{v}$, when added to **v**, gives **u**. So we could construct $\mathbf{u} - \mathbf{v}$ as in Figure 8(b) by means of the Triangle Law. Notice that if **u** and **v** both start from the same initial point, then $\mathbf{u} - \mathbf{v}$ connects the tip of **v** to the tip of **u**.

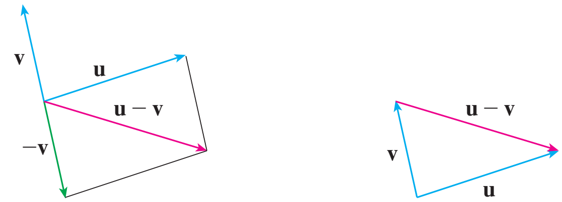


FIGURE 8 (a) (b)
Drawing $\mathbf{u} - \mathbf{v}$



FIGURE 9

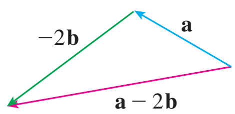


FIGURE 10

EXAMPLE 2 If **a** and **b** are the vectors shown in Figure 9, draw **a** − 2**b**.

SOLUTION We first draw the vector −2**b** pointing in the direction opposite to **b** and twice as long. We place it with its tail at the tip of **a** and then use the Triangle Law to draw **a** + (−2**b**) as in Figure 10. ■

Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11). These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

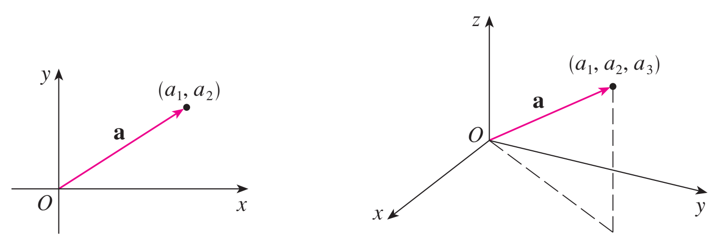


FIGURE 11 $\mathbf{a} = \langle a_1, a_2 \rangle$ $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

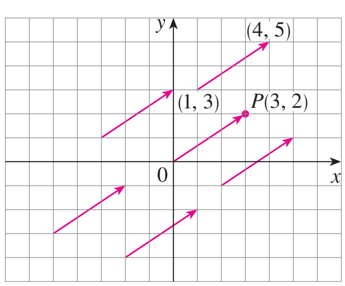


FIGURE 12
Representations of $\mathbf{a} = \langle 3, 2 \rangle$

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.) Let's consider any other representation $\overrightarrow{AB}$ of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$. Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following result.

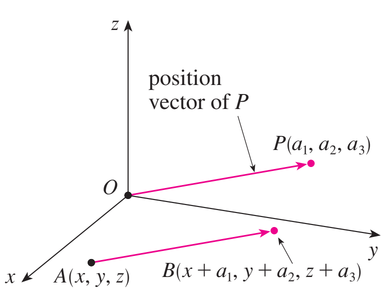


FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

EXAMPLE 3 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

SOLUTION By (1), the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$ ■

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

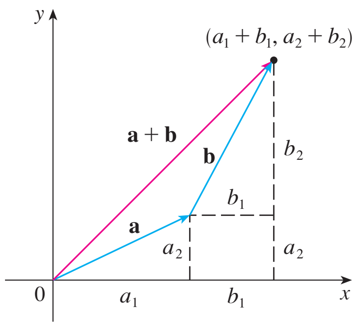


FIGURE 14

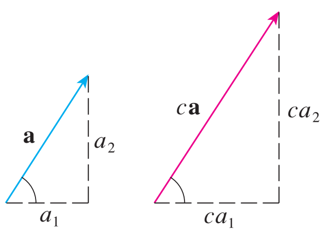


FIGURE 15

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. In other words, *to add algebraic vectors we add corresponding components.* Similarly, *to subtract vectors we subtract corresponding components.* From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2. So *to multiply a vector by a scalar we multiply each component by that scalar.*

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

EXAMPLE 4 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

SOLUTION

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 + (-2), 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle \end{aligned}$$

■

Vectors in n dimensions are used to list various quantities in an organized way. For instance, the components of a six-dimensional vector

$$\mathbf{p} = \langle p_1, p_2, p_3, p_4, p_5, p_6 \rangle$$

might represent the prices of six different ingredients required to make a particular product. Four-dimensional vectors $\langle x, y, z, t \rangle$ are used in relativity theory, where the first three components specify a position in space and the fourth represents time.

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we will later need to consider the set V_n of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\mathbf{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ **2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$ **4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ **6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7. $(cd)\mathbf{a} = c(d\mathbf{a})$ **8.** $1\mathbf{a} = \mathbf{a}$

These eight properties of vectors can be readily verified either geometrically or algebraically. For instance, Property 1 can be seen from Figure 4 (it's equivalent to the Parallelogram Law) or as follows for the case $n = 2$:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

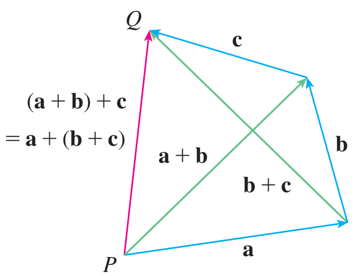


FIGURE 16

We can see why Property 2 (the associative law) is true by looking at Figure 16 and applying the Triangle Law several times: the vector $\overrightarrow{PQ}$ is obtained either by first constructing $\mathbf{a} + \mathbf{b}$ and then adding $\mathbf{c}$ or by adding $\mathbf{a}$ to the vector $\mathbf{b} + \mathbf{c}$.

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

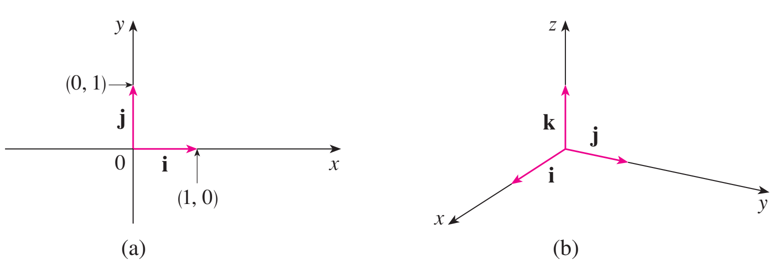
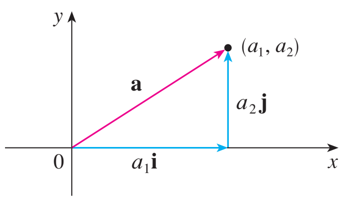
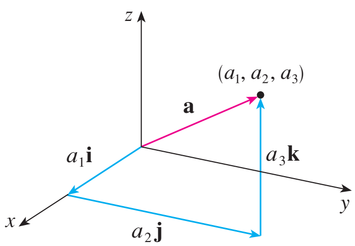


FIGURE 17
Standard basis vectors in V_2 and V_3



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

FIGURE 18

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of **i**, **j**, and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

3 $$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\,\mathbf{i} + a_2\,\mathbf{j}$$

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

EXAMPLE 5 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of **i**, **j**, and **k**.

SOLUTION Using Properties 1, 2, 5, 6, and 7 of vectors, we have

$$2\mathbf{a} + 3\mathbf{b} = 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k})$$

$$= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}$$ ■

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

4 $$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

EXAMPLE 6 Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

SOLUTION The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 4, the unit vector with the same direction is

$$\tfrac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}$$ ■

Gibbs

Josiah Willard Gibbs (1839–1903), a professor of mathematical physics at Yale College, published the first book on vectors, *Vector Analysis*, in 1881. More complicated objects, called quaternions, had earlier been invented by Hamilton as mathematical tools for describing space, but they weren't easy for scientists to use. Quaternions have a scalar part and a vector part. Gibb's idea was to use the vector part separately. Maxwell and Heaviside had similar ideas, but Gibb's approach has proved to be the most convenient way to study space.

■ Applications

Vectors are useful in many aspects of physics and engineering. In Chapter 13 we will see how they describe the velocity and acceleration of objects moving in space. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

EXAMPLE 7 A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

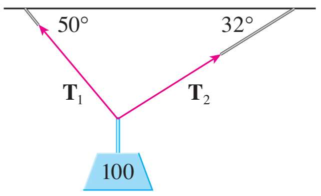


FIGURE 19

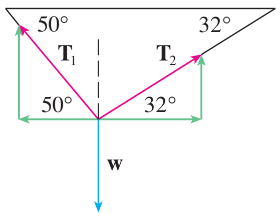


FIGURE 20

SOLUTION We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

$$\boxed{5} \qquad \mathbf{T}_1 = -|\mathbf{T}_1| \cos 50° \, \mathbf{i} + |\mathbf{T}_1| \sin 50° \, \mathbf{j}$$

$$\boxed{6} \qquad \mathbf{T}_2 = |\mathbf{T}_2| \cos 32° \, \mathbf{i} + |\mathbf{T}_2| \sin 32° \, \mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w} = -100\,\mathbf{j}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$$

Thus

$$\left(-|\mathbf{T}_1| \cos 50° + |\mathbf{T}_2| \cos 32°\right) \mathbf{i} + \left(|\mathbf{T}_1| \sin 50° + |\mathbf{T}_2| \sin 32°\right) \mathbf{j} = 100\,\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1| \cos 50° + |\mathbf{T}_2| \cos 32° = 0$$

$$|\mathbf{T}_1| \sin 50° + |\mathbf{T}_2| \sin 32° = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1| \sin 50° + \frac{|\mathbf{T}_1| \cos 50°}{\cos 32°} \sin 32° = 100$$

$$|\mathbf{T}_1| \left(\sin 50° + \cos 50° \, \frac{\sin 32°}{\cos 32°} \right) = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32° \cos 50°} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50°}{\cos 32°} \approx 64.91 \text{ lb}$$

Substituting these values in (5) and (6), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05\,\mathbf{i} + 65.60\,\mathbf{j}$$

$$\mathbf{T}_2 \approx 55.05\,\mathbf{i} + 34.40\,\mathbf{j}$$

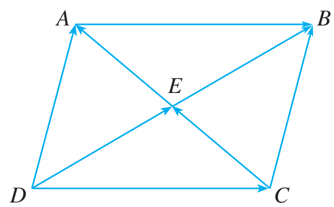
■

12.2 EXERCISES

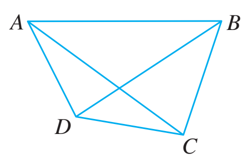
1. Are the following quantities vectors or scalars? Explain.
(a) The cost of a theater ticket
(b) The current in a river
(c) The initial flight path from Houston to Dallas
(d) The population of the world

2. What is the relationship between the point (4, 7) and the vector $\langle 4, 7 \rangle$? Illustrate with a sketch.

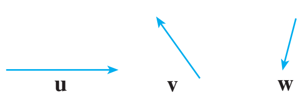
3. Name all the equal vectors in the parallelogram shown.



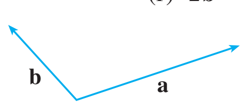
4. Write each combination of vectors as a single vector.
(a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{CD} + \overrightarrow{DB}$
(c) $\overrightarrow{DB} - \overrightarrow{AB}$ (d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$



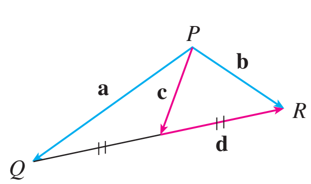
5. Copy the vectors in the figure and use them to draw the following vectors.
(a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{u} + \mathbf{w}$
(c) $\mathbf{v} + \mathbf{w}$ (d) $\mathbf{u} - \mathbf{v}$
(e) $\mathbf{v} + \mathbf{u} + \mathbf{w}$ (f) $\mathbf{u} - \mathbf{w} - \mathbf{v}$



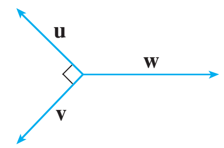
6. Copy the vectors in the figure and use them to draw the following vectors.
(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
(c) $\frac{1}{2}\mathbf{a}$ (d) $-3\mathbf{b}$
(e) $\mathbf{a} + 2\mathbf{b}$ (f) $2\mathbf{b} - \mathbf{a}$



7. In the figure, the tip of $\mathbf{c}$ and the tail of $\mathbf{d}$ are both the midpoint of QR. Express $\mathbf{c}$ and $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



8. If the vectors in the figure satisfy $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, what is $|\mathbf{w}|$?



9–14 Find a vector $\mathbf{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

9. $A(-2, 1)$, $B(1, 2)$ **10.** $A(-5, -1)$, $B(-3, 3)$
11. $A(3, -1)$, $B(2, 3)$ **12.** $A(3, 2)$, $B(1, 0)$
13. $A(0, 3, 1)$, $B(2, 3, -1)$ **14.** $A(0, 6, -1)$, $B(3, 4, 4)$

15–18 Find the sum of the given vectors and illustrate geometrically.

15. $\langle -1, 4 \rangle$, $\langle 6, -2 \rangle$ **16.** $\langle 3, -1 \rangle$, $\langle -1, 5 \rangle$
17. $\langle 3, 0, 1 \rangle$, $\langle 0, 8, 0 \rangle$ **18.** $\langle 1, 3, -2 \rangle$, $\langle 0, 0, 6 \rangle$

19–22 Find $\mathbf{a} + \mathbf{b}$, $4\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle -3, 4 \rangle$, $\mathbf{b} = \langle 9, -1 \rangle$
20. $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} - 2\mathbf{j}$
21. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$
22. $\mathbf{a} = \langle 8, 1, -4 \rangle$, $\mathbf{b} = \langle 5, -2, 1 \rangle$

23–25 Find a unit vector that has the same direction as the given vector.

23. $\langle 6, -2 \rangle$ **24.** $-5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
25. $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

26. Find the vector that has the same direction as $\langle 6, 2, -3 \rangle$ but has length 4.

27–28 What is the angle between the given vector and the positive direction of the x-axis?

27. $\mathbf{i} + \sqrt{3}\,\mathbf{j}$ **28.** $8\mathbf{i} + 6\mathbf{j}$

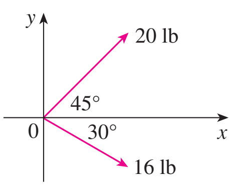
29. If $\mathbf{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive x-axis and $|\mathbf{v}| = 4$, find $\mathbf{v}$ in component form.

30. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

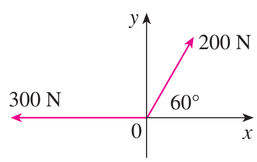
31. A quarterback throws a football with angle of elevation 40° and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

32–33 Find the magnitude of the resultant force and the angle it makes with the positive x-axis.

32.



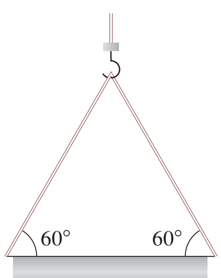
33.



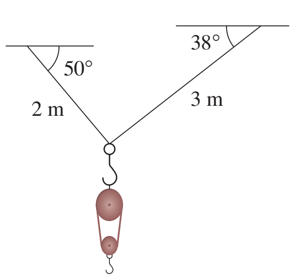
34. The magnitude of a velocity vector is called *speed*. Suppose that a wind is blowing *from* the direction N45°W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h. The *true course,* or *track,* of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.

35. A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.

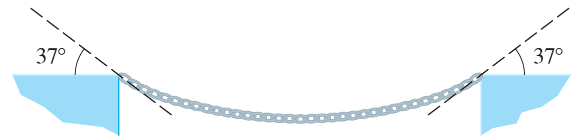
36. A crane suspends a 500-lb steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of 60° with the beam. Find the tension vector in each support cable and the magnitude of each tension.



37. A block-and-tackle pulley hoist is suspended in a warehouse by ropes of lengths 2 m and 3 m. The hoist weighs 350 N. The ropes, fastened at different heights, make angles of 50° and 38° with the horizontal. Find the tension in each rope and the magnitude of each tension.



38. The tension $\mathbf{T}$ at each end of a chain has magnitude 25 N (see the figure). What is the weight of the chain?



39. A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
(a) In what direction should he steer?
(b) How long will the trip take?

40. Three forces act on an object. Two of the forces are at an angle of 100° to each other and have magnitudes 25 N and 12 N. The third is perpendicular to the plane of these two forces and has magnitude 4 N. Calculate the magnitude of the force that would exactly counterbalance these three forces.

41. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

42. (a) Find the unit vectors that are parallel to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6, 1)$.
(b) Find the unit vectors that are perpendicular to the tangent line.
(c) Sketch the curve $y = 2\sin x$ and the vectors in parts (a) and (b), all starting at $(\pi/6, 1)$.

43. If A, B, and C are the vertices of a triangle, find

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$$

44. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

45. (a) Draw the vectors $\mathbf{a} = \langle 3, 2 \rangle$, $\mathbf{b} = \langle 2, -1 \rangle$, and $\mathbf{c} = \langle 7, 1 \rangle$.
(b) Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
(c) Use the sketch to estimate the values of s and t.
(d) Find the exact values of s and t.

46. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors that are not parallel and $\mathbf{c}$ is any vector in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Give a geometric argument to show that $\mathbf{c}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars s and t. Then give an argument using components.

47. If $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

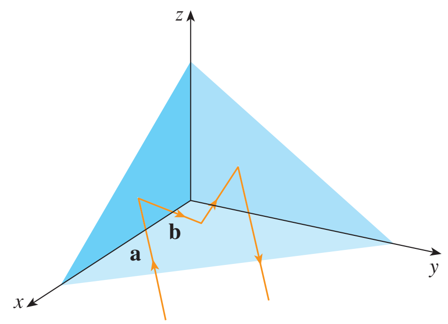
48. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{r}_1 = \langle x_1, y_1 \rangle$, and $\mathbf{r}_2 = \langle x_2, y_2 \rangle$, describe the set of all points (x, y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.

49. Figure 16 gives a geometric demonstration of Property 2 of vectors. Use components to give an algebraic proof of this fact for the case $n = 2$.

50. Prove Property 5 of vectors algebraically for the case $n = 3$. Then use similar triangles to give a geometric proof.

51. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

52. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams and an array of corner mirrors on the moon, to calculate very precisely the distance from the earth to the moon.)



12.3 The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of $\mathbf{a}$ and $\mathbf{b}$, we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

EXAMPLE 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\tfrac{1}{2} \right\rangle = (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$ ■

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5. $\mathbf{0} \cdot \mathbf{a} = 0$

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

The proofs of the remaining properties are left as exercises. ■

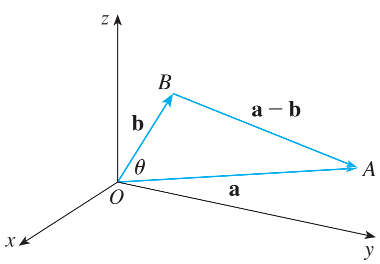


FIGURE 1

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \leqslant \theta \leqslant \pi$. In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

4 $$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

5 $$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \end{aligned}$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta$$

Thus $$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos \theta$$

or $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$ ■

EXAMPLE 2 If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

SOLUTION Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3) = 4 \cdot 6 \cdot \tfrac{1}{2} = 12$$ ■

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$$ ■

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

EXAMPLE 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7). ■

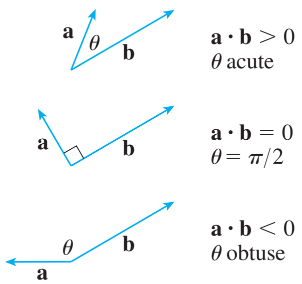


FIGURE 2

TEC Visual 12.3A shows an animation of Figure 2.

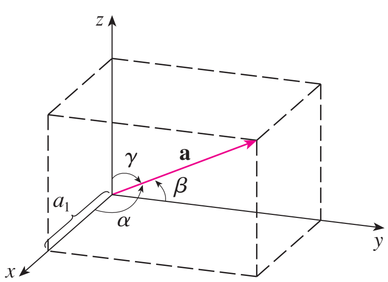


FIGURE 3

Because $\cos\theta > 0$ if $0 \leqslant \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \leqslant \pi$, we see that $\mathbf{a}\cdot\mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a}\cdot\mathbf{b}$ as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product $\mathbf{a}\cdot\mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then we have $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively. (See Figure 3.)

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain

8 $$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.)

Similarly, we also have

9 $$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

10 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We can also use Equations 8 and 9 to write

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma\rangle$$

$$= |\mathbf{a}|\langle \cos\alpha, \cos\beta, \cos\gamma\rangle$$

Therefore

11 $$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

EXAMPLE 5 Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74° \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58° \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37°$$

■

TEC Visual 12.3B shows how Figure 4 changes when we vary **a** and **b**.

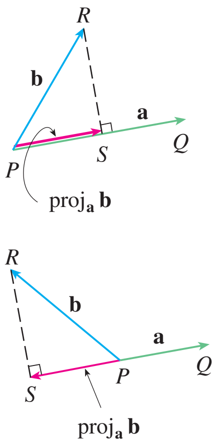


FIGURE 4
Vector projections

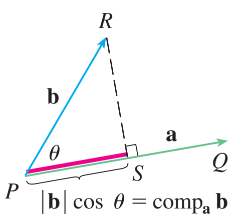


FIGURE 5
Scalar projection

Projections

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}} \mathbf{b}$. (You can think of it as a shadow of **b**).

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**. (See Figure 5.) This is denoted by $\text{comp}_{\mathbf{a}} \mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{a}|\big(|\mathbf{b}| \cos \theta\big)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad \text{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$ ■

One use of projections occurs in physics in calculating work. In Section 5.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along **D** and the distance moved:

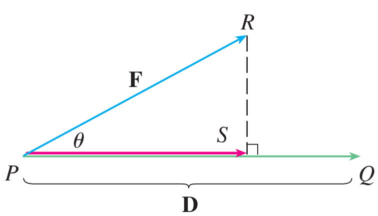


FIGURE 6

$$W = \big(|\mathbf{F}| \cos \theta\big)|\mathbf{D}|$$

But then, from Theorem 3, we have

12 $$W = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

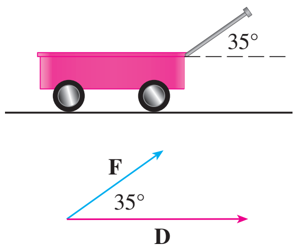


FIGURE 7

EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

SOLUTION If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35°$$

$$= (70)(100)\cos 35° \approx 5734\ \text{N}\cdot\text{m} = 5734\ \text{J}$$ ■

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2\rangle$, so by Equation 12, the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = \langle 3, 4, 5\rangle \cdot \langle 2, 5, 2\rangle$$

$$= 6 + 20 + 10 = 36$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J. ■

12.3 EXERCISES

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
(a) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ (b) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
(c) $|\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ (d) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$
(e) $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ (f) $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$

2–10 Find $\mathbf{a}\cdot\mathbf{b}$.

2. $\mathbf{a} = \langle 5, -2\rangle$, $\mathbf{b} = \langle 3, 4\rangle$

3. $\mathbf{a} = \langle 1.5, 0.4\rangle$, $\mathbf{b} = \langle -4, 6\rangle$

4. $\mathbf{a} = \langle 6, -2, 3\rangle$, $\mathbf{b} = \langle 2, 5, -1\rangle$

5. $\mathbf{a} = \langle 4, 1, \frac{1}{4}\rangle$, $\mathbf{b} = \langle 6, -3, -8\rangle$

6. $\mathbf{a} = \langle p, -p, 2p\rangle$, $\mathbf{b} = \langle 2q, q, -q\rangle$

7. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

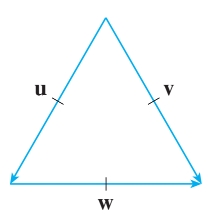
8. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$

9. $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is 30°

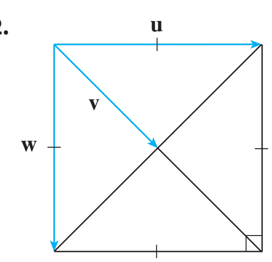
10. $|\mathbf{a}| = 80$, $|\mathbf{b}| = 50$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $3\pi/4$

11–12 If $\mathbf{u}$ is a unit vector, find $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\cdot\mathbf{w}$.

11.



12.



13. (a) Show that $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.
(b) Show that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$.

14. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$4 for a hamburger, \$2.50 for a hot dog, and \$1 for a soft drink. If $\mathbf{A} = \langle a, b, c\rangle$ and $\mathbf{P} = \langle 4, 2.5, 1\rangle$, what is the meaning of the dot product $\mathbf{A}\cdot\mathbf{P}$?

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15. $\mathbf{a} = \langle 4, 3\rangle$, $\mathbf{b} = \langle 2, -1\rangle$

16. $\mathbf{a} = \langle -2, 5\rangle$, $\mathbf{b} = \langle 5, 12\rangle$

17. $\mathbf{a} = \langle 1, -4, 1 \rangle$, $\mathbf{b} = \langle 0, 2, -2 \rangle$

18. $\mathbf{a} = \langle -1, 3, 4 \rangle$, $\mathbf{b} = \langle 5, 2, 1 \rangle$

19. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

20. $\mathbf{a} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + 2\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21. $P(2, 0)$, $Q(0, 3)$, $R(3, 4)$

22. $A(1, 0, -1)$, $B(3, -2, 0)$, $C(1, 3, 3)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle 9, 3 \rangle$, $\mathbf{b} = \langle -2, 6 \rangle$
(b) $\mathbf{a} = \langle 4, 5, -2 \rangle$, $\mathbf{b} = \langle 3, -1, 5 \rangle$
(c) $\mathbf{a} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$
(d) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$

24. (a) $\mathbf{u} = \langle -5, 4, -2 \rangle$, $\mathbf{v} = \langle 3, 4, -1 \rangle$
(b) $\mathbf{u} = 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
(c) $\mathbf{u} = \langle c, c, c \rangle$, $\mathbf{v} = \langle c, 0, -c \rangle$

25. Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

26. Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$, and $\langle 1, x, 0 \rangle$ is $45°$.

27. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

28. Find two unit vectors that make an angle of $60°$ with $\mathbf{v} = \langle 3, 4 \rangle$.

29–30 Find the acute angle between the lines.

29. $2x - y = 3$, $3x + y = 7$

30. $x + 2y = 7$, $5x - y = 2$

31–32 Find the acute angles between the curves at their points of intersection. (The angle between two curves is the angle between their tangent lines at the point of intersection.)

31. $y = x^2$, $y = x^3$

32. $y = \sin x$, $y = \cos x$, $0 \leqslant x \leqslant \pi/2$

33–37 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

33. $\langle 2, 1, 2 \rangle$

34. $\langle 6, 3, -2 \rangle$

35. $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

36. $\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

37. $\langle c, c, c \rangle$, where $c > 0$

38. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

39–44 Find the scalar and vector projections of **b** onto **a**.

39. $\mathbf{a} = \langle -5, 12 \rangle$, $\mathbf{b} = \langle 4, 6 \rangle$

40. $\mathbf{a} = \langle 1, 4 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$

41. $\mathbf{a} = \langle 4, 7, -4 \rangle$, $\mathbf{b} = \langle 3, -1, 1 \rangle$

42. $\mathbf{a} = \langle -1, 4, 8 \rangle$, $\mathbf{b} = \langle 12, 1, 2 \rangle$

43. $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

44. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{k}$

45. Show that the vector $\text{orth}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b}$ is orthogonal to **a**. (It is called an **orthogonal projection** of **b**.)

46. For the vectors in Exercise 40, find $\text{orth}_{\mathbf{a}}\,\mathbf{b}$ and illustrate by drawing the vectors **a**, **b**, $\text{proj}_{\mathbf{a}}\,\mathbf{b}$, and $\text{orth}_{\mathbf{a}}\,\mathbf{b}$.

47. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector **b** such that $\text{comp}_{\mathbf{a}}\,\mathbf{b} = 2$.

48. Suppose that **a** and **b** are nonzero vectors.
(a) Under what circumstances is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \text{comp}_{\mathbf{b}}\,\mathbf{a}$?
(b) Under what circumstances is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a}$?

49. Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

50. A tow truck drags a stalled car along a road. The chain makes an angle of $30°$ with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

51. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of $40°$ above the horizontal moves the sled 80 ft. Find the work done by the force.

52. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

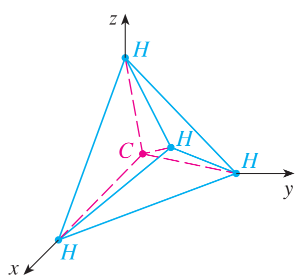
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

55. Find the angle between a diagonal of a cube and one of its edges.

56. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

57. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°. [*Hint:* Take the vertices of the tetrahedron to be the points (1, 0, 0), (0, 1, 0), (0, 0, 1), and (1, 1, 1), as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.]



58. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where **a**, **b**, and **c** are all nonzero vectors, show that **c** bisects the angle between **a** and **b**.

59. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

60. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

61. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$$

62. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

63. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 62.)

64. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors **u** and **v** must have the same length.

65. If θ is the angle between vectors **a** and **b**, show that

$$\text{proj}_{\mathbf{a}} \mathbf{b} \cdot \text{proj}_{\mathbf{b}} \mathbf{a} = (\mathbf{a} \cdot \mathbf{b}) \cos^2\theta$$

12.4 The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector **c** that is perpendicular to both **a** and **b**, as we will see in the next section and in Chapters 13 and 14. If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{1}$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \tag{2}$$

To eliminate c_3 we multiply (1) by b_3 and (2) by a_3 and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \tag{3}$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of (3) is

$$c_1 = a_2b_3 - a_3b_2 \qquad c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both **a** and **b** is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the *cross product* of **a** and **b** and is denoted by $\mathbf{a} \times \mathbf{b}$.

Hamilton

The cross product was invented by the Irish mathematician Sir William Rowan Hamilton (1805–1865), who had created a precursor of vectors, called quaternions. When he was five years old Hamilton could read Latin, Greek, and Hebrew. At age eight he added French and Italian and when ten he could read Arabic and Sanskrit. At the age of 21, while still an undergraduate at Trinity College in Dublin, Hamilton was appointed Professor of Astronomy at the university and Royal Astronomer of Ireland!

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

In order to make Definition 4 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Multiply across the diagonals and subtract.) For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\textbf{5} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\textbf{6} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

In view of the similarity between Equations 5 and 6, we often write

$$\textbf{7} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain

Equation 6. The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$ ■

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2a_3 - a_3a_2)\,\mathbf{i} - (a_1a_3 - a_3a_1)\,\mathbf{j} + (a_1a_2 - a_2a_1)\,\mathbf{k}$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$ ■

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. This is one of the most important properties of a cross product, so let's emphasize and verify it in the following theorem and give a formal proof.

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3$$

$$= 0$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. ■

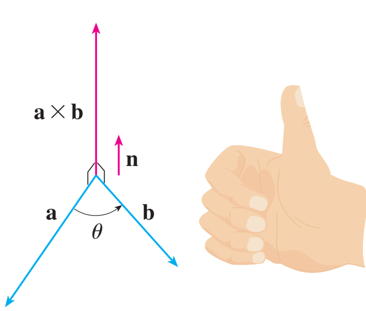


FIGURE 1
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule:* if the fingers of your right hand curl in the direction of

a rotation (through an angle less than 180°) from **a** to **b**, then your thumb points in the direction of **a** × **b**.

Now that we know the direction of the vector **a** × **b**, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 Theorem If θ is the angle between **a** and **b** (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

TEC Visual 12.4 shows how **a** × **b** changes as **b** changes.

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\
&\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \qquad \text{(by Theorem 12.3.3)} \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$ ■

Geometric characterization of **a** × **b**

Since a vector is completely determined by its magnitude and direction, we can now say that **a** × **b** is the vector that is perpendicular to both **a** and **b**, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* **a** × **b**.

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

PROOF Two nonzero vectors **a** and **b** are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. ■

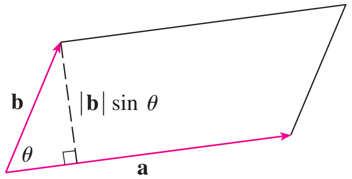


FIGURE 2

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2. If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|\big(|\mathbf{b}|\sin\theta\big) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE 3 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (12.2.1) that

$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\,\mathbf{i} - 15\,\mathbf{j} + 15\,\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, is also perpendicular to the plane. ■

EXAMPLE 4 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 3 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■

If we apply Theorems 8 and 9 to the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

⊘ Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products. The following theorem summarizes the properties of vector products.

11 Properties of the Cross Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

PROOF OF PROPERTY 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

12
$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\
&= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}
\end{aligned}$$

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Notice from Equation 12 that we can write the scalar triple product as a determinant:

13
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

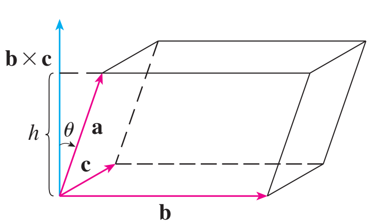


FIGURE 3

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (See Figure 3.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If we use the formula in (14) and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

EXAMPLE 5 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar.

SOLUTION We use Equation 13 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Therefore, by (14), the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ■

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **c**. Property 6 will be used to derive Kepler's First Law of planetary motion in Chapter 13. Its proof is left as Exercise 50.

Torque

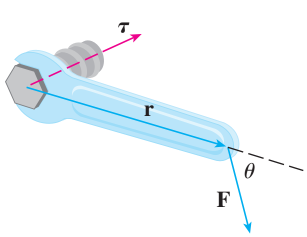


FIGURE 4

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.) The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 9, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathbf{F}| \sin \theta$. The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

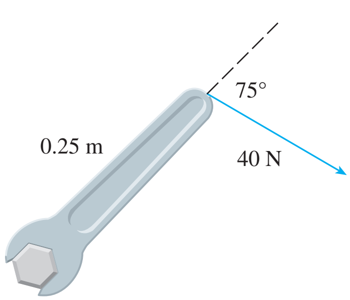


FIGURE 5

EXAMPLE 6 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75° = (0.25)(40) \sin 75°$$
$$= 10 \sin 75° \approx 9.66 \text{ N}\cdot\text{m}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\mathbf{n} \approx 9.66\,\mathbf{n}$$

where **n** is a unit vector directed down into the page (by the right-hand rule). ■

12.4 EXERCISES

1–7 Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both **a** and **b**.

1. $\mathbf{a} = \langle 2, 3, 0\rangle, \quad \mathbf{b} = \langle 1, 0, 5\rangle$

2. $\mathbf{a} = \langle 4, 3, -2\rangle, \quad \mathbf{b} = \langle 2, -1, 1\rangle$

3. $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

4. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

5. $\mathbf{a} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

6. $\mathbf{a} = t\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

7. $\mathbf{a} = \langle t, 1, 1/t\rangle, \quad \mathbf{b} = \langle t^2, t^2, 1\rangle$

8. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch **a**, **b**, and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

9–12 Find the vector, not with determinants, but by using properties of cross products.

9. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

10. $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

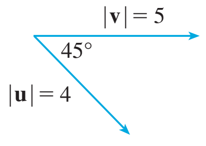
11. $(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})$

12. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

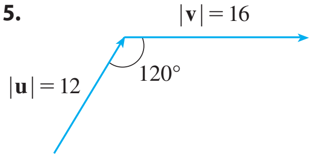
13. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
(d) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$
(e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$
(f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

14–15 Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

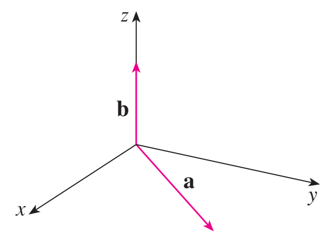
14.



15.



16. The figure shows a vector **a** in the xy-plane and a vector **b** in the direction of **k**. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
(a) Find $|\mathbf{a} \times \mathbf{b}|$.
(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.



17. If $\mathbf{a} = \langle 2, -1, 3\rangle$ and $\mathbf{b} = \langle 4, 2, 1\rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

18. If $\mathbf{a} = \langle 1, 0, 1\rangle$, $\mathbf{b} = \langle 2, 1, -1\rangle$, and $\mathbf{c} = \langle 0, 1, 3\rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

19. Find two unit vectors orthogonal to both $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 0\rangle$.

20. Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

21. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$ for any vector **a** in V_3.

22. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors **a** and **b** in V_3.

23–26 Prove the property of cross products (Theorem 11).

23. Property 1: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

24. Property 2: $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

25. Property 3: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

26. Property 4: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

27. Find the area of the parallelogram with vertices $A(-3, 0)$, $B(-1, 3)$, $C(5, 2)$, and $D(3, -1)$.

28. Find the area of the parallelogram with vertices $P(1, 0, 2)$, $Q(3, 3, 3)$, $R(7, 5, 8)$, and $S(5, 2, 7)$.

29–32 (a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

29. $P(1, 0, 1), \quad Q(-2, 1, 3), \quad R(4, 2, 5)$

30. $P(0, 0, -3), \quad Q(4, 2, 0), \quad R(3, 3, 1)$

31. $P(0, -2, 0), \quad Q(4, 1, -2), \quad R(5, 3, 1)$

32. $P(2, -3, 4)$, $Q(-1, -2, 2)$, $R(3, 1, -3)$

33–34 Find the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**.

33. $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle -1, 1, 2 \rangle$, $\mathbf{c} = \langle 2, 1, 4 \rangle$

34. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

35–36 Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

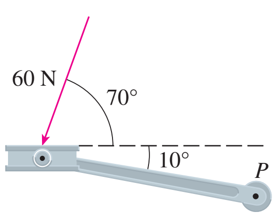
35. $P(-2, 1, 0)$, $Q(2, 3, 2)$, $R(1, 4, -1)$, $S(3, 6, 1)$

36. $P(3, 0, 1)$, $Q(-1, 2, 5)$, $R(5, 1, -1)$, $S(0, 4, 2)$

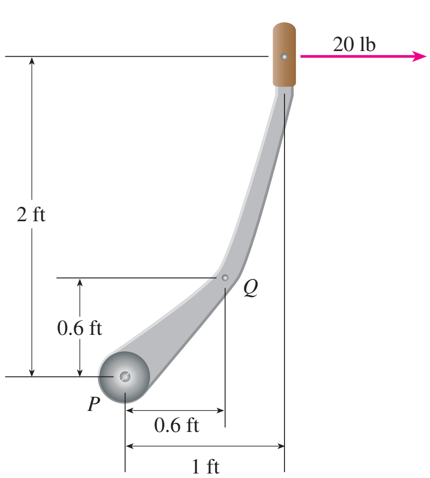
37. Use the scalar triple product to verify that the vectors $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ are coplanar.

38. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

39. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



40. (a) A horizontal force of 20 lb is applied to the handle of a gearshift lever as shown. Find the magnitude of the torque about the pivot point P.
(b) Find the magnitude of the torque about P if the same force is applied at the elbow Q of the lever.



41. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 N·m of torque to the bolt.

42. Let $\mathbf{v} = 5\mathbf{j}$ and let **u** be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

43. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between **a** and **b**.

44. (a) Find all vectors **v** such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, -5 \rangle$$

(b) Explain why there is no vector **v** such that

$$\langle 1, 2, 1 \rangle \times \mathbf{v} = \langle 3, 1, 5 \rangle$$

45. (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

46. (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.
(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

47. Show that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, show that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

49. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

50. Prove Property 6 of cross products, that is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

51. Use Exercise 50 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

52. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

53. Suppose that $\mathbf{a} \neq \mathbf{0}$.
(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

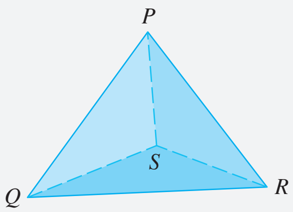
54. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice*.)
(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.
(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.
(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

DISCOVERY PROJECT THE GEOMETRY OF A TETRAHEDRON



A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in the figure.

1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.
(a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.
(b) Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.

3. Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the result of Problem 1, or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

12.5 Equations of Lines and Planes

Lines

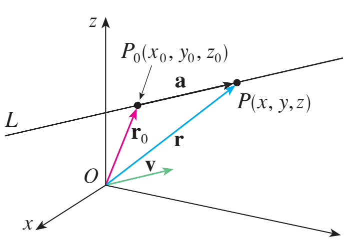


FIGURE 1

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. But, since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

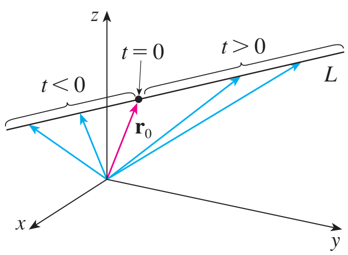


FIGURE 2

$$\boxed{1} \qquad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$. As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

If the vector $\mathbf{v}$ that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each value of the parameter t gives a point (x, y, z) on L.

2 Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Figure 3 shows the line L in Example 1 and its relation to the given point and to the vector that gives its direction.

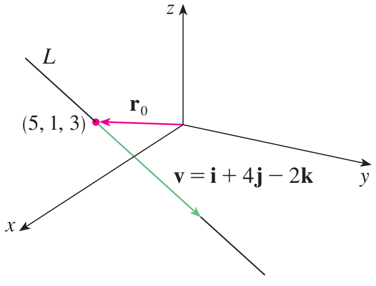


FIGURE 3

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

SOLUTION

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}$$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so (6, 5, 1) is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$. ■

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Or, if we stay with the point (5, 1, 3) but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a, b, or c is 0, we can solve each of these equations for t:

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

3
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators of Equations 3 are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

Figure 4 shows the line L in Example 2 and the point P where it intersects the xy-plane.

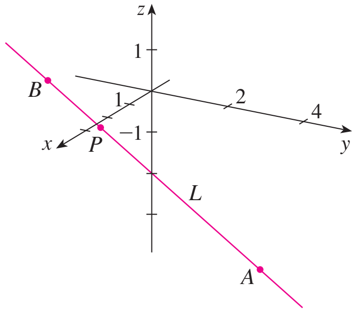


FIGURE 4

EXAMPLE 2
(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?

SOLUTION
(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations (2) are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$. ■

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Often, we need a description, not of an entire line, but of just a line segment. How, for instance, could we describe the line segment AB in Example 2? If we put $t = 0$ in the parametric equations in Example 2(a), we get the point $(2, 4, -3)$ and if we put $t = 1$ we get $(3, -1, 1)$. So the line segment AB is described by the parametric equations

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t \qquad 0 \leqslant t \leqslant 1$$

or by the corresponding vector equation

$$\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t \rangle \qquad 0 \leqslant t \leqslant 1$$

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \leqslant t \leqslant 1$.

4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

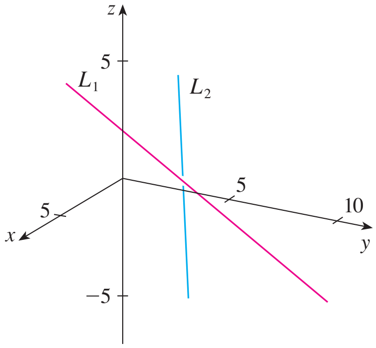


FIGURE 5

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$
$$-2 + 3t = 3 + s$$
$$4 - t = -3 + 4s$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines. ■

Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the

"direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.) The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

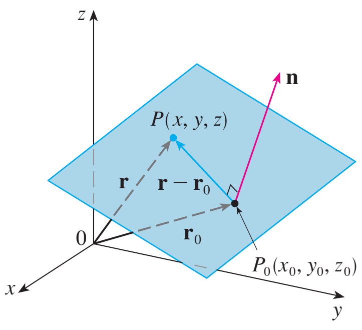


FIGURE 6

5 $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

6 $$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (5) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

7 A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

EXAMPLE 4 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

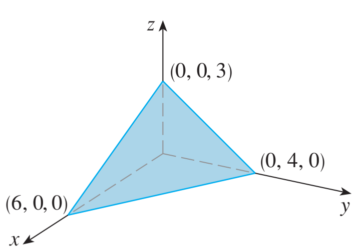


FIGURE 7

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ■

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

8 $$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation 8 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation (8) represents a plane with normal vector $\langle a, b, c \rangle$. (See Exercise 83.)

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle PQR.

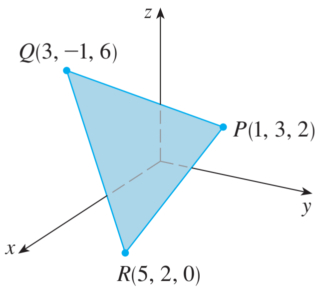


FIGURE 8

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

■

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$. ■

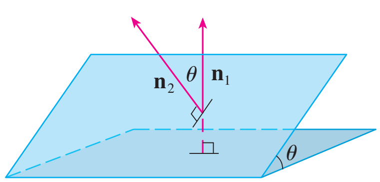


FIGURE 9

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in Figure 9).

Figure 10 shows the planes in Example 7 and their line of intersection L.

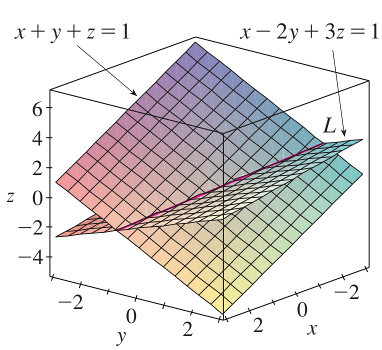


FIGURE 10

EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.

SOLUTION

(a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, Corollary 12.3.6 gives

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1 + 1 + 1}\,\sqrt{1 + 4 + 9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72°$$

(b) We first need to find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the

equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$, $y = 0$. So the point $(1, 0, 0)$ lies on L.

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\mathbf{v}$ parallel to L is given by the cross product

Another way to find the line of intersection is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\,\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

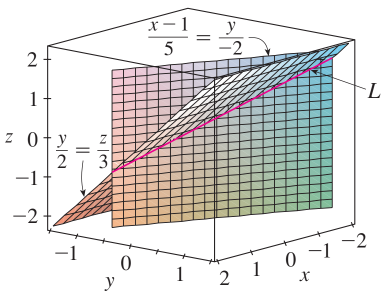


FIGURE 11

Figure 11 shows how the line L in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

NOTE Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was given as the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations that we found for L could be written as

$$\frac{x - 1}{5} = \frac{y}{-2} \qquad \text{and} \qquad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit L as the line of intersection of the planes $(x - 1)/5 = y/(-2)$ and $y/(-2) = z/(-3)$. (See Figure 11.)

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \text{and} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Distances

EXAMPLE 8 Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

SOLUTION Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 12.3.) Thus

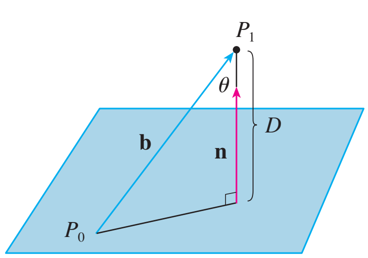


FIGURE 12

$$D = |\text{comp}_{\mathbf{n}}\,\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_0 + by_0 + cz_0 + d = 0$. Thus the formula for D can be written as

9
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $(\frac{1}{2}, 0, 0)$ is a point in this plane. By Formula 9, the distance between $(\frac{1}{2}, 0, 0)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$.

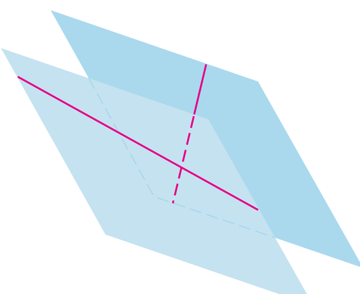


FIGURE 13
Skew lines, like those in Example 10, always lie on (nonidentical) parallel planes.

EXAMPLE 10 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2, which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \qquad \text{or} \qquad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

12.5 EXERCISES

1. Determine whether each statement is true or false in $\mathbb{R}^3$.
   (a) Two lines parallel to a third line are parallel.
   (b) Two lines perpendicular to a third line are parallel.
   (c) Two planes parallel to a third plane are parallel.
   (d) Two planes perpendicular to a third plane are parallel.
   (e) Two lines parallel to a plane are parallel.
   (f) Two lines perpendicular to a plane are parallel.
   (g) Two planes parallel to a line are parallel.
   (h) Two planes perpendicular to a line are parallel.
   (i) Two planes either intersect or are parallel.
   (j) Two lines either intersect or are parallel.
   (k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3} \rangle$

3. The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$

5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(4, 3, -1)$

7. The line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$

8. The line through the points $(1, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$

9. The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$

10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$

11. The line through $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$

12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

13. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?

14. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

15. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
    (b) Find the points in which the required line in part (a) intersects the coordinate planes.

16. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
    (b) In what points does this line intersect the coordinate planes?

17. Find a vector equation for the line segment from $(6, -1, 9)$ to $(7, 6, 0)$.

18. Find parametric equations for the line segment from $(-2, 18, 31)$ to $(11, -4, 48)$.

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = 3 + 2t, \quad y = 4 - t, \quad z = 1 + 3t$
    L_2: $x = 1 + 4s, \quad y = 3 - 2s, \quad z = 4 + 5s$

20. L_1: $x = 5 - 12t, \quad y = 3 + 9t, \quad z = 1 - 3t$
    L_2: $x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$

21. L_1: $\dfrac{x-2}{1} = \dfrac{y-3}{-2} = \dfrac{z-1}{-3}$
    L_2: $\dfrac{x-3}{1} = \dfrac{y+4}{3} = \dfrac{z-2}{-7}$

22. L_1: $\dfrac{x}{1} = \dfrac{y-1}{-1} = \dfrac{z-2}{3}$
    L_2: $\dfrac{x-2}{2} = \dfrac{y-3}{-2} = \dfrac{z}{7}$

23–40 Find an equation of the plane.

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5 \rangle$

24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. The plane through the point $(-1, \frac{1}{2}, 3)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

26. The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$

27. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

28. The plane through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$

29. The plane through the point $(1, \frac{1}{2}, \frac{1}{3})$ and parallel to the plane $x + y + z = 0$

30. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$

32. The plane through the origin and the points $(3, -2, 1)$ and $(1, 1, 1)$

33. The plane through the points $(2, 1, 2)$, $(3, -8, 6)$, and $(-2, -3, 1)$

34. The plane through the points $(3, 0, -1)$, $(-2, -2, 3)$, and $(7, 1, -4)$

35. The plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$

36. The plane that passes through the point $(6, -1, 3)$ and contains the line with symmetric equations $x/3 = y + 4 = z/2$

37. The plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$

38. The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$

39. The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$

40. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

41–44 Use intercepts to help sketch the plane.

41. $2x + 5y + z = 10$

42. $3x + y + 2z = 6$

43. $6x - 3y + 4z = 6$

44. $6x + 5y - 3z = 15$

45–47 Find the point at which the line intersects the given plane.

45. $x = 2 - 2t, \quad y = 3t, \quad z = 1 + t; \quad x + 2y - z = 7$

46. $x = t - 1, \quad y = 1 + 2t, \quad z = 3 - t; \quad 3x - y + 2z = 5$

47. $5x = y/2 = z + 2; \quad 10x - 7y + 3z + 24 = 0$

48. Where does the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ intersect the plane $2x + y - z = -2$?

49. Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

50. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

51–56 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Round to one decimal place.)

51. $x + 4y - 3z = 1, \quad -3x + 6y + 7z = 0$

52. $9x - 3y + 6z = 2, \quad 2y = 6x + 4z$

53. $x + 2y - z = 2, \quad 2x - 2y + z = 1$

54. $x - y + 3z = 1, \quad 3x + y - z = 2$

55. $2x - 3y = z, \quad 4x = 3 + 6y + 2z$

56. $5x + 2y + 3z = 2, \quad y = 4x - 6z$

57–58 (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

57. $x + y + z = 1, \quad x + 2y + 2z = 1$

58. $3x - 2y + z = 1, \quad 2x + y - 3z = 3$

59–60 Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1, \quad 4x + y + z = 6$

60. $z = 2x - y - 5, \quad z = 4x + 3y - 5$

61. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

62. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

63. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

64. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

65. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

66. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

67. Which of the following four planes are parallel? Are any of them identical?

P_1: $3x + 6y - 3z = 6$ $\qquad$ P_2: $4x - 12y + 8z = 5$

P_3: $9y = 1 + 3x + 6z$ $\qquad$ P_4: $z = x + 2y - 2$

68. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + 6t, \quad y = 1 - 3t, \quad z = 12t + 5$

L_2: $x = 1 + 2t, \quad y = t, \quad z = 1 + 4t$

L_3: $2x - 2 = 4 - 4y = z + 1$

L_4: $\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$

69–70 Use the formula in Exercise 12.4.45 to find the distance from the point to the given line.

69. $(4, 1, -2); \quad x = 1 + t, \quad y = 3 - 2t, \quad z = 4 - 3t$

70. $(0, 1, 3); \quad x = 2t, \quad y = 6 - 2t, \quad z = 3 + t$

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $x - 2y - 4z = 8$

73–74 Find the distance between the given parallel planes.

73. $2x - 3y + z = 4$, $4x - 6y + 2z = 3$

74. $6z = 4y - 2x$, $9z = 1 - 3x + 6y$

75. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

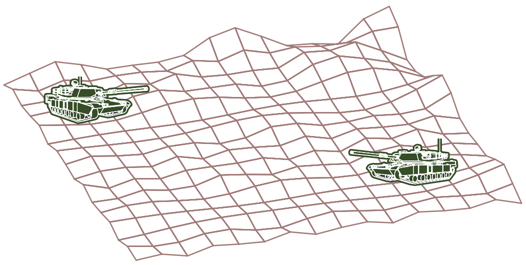
79. Let L_1 be the line through the origin and the point $(2, 0, -1)$. Let L_2 be the line through the points $(1, -1, 1)$ and $(4, 1, 3)$. Find the distance between L_1 and L_2.

80. Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes P_1 and P_2, where P_1 is the plane $x - y + 2z + 1 = 0$ and P_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$, and $(1, 2, 1)$. Calculate the distance between L_1 and L_2.

81. Two tanks are participating in a battle simulation. Tank A is at point $(325, 810, 561)$ and tank B is positioned at point $(765, 675, 599)$.

(a) Find parametric equations for the line of sight between the tanks.

(b) If we divide the line of sight into 5 equal segments, the elevations of the terrain at the four intermediate points from tank A to tank B are 549, 566, 586, and 589. Can the tanks see each other?



82. Give a geometric description of each family of planes.

(a) $x + y + z = c$ (b) $x + y + cz = 1$

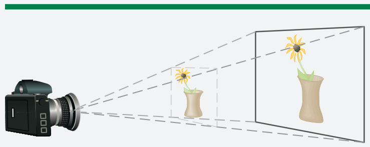
(c) $y\cos\theta + z\sin\theta = 1$

83. If a, b, and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c\rangle$ is a normal vector to the plane.

Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

LABORATORY PROJECT PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes.*

1. Suppose the screen is represented by a rectangle in the yz-plane with vertices $(0, \pm 400, 0)$ and $(0, \pm 400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?

2. If the clipped line segment is projected onto the screen window, identify the resulting line segment.

3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection onto the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.

4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering,* which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

12.6 Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes (in Section 12.5) and spheres (in Section 12.1). Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

EXAMPLE 1 Sketch the graph of the surface $z = x^2$.

SOLUTION Notice that the equation of the graph, $z = x^2$, doesn't involve y. This means that any vertical plane with equation $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So these vertical traces are parabolas. Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis. The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the y-axis.

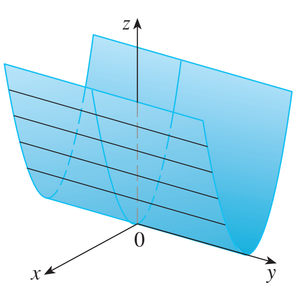


FIGURE 1
The surface $z = x^2$ is a parabolic cylinder.

We noticed that the variable y is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables x, y, or z is missing from the equation of a surface, then the surface is a cylinder.

EXAMPLE 2 Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$ (b) $y^2 + z^2 = 1$

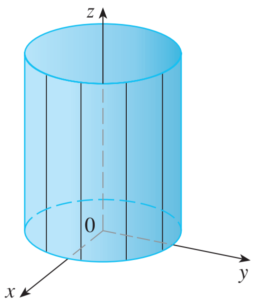


FIGURE 2
$x^2 + y^2 = 1$

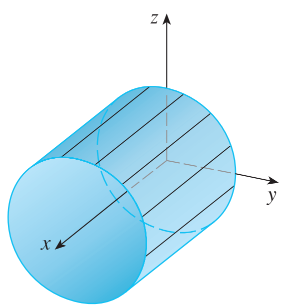


FIGURE 3
$y^2 + z^2 = 1$

SOLUTION
(a) Since z is missing and the equations $x^2 + y^2 = 1$, $z = k$ represent a circle with radius 1 in the plane $z = k$, the surface $x^2 + y^2 = 1$ is a circular cylinder whose axis is the z-axis. (See Figure 2.) Here the rulings are vertical lines.

(b) In this case x is missing and the surface is a circular cylinder whose axis is the x-axis. (See Figure 3.) It is obtained by taking the circle $y^2 + z^2 = 1$, $x = 0$ in the yz-plane and moving it parallel to the x-axis. ■

NOTE When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1$, $z = 0$.

Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane. (See Section 10.5 for a review of conic sections.)

EXAMPLE 3 Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

SOLUTION By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

Similarly, vertical traces parallel to the yz- and xz-planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k \qquad (\text{if } -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k \qquad (\text{if } -3 < k < 3)$$

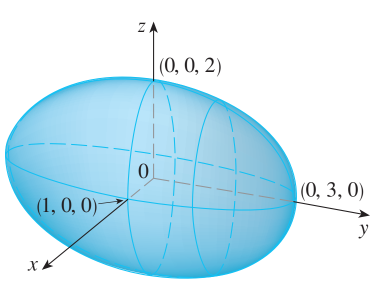


FIGURE 4
The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Figure 4 shows how drawing some traces indicates the shape of the surface. It's called an **ellipsoid** because all of its traces are ellipses. Notice that it is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of x, y, and z. ■

EXAMPLE 4 Use traces to sketch the surface $z = 4x^2 + y^2$.

SOLUTION If we put $x = 0$, we get $z = y^2$, so the yz-plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we

slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a family of ellipses. Knowing the shapes of the traces, we can sketch the graph in Figure 5. Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**. ■

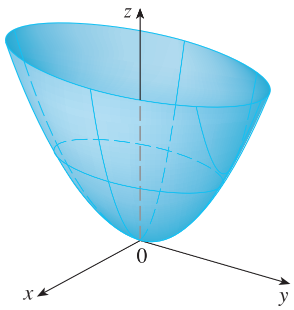
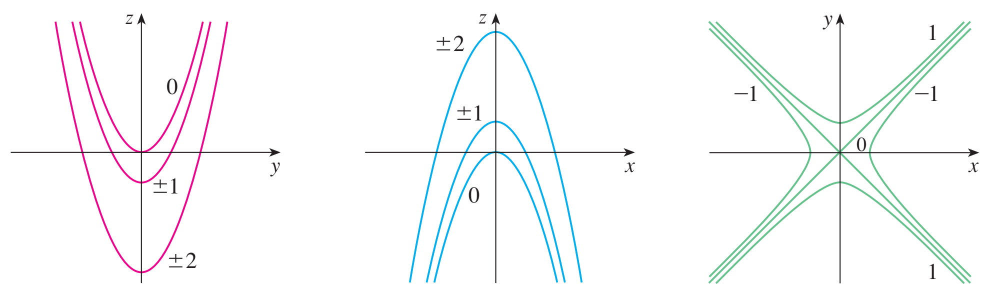


FIGURE 5
The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

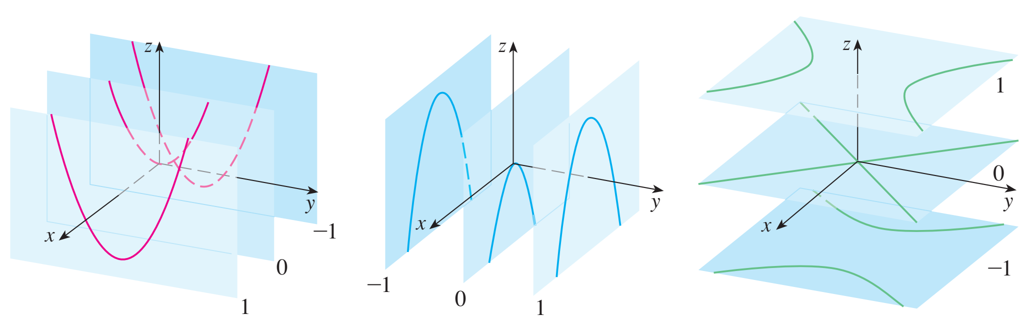
EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$. Traces in $y = k$ are $z = -x^2 + k^2$. Traces in $z = k$ are $y^2 - x^2 = k$.

FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.



Traces in $x = k$ Traces in $y = k$ Traces in $z = k$

FIGURE 7
Traces moved to their correct planes

TEC In Module 12.6A you can investigate how traces determine the shape of a surface.

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

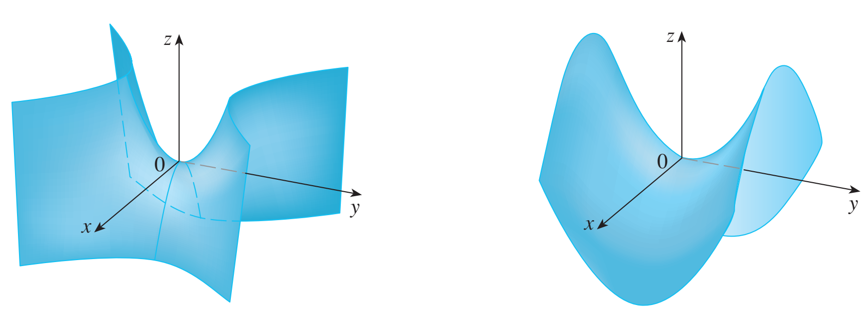


FIGURE 8
Two views of the surface $z = y^2 - x^2$, a hyperbolic paraboloid

■

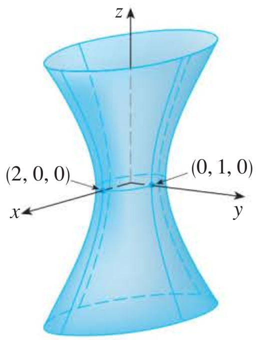


FIGURE 9

EXAMPLE 6 Sketch the surface $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$.

SOLUTION The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \qquad z = k$$

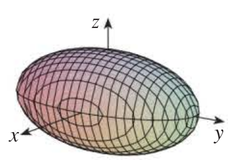
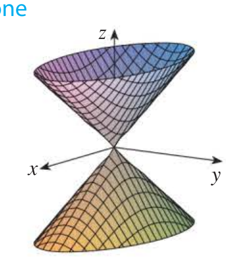
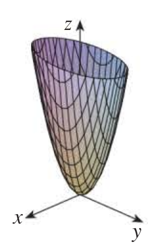
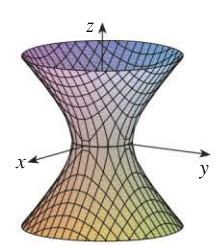
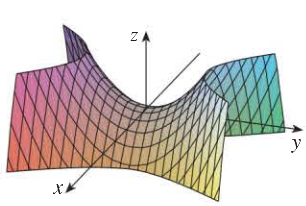
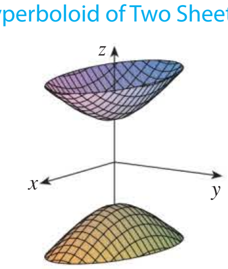
but the traces in the xz- and yz-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9. ■

The idea of using traces to draw a surface is employed in three-dimensional graphing software. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid  	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone  	$\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid  	$\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet  	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets  	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

TEC In Module 12.6B you can see how changing a, b, and c in Table 1 affects the shape of the quadric surface.

EXAMPLE 7 Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

SOLUTION Dividing by -4, we first put the equation in standard form:

$$-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$$

Comparing this equation with Table 1, we see that it represents a hyperboloid of two sheets, the only difference being that in this case the axis of the hyperboloid is the y-axis. The traces in the xy- and yz-planes are the hyperbolas

$$-x^2 + \frac{y^2}{4} = 1 \qquad z = 0 \qquad \text{and} \qquad \frac{y^2}{4} - \frac{z^2}{2} = 1 \qquad x = 0$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y = k$ for $|k| > 2$ are the ellipses

$$x^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1 \qquad y = k$$

which can be written as

$$\frac{x^2}{\dfrac{k^2}{4} - 1} + \frac{z^2}{2\left(\dfrac{k^2}{4} - 1\right)} = 1 \qquad y = k$$

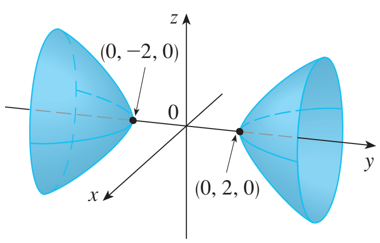


FIGURE 10
$4x^2 - y^2 + 2z^2 + 4 = 0$

These traces are used to make the sketch in Figure 10. ■

EXAMPLE 8 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic paraboloid. Here, however, the axis of the paraboloid is parallel to the y-axis, and it has been shifted so that its vertex is the point $(3, 1, 0)$. The traces in the plane $y = k$ $(k > 1)$ are the ellipses

$$(x - 3)^2 + 2z^2 = k - 1 \qquad y = k$$

The trace in the xy-plane is the parabola with equation $y = 1 + (x - 3)^2$, $z = 0$. The paraboloid is sketched in Figure 11.

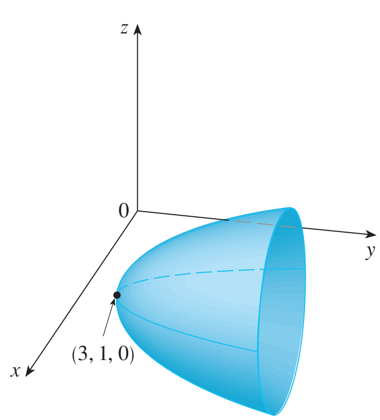


FIGURE 11
$x^2 + 2z^2 - 6x - y + 10 = 0$

■

Applications of Quadric Surfaces

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example. Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles. (See Exercise 49.)

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals. In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified. (The idea is explained in Problem 18 on page 202.) The same principle applies to microphones and satellite dishes in the shape of paraboloids.

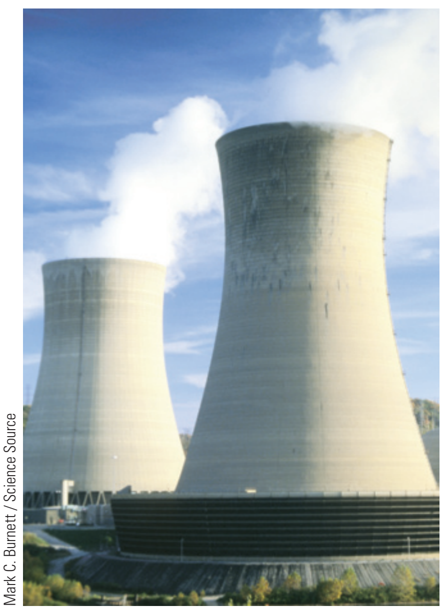
Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability. Pairs of hyperboloids are used to transmit rotational motion between skew axes. (The cogs of the gears are the generating lines of the hyperboloids. See Exercise 51.)

David Frazier / Spirit / Corbis

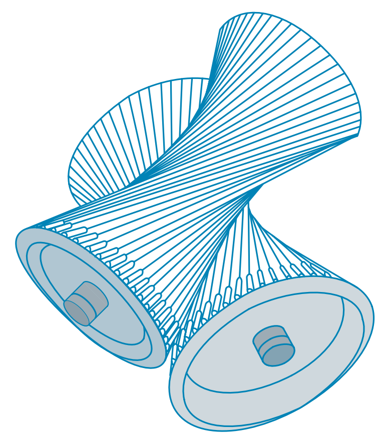


A satellite dish reflects signals to the focus of a paraboloid.

Mark C. Burnett / Science Source



Nuclear reactors have cooling towers in the shape of hyperboloids.



Hperboloids produce gear transmission.

12.6 EXERCISES

1. (a) What does the equation $y = x^2$ represent as a curve in $\mathbb{R}^2$?
(b) What does it represent as a surface in $\mathbb{R}^3$?
(c) What does the equation $z = y^2$ represent?

2. (a) Sketch the graph of $y = e^x$ as a curve in $\mathbb{R}^2$.
(b) Sketch the graph of $y = e^x$ as a surface in $\mathbb{R}^3$.
(c) Describe and sketch the surface $z = e^y$.

3–8 Describe and sketch the surface.

3. $x^2 + z^2 = 1$

4. $4x^2 + y^2 = 4$

5. $z = 1 - y^2$

6. $y = z^2$

7. $xy = 1$

8. $z = \sin y$

9. (a) Find and identify the traces of the quadric surface $x^2 + y^2 - z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of one sheet in Table 1.
(b) If we change the equation in part (a) to $x^2 - y^2 + z^2 = 1$, how is the graph affected?
(c) What if we change the equation in part (a) to $x^2 + y^2 + 2y - z^2 = 0$?

10. (a) Find and identify the traces of the quadric surface $-x^2 - y^2 + z^2 = 1$ and explain why the graph looks like the graph of the hyperboloid of two sheets in Table 1.
(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, what happens to the graph? Sketch the new graph.

11–20 Use traces to sketch and identify the surface.

11. $x = y^2 + 4z^2$

12. $4x^2 + 9y^2 + 9z^2 = 36$

13. $x^2 = 4y^2 + z^2$

14. $z^2 - 4x^2 - y^2 = 4$

15. $9y^2 + 4z^2 = x^2 + 36$

16. $3x^2 + y + 3z^2 = 0$

17. $\dfrac{x^2}{9} + \dfrac{y^2}{25} + \dfrac{z^2}{4} = 1$

18. $3x^2 - y^2 + 3z^2 = 0$

19. $y = z^2 - x^2$

20. $x = y^2 - z^2$

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

24. $-x^2 + y^2 - z^2 = 1$

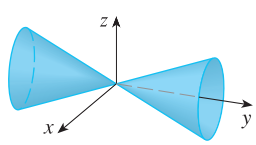
25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

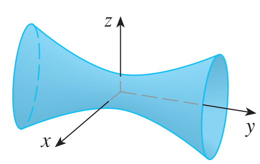
27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

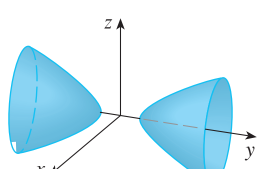
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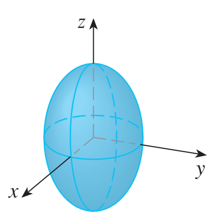
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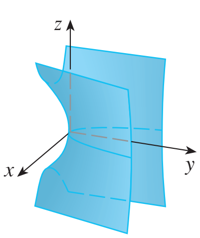
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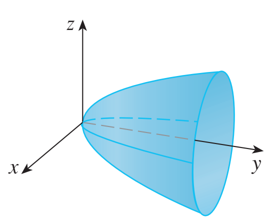
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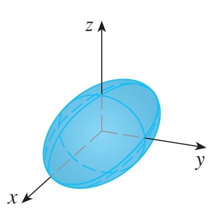
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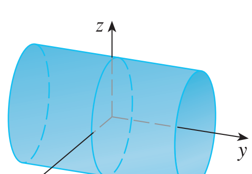
VI



VII

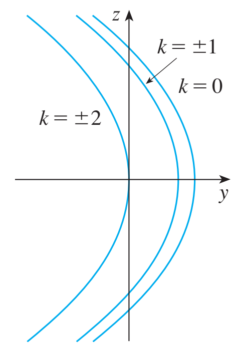


VIII

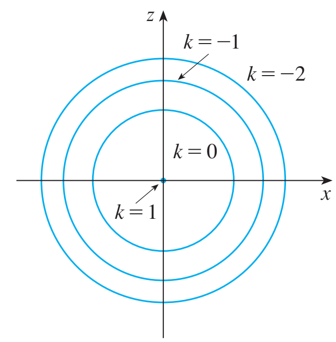


29–30 Sketch and identify a quadric surface that could have the traces shown.

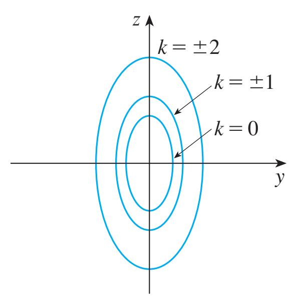
29. Traces in $x = k$



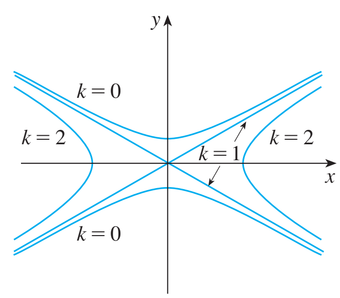
Traces in $y = k$



30. Traces in $x = k$



Traces in $z = k$



31–38 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

31. $y^2 = x^2 + \frac{1}{9}z^2$

32. $4x^2 - y + 2z^2 = 0$

33. $x^2 + 2y - 2z^2 = 0$

34. $y^2 = x^2 + 4z^2 + 4$

35. $x^2 + y^2 - 2x - 6y - z + 10 = 0$

36. $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$

37. $x^2 - y^2 + z^2 - 4x - 2z = 0$

38. $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$

39–42 Use a computer with three-dimensional graphing software to graph the surface. Experiment with viewpoints and with domains for the variables until you get a good view of the surface.

39. $-4x^2 - y^2 + z^2 = 1$

40. $x^2 - y^2 - z = 0$

41. $-4x^2 - y^2 + z^2 = 0$

42. $x^2 - 6x + 4y^2 - z = 0$

43. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leqslant z \leqslant 2$.

44. Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

45. Find an equation for the surface obtained by rotating the curve $y = \sqrt{x}$ about the x-axis.

46. Find an equation for the surface obtained by rotating the line $z = 2y$ about the z-axis.

47. Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

48. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

49. Traditionally, the earth's surface has been modeled as a sphere, but the World Geodetic System of 1984 (WGS-84) uses an ellipsoid as a more accurate model. It places the center of the earth at the origin and the north pole on the positive z-axis. The distance from the center to the poles is 6356.523 km and the distance to a point on the equator is 6378.137 km.

(a) Find an equation of the earth's surface as used by WGS-84.

(b) Curves of equal latitude are traces in the planes $z = k$. What is the shape of these curves?

(c) Meridians (curves of equal longitude) are traces in planes of the form $y = mx$. What is the shape of these meridians?

50. A cooling tower for a nuclear reactor is to be constructed in the shape of a hyperboloid of one sheet (see the photo on page 879). The diameter at the base is 280 m and the minimum diameter, 500 m above the base, is 200 m. Find an equation for the tower.

51. Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the lines with parametric equations $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$ and $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$ both lie entirely on this paraboloid. (This shows that the hyperbolic paraboloid is what is called a **ruled surface**; that is, it can be generated by the motion of a straight line. In fact, this exercise shows that through each point on the hyperbolic paraboloid there are two generating lines. The only other quadric surfaces that are ruled surfaces are cylinders, cones, and hyperboloids of one sheet.)

52. Show that the curve of intersection of the surfaces $x^2 + 2y^2 - z^2 + 3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane.

53. Graph the surfaces $z = x^2 + y^2$ and $z = 1 - y^2$ on a common screen using the domain $|x| \leqslant 1.2$, $|y| \leqslant 1.2$ and observe the curve of intersection of these surfaces. Show that the projection of this curve onto the xy-plane is an ellipse.

12 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. What is the difference between a vector and a scalar?

2. How do you add two vectors geometrically? How do you add them algebraically?

3. If **a** is a vector and c is a scalar, how is c**a** related to **a** geometrically? How do you find c**a** algebraically?

4. How do you find the vector from one point to another?

5. How do you find the dot product **a** · **b** of two vectors if you know their lengths and the angle between them? What if you know their components?

6. How are dot products useful?

7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate with diagrams.

8. How do you find the cross product **a** × **b** of two vectors if you know their lengths and the angle between them? What if you know their components?

9. How are cross products useful?

10. (a) How do you find the area of the parallelogram determined by **a** and **b**?

(b) How do you find the volume of the parallelepiped determined by **a**, **b**, and **c**?

11. How do you find a vector perpendicular to a plane?

12. How do you find the angle between two intersecting planes?

13. Write a vector equation, parametric equations, and symmetric equations for a line.

14. Write a vector equation and a scalar equation for a plane.

15. (a) How do you tell if two vectors are parallel?

(b) How do you tell if two vectors are perpendicular?

(c) How do you tell if two planes are parallel?

16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.

(b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.

17. (a) How do you find the distance from a point to a line?

(b) How do you find the distance from a point to a plane?

(c) How do you find the distance between two lines?

18. What are the traces of a surface? How do you find them?

19. Write equations in standard form of the six types of quadric surfaces.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then $\mathbf{u} \cdot \mathbf{v} = \langle u_1 v_1, u_2 v_2 \rangle$.

2. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

3. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

4. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

5. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

6. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

7. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

8. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 and any scalar k,

$$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$$

9. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 and any scalar k,

$$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$$

10. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

11. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

12. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V_3,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

13. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

14. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3, $(\mathbf{u} + \mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$.

15. The vector $\langle 3, -1, 2 \rangle$ is parallel to the plane

$$6x - 2y + 4z = 1$$

16. A linear equation $Ax + By + Cz + D = 0$ represents a line in space.

17. The set of points $\{(x, y, z) \mid x^2 + y^2 = 1\}$ is a circle.

18. In $\mathbb{R}^3$ the graph of $y = x^2$ is a paraboloid.

19. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

20. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

21. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

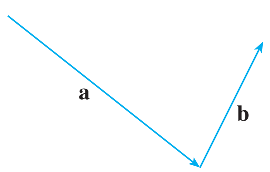
22. If $\mathbf{u}$ and $\mathbf{v}$ are in V_3, then $|\mathbf{u} \cdot \mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$.

EXERCISES

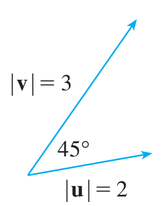
1. (a) Find an equation of the sphere that passes through the point $(6, -2, 3)$ and has center $(-1, 2, 1)$.
(b) Find the curve in which this sphere intersects the yz-plane.
(c) Find the center and radius of the sphere

$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

2. Copy the vectors in the figure and use them to draw each of the following vectors.
(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $-\frac{1}{2}\mathbf{a}$ (d) $2\mathbf{a} + \mathbf{b}$



3. If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in the figure, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it?



4. Calculate the given quantity if

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$
$$\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{j} - 5\mathbf{k}$$

(a) $2\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$
(c) $\mathbf{a} \cdot \mathbf{b}$ (d) $\mathbf{a} \times \mathbf{b}$
(e) $|\mathbf{b} \times \mathbf{c}|$ (f) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(g) $\mathbf{c} \times \mathbf{c}$ (h) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
(i) $\text{comp}_{\mathbf{a}}\,\mathbf{b}$ (j) $\text{proj}_{\mathbf{a}}\,\mathbf{b}$
(k) The angle between $\mathbf{a}$ and $\mathbf{b}$ (correct to the nearest degree)

5. Find the values of x such that the vectors $\langle 3, 2, x \rangle$ and $\langle 2x, 4, x \rangle$ are orthogonal.

6. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

7. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$. Find
(a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (b) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
(c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$ (d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$

8. Show that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in V_3, then

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$$

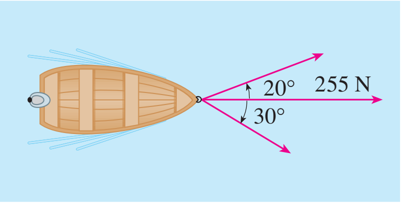
9. Find the acute angle between two diagonals of a cube.

10. Given the points $A(1, 0, 1)$, $B(2, 3, 0)$, $C(-1, 1, 4)$, and $D(0, 3, 2)$, find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

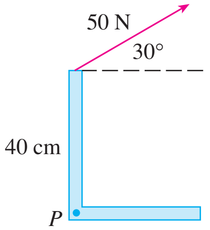
11. (a) Find a vector perpendicular to the plane through the points $A(1, 0, 0)$, $B(2, 0, -1)$, and $C(1, 4, 3)$.
(b) Find the area of triangle ABC.

12. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along the line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done if the distance is measured in meters and the force in newtons.

13. A boat is pulled onto shore using two ropes, as shown in the diagram. If a force of 255 N is needed, find the magnitude of the force in each rope.



14. Find the magnitude of the torque about P if a 50-N force is applied as shown.



15–17 Find parametric equations for the line.

15. The line through $(4, -1, 2)$ and $(1, 1, 5)$

16. The line through $(1, 0, -1)$ and parallel to the line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$

17. The line through $(-2, 2, 4)$ and perpendicular to the plane $2x - y + 5z = 12$

18–20 Find an equation of the plane.

18. The plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$

19. The plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$

20. The plane through $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z = 1 + 3t$

21. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.

22. Find the distance from the origin to the line $x = 1 + t$, $y = 2 - t$, $z = -1 + 2t$.

23. Determine whether the lines given by the symmetric equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and
$$\frac{x+1}{6} = \frac{y-3}{-1} = \frac{z+5}{2}$$

are parallel, skew, or intersecting.

24. (a) Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.
(b) Find, correct to the nearest degree, the angle between these planes.

25. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.

26. (a) Find an equation of the plane that passes through the points $A(2, 1, 1)$, $B(-1, -1, 10)$, and $C(1, 3, -4)$.
(b) Find symmetric equations for the line through B that is perpendicular to the plane in part (a).
(c) A second plane passes through $(2, 0, 4)$ and has normal vector $\langle 2, -4, -3 \rangle$. Show that the acute angle between the planes is approximately 43°.
(d) Find parametric equations for the line of intersection of the two planes.

27. Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

28–36 Identify and sketch the graph of each surface.

28. $x = 3$

29. $x = z$

30. $y = z^2$

31. $x^2 = y^2 + 4z^2$

32. $4x - y + 2z = 4$

33. $-4x^2 + y^2 - 4z^2 = 4$

34. $y^2 + z^2 = 1 + x^2$

35. $4x^2 + 4y^2 - 8y + z^2 = 0$

36. $x = y^2 + z^2 - 2y - 4z + 5$

37. An ellipsoid is created by rotating the ellipse $4x^2 + y^2 = 16$ about the x-axis. Find an equation of the ellipsoid.

38. A surface consists of all points P such that the distance from P to the plane $y = 1$ is twice the distance from P to the point $(0, -1, 0)$. Find an equation for this surface and identify it.

Problems Plus

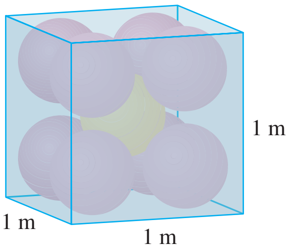


FIGURE FOR PROBLEM 1

1. Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius r. The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus the balls are tightly packed in the box (see the figure). Find r. (If you have trouble with this problem, read about the problem-solving strategy entitled *Use Analogy* on page 98.)

2. Let B be a solid box with length L, width W, and height H. Let S be the set of all points that are a distance at most 1 from some point of B. Express the volume of S in terms of L, W, and H.

3. Let L be the line of intersection of the planes $cx + y + z = c$ and $x - cy + cz = -1$, where c is a real number.
(a) Find symmetric equations for L.
(b) As the number c varies, the line L sweeps out a surface S. Find an equation for the curve of intersection of S with the horizontal plane $z = t$ (the trace of S in the plane $z = t$).
(c) Find the volume of the solid bounded by S and the planes $z = 0$ and $z = 1$.

4. A plane is capable of flying at a speed of 180 km/h in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time, the pilot notices that, due to the wind, the plane has actually traveled 80 km at an angle 5° east of north.
(a) What is the wind velocity?
(b) In what direction should the pilot have headed to reach the intended destination?

5. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors with $|\mathbf{v}_1| = 2$, $|\mathbf{v}_2| = 3$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 5$. Let $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1}\mathbf{v}_2$, $\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2}\mathbf{v}_3$, $\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3}\mathbf{v}_4$, and so on. Compute $\sum_{n=1}^{\infty} |\mathbf{v}_n|$.

6. Find an equation of the largest sphere that passes through the point $(-1, 1, 4)$ and is such that each of the points (x, y, z) inside the sphere satisfies the condition

$$x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$$

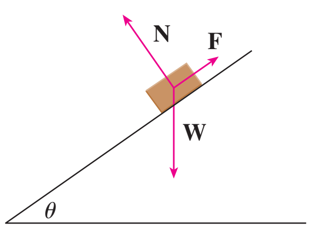


FIGURE FOR PROBLEM 7

7. Suppose a block of mass m is placed on an inclined plane, as shown in the figure. The block's descent down the plane is slowed by friction; if θ is not too large, friction will prevent the block from moving at all. The forces acting on the block are the weight $\mathbf{W}$, where $|\mathbf{W}| = mg$ (g is the acceleration due to gravity); the normal force $\mathbf{N}$ (the normal component of the reactionary force of the plane on the block), where $|\mathbf{N}| = n$; and the force $\mathbf{F}$ due to friction, which acts parallel to the inclined plane, opposing the direction of motion. If the block is at rest and θ is increased, $|\mathbf{F}|$ must also increase until ultimately $|\mathbf{F}|$ reaches its maximum, beyond which the block begins to slide. At this angle θ_s, it has been observed that $|\mathbf{F}|$ is proportional to n. Thus, when $|\mathbf{F}|$ is maximal, we can say that $|\mathbf{F}| = \mu_s n$, where μ_s is called the *coefficient of static friction* and depends on the materials that are in contact.
(a) Observe that $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$ and deduce that $\mu_s = \tan(\theta_s)$.
(b) Suppose that, for $\theta > \theta_s$, an additional outside force $\mathbf{H}$ is applied to the block, horizontally from the left, and let $|\mathbf{H}| = h$. If h is small, the block may still slide down the plane; if h is large enough, the block will move up the plane. Let $h_{\min}$ be the smallest value of h that allows the block to remain motionless (so that $|\mathbf{F}|$ is maximal).

By choosing the coordinate axes so that $\mathbf{F}$ lies along the x-axis, resolve each force into components parallel and perpendicular to the inclined plane and show that

$$h_{\min} \sin\theta + mg\cos\theta = n \qquad \text{and} \qquad h_{\min}\cos\theta + \mu_s n = mg\sin\theta$$

(c) Show that
$$h_{\min} = mg\tan(\theta - \theta_s)$$

Does this equation seem reasonable? Does it make sense for $\theta = \theta_s$? Does it make sense as $\theta \to 90°$? Explain.

(d) Let $h_{\max}$ be the largest value of h that allows the block to remain motionless. (In which direction is **F** heading?) Show that

$$h_{\max} = mg\tan(\theta + \theta_s)$$

Does this equation seem reasonable? Explain.

8. A solid has the following properties. When illuminated by rays parallel to the z-axis, its shadow is a circular disk. If the rays are parallel to the y-axis, its shadow is a square. If the rays are parallel to the x-axis, its shadow is an isosceles triangle. (In Exercise 12.1.48 you were asked to describe and sketch an example of such a solid, but there are many such solids.) Assume that the projection onto the xz-plane is a square whose sides have length 1.

(a) What is the volume of the largest such solid?

(b) Is there a smallest volume?

13 Vector Functions



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The paths of objects moving through space like the jet planes pictured here can be described by vector functions. In Section 13.1 we will see how to use these vector functions to determine whether or not two such objects will collide.

THE FUNCTIONS THAT WE HAVE been using so far have been real-valued functions. We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space. We will also use vector-valued functions to describe the motion of objects through space. In particular, we will use them to derive Kepler's laws of planetary motion.

13.1 Vector Functions and Space Curves

In general, a function is a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in V_3 denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

EXAMPLE 1 If

$$\mathbf{r}(t) = \left\langle t^3, \ln(3 - t), \sqrt{t} \right\rangle$$

then the component functions are

$$f(t) = t^3 \qquad g(t) = \ln(3 - t) \qquad h(t) = \sqrt{t}$$

By our usual convention, the domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined. The expressions t^3, $\ln(3 - t)$, and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geqslant 0$. Therefore the domain of $\mathbf{r}$ is the interval $[0, 3)$. ■

Limits and Continuity

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows.

If $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, this definition is equivalent to saying that the length and direction of the vector $\mathbf{r}(t)$ approach the length and direction of the vector $\mathbf{L}$.

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Equivalently, we could have used an ε-δ definition (see Exercise 54). Limits of vector functions obey the same rules as limits of real-valued functions (see Exercise 53).

EXAMPLE 2 Find $\lim_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \dfrac{\sin t}{t}\,\mathbf{k}$.

SOLUTION According to Definition 1, the limit of $\mathbf{r}$ is the vector whose components are the limits of the component functions of $\mathbf{r}$:

$$\lim_{t\to 0} \mathbf{r}(t) = \left[\lim_{t\to 0} (1 + t^3)\right]\mathbf{i} + \left[\lim_{t\to 0} te^{-t}\right]\mathbf{j} + \left[\lim_{t\to 0} \frac{\sin t}{t}\right]\mathbf{k}$$

$$= \mathbf{i} + \mathbf{k} \qquad \text{(by Equation 2.4.2)}$$ ■

A vector function $\mathbf{r}$ is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that $\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space Curves

There is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

2 $$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a **space curve**. The equations in (2) are called **parametric equations of *C*** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

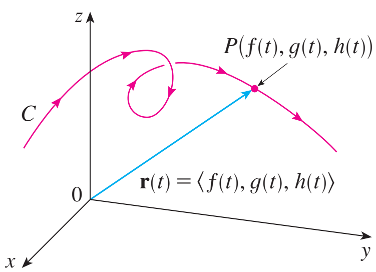


FIGURE 1
C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

EXAMPLE 3 Describe the curve defined by the vector function

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

SOLUTION The corresponding parametric equations are

$$x = 1 + t \qquad y = 2 + 5t \qquad z = -1 + 6t$$

which we recognize from Equations 12.5.2 as parametric equations of a line passing through the point $(1, 2, -1)$ and parallel to the vector $\langle 1, 5, 6 \rangle$. Alternatively, we could observe that the function can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$ and $\mathbf{v} = \langle 1, 5, 6 \rangle$, and this is the vector equation of a line as given by Equation 12.5.1. ■

TEC Visual 13.1A shows several curves being traced out by position vectors, including those in Figures 1 and 2.

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ (see Example 10.1.1) could also be described by the vector equation

$$\mathbf{r}(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)\,\mathbf{i} + (t + 1)\,\mathbf{j}$$

where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

EXAMPLE 4 Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

SOLUTION The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ for all values of t, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane. (The projection of the curve onto the xy-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. See Example 10.1.2.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**. ■

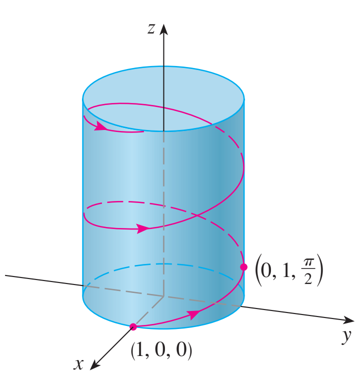


FIGURE 2



FIGURE 3
A double helix

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs. It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells). In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.

In Examples 3 and 4 we were given vector equations of curves and asked for a geometric description or sketch. In the next two examples we are given a geometric description of a curve and are asked to find parametric equations for the curve.

EXAMPLE 5 Find a vector equation and parametric equations for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

Figure 4 shows the line segment PQ in Example 5.

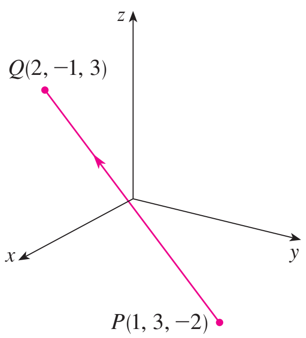


FIGURE 4

SOLUTION In Section 12.5 we found a vector equation for the line segment that joins the tip of the vector $\mathbf{r}_0$ to the tip of the vector $\mathbf{r}_1$:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

(See Equation 12.5.4.) Here we take $\mathbf{r}_0 = \langle 1, 3, -2 \rangle$ and $\mathbf{r}_1 = \langle 2, -1, 3 \rangle$ to obtain a vector equation of the line segment from P to Q:

$$\mathbf{r}(t) = (1 - t)\,\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle \qquad 0 \leqslant t \leqslant 1$$

or

$$\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle \qquad 0 \leqslant t \leqslant 1$$

The corresponding parametric equations are

$$x = 1 + t \qquad y = 3 - 4t \qquad z = -2 + 5t \qquad 0 \leqslant t \leqslant 1$$ ■

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

SOLUTION Figure 5 shows how the plane and the cylinder intersect, and Figure 6 shows the curve of intersection C, which is an ellipse.

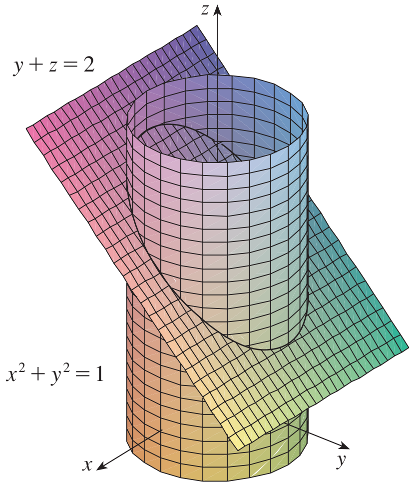


FIGURE 5

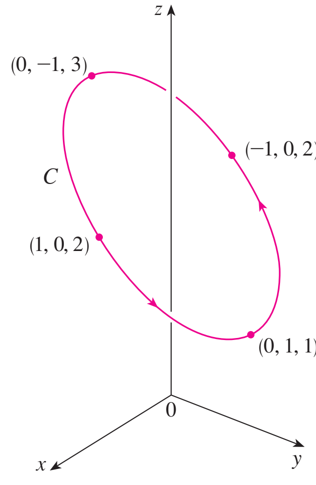


FIGURE 6

The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, $z = 0$. So we know from Example 10.1.2 that we can write

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

So we can write parametric equations for C as

$$x = \cos t \qquad y = \sin t \qquad z = 2 - \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2 - \sin t)\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

This equation is called a *parametrization* of the curve C. The arrows in Figure 6 indicate the direction in which C is traced as the parameter t increases. ■

Using Computers to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology. For instance, Figure 7 shows a computer-generated graph of the curve with parametric equations

$$x = (4 + \sin 20t)\cos t \qquad y = (4 + \sin 20t)\sin t \qquad z = \cos 20t$$

It's called a **toroidal spiral** because it lies on a torus. Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t)\cos t \qquad y = (2 + \cos 1.5t)\sin t \qquad z = \sin 1.5t$$

is graphed in Figure 8. It wouldn't be easy to plot either of these curves by hand.

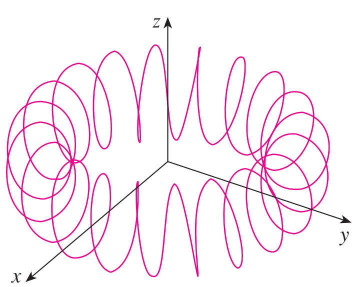


FIGURE 7
A toroidal spiral

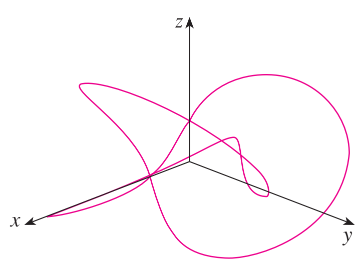


FIGURE 8
A trefoil knot

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 8. See Exercise 52.) The next example shows how to cope with this problem.

EXAMPLE 7 Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

SOLUTION We start by using the computer to plot the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$ for $-2 \leqslant t \leqslant 2$. The result is shown in Figure 9(a), but it's hard to see the true nature of the curve from that graph alone. Most three-dimensional

computer graphing programs allow the user to enclose a curve or surface in a box instead of displaying the coordinate axes. When we look at the same curve in a box in Figure 9(b), we have a much clearer picture of the curve. We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.

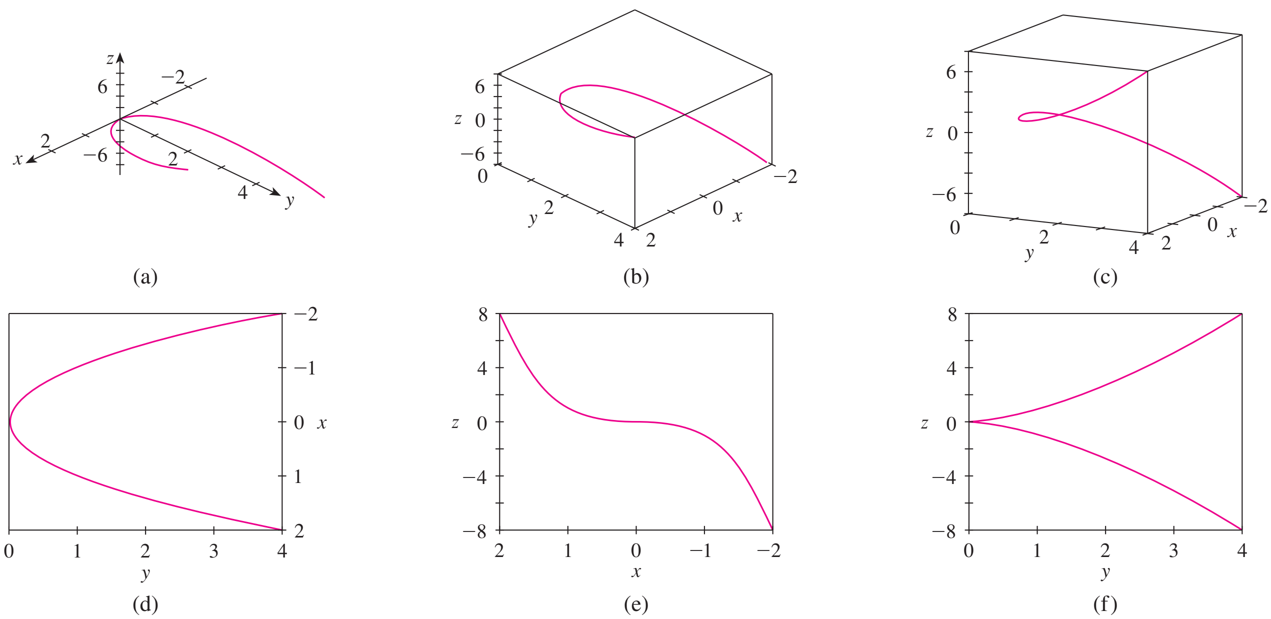


FIGURE 9 Views of the twisted cubic

TEC In Visual 13.1B you can rotate the box in Figure 9 to see the curve from any viewpoint.

We get an even better idea of the curve when we view it from different vantage points. Part (c) shows the result of rotating the box to give another viewpoint. Parts (d), (e), and (f) show the views we get when we look directly at a face of the box. In particular, part (d) shows the view from directly above the box. It is the projection of the curve onto the xy-plane, namely, the parabola $y = x^2$. Part (e) shows the projection onto the xz-plane, the cubic curve $z = x^3$. It's now obvious why the given curve is called a twisted cubic. ■

Another method of visualizing a space curve is to draw it on a surface. For instance, the twisted cubic in Example 7 lies on the parabolic cylinder $y = x^2$. (Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.) Figure 10 shows both the cylinder and the twisted cubic, and we see that the curve moves upward from the origin along the surface of the cylinder. We also used this method in Example 4 to visualize the helix lying on the circular cylinder (see Figure 2).

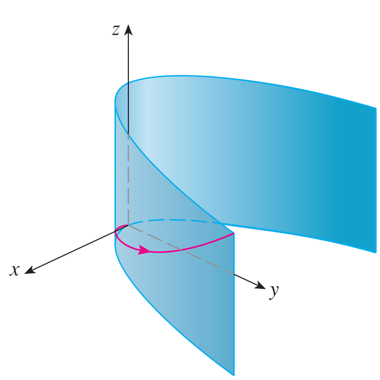


FIGURE 10

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)

TEC Visual 13.1C shows how curves arise as intersections of surfaces.

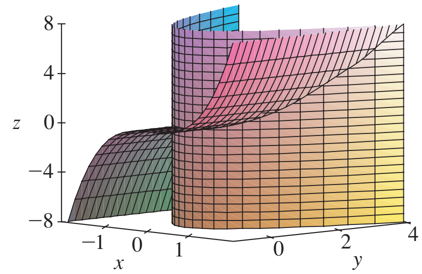
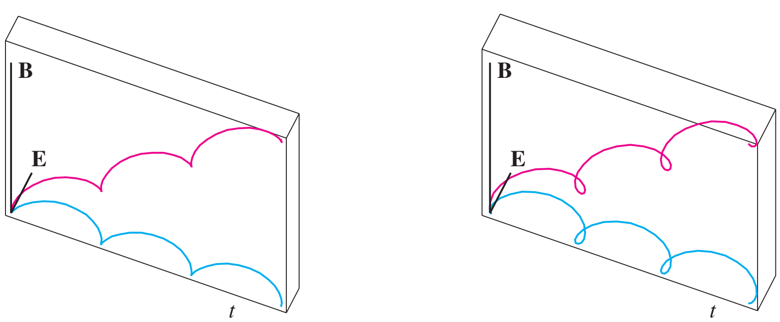


FIGURE 11

Some computer algebra systems provide us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 13 shows the curve of Figure 12(b) as rendered by the `tubeplot` command in Maple.

We have seen that an interesting space curve, the helix, occurs in the model of DNA. Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**. Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection onto the horizontal plane is the cycloid we studied in Section 10.1 [Figure 12(a)] or a curve whose projection is the trochoid investigated in Exercise 10.1.40 [Figure 12(b)].



(a) $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$

(b) $\mathbf{r}(t) = \langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \rangle$

FIGURE 12
Motion of a charged particle in orthogonally oriented electric and magnetic fields

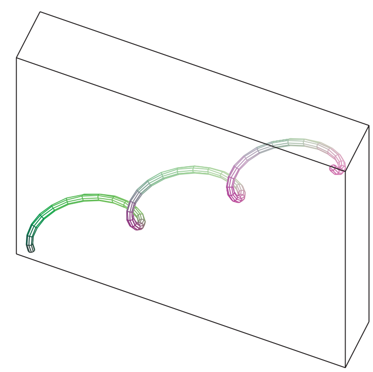


FIGURE 13

For further details concerning the physics involved and animations of the trajectories of the particles, see the following websites:

- www.physics.ucla.edu/plasma-exp/Beam/
- www.phy.ntnu.edu.tw/ntnujava/index.php?topic=36

13.1 EXERCISES

1–2 Find the domain of the vector function.

1. $\mathbf{r}(t) = \left\langle \ln(t+1), \dfrac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$

2. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{1}{t-2}\mathbf{k}$

3–6 Find the limit.

3. $\displaystyle\lim_{t\to 0} \left(e^{-3t}\,\mathbf{i} + \frac{t^2}{\sin^2 t}\,\mathbf{j} + \cos 2t\,\mathbf{k} \right)$

4. $\displaystyle\lim_{t\to 1} \left(\frac{t^2 - t}{t-1}\,\mathbf{i} + \sqrt{t+8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\,\mathbf{k} \right)$

5. $\lim_{t \to \infty} \left\langle \dfrac{1 + t^2}{1 - t^2}, \tan^{-1} t, \dfrac{1 - e^{-2t}}{t} \right\rangle$

6. $\lim_{t \to \infty} \left\langle te^{-t}, \dfrac{t^3 + t}{2t^3 - 1}, t \sin \dfrac{1}{t} \right\rangle$

7–14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$

9. $\mathbf{r}(t) = \langle t, 2 - t, 2t \rangle$

10. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$

12. $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + 2 \sin t\, \mathbf{j} + \mathbf{k}$

13. $\mathbf{r}(t) = t^2\, \mathbf{i} + t^4\, \mathbf{j} + t^6\, \mathbf{k}$

14. $\mathbf{r}(t) = \cos t\, \mathbf{i} - \cos t\, \mathbf{j} + \sin t\, \mathbf{k}$

15–16 Draw the projections of the curve on the three coordinate planes. Use these projections to help sketch the curve.

15. $\mathbf{r}(t) = \langle t, \sin t, 2 \cos t \rangle$

16. $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

17–20 Find a vector equation and parametric equations for the line segment that joins P to Q.

17. $P(2, 0, 0)$, $Q(6, 2, -2)$

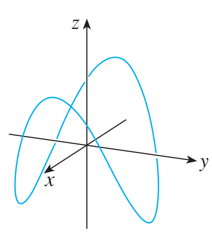
18. $P(-1, 2, -2)$, $Q(-3, 5, 1)$

19. $P(0, -1, 1)$, $Q(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

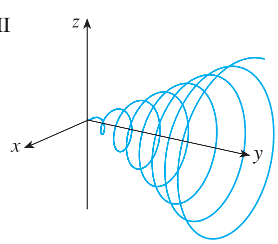
20. $P(a, b, c)$, $Q(u, v, w)$

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

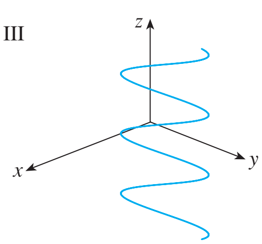
I



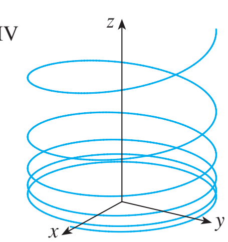
II



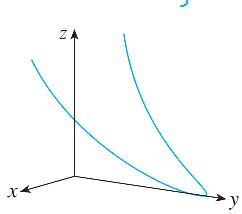
III



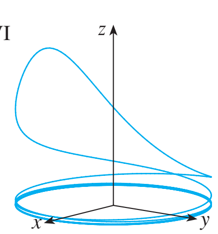
IV



V



VI



21. $x = t \cos t, \quad y = t, \quad z = t \sin t, \quad t \geqslant 0$

22. $x = \cos t, \quad y = \sin t, \quad z = 1/(1 + t^2)$

23. $x = t, \quad y = 1/(1 + t^2), \quad z = t^2$

24. $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$

25. $x = \cos 8t, \quad y = \sin 8t, \quad z = e^{0.8t}, \quad t \geqslant 0$

26. $x = \cos^2 t, \quad y = \sin^2 t, \quad z = t$

27. Show that the curve with parametric equations $x = t \cos t$, $y = t \sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

28. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

29. Find three different surfaces that contain the curve $\mathbf{r}(t) = 2t\, \mathbf{i} + e^t\, \mathbf{j} + e^{2t}\, \mathbf{k}$.

30. Find three different surfaces that contain the curve $\mathbf{r}(t) = t^2\, \mathbf{i} + \ln t\, \mathbf{j} + (1/t)\, \mathbf{k}$.

31. At what points does the curve $\mathbf{r}(t) = t\, \mathbf{i} + (2t - t^2)\, \mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

32. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

33–37 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

33. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

34. $\mathbf{r}(t) = \langle te^t, e^{-t}, t \rangle$

35. $\mathbf{r}(t) = \left\langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t \right\rangle$

36. $\mathbf{r}(t) = \langle \cos(8 \cos t) \sin t, \sin(8 \cos t) \sin t, \cos t \rangle$

37. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

38. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$. Explain its shape by graphing its projections onto the three coordinate planes.

39. Graph the curve with parametric equations

$$x = (1 + \cos 16t) \cos t$$

$$y = (1 + \cos 16t) \sin t$$

$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

40. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t}\, \cos t$$

$$y = \sqrt{1 - 0.25 \cos^2 10t}\, \sin t$$

$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.

41. Show that the curve with parametric equations $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the points (1, 4, 0) and (9, −8, 28) but not through the point (4, 7, −6).

42–46 Find a vector function that represents the curve of intersection of the two surfaces.

42. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

43. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

44. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

45. The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

46. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geqslant 0$, and the cylinder $x^2 + z^2 = 1$

47. Try to sketch by hand the curve of intersection of the circular cylinder $x^2 + y^2 = 4$ and the parabolic cylinder $z = x^2$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

48. Try to sketch by hand the curve of intersection of the parabolic cylinder $y = x^2$ and the top half of the ellipsoid $x^2 + 4y^2 + 4z^2 = 16$. Then find parametric equations for this curve and use these equations and a computer to graph the curve.

49. If two objects travel through space along two different curves, it's often important to know whether they will collide. (Will a missile hit its moving target? Will two aircraft collide?) The curves might intersect, but we need to know whether the objects are in the same position *at the same time*. Suppose the trajectories of two particles are given by the vector functions

$$\mathbf{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathbf{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geqslant 0$. Do the particles collide?

50. Two particles travel along the space curves

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$$

Do the particles collide? Do their paths intersect?

51. (a) Graph the curve with parametric equations

$$x = \tfrac{27}{26}\sin 8t - \tfrac{8}{39}\sin 18t$$
$$y = -\tfrac{27}{26}\cos 8t + \tfrac{8}{39}\cos 18t$$
$$z = \tfrac{144}{65}\sin 5t$$

(b) Show that the curve lies on the hyperboloid of one sheet $144x^2 + 144y^2 - 25z^2 = 100$.

52. The view of the trefoil knot shown in Figure 8 is accurate, but it doesn't reveal the whole story. Use the parametric equations

$$x = (2 + \cos 1.5t)\cos t$$
$$y = (2 + \cos 1.5t)\sin t$$
$$z = \sin 1.5t$$

to sketch the curve by hand as viewed from above, with gaps indicating where the curve passes over itself. Start by showing that the projection of the curve onto the xy-plane has polar coordinates $r = 2 + \cos 1.5t$ and $\theta = t$, so r varies between 1 and 3. Then show that z has maximum and minimum values when the projection is halfway between $r = 1$ and $r = 3$.

When you have finished your sketch, use a computer to draw the curve with viewpoint directly above and compare with your sketch. Then use the computer to draw the curve from several other viewpoints. You can get a better impression of the curve if you plot a tube with radius 0.2 around the curve. (Use the `tubeplot` command in Maple or the `tubecurve` or `Tube` command in Mathematica.)

53. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vector functions that possess limits as $t \to a$ and let c be a constant. Prove the following properties of limits.

(a) $\lim_{t\to a} [\mathbf{u}(t) + \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) + \lim_{t\to a} \mathbf{v}(t)$

(b) $\lim_{t\to a} c\mathbf{u}(t) = c \lim_{t\to a} \mathbf{u}(t)$

(c) $\lim_{t\to a} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \cdot \lim_{t\to a} \mathbf{v}(t)$

(d) $\lim_{t\to a} [\mathbf{u}(t) \times \mathbf{v}(t)] = \lim_{t\to a} \mathbf{u}(t) \times \lim_{t\to a} \mathbf{v}(t)$

54. Show that $\lim_{t\to a} \mathbf{r}(t) = \mathbf{b}$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |t - a| < \delta \quad \text{then} \quad |\mathbf{r}(t) - \mathbf{b}| < \varepsilon$$

13.2 Derivatives and Integrals of Vector Functions

Later in this chapter we are going to use vector functions to describe the motion of planets and other objects through space. Here we prepare the way by developing the calculus of vector functions.

Derivatives

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

1
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t+h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. We will also have occasion to consider the **unit tangent vector**, which is

Notice that when $0 < h < 1$, multiplying the secant vector by $1/h$ *stretches* the vector, as shown in Figure 1(b).

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

TEC Visual 13.2 shows an animation of Figure 1.

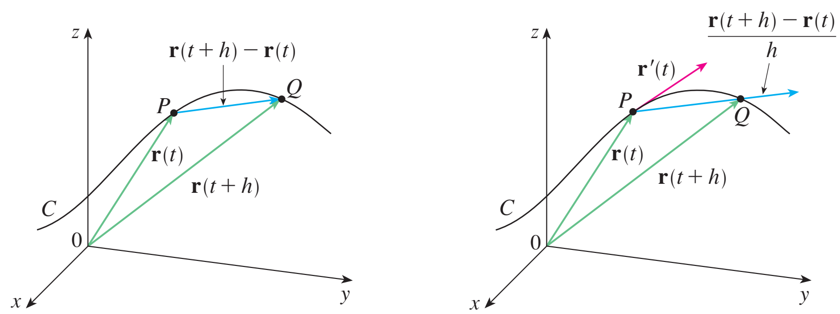


FIGURE 1

(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

The following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$.

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

PROOF

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle]$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t)\rangle$$ ■

EXAMPLE 1
(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.
(b) Find the unit tangent vector at the point where $t = 0$.

SOLUTION
(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}}\,\mathbf{j} + \frac{2}{\sqrt{5}}\,\mathbf{k}$$ ■

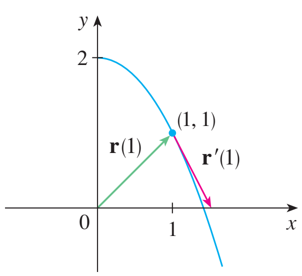


FIGURE 2

Notice from Figure 2 that the tangent vector points in the direction of increasing t. (See Exercise 58.)

EXAMPLE 2 For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

SOLUTION We have

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\,\mathbf{i} - \mathbf{j} \qquad \text{and} \qquad \mathbf{r}'(1) = \frac{1}{2}\,\mathbf{i} - \mathbf{j}$$

The curve is a plane curve and elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives $y = 2 - x^2$, $x \geqslant 0$. In Figure 2 we draw the position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point $(1, 1)$. ■

EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1\rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1\rangle$, so by Equations 12.5.2 its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t$$ ■

The helix and the tangent line in Example 3 are shown in Figure 3.

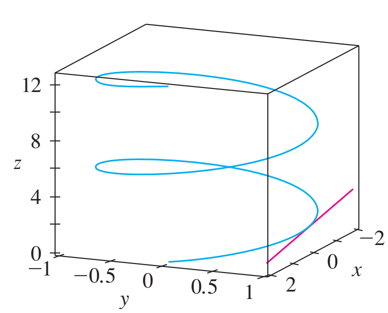


FIGURE 3

In Section 13.4 we will see how $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ can be interpreted as the velocity and acceleration vectors of a particle moving through space with position vector $\mathbf{r}(t)$ at time t.

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$. For instance, the second derivative of the function in Example 3 is

$$\mathbf{r}''(t) = \langle -2\cos t, -\sin t, 0\rangle$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ (Chain Rule)

This theorem can be proved either directly from Definition 1 or by using Theorem 2 and the corresponding differentiation formulas for real-valued functions. The proof of Formula 4 follows; the remaining formulas are left as exercises.

PROOF OF FORMULA 4 Let

$$\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle \qquad \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t)\rangle$$

Then

$$\mathbf{u}(t)\cdot\mathbf{v}(t) = f_1(t)\,g_1(t) + f_2(t)\,g_2(t) + f_3(t)\,g_3(t) = \sum_{i=1}^{3} f_i(t)\,g_i(t)$$

so the ordinary Product Rule gives

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] &= \frac{d}{dt}\sum_{i=1}^{3} f_i(t)\,g_i(t) = \sum_{i=1}^{3}\frac{d}{dt}[f_i(t)\,g_i(t)]\\ &= \sum_{i=1}^{3}[f_i'(t)\,g_i(t) + f_i(t)\,g_i'(t)]\\ &= \sum_{i=1}^{3} f_i'(t)\,g_i(t) + \sum_{i=1}^{3} f_i(t)\,g_i'(t)\\ &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)\end{aligned}$$

■

EXAMPLE 4 Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

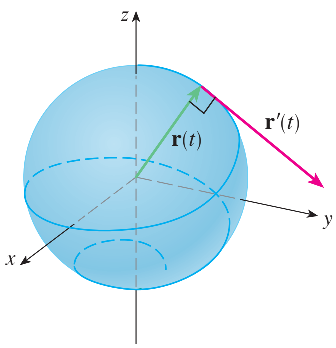


FIGURE 4

SOLUTION Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.) ■

Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows. (We use the notation of Chapter 4.)

$$\int_a^b \mathbf{r}(t)\, dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\, \Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\, \Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\, \Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\, \Delta t \right) \mathbf{k} \right]$$

and so

$$\int_a^b \mathbf{r}(t)\, dt = \left(\int_a^b f(t)\, dt \right) \mathbf{i} + \left(\int_a^b g(t)\, dt \right) \mathbf{j} + \left(\int_a^b h(t)\, dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\, dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\, dt$ for indefinite integrals (antiderivatives).

EXAMPLE 5 If $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$, then

$$\int \mathbf{r}(t)\, dt = \left(\int 2\cos t\, dt \right) \mathbf{i} + \left(\int \sin t\, dt \right) \mathbf{j} + \left(\int 2t\, dt \right) \mathbf{k}$$

$$= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration, and

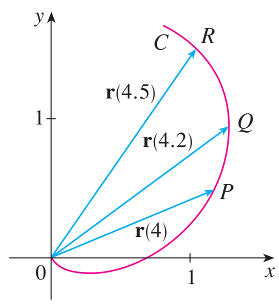
$$\int_0^{\pi/2} \mathbf{r}(t)\, dt = \Big[2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k}\Big]_0^{\pi/2} = 2\,\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\,\mathbf{k}$$ ■

13.2 EXERCISES

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.
(a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.
(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.
(d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t \rangle$, $0 \leqslant t \leqslant 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.
(b) Draw the vector $\mathbf{r}'(1)$ starting at (1, 1), and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3–8
(a) Sketch the plane curve with the given vector equation.
(b) Find $\mathbf{r}'(t)$.
(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle, \quad t = -1$

4. $\mathbf{r}(t) = \langle t^2, t^3 \rangle, \quad t = 1$

5. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^{t}\,\mathbf{j}, \quad t = 0$

6. $\mathbf{r}(t) = e^{t}\,\mathbf{i} + 2t\,\mathbf{j}, \quad t = 0$

7. $\mathbf{r}(t) = 4 \sin t\,\mathbf{i} - 2 \cos t\,\mathbf{j}, \quad t = 3\pi/4$

8. $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}, \quad t = -\pi/3$

9–16 Find the derivative of the vector function.

9. $\mathbf{r}(t) = \left\langle \sqrt{t - 2}, 3, 1/t^2 \right\rangle$

10. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t \rangle$

11. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos(t^2)\,\mathbf{j} + \sin^2 t\,\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{1 + t}\,\mathbf{i} + \dfrac{t}{1 + t}\,\mathbf{j} + \dfrac{t^2}{1 + t}\,\mathbf{k}$

13. $\mathbf{r}(t) = t \sin t\,\mathbf{i} + e^{t} \cos t\,\mathbf{j} + \sin t \cos t\,\mathbf{k}$

14. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k}$

15. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c}$

16. $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c})$

17–20 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

17. $\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \right\rangle, \quad t = 2$

18. $\mathbf{r}(t) = \langle \tan^{-1} t, 2e^{2t}, 8te^{t} \rangle, \quad t = 0$

19. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2 \sin 2t\,\mathbf{k}, \quad t = 0$

20. $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}, \quad t = \pi/4$

21. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

22. If $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = t^2 + 1, \quad y = 4\sqrt{t}, \quad z = e^{t^2 - t}; \quad (2, 4, 1)$

24. $x = \ln(t + 1), \quad y = t \cos 2t, \quad z = 2^{t}; \quad (0, 0, 1)$

25. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t}; \quad (1, 0, 1)$

26. $x = \sqrt{t^2 + 3}, \quad y = \ln(t^2 + 3), \quad z = t; \quad (2, \ln 4, 1)$

27. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point (3, 4, 2).

28. Find the point on the curve $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, e^{t} \rangle$, $0 \leqslant t \leqslant \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

CAS **29–31** Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

29. $x = t, \; y = e^{-t}, \; z = 2t - t^2; \quad (0, 1, 0)$

30. $x = 2 \cos t, \; y = 2 \sin t, \; z = 4 \cos 2t; \quad \left(\sqrt{3}, 1, 2\right)$

31. $x = t \cos t, \; y = t, \; z = t \sin t; \quad (-\pi, \pi, 0)$

32. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.
(b) Illustrate by graphing the curve and both tangent lines.

33. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

34. At what point do the curves $\mathbf{r}_1(t) = \langle t, 1 - t, 3 + t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3 - s, s - 2, s^2 \rangle$ intersect? Find their angle of intersection correct to the nearest degree.

35–40 Evaluate the integral.

35. $\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt$

36. $\int_1^4 (2t^{3/2}\,\mathbf{i} + (t + 1)\sqrt{t}\,\mathbf{k})\,dt$

37. $\int_0^1 \left(\dfrac{1}{t+1}\,\mathbf{i} + \dfrac{1}{t^2+1}\,\mathbf{j} + \dfrac{t}{t^2+1}\,\mathbf{k} \right) dt$

38. $\int_0^{\pi/4} (\sec t \tan t\,\mathbf{i} + t \cos 2t\,\mathbf{j} + \sin^2 2t \cos 2t\,\mathbf{k})\,dt$

39. $\int (\sec^2 t\,\mathbf{i} + t(t^2 + 1)^3\,\mathbf{j} + t^2 \ln t\,\mathbf{k})\,dt$

40. $\int \left(te^{2t}\,\mathbf{i} + \dfrac{t}{1-t}\,\mathbf{j} + \dfrac{1}{\sqrt{1-t^2}}\,\mathbf{k} \right) dt$

---

41. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$.

42. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + te^t\,\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

43. Prove Formula 1 of Theorem 3.

44. Prove Formula 3 of Theorem 3.

45. Prove Formula 5 of Theorem 3.

46. Prove Formula 6 of Theorem 3.

47. If $\mathbf{u}(t) = \langle \sin t, \cos t, t \rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t \rangle$, use Formula 4 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)]$$

48. If $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 47, use Formula 5 of Theorem 3 to find

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)]$$

49. Find $f'(2)$, where $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1 \rangle$, $\mathbf{u}'(2) = \langle 3, 0, 4 \rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3 \rangle$.

50. If $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$, where $\mathbf{u}$ and $\mathbf{v}$ are the vector functions in Exercise 49, find $\mathbf{r}'(2)$.

51. If $\mathbf{r}(t) = \mathbf{a} \cos \omega t + \mathbf{b} \sin \omega t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega\mathbf{a} \times \mathbf{b}$.

52. If $\mathbf{r}$ is the vector function in Exercise 51, show that $\mathbf{r}''(t) + \omega^2\mathbf{r}(t) = \mathbf{0}$.

53. Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

54. Find an expression for $\dfrac{d}{dt}[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t))]$.

55. If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\dfrac{d}{dt}|\mathbf{r}(t)| = \dfrac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

[*Hint:* $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$]

56. If a curve has the property that the position vector $\mathbf{r}(t)$ is always perpendicular to the tangent vector $\mathbf{r}'(t)$, show that the curve lies on a sphere with center the origin.

57. If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

58. Show that the tangent vector to a curve defined by a vector function $\mathbf{r}(t)$ points in the direction of increasing t. [*Hint:* Refer to Figure 1 and consider the cases $h > 0$ and $h < 0$ separately.]

13.3 Arc Length and Curvature

Length of a Curve

In Section 10.2 we defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$\boxed{1} \qquad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

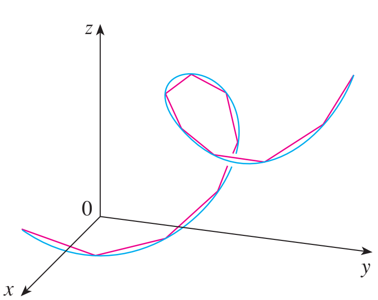


FIGURE 1
The length of a space curve is the limit of lengths of inscribed polygons.

The length of a space curve is defined in exactly the same way (see Figure 1). Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leqslant t \leqslant b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown

that its length is

2

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

3

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

In the next section we will see that if $\mathbf{r}(t)$ is the position vector of a moving object at time t, then $\mathbf{r}'(t)$ is the velocity vector and $|\mathbf{r}'(t)|$ is the speed. Thus Equation 3 says that to compute distance traveled, we integrate speed.

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Figure 2 shows the arc of the helix whose length is computed in Example 1.

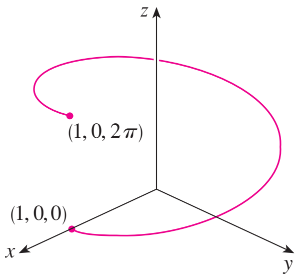


FIGURE 2

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

SOLUTION Since $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1, 0, 0)$ to $(1, 0, 2\pi)$ is described by the parameter interval $0 \leqslant t \leqslant 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$ ■

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

4 $$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leqslant t \leqslant 2$$

could also be represented by the function

5 $$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leqslant u \leqslant \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$. We say that Equations 4 and 5 are **parametrizations** of the curve C. If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer. In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

The Arc Length Function

Now we suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \qquad a \leqslant t \leqslant b$$

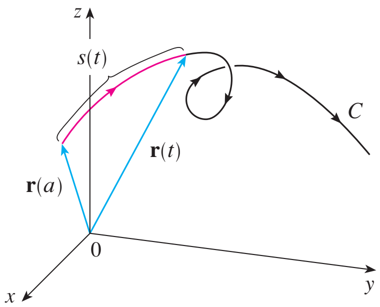


FIGURE 3

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its **arc length function** s by

6
$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.) If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

7
$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$. Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

EXAMPLE 2 Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

SOLUTION The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From Example 1 we have

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

and so
$$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

Therefore $t = s/\sqrt{2}$ and the required reparametrization is obtained by substituting for t:

$$\mathbf{r}(t(s)) = \cos\!\left(s/\sqrt{2}\right)\mathbf{i} + \sin\!\left(s/\sqrt{2}\right)\mathbf{j} + \left(s/\sqrt{2}\right)\mathbf{k}$$ ■

Curvature

TEC Visual 13.3A shows animated unit tangent vectors, like those in Figure 4, for a variety of plane curves and space curves.

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

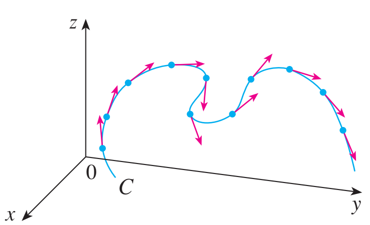


FIGURE 4
Unit tangent vectors at equally spaced points on C

and indicates the direction of the curve. From Figure 4 you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\mathbf{T}$.

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule (Theorem 13.2.3, Formula 6) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9
$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

SOLUTION We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t \, \mathbf{i} + a \sin t \, \mathbf{j}$$

Therefore $\qquad \mathbf{r}'(t) = -a \sin t \, \mathbf{i} + a \cos t \, \mathbf{j} \qquad \text{and} \qquad |\mathbf{r}'(t)| = a$

so
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t \, \mathbf{i} + \cos t \, \mathbf{j}$$

and
$$\mathbf{T}'(t) = -\cos t \, \mathbf{i} - \sin t \, \mathbf{j}$$

This gives $|\mathbf{T}'(t)| = 1$, so using Formula 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

■

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition. We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 12.4.2), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$

Now $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal by Example 13.2.4. Therefore, by Theorem 12.4.9,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|$$

Thus
$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$
■

EXAMPLE 4 Find the curvature of the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

SOLUTION We first compute the required ingredients:

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \qquad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\,\mathbf{i} - 6t\,\mathbf{j} + 2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

Theorem 10 then gives

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

At the origin, where $t = 0$, the curvature is $\kappa(0) = 2$. ■

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\,\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11
$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

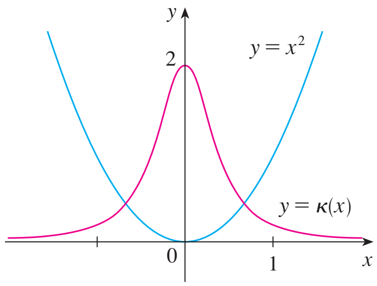


FIGURE 5
The parabola $y = x^2$ and its curvature function

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points (0, 0), (1, 1), and (2, 4).

SOLUTION Since $y' = 2x$ and $y'' = 2$, Formula 11 gives

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

The curvature at (0, 0) is $\kappa(0) = 2$. At (1, 1) it is $\kappa(1) = 2/5^{3/2} \approx 0.18$. At (2, 4) it is $\kappa(2) = 2/17^{3/2} \approx 0.03$. Observe from the expression for $\kappa(x)$ or the graph of κ in Figure 5 that $\kappa(x) \to 0$ as $x \to \pm\infty$. This corresponds to the fact that the parabola appears to become flatter as $x \to \pm\infty$. ■

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Example 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

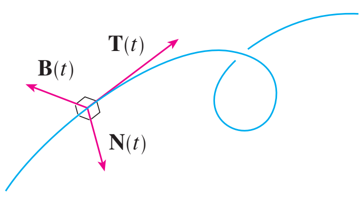


FIGURE 6

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. (See Figure 6.)

Figure 7 illustrates Example 6 by showing the vectors **T**, **N**, and **B** at two locations on the helix. In general, the vectors **T**, **N**, and **B**, starting at the various points on a curve, form a set of orthogonal vectors, called the **TNB** frame, that moves along the curve as t varies. This **TNB** frame plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

EXAMPLE 6 Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

SOLUTION We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, -\sin t, 0\rangle$$

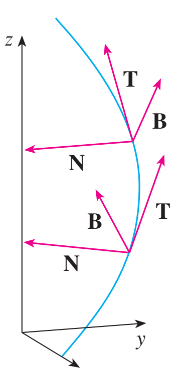


FIGURE 7

This shows that the normal vector at any point on the helix is horizontal and points toward the z-axis. The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$$ ■

TEC Visual 13.3B shows how the **TNB** frame moves along several curves.

The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P. The name comes from the Latin *osculum*, meaning "kiss." It is the plane that comes closest to containing the part of the curve near P. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which **N** points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P. It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

EXAMPLE 7 Find equations of the normal plane and osculating plane of the helix in Example 6 at the point $P(0, 1, \pi/2)$.

Figure 8 shows the helix and the osculating plane in Example 7.

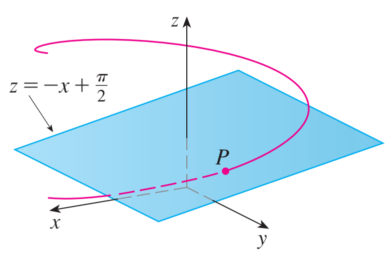


FIGURE 8

SOLUTION The point P corresponds to $t = \pi/2$ and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -1, 0, 1 \rangle$, so an equation is

$$-1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \qquad \text{or} \qquad z = x + \frac{\pi}{2}$$

The osculating plane at P contains the vectors **T** and **N**, so its normal vector is $\mathbf{T} \times \mathbf{N} = \mathbf{B}$. From Example 6 we have

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}} \langle \sin t, \, -\cos t, \, 1 \rangle \qquad \mathbf{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

A simpler normal vector is $\langle 1, 0, 1 \rangle$, so an equation of the osculating plane is

$$1(x - 0) + 0(y - 1) + 1\left(z - \frac{\pi}{2}\right) = 0 \qquad \text{or} \qquad z = -x + \frac{\pi}{2}$$

■

EXAMPLE 8 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

SOLUTION From Example 5, the curvature of the parabola at the origin is $\kappa(0) = 2$. So the radius of the osculating circle at the origin is $1/\kappa = \frac{1}{2}$ and its center is $\left(0, \frac{1}{2}\right)$. Its equation is therefore

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

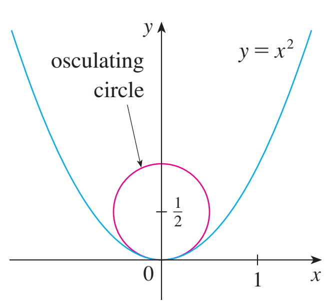


FIGURE 9
Notice that the circle and the parabola appear to bend similarly at the origin.

For the graph in Figure 9 we use parametric equations of this circle:

$$x = \tfrac{1}{2} \cos t \qquad y = \tfrac{1}{2} + \tfrac{1}{2} \sin t$$

■

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

TEC Visual 13.3C shows how the osculating circle changes as a point moves along a curve.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

13.3 EXERCISES

1–6 Find the length of the curve.

1. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle, \quad -5 \leqslant t \leqslant 5$

2. $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle, \quad 0 \leqslant t \leqslant 1$

3. $\mathbf{r}(t) = \sqrt{2}\, t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

4. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/4$

5. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

6. $\mathbf{r}(t) = t^2\,\mathbf{i} + 9t\,\mathbf{j} + 4t^{3/2}\,\mathbf{k}, \quad 1 \leqslant t \leqslant 4$

7–9 Find the length of the curve correct to four decimal places. (Use a calculator to approximate the integral.)

7. $\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle, \quad 0 \leqslant t \leqslant 2$

8. $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t}\rangle, \quad 1 \leqslant t \leqslant 3$

9. $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t\rangle$, from $(1, 0, 0)$ to $(1, 4, 0)$

10. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. Find the total length of this curve correct to four decimal places.

11. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

12. Find, correct to four decimal places, the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

13–14 (a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P, and (b) find the point 4 units along the curve (in the direction of increasing t) from P.

13. $\mathbf{r}(t) = (5 - t)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + 3t\,\mathbf{k}, \quad P(4, 1, 3)$

14. $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k}, \quad P\left(0, 1, \sqrt{2}\right)$

15. Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3\sin t$, $y = 4t$, $z = 3\cos t$ in the positive direction. Where are you now?

16. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2 + 1} - 1\right)\mathbf{i} + \frac{2t}{t^2 + 1}\,\mathbf{j}$$

with respect to arc length measured from the point $(1, 0)$ in the direction of increasing t. Express the reparametrization in its simplest form. What can you conclude about the curve?

17–20
(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
(b) Use Formula 9 to find the curvature.

17. $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$

18. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle, \quad t > 0$

19. $\mathbf{r}(t) = \left\langle \sqrt{2}\,t, e^t, e^{-t}\right\rangle$

20. $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, t^2\right\rangle$

21–23 Use Theorem 10 to find the curvature.

21. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k}$

22. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k}$

23. $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k}$

24. Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t\ln t\rangle$ at the point $(1, 0, 0)$.

25. Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at the point $(1, 1, 1)$.

26. Graph the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin 5t$ and find the curvature at the point $(1, 0, 0)$.

27–29 Use Formula 11 to find the curvature.

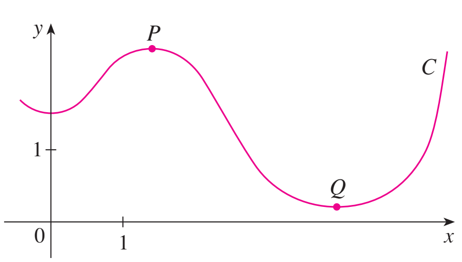
27. $y = x^4$ **28.** $y = \tan x$ **29.** $y = xe^x$

30–31 At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

30. $y = \ln x$ **31.** $y = e^x$

32. Find an equation of a parabola that has curvature 4 at the origin.

33. (a) Is the curvature of the curve C shown in the figure greater at P or at Q? Explain.
(b) Estimate the curvature at P and at Q by sketching the osculating circles at those points.



34–35 Use a graphing calculator or computer to graph both the curve and its curvature function $\kappa(x)$ on the same screen. Is the graph of κ what you would expect?

34. $y = x^4 - 2x^2$ **35.** $y = x^{-2}$

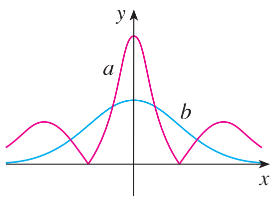
CAS **36–37** Plot the space curve and its curvature function $\kappa(t)$. Comment on how the curvature reflects the shape of the curve.

36. $\mathbf{r}(t) = \langle t - \sin t,\, 1 - \cos t,\, 4\cos(t/2)\rangle, \quad 0 \leqslant t \leqslant 8\pi$

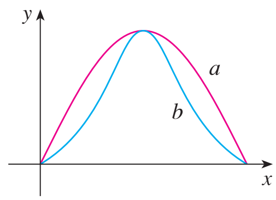
37. $\mathbf{r}(t) = \left\langle te^t,\, e^{-t},\, \sqrt{2}\,t\right\rangle, \quad -5 \leqslant t \leqslant 5$

38–39 Two graphs, a and b, are shown. One is a curve $y = f(x)$ and the other is the graph of its curvature function $y = \kappa(x)$. Identify each curve and explain your choices.

38.



39.



CAS **40.** (a) Graph the curve $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t\rangle$. At how many points on the curve does it appear that the curvature has a local or absolute maximum?
(b) Use a CAS to find and graph the curvature function. Does this graph confirm your conclusion from part (a)?

CAS **41.** The graph of $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t,\, 1 - \frac{3}{2}\cos t,\, t\right\rangle$ is shown in Figure 13.1.12(b). Where do you think the curvature is largest? Use a CAS to find and graph the curvature function. For which values of t is the curvature largest?

42. Use Theorem 10 to show that the curvature of a plane parametric curve $x = f(t)$, $y = g(t)$ is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t.

43–45 Use the formula in Exercise 42 to find the curvature.

43. $x = t^2, \quad y = t^3$

44. $x = a\cos \omega t, \quad y = b \sin \omega t$

45. $x = e^t \cos t, \quad y = e^t \sin t$

46. Consider the curvature at $x = 0$ for each member of the family of functions $f(x) = e^{cx}$. For which members is $\kappa(0)$ largest?

47–48 Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given point.

47. $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t\right\rangle, \quad \left(1, \frac{2}{3}, 1\right)$

48. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t\rangle, \quad (1, 0, 0)$

49–50 Find equations of the normal plane and osculating plane of the curve at the given point.

49. $x = \sin 2t,\; y = -\cos 2t,\; z = 4t; \quad (0, 1, 2\pi)$

50. $x = \ln t,\; y = 2t,\; z = t^2; \quad (0, 2, 1)$

51. Find equations of the osculating circles of the ellipse $9x^2 + 4y^2 = 36$ at the points $(2, 0)$ and $(0, 3)$. Use a graphing calculator or computer to graph the ellipse and both osculating circles on the same screen.

52. Find equations of the osculating circles of the parabola $y = \frac{1}{2}x^2$ at the points $(0, 0)$ and $\left(1, \frac{1}{2}\right)$. Graph both osculating circles and the parabola on the same screen.

53. At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

CAS **54.** Is there a point on the curve in Exercise 53 where the osculating plane is parallel to the plane $x + y + z = 1$? [*Note:* You will need a CAS for differentiating, for simplifying, and for computing a cross product.]

55. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1, 1, 1)$.

56. Show that the osculating plane at every point on the curve $\mathbf{r}(t) = \left\langle t + 2, 1 - t, \frac{1}{2}t^2\right\rangle$ is the same plane. What can you conclude about the curve?

57. Show that at every point on the curve

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t\rangle$$

the angle between the unit tangent vector and the z-axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

58. The *rectifying plane* of a curve at a point is the plane that contains the vectors $\mathbf{T}$ and $\mathbf{B}$ at that point. Find the rectifying plane of the curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}$ at the point $\left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$.

59. Show that the curvature κ is related to the tangent and normal vectors by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

60. Show that the curvature of a plane curve is $\kappa = |d\phi/ds|$, where ϕ is the angle between $\mathbf{T}$ and $\mathbf{i}$; that is, ϕ is the angle of inclination of the tangent line. (This shows that the definition of curvature is consistent with the definition for plane curves given in Exercise 10.2.69.)

61. (a) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$.
(b) Show that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{T}$.
(c) Deduce from parts (a) and (b) that $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ for some number $\tau(s)$ called the **torsion** of the curve. (The torsion measures the degree of twisting of a curve.)
(d) Show that for a plane curve the torsion is $\tau(s) = 0$.

62. The following formulas, called the **Frenet-Serret formulas**, are of fundamental importance in differential geometry:

1. $d\mathbf{T}/ds = \kappa\mathbf{N}$

2. $d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B}$

3. $d\mathbf{B}/ds = -\tau\mathbf{N}$

(Formula 1 comes from Exercise 59 and Formula 3 comes from Exercise 61.) Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ to deduce Formula 2 from Formulas 1 and 3.

63. Use the Frenet-Serret formulas to prove each of the following. (Primes denote derivatives with respect to t. Start as in the proof of Theorem 10.)

(a) $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$

(b) $\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{B}$

(c) $\mathbf{r}''' = [s''' - \kappa^2(s')^3]\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$

(d) $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$

64. Show that the circular helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, where a and b are positive constants, has constant curvature and constant torsion. [Use the result of Exercise 63(d).]

65. Use the formula in Exercise 63(d) to find the torsion of the curve $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle$.

66. Find the curvature and torsion of the curve $x = \sinh t$, $y = \cosh t$, $z = t$ at the point $(0, 1, 0)$.

67. The DNA molecule has the shape of a double helix (see Figure 3 on page 890). The radius of each helix is about 10 angstroms ($1\text{ Å} = 10^{-8}$ cm). Each helix rises about 34 Å during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

68. Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x-axis is to be joined smoothly to a track along the line $y = 1$ for $x \geqslant 1$.
(a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geqslant 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.
(b) Graph F.

13.4 Motion in Space: Velocity and Acceleration

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve. In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h, the vector

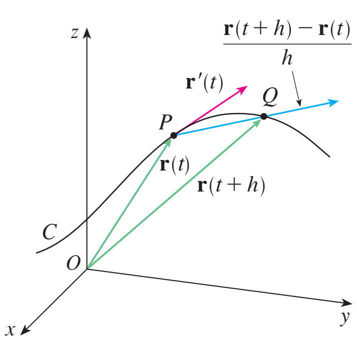


FIGURE 1

1
$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector (1) gives the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

2
$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because, from (2) and from Equation 13.3.7, we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

EXAMPLE 1 The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

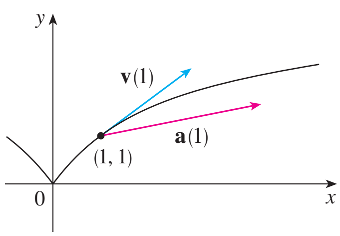


FIGURE 2

TEC Visual 13.4 shows animated velocity and acceleration vectors for objects moving along various curves.

SOLUTION The velocity and acceleration at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

When $t = 1$, we have

$$\mathbf{v}(1) = 3\,\mathbf{i} + 2\,\mathbf{j} \qquad \mathbf{a}(1) = 6\,\mathbf{i} + 2\,\mathbf{j} \qquad |\mathbf{v}(1)| = \sqrt{13}$$

These velocity and acceleration vectors are shown in Figure 2. ■

Figure 3 shows the path of the particle in Example 2 with the velocity and acceleration vectors when $t = 1$.

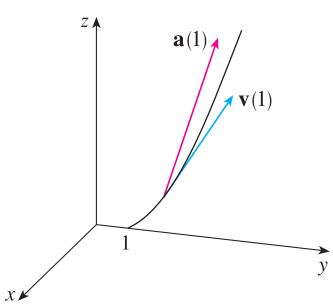


FIGURE 3

EXAMPLE 2 Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

SOLUTION

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, e^t, (1 + t)e^t \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, e^t, (2 + t)e^t \rangle$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1 + t)^2 e^{2t}}$$ ■

The vector integrals that were introduced in Section 13.2 can be used to find position vectors when velocity or acceleration vectors are known, as in the next example.

EXAMPLE 3 A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

SOLUTION Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k})\,dt$$

$$= 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{C}$$

To determine the value of the constant vector $\mathbf{C}$, we use the fact that $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. The preceding equation gives $\mathbf{v}(0) = \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and

$$\mathbf{v}(t) = 2t^2\,\mathbf{i} + 3t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$= (2t^2 + 1)\,\mathbf{i} + (3t^2 - 1)\,\mathbf{j} + (t + 1)\,\mathbf{k}$$

The expression for $\mathbf{r}(t)$ that we obtained in Example 3 was used to plot the path of the particle in Figure 4 for $0 \leq t \leq 3$.

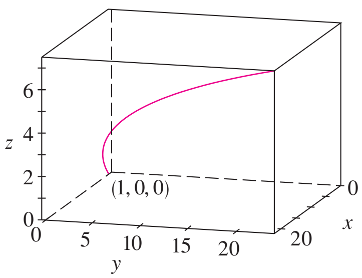


FIGURE 4

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we have

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt$$

$$= \int [(2t^2 + 1)\,\mathbf{i} + (3t^2 - 1)\,\mathbf{j} + (t + 1)\,\mathbf{k}]\, dt$$

$$= \left(\tfrac{2}{3}t^3 + t\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\tfrac{1}{2}t^2 + t\right)\mathbf{k} + \mathbf{D}$$

Putting $t = 0$, we find that $\mathbf{D} = \mathbf{r}(0) = \mathbf{i}$, so the position at time t is given by

$$\mathbf{r}(t) = \left(\tfrac{2}{3}t^3 + t + 1\right)\mathbf{i} + (t^3 - t)\,\mathbf{j} + \left(\tfrac{1}{2}t^2 + t\right)\mathbf{k}$$ ■

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\, du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\, du$$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**. The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

EXAMPLE 4 An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

The object moving with position P has angular speed $\omega = d\theta/dt$, where θ is the angle shown in Figure 5.

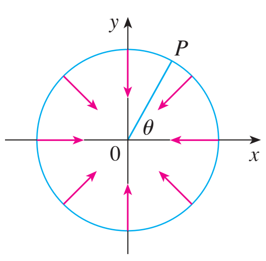


FIGURE 5

SOLUTION To find the force, we first need to know the acceleration:

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a\omega\sin\omega t\,\mathbf{i} + a\omega\cos\omega t\,\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -a\omega^2\cos\omega t\,\mathbf{i} - a\omega^2\sin\omega t\,\mathbf{j}$$

Therefore Newton's Second Law gives the force as

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2(a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j})$$

Notice that $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$. This shows that the force acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and therefore points toward the origin (see Figure 5). Such a force is called a *centripetal* (center-seeking) force. ■

Projectile Motion

EXAMPLE 5 A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. (See Figure 6.) Assuming that air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?

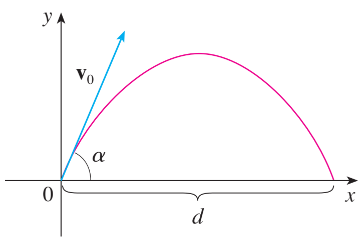


FIGURE 6

SOLUTION We set up the axes so that the projectile starts at the origin. Since the force due to gravity acts downward, we have

$$\mathbf{F} = m\mathbf{a} = -mg\,\mathbf{j}$$

where $g = |\mathbf{a}| \approx 9.8 \text{ m/s}^2$. Thus

$$\mathbf{a} = -g\,\mathbf{j}$$

Since $\mathbf{v}'(t) = \mathbf{a}$, we have

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$$

where $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$. Therefore

$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0$$

Integrating again, we obtain

$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{D}$$

But $\mathbf{D} = \mathbf{r}(0) = \mathbf{0}$, so the position vector of the projectile is given by

3
$$\mathbf{r}(t) = -\tfrac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0$$

If we write $|\mathbf{v}_0| = v_0$ (the initial speed of the projectile), then

$$\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$$

and Equation 3 becomes

$$\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \tfrac{1}{2}gt^2\right]\mathbf{j}$$

The parametric equations of the trajectory are therefore

If you eliminate t from Equations 4, you will see that y is a quadratic function of x. So the path of the projectile is part of a parabola.

4
$$x = (v_0 \cos\alpha)t \qquad y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

The horizontal distance d is the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = (2v_0 \sin\alpha)/g$. This second value of t then gives

$$d = x = (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} = \frac{v_0^2(2 \sin\alpha \cos\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

Clearly, d has its maximum value when $\sin 2\alpha = 1$, that is, $\alpha = 45°$. ■

EXAMPLE 6 A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

SOLUTION If we place the origin at ground level, then the initial position of the projectile is (0, 10) and so we need to adjust Equations 4 by adding 10 to the expression for y. With $v_0 = 150$ m/s, $\alpha = 45°$, and $g = 9.8 \text{ m/s}^2$, we have

$$x = 150 \cos(45°)t = 75\sqrt{2}\,t$$

$$y = 10 + 150 \sin(45°)t - \tfrac{1}{2}(9.8)t^2 = 10 + 75\sqrt{2}\,t - 4.9t^2$$

Impact occurs when $y = 0$, that is, $4.9t^2 - 75\sqrt{2}\,t - 10 = 0$. Using the quadratic formula to solve this equation (and taking only the positive value of t), we get

$$t = \frac{75\sqrt{2} + \sqrt{11{,}250 + 196}}{9.8} \approx 21.74$$

Then $x \approx 75\sqrt{2}\,(21.74) \approx 2306$, so the projectile hits the ground about 2306 m away.

The velocity of the projectile is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 75\sqrt{2}\,\mathbf{i} + \left(75\sqrt{2} - 9.8t\right)\mathbf{j}$$

So its speed at impact is

$$|\mathbf{v}(21.74)| = \sqrt{(75\sqrt{2})^2 + (75\sqrt{2} - 9.8 \cdot 21.74)^2} \approx 151 \text{ m/s}$$

Tangential and Normal Components of Acceleration

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal. If we write $v = |\mathbf{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so

$$\mathbf{v} = v\mathbf{T}$$

If we differentiate both sides of this equation with respect to t, we get

5 $$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

If we use the expression for the curvature given by Equation 13.3.9, then we have

6 $$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \qquad \text{so} \qquad |\mathbf{T}'| = \kappa v$$

The unit normal vector was defined in the preceding section as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so (6) gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$$

and Equation 5 becomes

7 $$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}$$

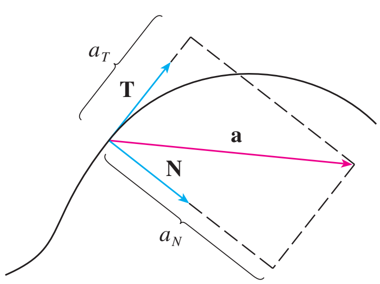


FIGURE 7

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

8 $$a_T = v' \qquad \text{and} \qquad a_N = \kappa v^2$$

This resolution is illustrated in Figure 7.

Let's look at what Formula 7 says. The first thing to notice is that the binormal vector **B** is absent. No matter how an object moves through space, its acceleration always lies in the plane of **T** and **N** (the osculating plane). (Recall that **T** gives the direction of motion and **N** points in the direction the curve is turning.) Next we notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed. This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature κ, so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door. High speed around the turn has the same effect; in fact, if you double your speed, a_N is increased by a factor of 4.

Although we have expressions for the tangential and normal components of acceleration in Equations 8, it's desirable to have expressions that depend only on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$. To this end we take the dot product of $\mathbf{v} = v\mathbf{T}$ with $\mathbf{a}$ as given by Equation 7:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= v\mathbf{T}\cdot(v'\mathbf{T} + \kappa v^2\mathbf{N})\\ &= vv'\mathbf{T}\cdot\mathbf{T} + \kappa v^3\mathbf{T}\cdot\mathbf{N}\\ &= vv' \qquad (\text{since } \mathbf{T}\cdot\mathbf{T} = 1 \text{ and } \mathbf{T}\cdot\mathbf{N} = 0)\end{aligned}$$

Therefore

9
$$a_T = v' = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Using the formula for curvature given by Theorem 13.3.10, we have

10
$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 7 A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3\rangle$. Find the tangential and normal components of acceleration.

SOLUTION

$$\begin{aligned}\mathbf{r}(t) &= t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}\\ \mathbf{r}'(t) &= 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}\\ \mathbf{r}''(t) &= 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}\\ |\mathbf{r}'(t)| &= \sqrt{8t^2 + 9t^4}\end{aligned}$$

Therefore Equation 9 gives the tangential component as

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

Since
$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2t & 2t & 3t^2\\ 2 & 2 & 6t\end{vmatrix} = 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

Equation 10 gives the normal component as

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2 + 9t^4}}$$ ■

Kepler's Laws of Planetary Motion

We now describe one of the great accomplishments of calculus by showing how the material of this chapter can be used to prove Kepler's laws of planetary motion. After 20 years of studying the astronomical observations of the Danish astronomer Tycho Brahe, the German mathematician and astronomer Johannes Kepler (1571–1630) formulated the following three laws.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

In his book *Principia Mathematica* of 1687, Sir Isaac Newton was able to show that these three laws are consequences of two of his own laws, the Second Law of Motion and the Law of Universal Gravitation. In what follows we prove Kepler's First Law. The remaining laws are left as exercises (with hints).

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it. We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around the earth or a comet moving around a star.) The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$. We use the following laws of Newton:

$$\text{Second Law of Motion:} \quad \mathbf{F} = m\mathbf{a}$$

$$\text{Law of Gravitation:} \quad \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$$

where $\mathbf{F}$ is the gravitational force on the planet, m and M are the masses of the planet and the sun, G is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = (1/r)\mathbf{r}$ is the unit vector in the direction of $\mathbf{r}$.

We first show that the planet moves in one plane. By equating the expressions for $\mathbf{F}$ in Newton's two laws, we find that

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

and so $\mathbf{a}$ is parallel to $\mathbf{r}$. It follows that $\mathbf{r} \times \mathbf{a} = \mathbf{0}$. We use Formula 5 in Theorem 13.2.3 to write

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0} \end{aligned}$$

Therefore

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where $\mathbf{h}$ is a constant vector. (We may assume that $\mathbf{h} \neq \mathbf{0}$; that is, $\mathbf{r}$ and $\mathbf{v}$ are not parallel.) This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to $\mathbf{h}$ for all values of t, so the planet always lies in the plane through the origin perpendicular to $\mathbf{h}$. Thus the orbit of the planet is a plane curve.

To prove Kepler's First Law we rewrite the vector $\mathbf{h}$ as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\ &= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}') \end{aligned}$$

Then

$$\mathbf{a} \times \mathbf{h} = \frac{-GM}{r^2}\mathbf{u} \times (r^2\mathbf{u} \times \mathbf{u}') = -GM\,\mathbf{u} \times (\mathbf{u} \times \mathbf{u}')$$

$$= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}'] \qquad \text{(by Theorem 12.4.11, Property 6)}$$

But $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and, since $|\mathbf{u}(t)| = 1$, it follows from Example 13.2.4 that

$$\mathbf{u} \cdot \mathbf{u}' = 0$$

Therefore $$\mathbf{a} \times \mathbf{h} = GM\,\mathbf{u}'$$

and so $$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} = \mathbf{a} \times \mathbf{h} = GM\,\mathbf{u}'$$

Integrating both sides of this equation, we get

11 $$\mathbf{v} \times \mathbf{h} = GM\,\mathbf{u} + \mathbf{c}$$

where **c** is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector **k** points in the direction of the vector **h**. Then the planet moves in the xy-plane. Since both $\mathbf{v} \times \mathbf{h}$ and **u** are perpendicular to **h**, Equation 11 shows that **c** lies in the xy-plane. This means that we can choose the x- and y-axes so that the vector **i** lies in the direction of **c**, as shown in Figure 8.

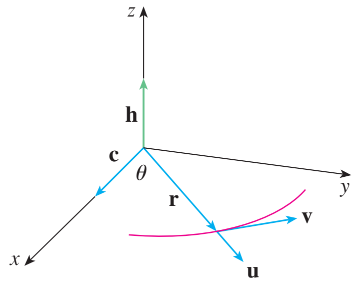


FIGURE 8

If θ is the angle between **c** and **r**, then (r, θ) are polar coordinates of the planet. From Equation 11 we have

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = \mathbf{r} \cdot (GM\,\mathbf{u} + \mathbf{c}) = GM\,\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c}$$

$$= GMr\,\mathbf{u} \cdot \mathbf{u} + |\mathbf{r}||\mathbf{c}|\cos\theta = GMr + rc\cos\theta$$

where $c = |\mathbf{c}|$. Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c\cos\theta} = \frac{1}{GM}\,\frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos\theta}$$

where $e = c/(GM)$. But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$. So

$$r = \frac{h^2/(GM)}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta}$$

Writing $d = h^2/c$, we obtain the equation

12 $$r = \frac{ed}{1 + e\cos\theta}$$

Comparing with Theorem 10.6.6, we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity e. We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.

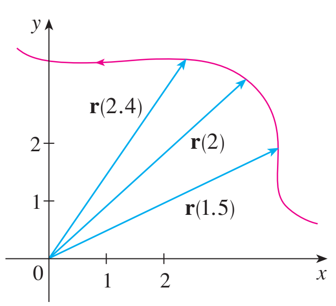
This completes the derivation of Kepler's First Law. We will guide you through the derivation of the Second and Third Laws in the Applied Project on page 920. The proofs of these three laws show that the methods of this chapter provide a powerful tool for describing some of the laws of nature.

13.4 EXERCISES

1. The table gives coordinates of a particle moving through space along a smooth curve.
(a) Find the average velocities over the time intervals [0, 1], [0.5, 1], [1, 2], and [1, 1.5].
(b) Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t.
(a) Draw a vector that represents the average velocity of the particle over the time interval $2 \leqslant t \leqslant 2.4$.
(b) Draw a vector that represents the average velocity over the time interval $1.5 \leqslant t \leqslant 2$.
(c) Write an expression for the velocity vector $\mathbf{v}(2)$.
(d) Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.



3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

3. $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle, \quad t = 2$

4. $\mathbf{r}(t) = \langle t^2, 1/t^2 \rangle, \quad t = 1$

5. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \pi/3$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad t = 0$

7. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}, \quad t = 1$

8. $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

9–14 Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle$

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

12. $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$

13. $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$

14. $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle, \quad t \geqslant 0$

15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15. $\mathbf{a}(t) = 2\,\mathbf{i} + 2t\,\mathbf{k}, \quad \mathbf{v}(0) = 3\,\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

16. $\mathbf{a}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k},$
$\mathbf{v}(0) = -\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} - 4\,\mathbf{k}$

17–18
(a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.
(b) Use a computer to graph the path of the particle.

17. $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}$

18. $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

20. What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$?

21. A force with magnitude 20 N acts directly upward from the xy-plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t.

22. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

23. A projectile is fired with an initial speed of 200 m/s and angle of elevation 60°. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

24. Rework Exercise 23 if the projectile is fired from a position 100 m above the ground.

25. A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

26. A projectile is fired from a tank with initial speed 400 m/s. Find two angles of elevation that can be used to hit a target 3000 m away.

27. A rifle is fired with angle of elevation 36°. What is the muzzle speed if the maximum height of the bullet is 1600 ft?

28. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

29. A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)

30. Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.

31. A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis). The initial velocity is $50\,\mathbf{i} + 80\,\mathbf{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/s^2, so the acceleration vector is $\mathbf{a} = -4\,\mathbf{j} - 32\,\mathbf{k}$. Where does the ball land and with what speed?

32. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

33. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{400}x(40 - x)$.

(a) A boat proceeds at a constant speed of 5 m/s from a point A on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.

(b) Suppose we would like to pilot the boat to land at the point B on the east bank directly opposite A. If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?

34. Another reasonable model for the water speed of the river in Exercise 33 is a sine function: $f(x) = 3\sin(\pi x/40)$. If a boater would like to cross the river from A to B with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.

35. A particle has position function $\mathbf{r}(t)$. If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, where $\mathbf{c}$ is a constant vector, describe the path of the particle.

36. (a) If a particle moves along a straight line, what can you say about its acceleration vector?

(b) If a particle moves with constant speed along a curve, what can you say about its acceleration vector?

37–40 Find the tangential and normal components of the acceleration vector.

37. $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + t^3\,\mathbf{j}, \quad t \geqslant 0$

38. $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \left(\frac{2}{3}t^3 - 2t\right)\mathbf{j}$

39. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$

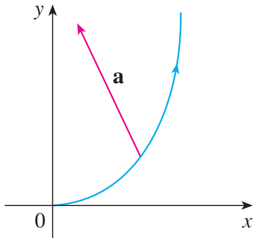
40. $\mathbf{r}(t) = t\,\mathbf{i} + 2e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}$

41–42 Find the tangential and normal components of the acceleration vector at the given point.

41. $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t^2 + 3t)\,\mathbf{j} + 4\sqrt{t}\,\mathbf{k}, \quad (0, 4, 4)$

42. $\mathbf{r}(t) = \dfrac{1}{t}\,\mathbf{i} + \dfrac{1}{t^2}\,\mathbf{j} + \dfrac{1}{t^3}\,\mathbf{k}, \quad (1, 1, 1)$

43. The magnitude of the acceleration vector $\mathbf{a}$ is 10 cm/s^2. Use the figure to estimate the tangential and normal components of $\mathbf{a}$.



44. If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its **angular momentum** is defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$ and its **torque** as $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t, then $\mathbf{L}(t)$ is constant. (This is the *law of conservation of angular momentum.*)

45. The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t)\,\mathbf{i} + (2 + \ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\mathbf{k}$$

and the coordinates of a space station are (6, 4, 9). The captain wants the spaceship to coast into the space station. When should the engines be turned off?

46. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t. If the exhaust gases escape with velocity $\mathbf{v}_e$ relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e$$

(a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln\dfrac{m(0)}{m(t)}\mathbf{v}_e$.

(b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?

APPLIED PROJECT KEPLER'S LAWS

Johannes Kepler stated the following three laws of planetary motion on the basis of massive amounts of data on the positions of the planets at various times.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

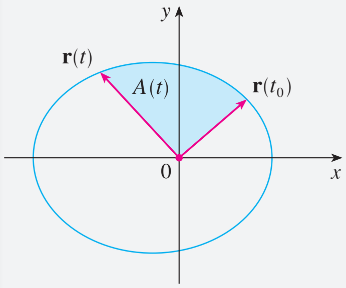
Kepler formulated these laws because they fitted the astronomical data. He wasn't able to see why they were true or how they related to each other. But Sir Isaac Newton, in his *Principia Mathematica* of 1687, showed how to deduce Kepler's three laws from two of Newton's own laws, the Second Law of Motion and the Law of Universal Gravitation. In Section 13.4 we proved Kepler's First Law using the calculus of vector functions. In this project we guide you through the proofs of Kepler's Second and Third Laws and explore some of their consequences.

1. Use the following steps to prove Kepler's Second Law. The notation is the same as in the proof of the First Law in Section 13.4. In particular, use polar coordinates so that $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$.

(a) Show that $\mathbf{h} = r^2 \dfrac{d\theta}{dt}\,\mathbf{k}$.

(b) Deduce that $r^2 \dfrac{d\theta}{dt} = h$.

(c) If $A = A(t)$ is the area swept out by the radius vector $\mathbf{r} = \mathbf{r}(t)$ in the time interval $[t_0, t]$ as in the figure, show that



$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt}$$

(d) Deduce that

$$\frac{dA}{dt} = \tfrac{1}{2}h = \text{constant}$$

This says that the rate at which A is swept out is constant and proves Kepler's Second Law.

2. Let T be the period of a planet about the sun; that is, T is the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes of the ellipse are $2a$ and $2b$.

(a) Use part (d) of Problem 1 to show that $T = 2\pi ab/h$.

(b) Show that $\dfrac{h^2}{GM} = ed = \dfrac{b^2}{a}$.

(c) Use parts (a) and (b) to show that $T^2 = \dfrac{4\pi^2}{GM}a^3$.

This proves Kepler's Third Law. [Notice that the proportionality constant $4\pi^2/(GM)$ is independent of the planet.]

3. The period of the earth's orbit is approximately 365.25 days. Use this fact and Kepler's Third Law to find the length of the major axis of the earth's orbit. You will need the mass of the sun, $M = 1.99 \times 10^{30}$ kg, and the gravitational constant, $G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

4. It's possible to place a satellite into orbit about the earth so that it remains fixed above a given location on the equator. Compute the altitude that is needed for such a satellite. The earth's mass is 5.98×10^{24} kg; its radius is 6.37×10^6 m. (This orbit is called the Clarke Geosynchronous Orbit after Arthur C. Clarke, who first proposed the idea in 1945. The first such satellite, *Syncom II,* was launched in July 1963.)

13 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. What is a vector function? How do you find its derivative and its integral?

2. What is the connection between vector functions and space curves?

3. How do you find the tangent vector to a smooth curve at a point? How do you find the tangent line? The unit tangent vector?

4. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating the following vector functions.
(a) $\mathbf{u}(t) + \mathbf{v}(t)$ (b) $c\mathbf{u}(t)$ (c) $f(t)\,\mathbf{u}(t)$
(d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$ (e) $\mathbf{u}(t) \times \mathbf{v}(t)$ (f) $\mathbf{u}(f(t))$

5. How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?

6. (a) What is the definition of curvature?
(b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.
(c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.
(d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.

7. (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.
(b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?

8. (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?
(b) Write the acceleration in terms of its tangential and normal components.

9. State Kepler's Laws.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. The curve with vector equation $\mathbf{r}(t) = t^3\,\mathbf{i} + 2t^3\,\mathbf{j} + 3t^3\,\mathbf{k}$ is a line.

2. The curve $\mathbf{r}(t) = \langle 0, t^2, 4t \rangle$ is a parabola.

3. The curve $\mathbf{r}(t) = \langle 2t, 3 - t, 0 \rangle$ is a line that passes through the origin.

4. The derivative of a vector function is obtained by differentiating each component function.

5. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}'(t)$$

6. If $\mathbf{r}(t)$ is a differentiable vector function, then

$$\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$$

7. If $\mathbf{T}(t)$ is the unit tangent vector of a smooth curve, then the curvature is $\kappa = |d\mathbf{T}/dt|$.

8. The binormal vector is $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$.

9. Suppose f is twice continuously differentiable. At an inflection point of the curve $y = f(x)$, the curvature is 0.

10. If $\kappa(t) = 0$ for all t, the curve is a straight line.

11. If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is a constant.

12. If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

13. The osculating circle of a curve C at a point has the same tangent vector, normal vector, and curvature as C at that point.

14. Different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

EXERCISES

1. (a) Sketch the curve with vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \qquad t \geqslant 0$$

(b) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

2. Let $\mathbf{r}(t) = \langle \sqrt{2 - t}, (e^t - 1)/t, \ln(t + 1) \rangle$.
(a) Find the domain of $\mathbf{r}$.
(b) Find $\lim_{t \to 0} \mathbf{r}(t)$.
(c) Find $\mathbf{r}'(t)$.

3. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$.

4. Find parametric equations for the tangent line to the curve $x = 2 \sin t$, $y = 2 \sin 2t$, $z = 2 \sin 3t$ at the point $(1, \sqrt{3}, 2)$. Graph the curve and the tangent line on a common screen.

5. If $\mathbf{r}(t) = t^2\,\mathbf{i} + t \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k}$, evaluate $\int_0^1 \mathbf{r}(t)\,dt$.

6. Let C be the curve with equations $x = 2 - t^3$, $y = 2t - 1$, $z = \ln t$. Find (a) the point where C intersects the xz-plane, (b) parametric equations of the tangent line at $(1, 1, 0)$, and (c) an equation of the normal plane to C at $(1, 1, 0)$.

7. Use Simpson's Rule with $n = 6$ to estimate the length of the arc of the curve with equations $x = t^2$, $y = t^3$, $z = t^4$, $0 \leqslant t \leqslant 3$.

8. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$, $0 \leqslant t \leqslant 1$.

9. The helix $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1 + t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1, 0, 0)$. Find the angle of intersection of these curves.

10. Reparametrize the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t \cos t\,\mathbf{k}$ with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

11. For the curve given by $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$, $0 \leqslant t \leqslant \pi/2$, find
(a) the unit tangent vector,
(b) the unit normal vector,
(c) the unit binormal vector, and
(d) the curvature.

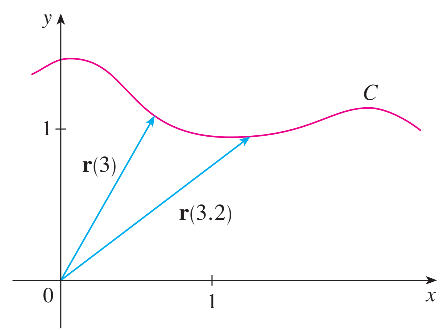
12. Find the curvature of the ellipse $x = 3 \cos t$, $y = 4 \sin t$ at the points $(3, 0)$ and $(0, 4)$.

13. Find the curvature of the curve $y = x^4$ at the point $(1, 1)$.

14. Find an equation of the osculating circle of the curve $y = x^4 - x^2$ at the origin. Graph both the curve and its osculating circle.

15. Find an equation of the osculating plane of the curve $x = \sin 2t$, $y = t$, $z = \cos 2t$ at the point $(0, \pi, 1)$.

16. The figure shows the curve C traced by a particle with position vector $\mathbf{r}(t)$ at time t.
(a) Draw a vector that represents the average velocity of the particle over the time interval $3 \leqslant t \leqslant 3.2$.
(b) Write an expression for the velocity $\mathbf{v}(3)$.
(c) Write an expression for the unit tangent vector $\mathbf{T}(3)$ and draw it.



17. A particle moves with position function $\mathbf{r}(t) = t \ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$. Find the velocity, speed, and acceleration of the particle.

18. Find the velocity, speed, and acceleration of a particle moving with position function $\mathbf{r}(t) = (2t^2 - 3)\,\mathbf{i} + 2t\,\mathbf{j}$. Sketch the path of the particle and draw the position, velocity, and acceleration vectors for $t = 1$.

19. A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k}$. Find its position function.

20. An athlete throws a shot at an angle of $45°$ to the horizontal at an initial speed of 43 ft/s. It leaves his hand 7 ft above the ground.
(a) Where is the shot 2 seconds later?
(b) How high does the shot go?
(c) Where does the shot land?

21. A projectile is launched with an initial speed of 40 m/s from the floor of a tunnel whose height is 30 m. What angle of elevation should be used to achieve the maximum possible horizontal range of the projectile? What is the maximum range?

22. Find the tangential and normal components of the acceleration vector of a particle with position function

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$$

23. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radius so that its position at time t, $t \geqslant 0$, is given by

$\mathbf{r}(t) = t\,\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \cos \omega t\,\mathbf{i} + \sin \omega t\,\mathbf{j}$$

(a) Show that the velocity $\mathbf{v}$ of the particle is

$$\mathbf{v} = \cos \omega t\,\mathbf{i} + \sin \omega t\,\mathbf{j} + t\,\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

(b) Show that the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$$

where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the edge of the disk. The extra term $2\,\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving merry-go-round.

(c) Determine the Coriolis acceleration of a particle that moves on a rotating disk according to the equation

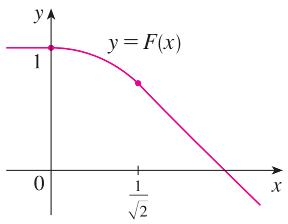
$$\mathbf{r}(t) = e^{-t}\cos \omega t\,\mathbf{i} + e^{-t}\sin \omega t\,\mathbf{j}$$

24. In designing *transfer curves* to connect sections of straight railroad tracks, it's important to realize that the acceleration of the train should be continuous so that the reactive force exerted by the train on the track is also continuous. Because of the formulas for the components of acceleration in Section 13.4, this will be the case if the curvature varies continuously.

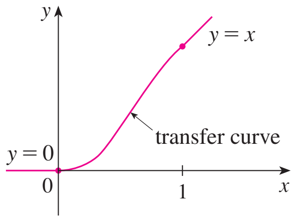
(a) A logical candidate for a transfer curve to join existing tracks given by $y = 1$ for $x \leqslant 0$ and $y = \sqrt{2} - x$ for $x \geqslant 1/\sqrt{2}$ might be the function $f(x) = \sqrt{1 - x^2}$, $0 < x < 1/\sqrt{2}$, whose graph is the arc of the circle shown in the figure. It looks reasonable at first glance. Show that the function

$$F(x) = \begin{cases} 1 & \text{if } x \leqslant 0 \\ \sqrt{1 - x^2} & \text{if } 0 < x < 1/\sqrt{2} \\ \sqrt{2} - x & \text{if } x \geqslant 1/\sqrt{2} \end{cases}$$

is continuous and has continuous slope, but does not have continuous curvature. Therefore f is not an appropriate transfer curve.



(b) Find a fifth-degree polynomial to serve as a transfer curve between the following straight line segments: $y = 0$ for $x \leqslant 0$ and $y = x$ for $x \geqslant 1$. Could this be done with a fourth-degree polynomial? Use a graphing calculator or computer to sketch the graph of the "connected" function and check to see that it looks like the one in the figure.



Problems Plus

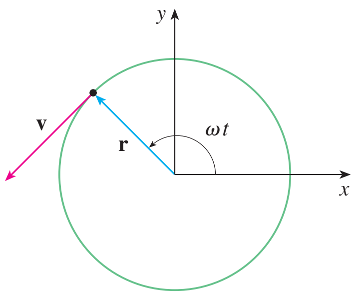


FIGURE FOR PROBLEM 1

1. A particle P moves with constant angular speed ω around a circle whose center is at the origin and whose radius is R. The particle is said to be in *uniform circular motion*. Assume that the motion is counterclockwise and that the particle is at the point $(R, 0)$ when $t = 0$. The position vector at time $t \geqslant 0$ is $\mathbf{r}(t) = R \cos \omega t \, \mathbf{i} + R \sin \omega t \, \mathbf{j}$.

(a) Find the velocity vector $\mathbf{v}$ and show that $\mathbf{v} \cdot \mathbf{r} = 0$. Conclude that $\mathbf{v}$ is tangent to the circle and points in the direction of the motion.

(b) Show that the speed $|\mathbf{v}|$ of the particle is the constant ωR. The *period* T of the particle is the time required for one complete revolution. Conclude that

$$T = \frac{2\pi R}{|\mathbf{v}|} = \frac{2\pi}{\omega}$$

(c) Find the acceleration vector $\mathbf{a}$. Show that it is proportional to $\mathbf{r}$ and that it points toward the origin. An acceleration with this property is called a *centripetal acceleration.* Show that the magnitude of the acceleration vector is $|\mathbf{a}| = R\omega^2$.

(d) Suppose that the particle has mass m. Show that the magnitude of the force $\mathbf{F}$ that is required to produce this motion, called a *centripetal force,* is

$$|\mathbf{F}| = \frac{m|\mathbf{v}|^2}{R}$$

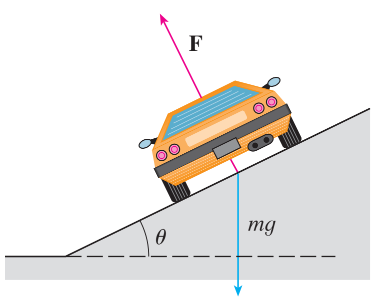


FIGURE FOR PROBLEM 2

2. A circular curve of radius R on a highway is banked at an angle θ so that a car can safely traverse the curve without skidding when there is no friction between the road and the tires. The loss of friction could occur, for example, if the road is covered with a film of water or ice. The rated speed v_R of the curve is the maximum speed that a car can attain without skidding. Suppose a car of mass m is traversing the curve at the rated speed v_R. Two forces are acting on the car: the vertical force, mg, due to the weight of the car, and a force $\mathbf{F}$ exerted by, and normal to, the road (see the figure).

The vertical component of $\mathbf{F}$ balances the weight of the car, so that $|\mathbf{F}| \cos \theta = mg$. The horizontal component of $\mathbf{F}$ produces a centripetal force on the car so that, by Newton's Second Law and part (d) of Problem 1,

$$|\mathbf{F}| \sin \theta = \frac{mv_R^2}{R}$$

(a) Show that $v_R^2 = Rg \tan \theta$.

(b) Find the rated speed of a circular curve with radius 400 ft that is banked at an angle of 12°.

(c) Suppose the design engineers want to keep the banking at 12°, but wish to increase the rated speed by 50%. What should the radius of the curve be?

3. A projectile is fired from the origin with angle of elevation α and initial speed v_0. Assuming that air resistance is negligible and that the only force acting on the projectile is gravity, g, we showed in Example 13.4.5 that the position vector of the projectile is

$$\mathbf{r}(t) = (v_0 \cos \alpha)t \, \mathbf{i} + \left[(v_0 \sin \alpha)t - \tfrac{1}{2}gt^2\right] \mathbf{j}$$

We also showed that the maximum horizontal distance of the projectile is achieved when $\alpha = 45°$ and in this case the range is $R = v_0^2/g$.

(a) At what angle should the projectile be fired to achieve maximum height and what is the maximum height?

(b) Fix the initial speed v_0 and consider the parabola $x^2 + 2Ry - R^2 = 0$, whose graph is shown in the figure at the left. Show that the projectile can hit any target inside or on the boundary of the region bounded by the parabola and the x-axis, and that it can't hit any target outside this region.

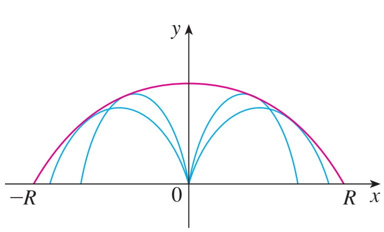
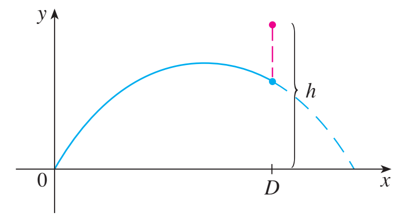


FIGURE FOR PROBLEM 3

(c) Suppose that the gun is elevated to an angle of inclination α in order to aim at a target that is suspended at a height h directly over a point D units downrange (see the figure below). The target is released at the instant the gun is fired. Show that the projectile always hits the target, regardless of the value v_0, provided the projectile does not hit the ground "before" D.



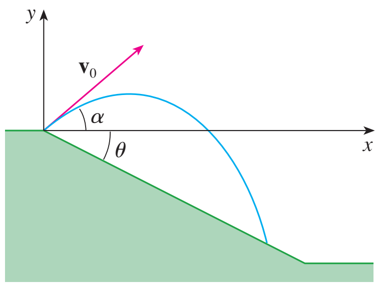


FIGURE FOR PROBLEM 4

4. (a) A projectile is fired from the origin down an inclined plane that makes an angle θ with the horizontal. The angle of elevation of the gun and the initial speed of the projectile are α and v_0, respectively. Find the position vector of the projectile and the parametric equations of the path of the projectile as functions of the time t. (Ignore air resistance.)

(b) Show that the angle of elevation α that will maximize the downhill range is the angle halfway between the plane and the vertical.

(c) Suppose the projectile is fired up an inclined plane whose angle of inclination is θ. Show that, in order to maximize the (uphill) range, the projectile should be fired in the direction halfway between the plane and the vertical.

(d) In a paper presented in 1686, Edmond Halley summarized the laws of gravity and projectile motion and applied them to gunnery. One problem he posed involved firing a projectile to hit a target a distance R up an inclined plane. Show that the angle at which the projectile should be fired to hit the target but use the least amount of energy is the same as the angle in part (c). (Use the fact that the energy needed to fire the projectile is proportional to the square of the initial speed, so minimizing the energy is equivalent to minimizing the initial speed.)

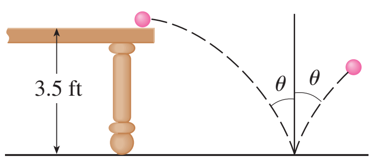


FIGURE FOR PROBLEM 5

5. A ball rolls off a table with a speed of 2 ft/s. The table is 3.5 ft high.

(a) Determine the point at which the ball hits the floor and find its speed at the instant of impact.

(b) Find the angle θ between the path of the ball and the vertical line drawn through the point of impact (see the figure).

(c) Suppose the ball rebounds from the floor at the same angle with which it hits the floor, but loses 20% of its speed due to energy absorbed by the ball on impact. Where does the ball strike the floor on the second bounce?

6. Find the curvature of the curve with parametric equations

$$x = \int_0^t \sin\left(\tfrac{1}{2}\pi\theta^2\right) d\theta \qquad y = \int_0^t \cos\left(\tfrac{1}{2}\pi\theta^2\right) d\theta$$

7. If a projectile is fired with angle of elevation α and initial speed v, then parametric equations for its trajectory are

$$x = (v \cos \alpha)t \qquad y = (v \sin \alpha)t - \tfrac{1}{2}gt^2$$

(See Example 13.4.5.) We know that the range (horizontal distance traveled) is maximized when $\alpha = 45°$. What value of α maximizes the total distance traveled by the projectile? (State your answer correct to the nearest degree.)

8. A cable has radius r and length L and is wound around a spool with radius R without overlapping. What is the shortest length along the spool that is covered by the cable?

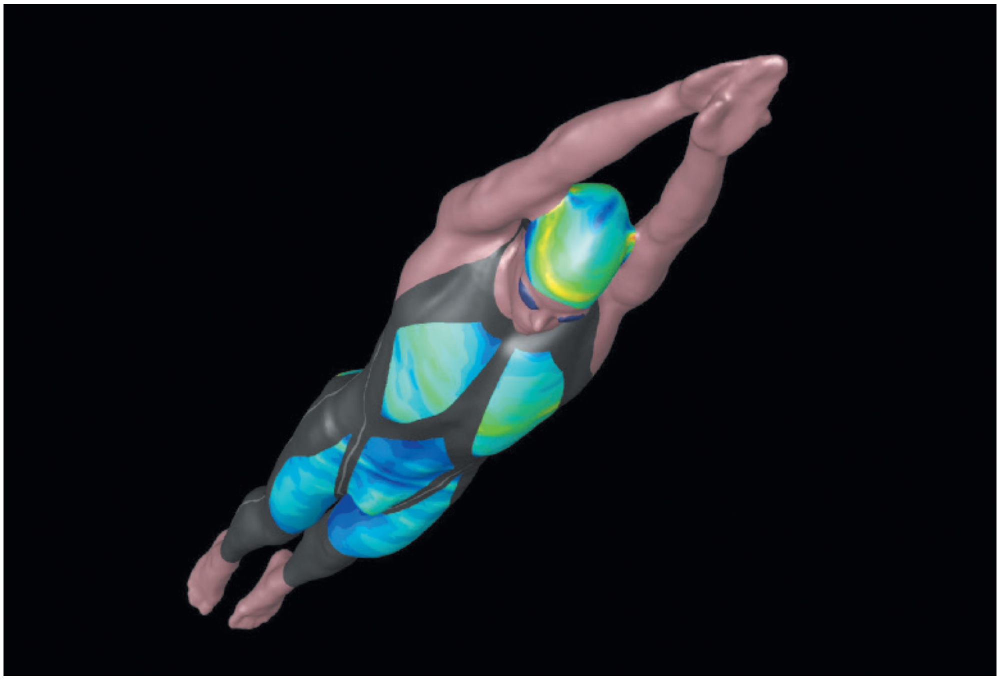
9. Show that the curve with vector equation

$$\mathbf{r}(t) = \langle a_1t^2 + b_1t + c_1,\, a_2t^2 + b_2t + c_2,\, a_3t^2 + b_3t + c_3 \rangle$$

lies in a plane and find an equation of the plane.

14 Partial Derivatives

In 2008 Speedo introduced the LZR Racer and, because it reduced drag in the water, many swimming records were broken. In the project on page 976 you are asked to use partial derivatives to explain why a small decrease in drag can have a big effect on performance.



Courtesy Speedo and ANSYS, Inc.

SO FAR WE HAVE DEALT with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables, so in this chapter we turn our attention to functions of several variables and extend the basic ideas of differential calculus to such functions.

14.1 Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point. We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

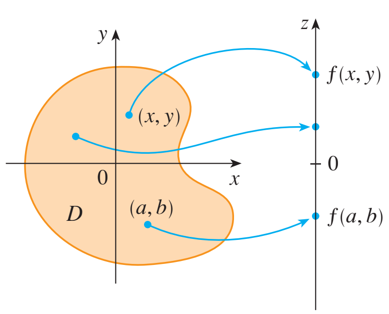


FIGURE 1

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy-plane and the range is a set of numbers on a real line, shown as a z-axis. For instance, if $f(x, y)$ represents the temperature at a point (x, y) in a flat metal plate with the shape of D, we can think of the z-axis as a thermometer displaying the recorded temperatures.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

SOLUTION

(a) $f(3, 2) = \dfrac{\sqrt{3 + 2 + 1}}{3 - 1} = \dfrac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geqslant 0,\ x \neq 1\}$$

The inequality $x + y + 1 \geqslant 0$, or $y \geqslant -x - 1$, describes the points that lie on or

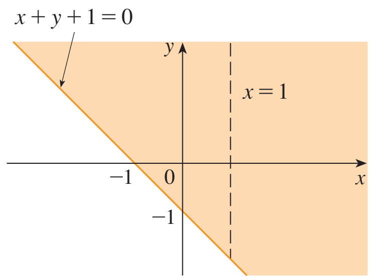


FIGURE 2
Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

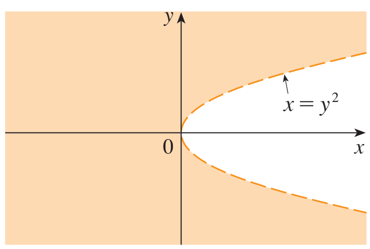


FIGURE 3
Domain of $f(x, y) = x \ln(y^2 - x)$

above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain. (See Figure 2.)

(b)
$$f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$. (See Figure 3.) ■

Not all functions can be represented by explicit formulas. The function in the next example is described verbally and by numerical estimates of its values.

The Wind-Chill Index
The wind-chill index measures how cold it feels when it's windy. It is based on a model of how fast a human face loses heat. It was developed through clinical trials in which volunteers were exposed to a variety of temperatures and wind speeds in a refrigerated wind tunnel.

EXAMPLE 2 In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write $W = f(T, v)$. Table 1 records values of W compiled by the US National Weather Service and the Meteorological Service of Canada.

Table 1 Wind-chill index as a function of air temperature and wind speed

Wind speed (km/h); rows: Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For instance, the table shows that if the temperature is $-5°C$ and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about $-15°C$ with no wind. So

$$f(-5, 50) = -15$$ ■

EXAMPLE 3 In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate. The function they used to model production was of the form

1
$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is

Table 2

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

the amount of capital invested (the monetary worth of all machinery, equipment, and buildings). In Section 14.3 we will show how the form of Equation 1 follows from certain economic assumptions.

Cobb and Douglas used economic data published by the government to obtain Table 2. They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

$$P(L, K) = 1.01L^{0.75}K^{0.25} \tag{2}$$

(See Exercise 81 for the details.)

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$
$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function (1) has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) \mid L \geqslant 0, K \geqslant 0\}$ because L and K represent labor and capital and are therefore never negative. ■

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center (0, 0) and radius 3. (See Figure 4.) The range of g is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\right\}$$

Since z is a positive square root, $z \geqslant 0$. Also, because $9 - x^2 - y^2 \leqslant 9$, we have

$$\sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$ ■

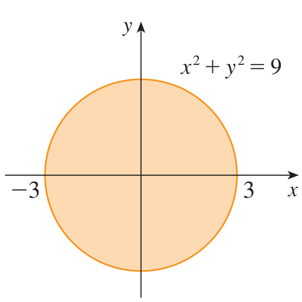


FIGURE 4
Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

> **Definition** If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

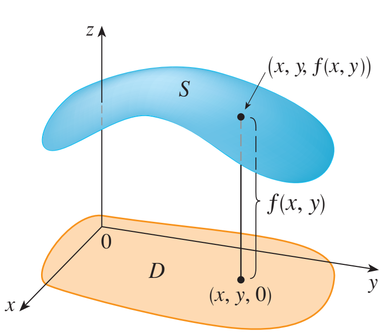


FIGURE 5

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

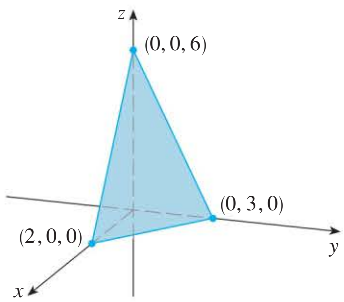


FIGURE 6

EXAMPLE 5 Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

SOLUTION The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant in Figure 6. ■

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

which is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

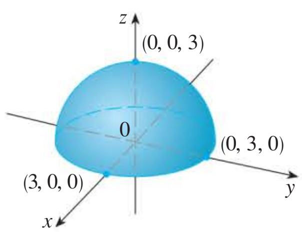


FIGURE 7
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geqslant 0$, the graph of g is just the top half of this sphere (see Figure 7). ■

NOTE An entire sphere can't be represented by a single function of x and y. As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

EXAMPLE 7 Use a computer to draw the graph of the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$.

SOLUTION Figure 8 shows the graph of P for values of the labor L and capital K that lie between 0 and 300. The computer has drawn the surface by plotting vertical traces. We see from these traces that the value of the production P increases as either L or K increases, as is to be expected.

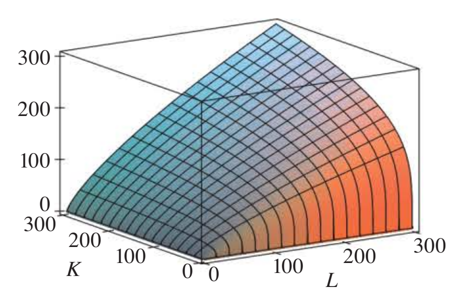


FIGURE 8

■

EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^2$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geqslant 0$ and $y^2 \geqslant 0$, so $h(x, y) \geqslant 0$ for all x

and y.] The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 12.6.4. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

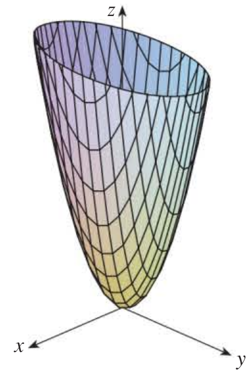
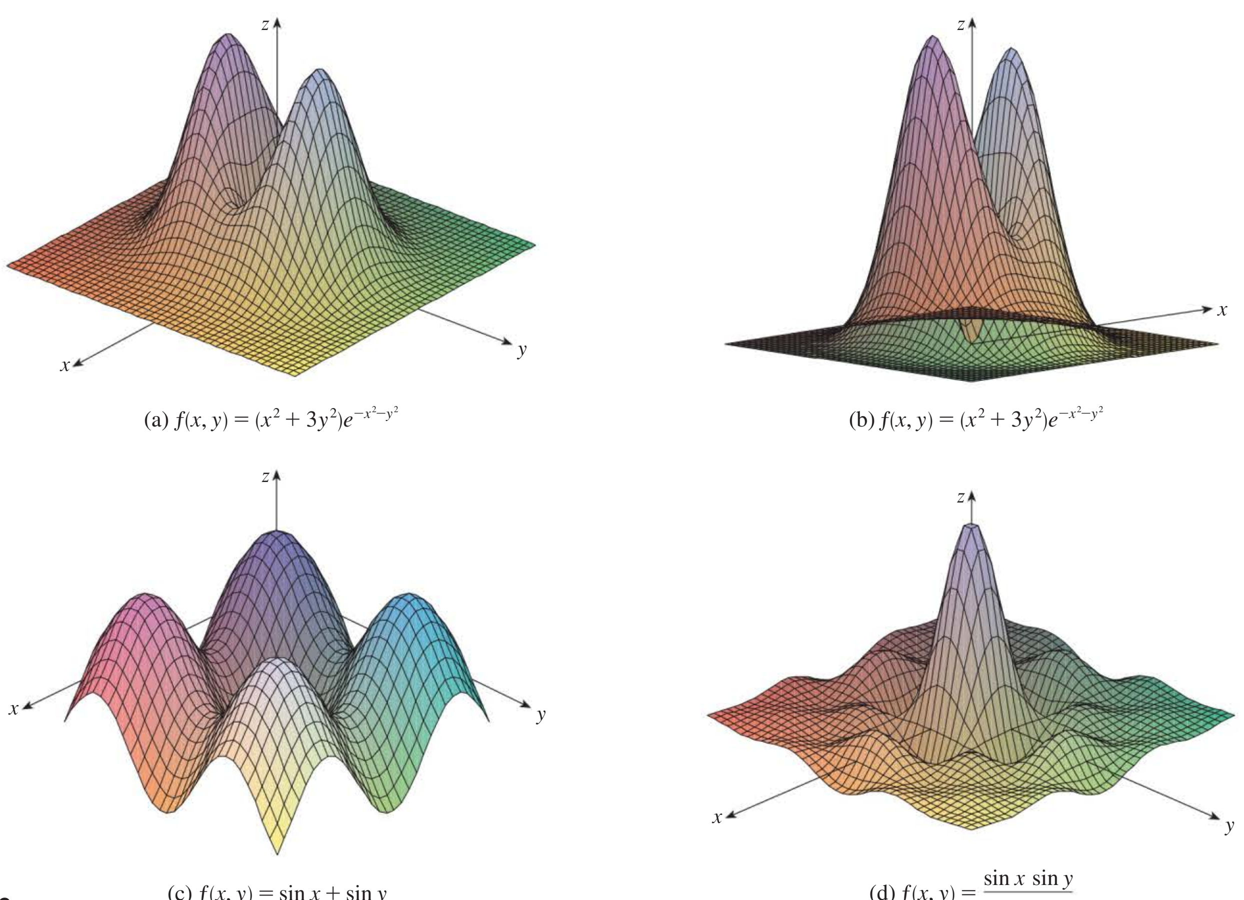


FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$

Computer programs are readily available for graphing functions of two variables. In most such programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.

Figure 10 shows computer-generated graphs of several functions. Notice that we get an especially good picture of a function when rotation is used to give views from dif-



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(c) $f(x, y) = \sin x + \sin y$

(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

FIGURE 10

ferent vantage points. In parts (a) and (b) the graph of f is very flat and close to the xy-plane except near the origin; this is because $e^{-x^2-y^2}$ is very small when x or y is large.

Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves*, or *level curves*.

> **Definition** The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

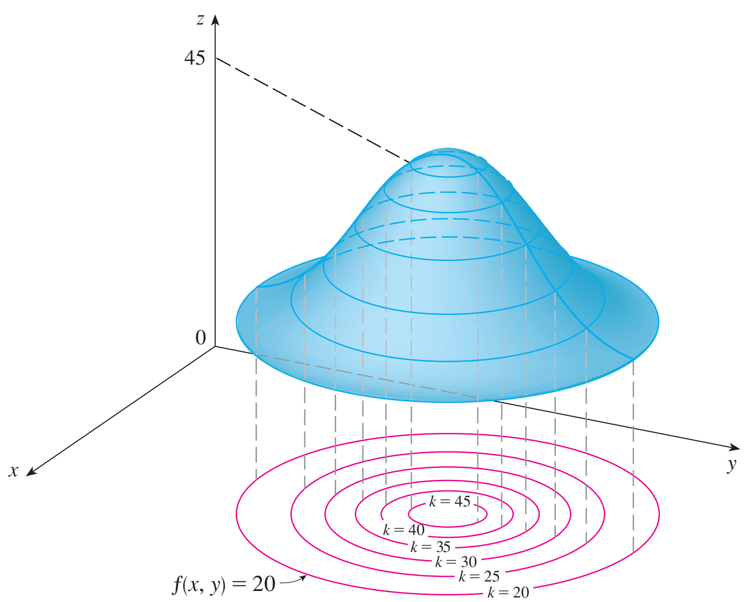


FIGURE 11

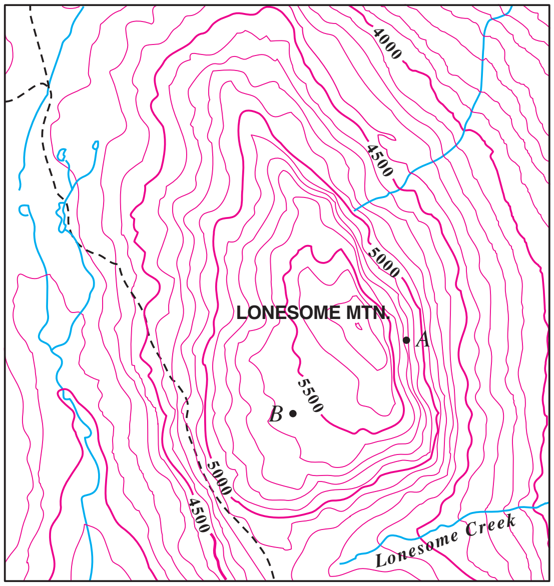


FIGURE 12

TEC Visual 14.1A animates Figure 11 by showing level curves being lifted up to graphs of functions.

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12. The level curves are curves of constant elevation above sea level. If you walk along one of these contour lines, you neither ascend nor descend. Another common example is the temperature function introduced in the opening paragraph of this section. Here the level curves are called **isothermals** and join loca-

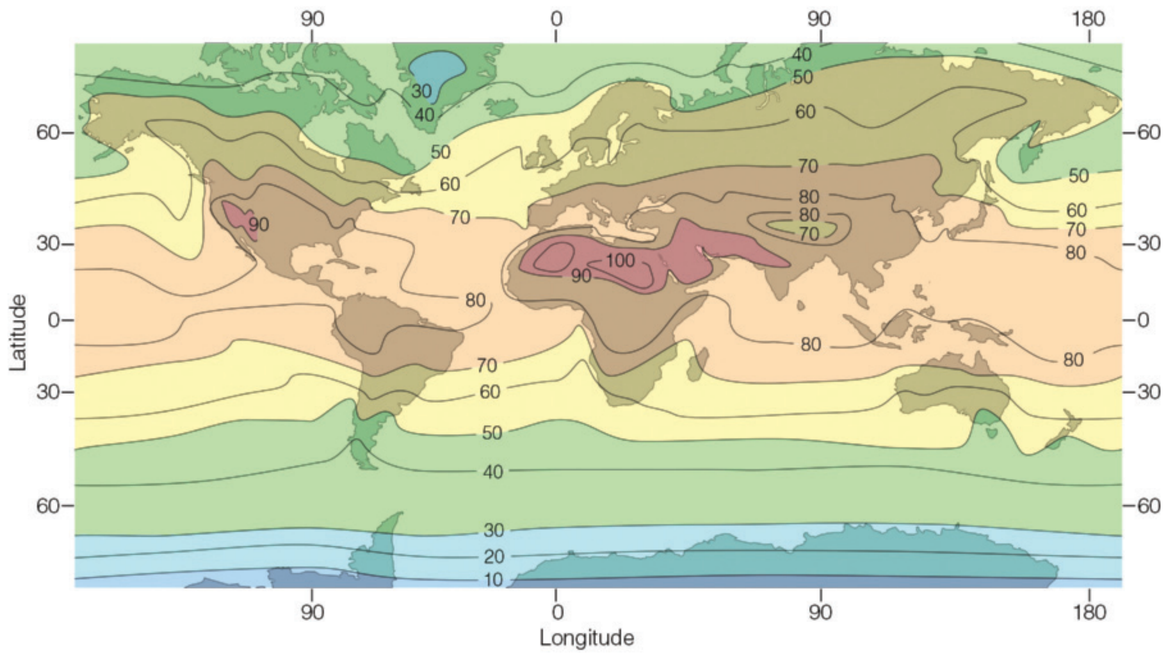


FIGURE 13 Average air temperature near sea level in July (°F)

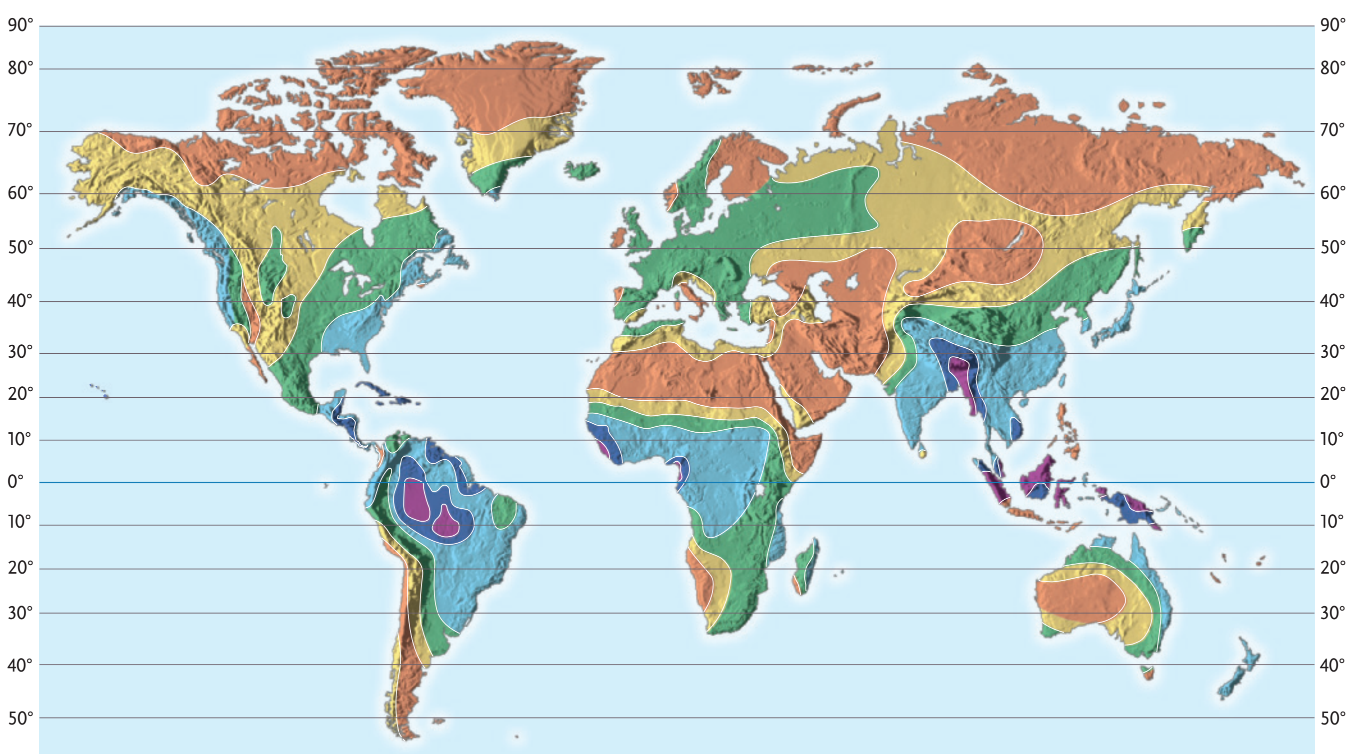


FIGURE 14
Precipitation

KEY

Precipitation (cm/yr)

Under 25	50 to 100	200 to 250
25 to 50	100 to 200	Over 250

tions with the same temperature. Figure 13 shows a weather map of the world indicating the average July temperatures. The isothermals are the curves that separate the colored bands.

In weather maps of atmospheric pressure at a given time as a function of longitude and latitude, the level curves are called **isobars** and join locations with the same pressure. (See Exercise 34.) Surface winds tend to flow from areas of high pressure across the isobars toward areas of low pressure, and are strongest where the isobars are tightly packed.

A contour map of world-wide precipitation is shown in Figure 14. Here the level curves are not labeled but they separate the colored regions and the amount of precipitation in each region is indicated in the color key.

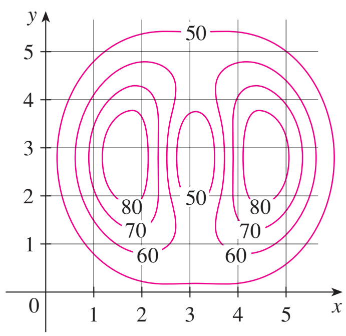


FIGURE 15

EXAMPLE 9 A contour map for a function f is shown in Figure 15. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

SOLUTION The point $(1, 3)$ lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that $\qquad f(4, 5) \approx 56$ ■

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 16. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

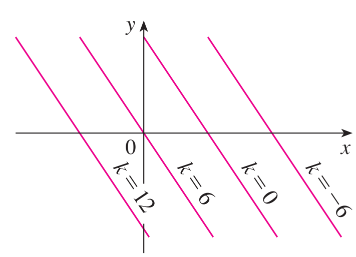


FIGURE 16
Contour map of
$f(x, y) = 6 - 3x - 2y$

■

EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \qquad \text{or} \qquad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases

$k = 0, 1, 2, 3$ are shown in Figure 17. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 14.1A.)

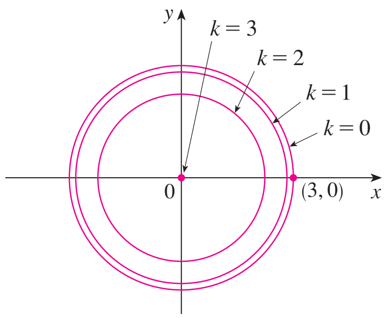


FIGURE 17
Contour map of
$g(x, y) = \sqrt{9 - x^2 - y^2}$

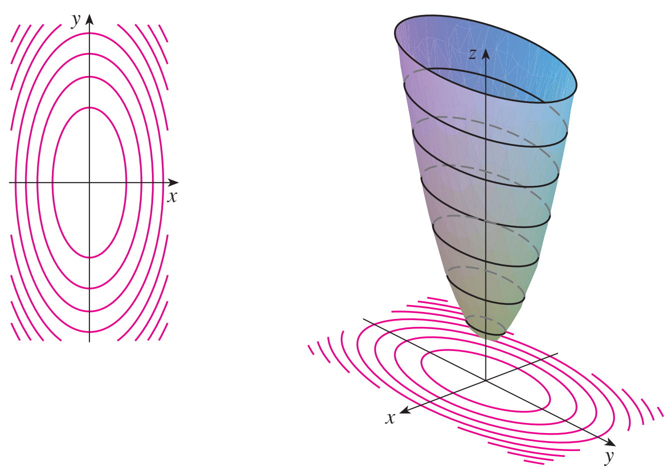
EXAMPLE 12 Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

SOLUTION The level curves are

$$4x^2 + y^2 + 1 = k \qquad \text{or} \qquad \frac{x^2}{\frac{1}{4}(k - 1)} + \frac{y^2}{k - 1} = 1$$

which, for $k > 1$, describes a family of ellipses with semiaxes $\frac{1}{2}\sqrt{k - 1}$ and $\sqrt{k - 1}$. Figure 18(a) shows a contour map of h drawn by a computer. Figure 18(b) shows these level curves lifted up to the graph of h (an elliptic paraboloid) where they become horizontal traces. We see from Figure 18 how the graph of h is put together from the level curves.

TEC Visual 14.1B demonstrates the connection between surfaces and their contour maps.



(a) Contour map

(b) Horizontal traces are raised level curves

FIGURE 18
The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

EXAMPLE 13 Plot level curves for the Cobb-Douglas production function of Example 3.

SOLUTION In Figure 19 we use a computer to draw a contour plot for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

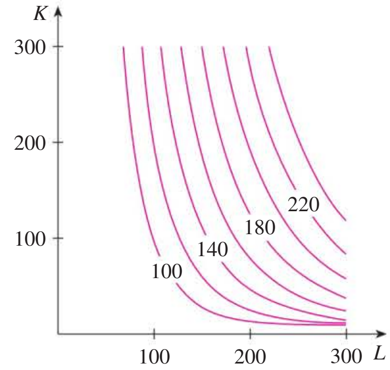
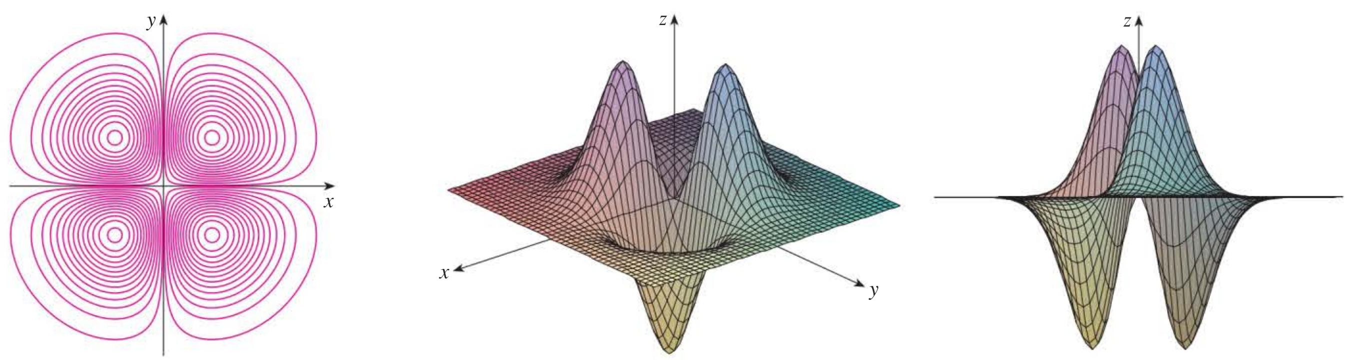


FIGURE 19

Level curves are labeled with the value of the production P. For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P, as L increases K decreases, and vice versa. ■

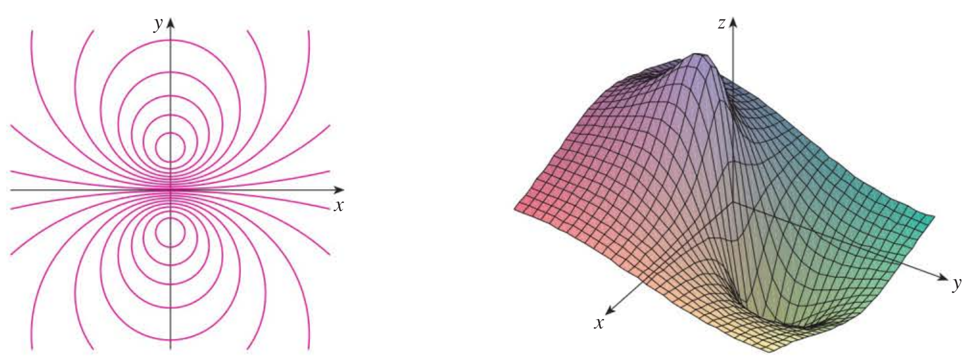
For some purposes, a contour map is more useful than a graph. That is certainly true in Example 13. (Compare Figure 19 with Figure 8.) It is also true in estimating function values, as in Example 9.

Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs. Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.



(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

FIGURE 20

■ Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

EXAMPLE 14 Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

SOLUTION The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$. ■

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 15 Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

SOLUTION The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. (See Figure 21.) Thus, as (x, y, z) varies over any sphere with center O, the value of $f(x, y, z)$ remains fixed. ■

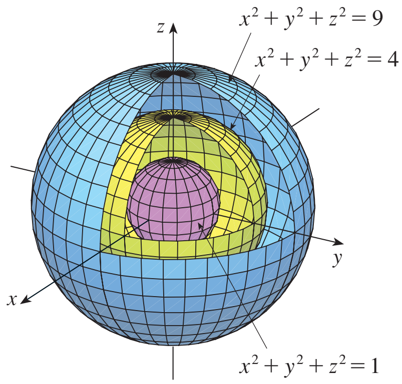


FIGURE 21

Functions of any number of variables can be considered. A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad \boxed{3}$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

14.1 EXERCISES

1. In Example 2 we considered the function $W = f(T, v)$, where W is the wind-chill index, T is the actual temperature, and v is the wind speed. A numerical representation is given in Table 1 on page 929.

(a) What is the value of $f(-15, 40)$? What is its meaning?

(b) Describe in words the meaning of the question "For what value of v is $f(-20, v) = -30$?" Then answer the question.

(c) Describe in words the meaning of the question "For what value of T is $f(T, 20) = -49$?" Then answer the question.

(d) What is the meaning of the function $W = f(-5, v)$? Describe the behavior of this function.

(e) What is the meaning of the function $W = f(T, 50)$? Describe the behavior of this function.

2. The *temperature-humidity index* I (or humidex, for short) is the perceived air temperature when the actual temperature is T and the relative humidity is h, so we can write $I = f(T, h)$. The following table of values of I is an excerpt from a table compiled by the National Oceanic & Atmospheric Administration.

Table 3 Apparent temperature as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F)

T \ h	20	30	40	50	60	70
80	77	78	79	81	82	83
85	82	84	86	88	90	93
90	87	90	93	96	100	106
95	93	96	101	107	114	124
100	99	104	110	120	132	144

(a) What is the value of $f(95, 70)$? What is its meaning?

(b) For what value of h is $f(90, h) = 100$?

(c) For what value of T is $f(T, 50) = 88$?

(d) What are the meanings of the functions $I = f(80, h)$ and $I = f(100, h)$? Compare the behavior of these two functions of h.

3. A manufacturer has modeled its yearly production function P (the monetary value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Find $P(120, 20)$ and interpret it.

4. Verify for the Cobb-Douglas production function

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

discussed in Example 3 that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

5. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.

(a) Find $f(160, 70)$ and interpret it.

(b) What is your own surface area?

6. The wind-chill index W discussed in Example 2 has been modeled by the following function:

$$W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Check to see how closely this model agrees with the values in Table 1 for a few values of T and v.

7. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in Table 4.

(a) What is the value of $f(40, 15)$? What is its meaning?

(b) What is the meaning of the function $h = f(30, t)$? Describe the behavior of this function.

(c) What is the meaning of the function $h = f(v, 30)$? Describe the behavior of this function.

Table 4

Duration (hours)

Wind speed (knots)

v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

8. A company makes three sizes of cardboard boxes: small, medium, and large. It costs \$2.50 to make a small box,

\$4.00 for a medium box, and \$4.50 for a large box. Fixed costs are \$8000.

(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.

(b) Find $f(3000, 5000, 4000)$ and interpret it.

(c) What is the domain of f?

9. Let $g(x, y) = \cos(x + 2y)$.

(a) Evaluate $g(2, -1)$.

(b) Find the domain of g.

(c) Find the range of g.

10. Let $F(x, y) = 1 + \sqrt{4 - y^2}$.

(a) Evaluate $F(3, 1)$.

(b) Find and sketch the domain of F.

(c) Find the range of F.

11. Let $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} + \ln(4 - x^2 - y^2 - z^2)$.

(a) Evaluate $f(1, 1, 1)$.

(b) Find and describe the domain of f.

12. Let $g(x, y, z) = x^3 y^2 z \sqrt{10 - x - y - z}$.

(a) Evaluate $g(1, 2, 3)$.

(b) Find and describe the domain of g.

13–22 Find and sketch the domain of the function.

13. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

14. $f(x, y) = \sqrt[4]{x - 3y}$

15. $f(x, y) = \ln(9 - x^2 - 9y^2)$

16. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

17. $g(x, y) = \dfrac{x - y}{x + y}$

18. $g(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$

19. $f(x, y) = \dfrac{\sqrt{y - x^2}}{1 - x^2}$

20. $f(x, y) = \sin^{-1}(x + y)$

21. $f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$

22. $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$

23–31 Sketch the graph of the function.

23. $f(x, y) = y$

24. $f(x, y) = x^2$

25. $f(x, y) = 10 - 4x - 5y$

26. $f(x, y) = \cos y$

27. $f(x, y) = \sin x$

28. $f(x, y) = 2 - x^2 - y^2$

29. $f(x, y) = x^2 + 4y^2 + 1$

30. $f(x, y) = \sqrt{4x^2 + y^2}$

31. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

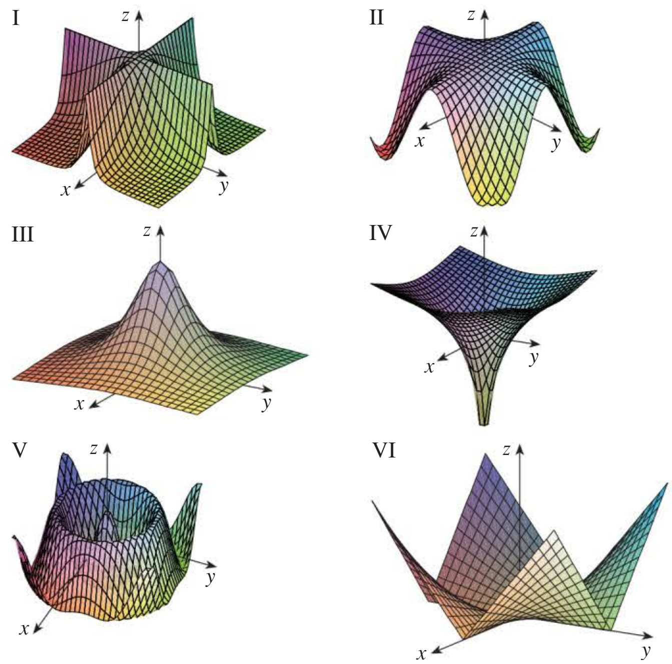
(b) $f(x, y) = \dfrac{1}{1 + x^2 y^2}$

(c) $f(x, y) = \ln(x^2 + y^2)$

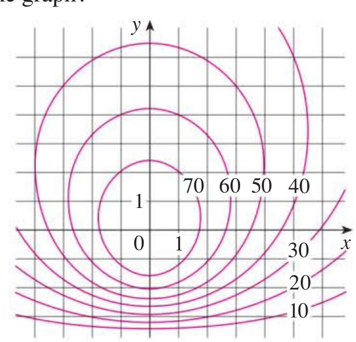
(d) $f(x, y) = \cos\sqrt{x^2 + y^2}$

(e) $f(x, y) = |xy|$

(f) $f(x, y) = \cos(xy)$



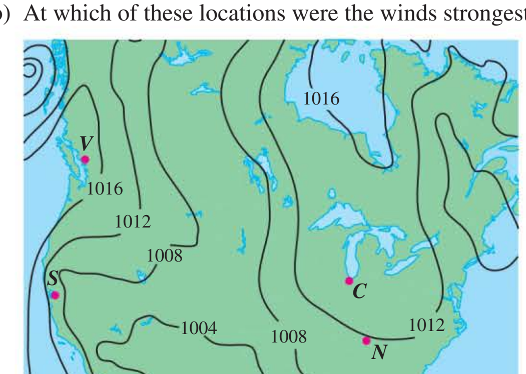
33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?



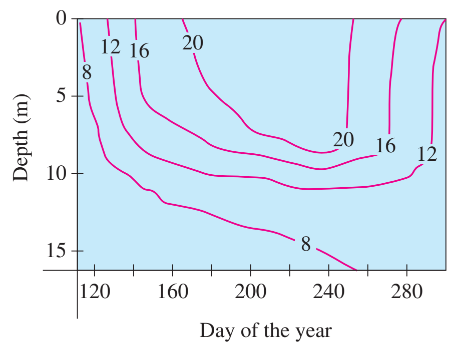
34. Shown is a contour map of atmospheric pressure in North America on August 12, 2008. On the level curves (called isobars) the pressure is indicated in millibars (mb).

(a) Estimate the pressure at C (Chicago), N (Nashville), S (San Francisco), and V (Vancouver).

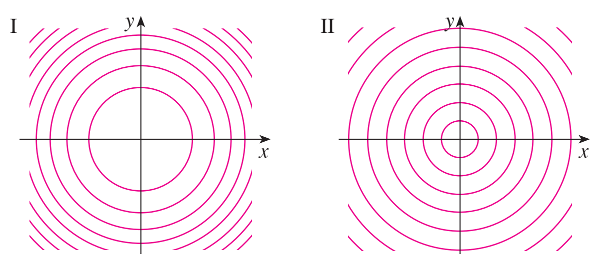
(b) At which of these locations were the winds strongest?



35. Level curves (isothermals) are shown for the typical water temperature (in °C) in Long Lake (Minnesota) as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.

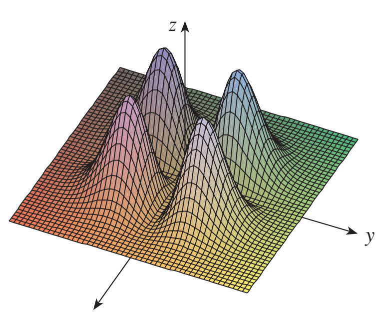


36. Two contour maps are shown. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



37. Locate the points A and B on the map of Lonesome Mountain (Figure 12). How would you describe the terrain near A? Near B?

38. Make a rough sketch of a contour map for the function whose graph is shown.



39. The *body mass index* (BMI) of a person is defined by

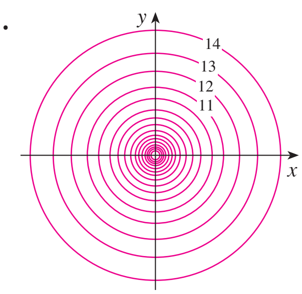
$$B(m, h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the height (in meters). Draw the level curves $B(m, h) = 18.5$, $B(m, h) = 25$, $B(m, h) = 30$, and $B(m, h) = 40$. A rough guideline is that a person is underweight if the BMI is less than 18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Shade the region corresponding to optimal BMI. Does someone who weighs 62 kg and is 152 cm tall fall into this category?

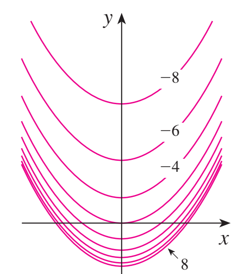
40. The body mass index is defined in Exercise 39. Draw the level curve of this function corresponding to someone who is 200 cm tall and weighs 80 kg. Find the weights and heights of two other people with that same level curve.

41–44 A contour map of a function is shown. Use it to make a rough sketch of the graph of f.

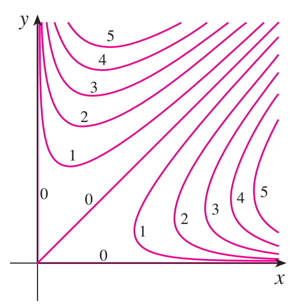
41.



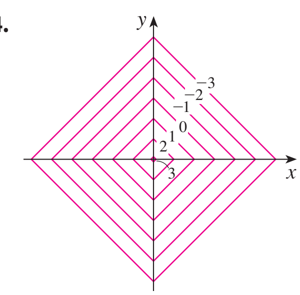
42.



43.



44.



45–52 Draw a contour map of the function showing several level curves.

45. $f(x, y) = x^2 - y^2$

46. $f(x, y) = xy$

47. $f(x, y) = \sqrt{x} + y$

48. $f(x, y) = \ln(x^2 + 4y^2)$

49. $f(x, y) = ye^x$

50. $f(x, y) = y - \arctan x$

51. $f(x, y) = \sqrt[3]{x^2 + y^2}$

52. $f(x, y) = y/(x^2 + y^2)$

53–54 Sketch both a contour map and a graph of the function and compare them.

53. $f(x, y) = x^2 + 9y^2$

54. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

55. A thin metal plate, located in the xy-plane, has temperature $T(x, y)$ at the point (x, y). Sketch some level curves (isothermals) if the temperature function is given by

$$T(x, y) = \frac{100}{1 + x^2 + 2y^2}$$

56. If $V(x, y)$ is the electric potential at a point (x, y) in the xy-plane, then the level curves of V are called *equipotential curves* because at all points on such a curve the electric potential is the same. Sketch some equipotential curves if $V(x, y) = c/\sqrt{r^2 - x^2 - y^2}$, where c is a positive constant.

57–60 Use a computer to graph the function using various domains and viewpoints. Get a printout of one that, in your opinion, gives a good view. If your software also produces level curves, then plot some contour lines of the same function and compare with the graph.

57. $f(x, y) = xy^2 - x^3$ (monkey saddle)

58. $f(x, y) = xy^3 - yx^3$ (dog saddle)

59. $f(x, y) = e^{-(x^2+y^2)/3}(\sin(x^2) + \cos(y^2))$

60. $f(x, y) = \cos x \cos y$

61–66 Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.

61. $z = \sin(xy)$

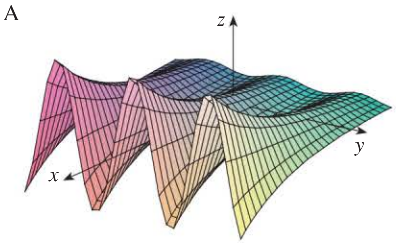
62. $z = e^x \cos y$

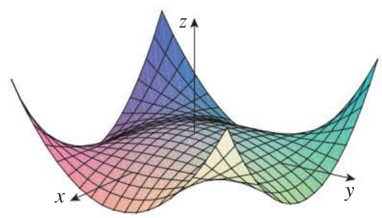
63. $z = \sin(x - y)$

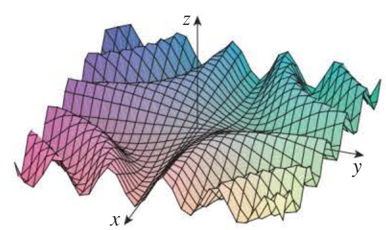
64. $z = \sin x - \sin y$

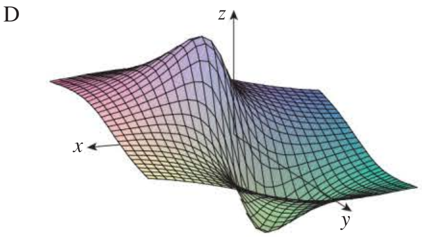
65. $z = (1 - x^2)(1 - y^2)$

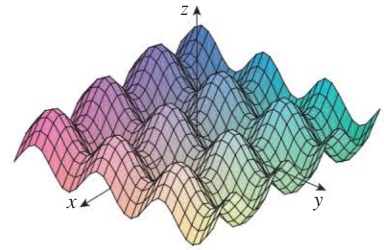
66. $z = \dfrac{x - y}{1 + x^2 + y^2}$

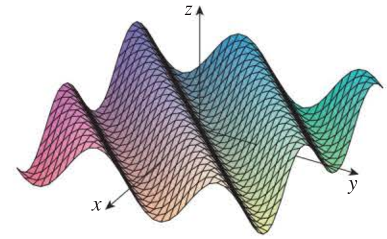
A


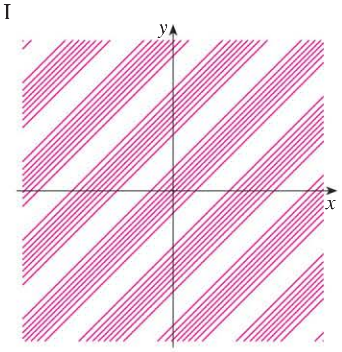
B


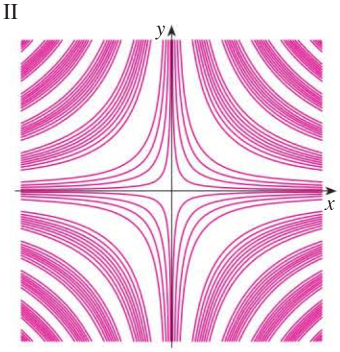
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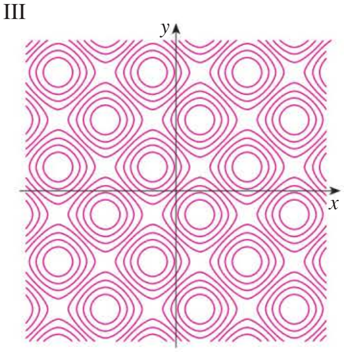
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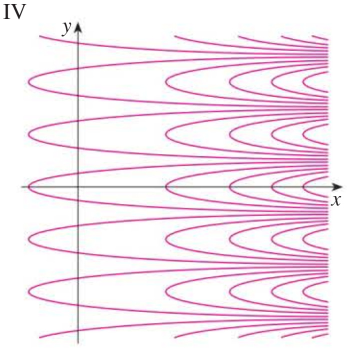
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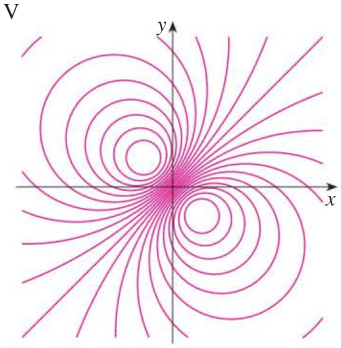
F


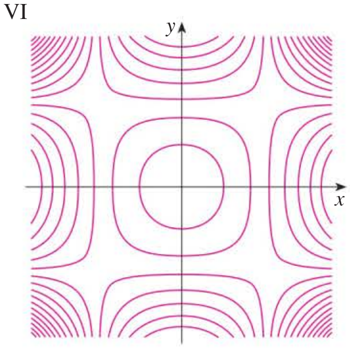
I


II


III


IV


V


VI


67–70 Describe the level surfaces of the function.

67. $f(x, y, z) = x + 3y + 5z$

68. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

69. $f(x, y, z) = y^2 + z^2$

70. $f(x, y, z) = x^2 - y^2 - z^2$

71–72 Describe how the graph of g is obtained from the graph of f.

71. (a) $g(x, y) = f(x, y) + 2$
(b) $g(x, y) = 2f(x, y)$
(c) $g(x, y) = -f(x, y)$
(d) $g(x, y) = 2 - f(x, y)$

72. (a) $g(x, y) = f(x - 2, y)$
(b) $g(x, y) = f(x, y + 2)$
(c) $g(x, y) = f(x + 3, y - 4)$

73–74 Use a computer to graph the function using various domains and viewpoints. Get a printout that gives a good view of the "peaks and valleys." Would you say the function has a maximum value? Can you identify any points on the graph that you might consider to be "local maximum points"? What about "local minimum points"?

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$

74. $f(x, y) = xye^{-x^2-y^2}$

75–76 Graph the function using various domains and viewpoints. Comment on the limiting behavior of the function. What happens as both x and y become large? What happens as (x, y) approaches the origin?

75. $f(x, y) = \dfrac{x + y}{x^2 + y^2}$ **76.** $f(x, y) = \dfrac{xy}{x^2 + y^2}$

77. Investigate the family of functions $f(x, y) = e^{cx^2+y^2}$. How does the shape of the graph depend on c?

78. Use a computer to investigate the family of surfaces

$$z = (ax^2 + by^2)e^{-x^2-y^2}$$

How does the shape of the graph depend on the numbers a and b?

79. Use a computer to investigate the family of surfaces $z = x^2 + y^2 + cxy$. In particular, you should determine the transitional values of c for which the surface changes from one type of quadric surface to another.

80. Graph the functions

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = e^{\sqrt{x^2+y^2}}$$

$$f(x, y) = \ln\sqrt{x^2 + y^2}$$

$$f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$$

and

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if g is a function of one variable, how is the graph of

$$f(x, y) = g\left(\sqrt{x^2 + y^2}\right)$$

obtained from the graph of g?

81. (a) Show that, by taking logarithms, the general Cobb-Douglas function $P = bL^\alpha K^{1-\alpha}$ can be expressed as

$$\ln \frac{P}{K} = \ln b + \alpha \ln \frac{L}{K}$$

(b) If we let $x = \ln(L/K)$ and $y = \ln(P/K)$, the equation in part (a) becomes the linear equation $y = \alpha x + \ln b$. Use Table 2 (in Example 3) to make a table of values of $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. Then use a graphing calculator or computer to find the least squares regression line through the points $(\ln(L/K), \ln(P/K))$.

(c) Deduce that the Cobb-Douglas production function is $P = 1.01L^{0.75}K^{0.25}$.

14.2 Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \qquad \text{and} \qquad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 Values of $f(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f. In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x\to a \\ y\to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by

making the distance from (x, y) to (a, b) sufficiently small (but not 0). Figure 1 illustrates Definition 1 by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

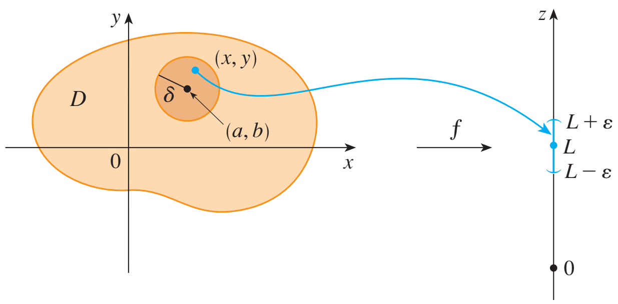


FIGURE 1

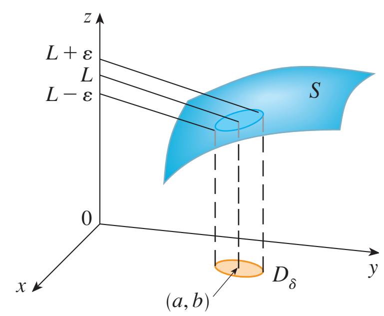


FIGURE 2

Another illustration of Definition 1 is given in Figure 2 where the surface S is the graph of f. If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$.

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. We recall from Chapter 1 that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

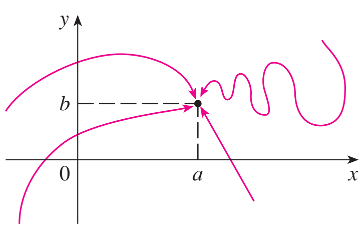


FIGURE 3

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

> If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

EXAMPLE 1 Show that $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

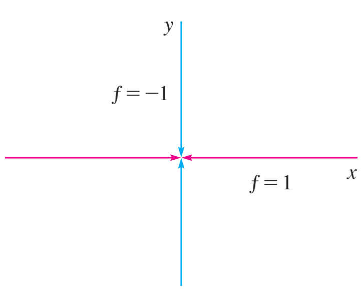


FIGURE 4

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.) Since f has two different limits along two different lines, the given limit

does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.) ■

EXAMPLE 2 If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x, y)\to(0,0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

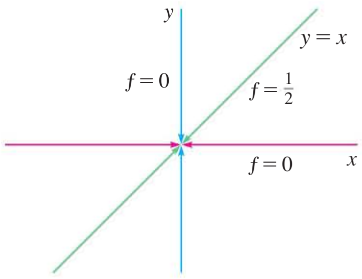


FIGURE 5

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach (0, 0) along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\qquad f(x, y) \to \frac{1}{2} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ■

Figure 6 sheds some light on Example 2. The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = \frac{1}{2}$ for all points (x, y) on that line except the origin.

TEC In Visual 14.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

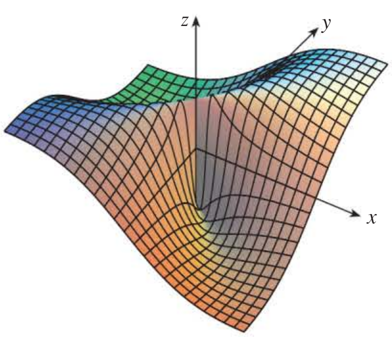


FIGURE 6

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\lim_{(x, y)\to(0,0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any line through the origin. If the line is not the y-axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $\qquad f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = mx$

We get the same result as $(x, y) \to (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show that the given

Figure 7 shows the graph of the function in Example 3. Notice the ridge above the parabola $x = y^2$.

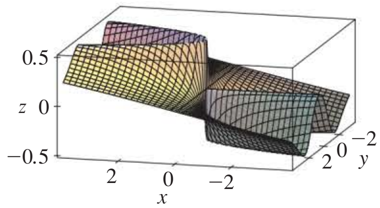


FIGURE 7

limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $\qquad f(x, y) \to \frac{1}{2} \qquad$ as $\qquad (x, y) \to (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist. ■

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 1.6 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

2 $$\lim_{(x, y) \to (a, b)} x = a \qquad \lim_{(x, y) \to (a, b)} y = b \qquad \lim_{(x, y) \to (a, b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE 4 Find $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{3x^2 y}{x^2 + y^2}$ if it exists.

SOLUTION As in Example 3, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| < \varepsilon$$

that is, $$\text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta \quad \text{then} \quad \frac{3x^2 |y|}{x^2 + y^2} < \varepsilon$$

But $x^2 \leqslant x^2 + y^2$ since $y^2 \geqslant 0$, so $x^2/(x^2 + y^2) \leqslant 1$ and therefore

3 $$\frac{3x^2 |y|}{x^2 + y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2 + y^2}$$

Thus if we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2 y}{x^2 + y^2} - 0 \right| \leqslant 3\sqrt{x^2 + y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by Definition 1,

$$\lim_{(x, y) \to (0, 0)} \frac{3x^2 y}{x^2 + y^2} = 0$$ ■

Another way to do Example 4 is to use the Squeeze Theorem instead of Definition 1. From (2) it follows that

$$\lim_{(x, y) \to (0, 0)} 3|y| = 0$$

and so the first inequality in (3) shows that the given limit is 0.

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy. It can be accomplished by direct substitution because the defining property of a continuous

function is $\lim_{x \to a} f(x) = f(a)$. Continuous functions of two variables are also defined by the direct substitution property.

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Let's use this fact to give examples of continuous functions.

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials. For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

The limits in (2) show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

EXAMPLE 5 Evaluate $\displaystyle\lim_{(x, y) \to (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

SOLUTION Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x, y) \to (1, 2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$ ■

EXAMPLE 6 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at (0, 0) because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y)\to(0, 0)} g(x, y)$ does not exist (see Example 1). ■

Figure 8 shows the graph of the continuous function in Example 8.

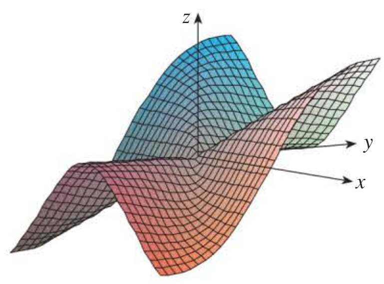


FIGURE 8

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there. Also, from Example 4, we have

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{(x, y)\to(0, 0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0)$$

Therefore f is continuous at $(0, 0)$, and so it is continuous on $\mathbb{R}^2$. ■

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

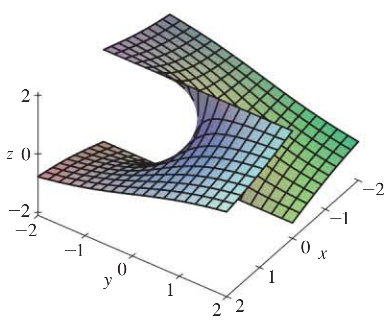


FIGURE 9
The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE 9 Where is the function $h(x, y) = \arctan(y/x)$ continuous?

SOLUTION The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x, y)) = \arctan(y/x) = h(x, y)$$

is continuous except where $x = 0$. The graph in Figure 9 shows the break in the graph of h above the y-axis. ■

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$$

$$\text{then } \quad |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z)\to(a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and (5) is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y \rangle$, $\mathbf{a} = \langle a, b \rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x - a)^2 + (y - b)^2}$, so (5) becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a, b, c \rangle$, and (5) becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 EXERCISES

1. Suppose that $\lim_{(x, y) \to (3, 1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?

2. Explain why each function is continuous or discontinuous.
(a) The outdoor temperature as a function of longitude, latitude, and time
(b) Elevation (height above sea level) as a function of longitude, latitude, and time
(c) The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Then explain why your guess is correct.

3. $f(x, y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

4. $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$

5–22 Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x, y) \to (3, 2)} (x^2y^3 - 4y^2)$

6. $\lim_{(x, y) \to (2, -1)} \dfrac{x^2y + xy^2}{x^2 - y^2}$

7. $\lim_{(x, y) \to (\pi, \pi/2)} y \sin(x - y)$

8. $\lim_{(x, y) \to (3, 2)} e^{\sqrt{2x - y}}$

9. $\lim_{(x, y) \to (0, 0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$

10. $\lim_{(x, y) \to (0, 0)} \dfrac{5y^4 \cos^2 x}{x^4 + y^4}$

11. $\lim_{(x, y) \to (0, 0)} \dfrac{y^2 \sin^2 x}{x^4 + y^4}$

12. $\lim_{(x, y) \to (1, 0)} \dfrac{xy - y}{(x - 1)^2 + y^2}$

13. $\lim_{(x, y) \to (0, 0)} \dfrac{xy}{\sqrt{x^2 + y^2}}$

14. $\lim_{(x, y) \to (0, 0)} \dfrac{x^3 - y^3}{x^2 + xy + y^2}$

15. $\lim_{(x, y) \to (0, 0)} \dfrac{xy^2 \cos y}{x^2 + y^4}$

16. $\lim_{(x, y) \to (0, 0)} \dfrac{xy^4}{x^4 + y^4}$

17. $\lim_{(x, y) \to (0, 0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

18. $\lim_{(x, y) \to (0, 0)} \dfrac{xy^4}{x^2 + y^8}$

19. $\lim_{(x, y, z) \to (\pi, 0, 1/3)} e^{y^2} \tan(xz)$

20. $\lim_{(x, y, z) \to (0, 0, 0)} \dfrac{xy + yz}{x^2 + y^2 + z^2}$

21. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

22. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$

23–24 Use a computer graph of the function to explain why the limit does not exist.

23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ **24.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

25–26 Find $h(x, y) = g(f(x, y))$ and the set of points at which h is continuous.

25. $g(t) = t^2 + \sqrt{t}$, $f(x, y) = 2x + 3y - 6$

26. $g(t) = t + \ln t$, $f(x, y) = \dfrac{1 - xy}{1 + x^2y^2}$

27–28 Graph the function and observe where it is discontinuous. Then use the formula to explain what you have observed.

27. $f(x, y) = e^{1/(x-y)}$ **28.** $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$

29–38 Determine the set of points at which the function is continuous.

29. $F(x, y) = \dfrac{xy}{1 + e^{x-y}}$ **30.** $F(x, y) = \cos\sqrt{1 + x - y}$

31. $F(x, y) = \dfrac{1 + x^2 + y^2}{1 - x^2 - y^2}$ **32.** $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

33. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

34. $G(x, y) = \ln(1 + x - y)$

35. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

36. $f(x, y, z) = \sqrt{y - x^2}\,\ln z$

37. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

38. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

39–41 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

39. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

40. $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$

41. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$

42. At the beginning of this section we considered the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and guessed on the basis of numerical evidence that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$. Use polar coordinates to confirm the value of the limit. Then graph the function.

43. Graph and discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

44. Let

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any path through $(0, 0)$ of the form $y = mx^a$ with $0 < a < 4$.
(b) Despite part (a), show that f is discontinuous at $(0, 0)$.
(c) Show that f is discontinuous on two entire curves.

45. Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$. [*Hint:* Consider $|\mathbf{x} - \mathbf{a}|^2 = (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.]

46. If $\mathbf{c} \in V_n$, show that the function f given by $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is continuous on $\mathbb{R}^n$.

14.3 Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined

effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F) T \\ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$, we are considering the heat index as a function of the single variable T for a fixed value of H. Let's write $g(T) = f(T, 70)$. Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%. The derivative of g when $T = 96°\text{F}$ is the rate of change of I with respect to T when $T = 96°\text{F}$:

$$g'(96) = \lim_{h \to 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \to 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

We can approximate $g'(96)$ using the values in Table 1 by taking $h = 2$ and -2:

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75. This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of $T = 96°\text{F}$. The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°\text{F}$. The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \to 0} \frac{G(70 + h) - G(70)}{h} = \lim_{h \to 0} \frac{f(96, 70 + h) - f(96, 70)}{h}$$

By taking $h = 5$ and -5, we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at (*a, b*)** and denote it by $f_x(a, b)$. Thus

1 $$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2 $$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of *f* with respect to *y* at (*a, b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96$°F and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$. But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 1 If $f(x, y) = x^3 + x^2 y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

SOLUTION Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2 y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$ ■

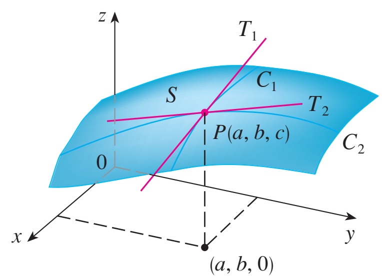


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

SOLUTION We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

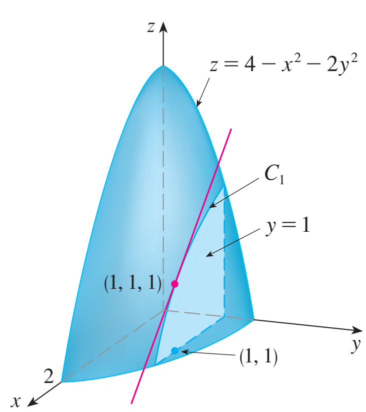


FIGURE 2

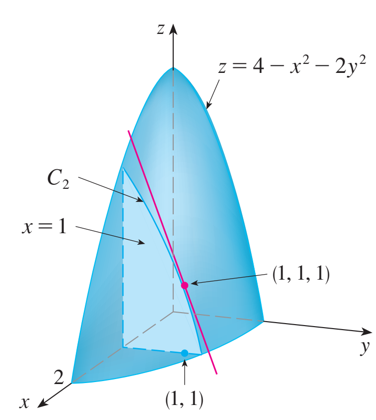


FIGURE 3

Figure 4 is a computer-drawn counterpart to Figure 2. Part (a) shows the plane $y = 1$ intersecting the surface to form the curve C_1 and part (b) shows C_1 and T_1. [We have used the vector equations $\mathbf{r}(t) = \langle t, 1, 2 - t^2 \rangle$ for C_1 and $\mathbf{r}(t) = \langle 1 + t, 1, 1 - 2t \rangle$ for T_1.] Similarly, Figure 5 corresponds to Figure 3.

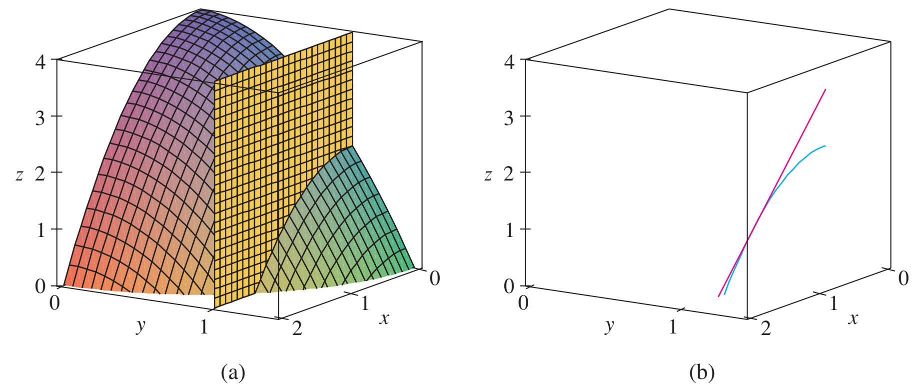


FIGURE 4

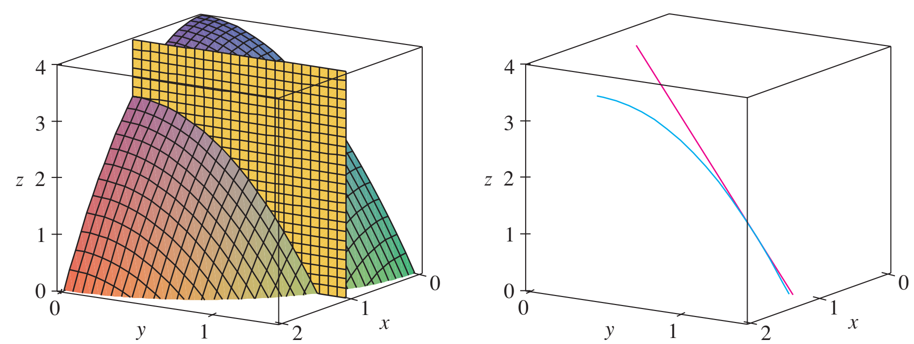


FIGURE 5

EXAMPLE 3 In Exercise 14.1.39 we defined the body mass index of a person as

$$B(m, h) = \frac{m}{h^2}$$

Calculate the partial derivatives of B for a young man with $m = 64$ kg and $h = 1.68$ m and interpret them.

SOLUTION Regarding h as a constant, we see that the partial derivative with respect to m is

$$\frac{\partial B}{\partial m}(m, h) = \frac{\partial}{\partial m}\left(\frac{m}{h^2}\right) = \frac{1}{h^2}$$

so
$$\frac{\partial B}{\partial m}(64, 1.68) = \frac{1}{(1.68)^2} \approx 0.35 \text{ (kg/m}^2\text{)/kg}$$

This is the rate at which the man's BMI increases with respect to his weight when he weighs 64 kg and his height is 1.68 m. So if his weight increases by a small amount, one kilogram for instance, and his height remains unchanged, then his BMI will increase by about 0.35.

Now we regard m as a constant. The partial derivative with respect to h is

$$\frac{\partial B}{\partial h}(m, h) = \frac{\partial}{\partial h}\left(\frac{m}{h^2}\right) = m\left(-\frac{2}{h^3}\right) = -\frac{2m}{h^3}$$

so

$$\frac{\partial B}{\partial h}(64, 1.68) = -\frac{2 \cdot 64}{(1.68)^3} \approx -27 \text{ (kg/m}^2\text{)/m}$$

This is the rate at which the man's BMI increases with respect to his height when he weighs 64 kg and his height is 1.68 m. So if the man is still growing and his weight stays unchanged while his height increases by a small amount, say 1 cm, then his BMI will *decrease* by about $27(0.01) = 0.27$. ■

EXAMPLE 4 If $f(x, y) = \sin\left(\dfrac{x}{1 + y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1 + y}\right) = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{1}{1 + y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1 + y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1 + y}\right) = -\cos\left(\frac{x}{1 + y}\right) \cdot \frac{x}{(1 + y)^2}$$ ■

Some computer software can plot surfaces defined by implicit equations in three variables. Figure 6 shows such a plot of the surface defined by the equation in Example 5.



FIGURE 6

EXAMPLE 5 Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

SOLUTION To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$ ■

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with

respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

EXAMPLE 6 Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

SOLUTION Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,
$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$
■

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\, \partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

EXAMPLE 7 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

SOLUTION In Example 1 we found that

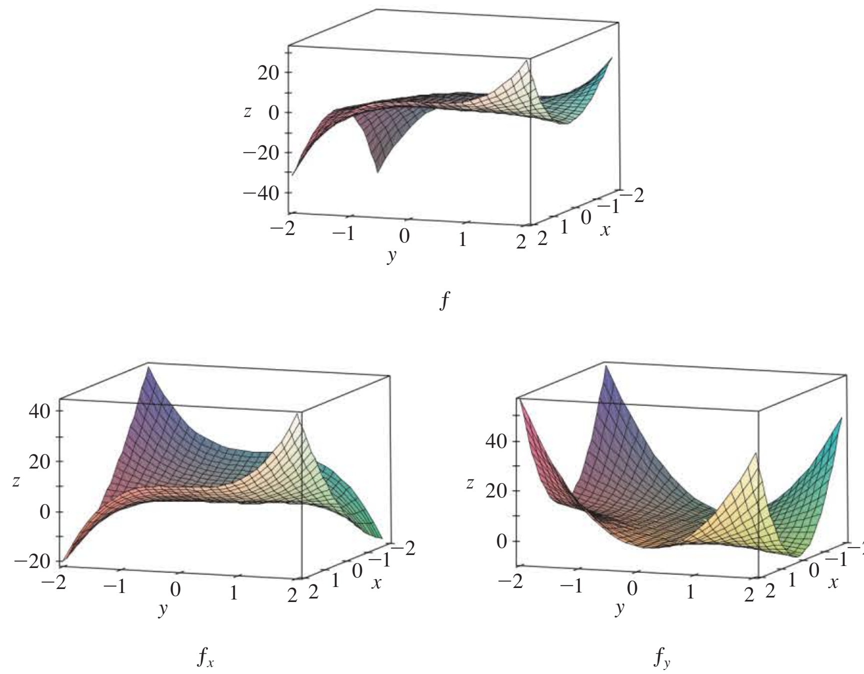
$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2 y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2 y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2 y^2 - 4y) = 6x^2 y - 4$$
■

Figure 7 shows the graph of the function f in Example 7 and the graphs of its first- and second-order partial derivatives for $-2 \leqslant x \leqslant 2$, $-2 \leqslant y \leqslant 2$. Notice that these graphs are consistent with our interpretations of f_x and f_y as slopes of tangent lines to traces of the graph of f. For instance, the graph of f decreases if we start at $(0, -2)$ and move in the positive x-direction. This is reflected in the negative values of f_x. You should compare the graphs of f_{yx} and f_{yy} with the graph of f_y to see the relationships.



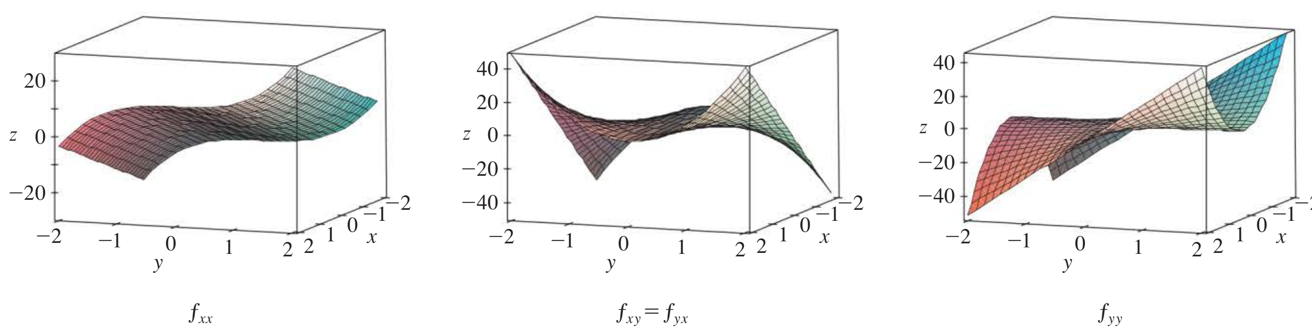


FIGURE 7

Notice that $f_{xy} = f_{yx}$ in Example 7. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice. The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$. The proof is given in Appendix F.

Clairaut

Alexis Clairaut was a child prodigy in mathematics: he read l'Hospital's textbook on calculus when he was ten and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure,* which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

EXAMPLE 8 Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

SOLUTION

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

■

Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

EXAMPLE 9 Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

SOLUTION We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

So

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation. ■

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 8), then $u(x, t)$ satisfies the wave equation. Here the constant a depends on the density of the string and on the tension in the string.

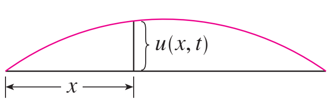


FIGURE 8

EXAMPLE 10 Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

SOLUTION

$$u_x = \cos(x - at) \qquad u_t = -a\cos(x - at)$$

$$u_{xx} = -\sin(x - at) \qquad u_{tt} = -a^2\sin(x - at) = a^2 u_{xx}$$

So u satisfies the wave equation. ■

Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

5
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one place it occurs is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z), then it satisfies Equation 5. The strength of the magnetic field indicates the distribution of iron-rich minerals and reflects different rock types and the location of faults. Figure 9 shows a contour map of the earth's magnetic field as recorded from an aircraft carrying a magnetometer and flying 200 m above the surface of the ground. The contour map is enhanced by color-coding of the regions between the level curves.

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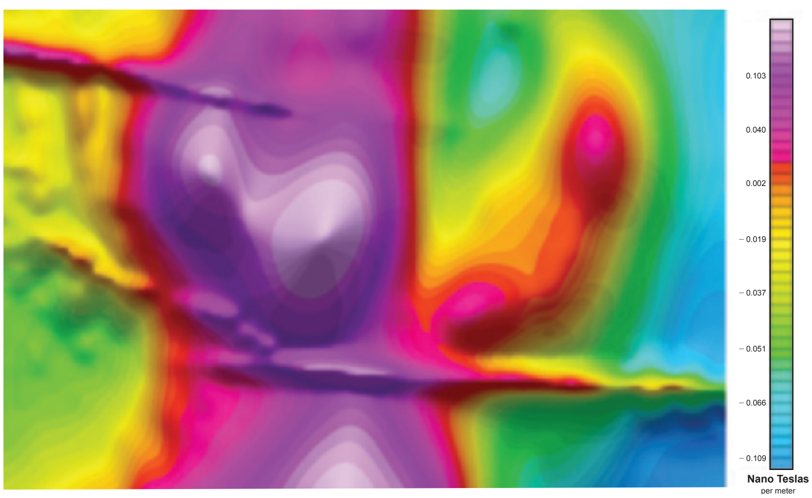


FIGURE 9
Magnetic field strength of the earth

Figure 10 shows a contour map for the second-order partial derivative of u in the vertical direction, that is, u_{zz}. It turns out that the values of the partial derivatives u_{xx} and u_{yy} are relatively easily measured from a map of the magnetic field. Then values of u_{zz} can be calculated from Laplace's equation (5).

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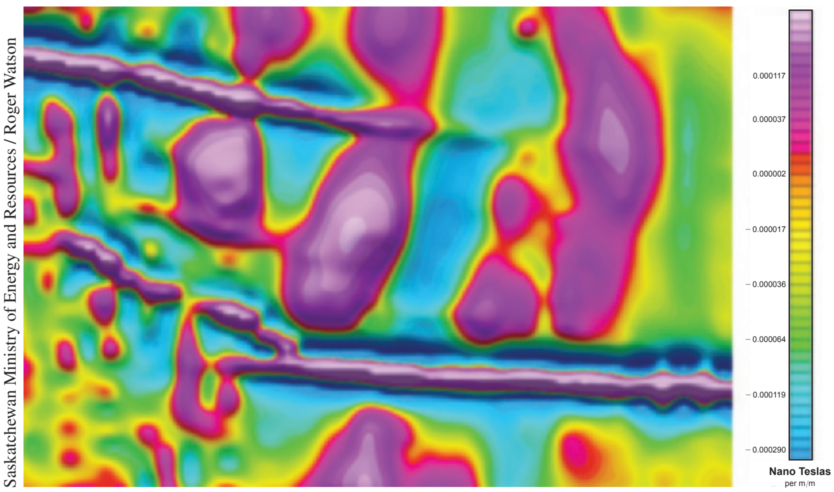


FIGURE 10
Second vertical derivative of the magnetic field

The Cobb-Douglas Production Function

In Example 14.1.3 we described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K. Here we use partial derivatives to show how the particular form of their model follows from certain assumptions they made about the economy.

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\partial P/\partial L$ is the rate at which production changes with respect to the amount of labor. Economists call it the marginal production with respect to labor or the **marginal productivity of labor**. Likewise, the partial derivative $\partial P/\partial K$ is the rate of change of production with respect to capital and is called the **marginal productivity of capital**. In these terms, the assumptions made by Cobb and Douglas can be stated as follows.

(i) If either labor or capital vanishes, then so will production.

(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.

(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

Because the production per unit of labor is P/L, assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α. If we keep K constant ($K = K_0$), then this partial differential equation becomes an ordinary differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L} \tag{6}$$

If we solve this separable differential equation by the methods of Section 9.3 (see also Exercise 85), we get

$$P(L, K_0) = C_1(K_0)L^{\alpha} \tag{7}$$

Notice that we have written the constant C_1 as a function of K_0 because it could depend on the value of K_0.

Similarly, assumption (iii) says that

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

and we can solve this differential equation to get

$$P(L_0, K) = C_2(L_0)K^{\beta} \tag{8}$$

Comparing Equations 7 and 8, we have

$$P(L, K) = bL^{\alpha}K^{\beta} \tag{9}$$

where b is a constant that is independent of both L and K. Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Notice from Equation 9 that if labor and capital are both increased by a factor m, then

$$P(mL, mK) = b(mL)^\alpha(mK)^\beta = m^{\alpha+\beta}bL^\alpha K^\beta = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m. That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

This is the Cobb-Douglas production function that we discussed in Section 14.1.

14.3 EXERCISES

1. The temperature T (in °C) at a location in the Northern Hemisphere depends on the longitude x, latitude y, and time t, so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$?

(b) Honolulu has longitude 158°W and latitude 21°N. Suppose that at 9:00 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(158, 21, 9)$, $f_y(158, 21, 9)$, and $f_t(158, 21, 9)$ to be positive or negative? Explain.

2. At the beginning of this section we discussed the function $I = f(T, H)$, where I is the heat index, T is the temperature, and H is the relative humidity. Use Table 1 to estimate $f_T(92, 60)$ and $f_H(92, 60)$. What are the practical interpretations of these values?

3. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1.

Wind speed (km/h)

Actual temperature (°C) T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

(a) Estimate the values of $f_T(-15, 30)$ and $f_v(-15, 30)$. What are the practical interpretations of these values?

(b) In general, what can you say about the signs of $\partial W/\partial T$ and $\partial W/\partial v$?

(c) What appears to be the value of the following limit?

$$\lim_{v \to \infty} \frac{\partial W}{\partial v}$$

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

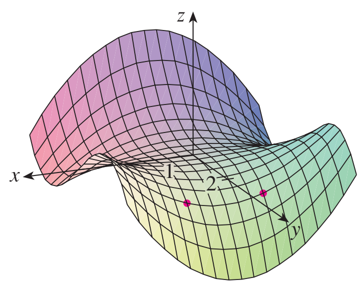
(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) What appears to be the value of the following limit?

$$\lim_{t \to \infty} \frac{\partial h}{\partial t}$$

5–8 Determine the signs of the partial derivatives for the function f whose graph is shown.



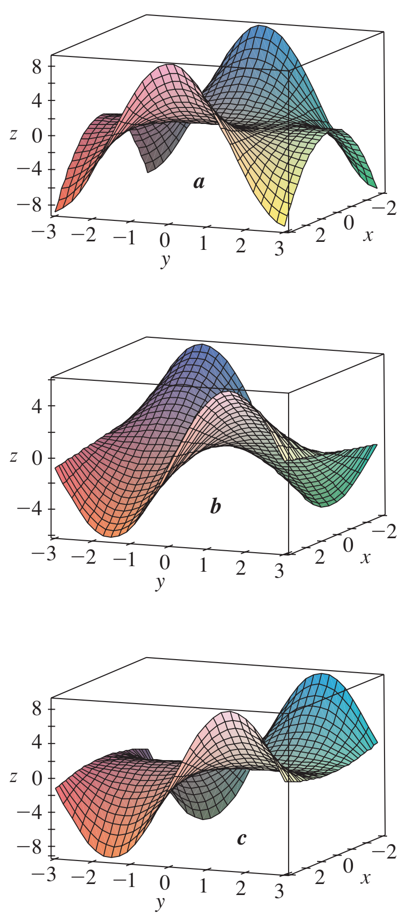
5. (a) $f_x(1, 2)$ (b) $f_y(1, 2)$

6. (a) $f_x(-1, 2)$ (b) $f_y(-1, 2)$

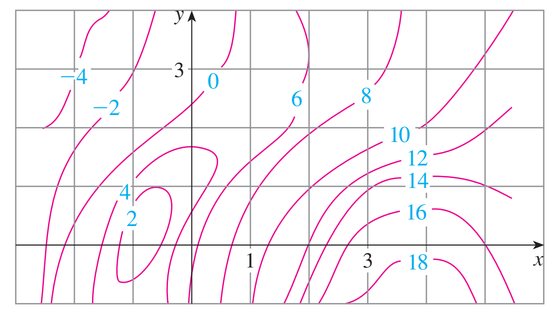
7. (a) $f_{xx}(-1, 2)$ (b) $f_{yy}(-1, 2)$

8. (a) $f_{xy}(1, 2)$ (b) $f_{xy}(-1, 2)$

9. The following surfaces, labeled a, b, and c, are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and give reasons for your choices.



10. A contour map is given for a function f. Use it to estimate $f_x(2, 1)$ and $f_y(2, 1)$.



11. If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

12. If $f(x, y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1, 0)$ and $f_y(1, 0)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

13–14 Find f_x and f_y and graph f, f_x, and f_y with domains and viewpoints that enable you to see the relationships between them.

13. $f(x, y) = x^2y^3$

14. $f(x, y) = \dfrac{y}{1 + x^2y^2}$

15–40 Find the first partial derivatives of the function.

15. $f(x, y) = x^4 + 5xy^3$

16. $f(x, y) = x^2y - 3y^4$

17. $f(x, t) = t^2e^{-x}$

18. $f(x, t) = \sqrt{3x + 4t}$

19. $z = \ln(x + t^2)$

20. $z = x\sin(xy)$

21. $f(x, y) = \dfrac{x}{y}$

22. $f(x, y) = \dfrac{x}{(x + y)^2}$

23. $f(x, y) = \dfrac{ax + by}{cx + dy}$

24. $w = \dfrac{e^v}{u + v^2}$

25. $g(u, v) = (u^2v - v^3)^5$

26. $u(r, \theta) = \sin(r\cos\theta)$

27. $R(p, q) = \tan^{-1}(pq^2)$

28. $f(x, y) = x^y$

29. $F(x, y) = \int_y^x \cos(e^t)\, dt$

30. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1}\, dt$

31. $f(x, y, z) = x^3yz^2 + 2yz$

32. $f(x, y, z) = xy^2e^{-xz}$

33. $w = \ln(x + 2y + 3z)$

34. $w = y\tan(x + 2z)$

35. $p = \sqrt{t^4 + u^2\cos v}$

36. $u = x^{y/z}$

37. $h(x, y, z, t) = x^2y\cos(z/t)$

38. $\phi(x, y, z, t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

39. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

40. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

41–44 Find the indicated partial derivative.

41. $R(s, t) = te^{s/t}$; $R_t(0, 1)$

42. $f(x, y) = y \sin^{-1}(xy)$; $f_y(1, \frac{1}{2})$

43. $f(x, y, z) = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}$; $f_y(1, 2, 2)$

44. $f(x, y, z) = x^{yz}$; $f_z(e, 1, 0)$

45–46 Use the definition of partial derivatives as limits (4) to find $f_x(x, y)$ and $f_y(x, y)$.

45. $f(x, y) = xy^2 - x^3y$

46. $f(x, y) = \dfrac{x}{x + y^2}$

47–50 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

47. $x^2 + 2y^2 + 3z^2 = 1$

48. $x^2 - y^2 + z^2 - 2z = 4$

49. $e^z = xyz$

50. $yz + x \ln y = z^2$

51–52 Find $\partial z/\partial x$ and $\partial z/\partial y$.

51. (a) $z = f(x) + g(y)$ (b) $z = f(x + y)$

52. (a) $z = f(x)g(y)$ (b) $z = f(xy)$
(c) $z = f(x/y)$

53–58 Find all the second partial derivatives.

53. $f(x, y) = x^4y - 2x^3y^2$

54. $f(x, y) = \ln(ax + by)$

55. $z = \dfrac{y}{2x + 3y}$

56. $T = e^{-2r} \cos \theta$

57. $v = \sin(s^2 - t^2)$

58. $w = \sqrt{1 + uv^2}$

59–62 Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

59. $u = x^4y^3 - y^4$

60. $u = e^{xy} \sin y$

61. $u = \cos(x^2y)$

62. $u = \ln(x + 2y)$

63–70 Find the indicated partial derivative(s).

63. $f(x, y) = x^4y^2 - x^3y$; f_{xxx}, f_{xyx}

64. $f(x, y) = \sin(2x + 5y)$; f_{yxy}

65. $f(x, y, z) = e^{xyz^2}$; f_{xyz}

66. $g(r, s, t) = e^r \sin(st)$; g_{rst}

67. $W = \sqrt{u + v^2}$; $\dfrac{\partial^3 W}{\partial u^2 \, \partial v}$

68. $V = \ln(r + s^2 + t^3)$; $\dfrac{\partial^3 V}{\partial r \, \partial s \, \partial t}$

69. $w = \dfrac{x}{y + 2z}$; $\dfrac{\partial^3 w}{\partial z \, \partial y \, \partial x}$, $\dfrac{\partial^3 w}{\partial x^2 \, \partial y}$

70. $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x \, \partial y^2 \, \partial z^3}$

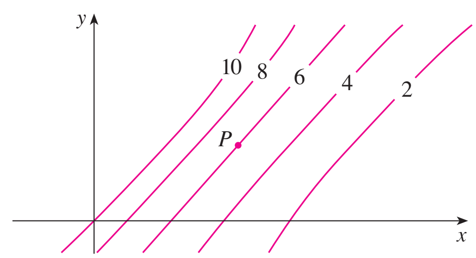
71. If $f(x, y, z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}. [*Hint:* Which order of differentiation is easiest?]

72. If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}. [*Hint:* Use a different order of differentiation for each term.]

73. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$, $f_x(3, 2.2)$, and $f_{xy}(3, 2)$.

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

74. Level curves are shown for a function f. Determine whether the following partial derivatives are positive or negative at the point P.
(a) f_x (b) f_y (c) f_{xx}
(d) f_{xy} (e) f_{yy}



75. Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the *heat conduction equation* $u_t = \alpha^2 u_{xx}$.

76. Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
(a) $u = x^2 + y^2$ (b) $u = x^2 - y^2$
(c) $u = x^3 + 3xy^2$ (d) $u = \ln \sqrt{x^2 + y^2}$
(e) $u = \sin x \cosh y + \cos x \sinh y$
(f) $u = e^{-x} \cos y - e^{-y} \cos x$

77. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

78. Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.
(a) $u = \sin(kx) \sin(akt)$ (b) $u = t/(a^2t^2 - x^2)$
(c) $u = (x - at)^6 + (x + at)^6$
(d) $u = \sin(x - at) + \ln(x + at)$

79. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

is a solution of the wave equation given in Exercise 78.

80. If $u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$, where $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u$$

81. The *diffusion equation*

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

where D is a positive constant, describes the diffusion of heat through a solid, or the concentration of a pollutant at time t at a distance x from the source of the pollution, or the invasion of alien species into a new habitat. Verify that the function

$$c(x, t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/(4Dt)}$$

is a solution of the diffusion equation.

82. The temperature at a point (x, y) on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where T is measured in °C and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in (a) the x-direction and (b) the y-direction.

83. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find $\partial R/\partial R_1$.

84. Show that the Cobb-Douglas production function $P = bL^{\alpha}K^{\beta}$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$$

85. Show that the Cobb-Douglas production function satisfies $P(L, K_0) = C_1(K_0)L^{\alpha}$ by solving the differential equation

$$\frac{dP}{dL} = \alpha\frac{P}{L}$$

(See Equation 6.)

86. Cobb and Douglas used the equation $P(L, K) = 1.01L^{0.75}K^{0.25}$ to model the American economy from 1899 to 1922, where L is the amount of labor and K is the amount of capital. (See Example 14.1.3.)

(a) Calculate P_L and P_K.

(b) Find the marginal productivity of labor and the marginal productivity of capital in the year 1920, when $L = 194$ and $K = 407$ (compared with the assigned values $L = 100$ and $K = 100$ in 1899). Interpret the results.

(c) In the year 1920, which would have benefited production more, an increase in capital investment or an increase in spending on labor?

87. The *van der Waals equation* for n moles of a gas is

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. Calculate $\partial T/\partial P$ and $\partial P/\partial V$.

88. The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

89. For the ideal gas of Exercise 88, show that

$$T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = mR$$

90. The wind-chill index is modeled by the function

$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature (°C) and v is the wind speed (km/h). When $T = -15$°C and $v = 30$ km/h, by how much would you expect the apparent temperature W to drop if the actual temperature decreases by 1°C? What if the wind speed increases by 1 km/h?

91. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. Calculate and interpret the partial derivatives.

(a) $\dfrac{\partial S}{\partial w}(160, 70)$ (b) $\dfrac{\partial S}{\partial h}(160, 70)$

92. One of Poiseuille's laws states that the resistance of blood flowing through an artery is

$$R = C\frac{L}{r^4}$$

where L and r are the length and radius of the artery and C is a positive constant determined by the viscosity of the blood. Calculate $\partial R/\partial L$ and $\partial R/\partial r$ and interpret them.

93. In the project on page 271 we expressed the power needed by a bird during its flapping mode as

$$P(v, x, m) = Av^3 + \frac{B(mg/x)^2}{v}$$

where A and B are constants specific to a species of bird, v is the velocity of the bird, m is the mass of the bird, and x is the fraction of the flying time spent in flapping mode. Calculate $\partial P/\partial v$, $\partial P/\partial x$, and $\partial P/\partial m$ and interpret them.

94. The average energy E (in kcal) needed for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where m is the body mass of the lizard (in grams) and v is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and interpret your answers.

Source: C. Robbins, *Wildlife Feeding and Nutrition*, 2d ed. (San Diego: Academic Press, 1993).

95. The kinetic energy of a body with mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$$

96. If a, b, c are the sides of a triangle and A, B, C are the opposite angles, find $\partial A/\partial a$, $\partial A/\partial b$, $\partial A/\partial c$ by implicit differentiation of the Law of Cosines.

97. You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

98. The paraboloid $z = 6 - x - x^2 - 2y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations for the tangent line to this parabola at the point $(1, 2, -4)$. Use a computer to graph the paraboloid, the parabola, and the tangent line on the same screen.

99. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

100. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in feet) can be modeled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and λ is a positive constant.

(a) Find $\partial T/\partial x$. What is its physical significance?

(b) Find $\partial T/\partial t$. What is its physical significance?

(c) Show that T satisfies the heat equation $T_t = kT_{xx}$ for a certain constant k.

(d) If $\lambda = 0.2$, $T_0 = 0$, and $T_1 = 10$, use a computer to graph $T(x, t)$.

(e) What is the physical significance of the term $-\lambda x$ in the expression $\sin(\omega t - \lambda x)$?

101. Use Clairaut's Theorem to show that if the third-order partial derivatives of f are continuous, then

$$f_{xyy} = f_{yxy} = f_{yyx}$$

102. (a) How many nth-order partial derivatives does a function of two variables have?

(b) If these partial derivatives are all continuous, how many of them can be distinct?

(c) Answer the question in part (a) for a function of three variables.

103. If

$$f(x, y) = x(x^2 + y^2)^{-3/2}e^{\sin(x^2 y)}$$

find $f_x(1, 0)$. [*Hint:* Instead of finding $f_x(x, y)$ first, note that it's easier to use Equation 1 or Equation 2.]

104. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

105. Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use a computer to graph f.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.

(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

14.4 Tangent Planes and Linear Approximations

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. (See Section 2.9.) Here we develop similar ideas in three dimensions. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables. We also extend the idea of a differential to functions of two or more variables.

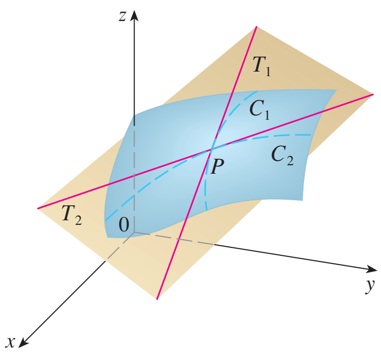


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

We will see in Section 14.6 that if C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane. Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

1
$$z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a. But from Section 14.3 we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

■

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restrict-

TEC Visual 14.4 shows an animation of Figures 2 and 3.

ing the domain of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.

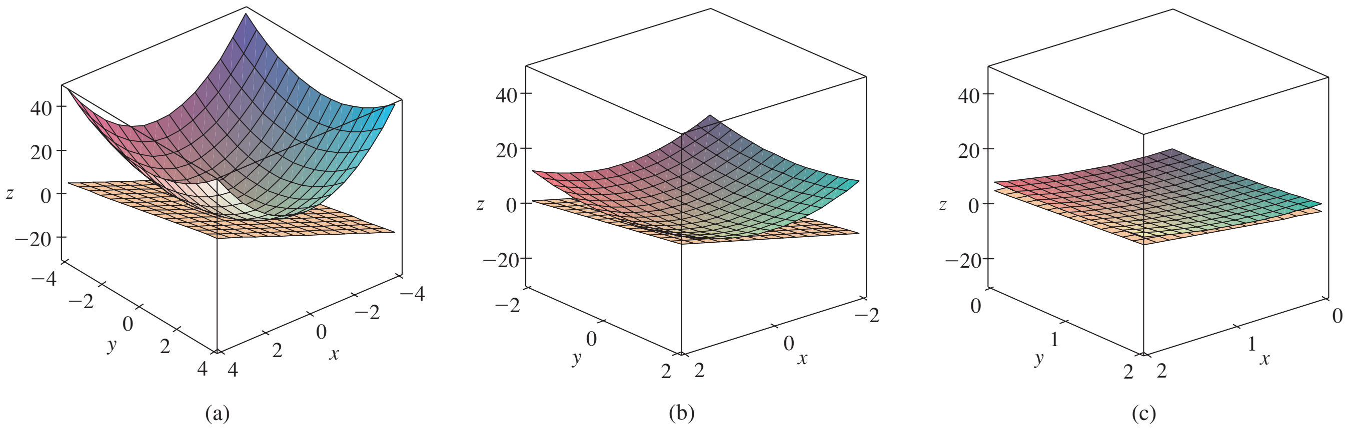


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

In Figure 3 we corroborate this impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$. Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

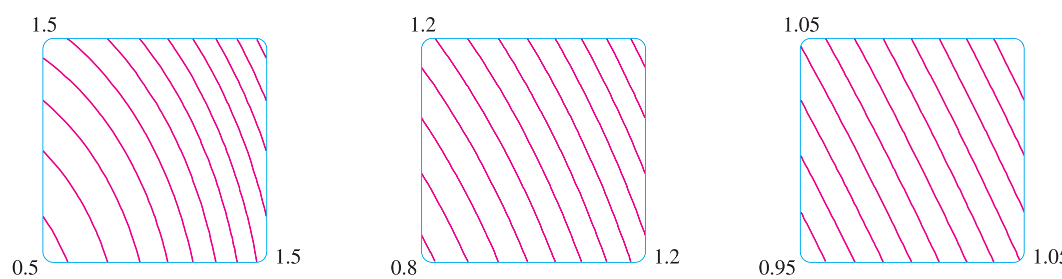


FIGURE 3
Zooming in toward (1, 1) on a contour map of $f(x, y) = 2x^2 + y^2$

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, in view of the visual evidence in Figures 2 and 3, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$. But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation. In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

3 $$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

4 $$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

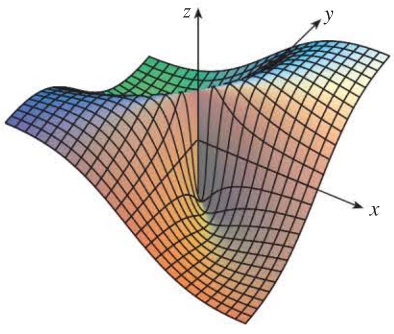


FIGURE 4
$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

You can verify (see Exercise 46) that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous. The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$. So a function of two variables can behave badly even though both of its partial derivatives exist. To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

In Chapter 2 we showed that if f is differentiable at a, then

This is Equation 2.5.5.

5 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

6 $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. By analogy with (5) we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation (4) is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem 8 is proved in Appendix F.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Figure 5 shows the graphs of the function f and its linearization L in Example 2.

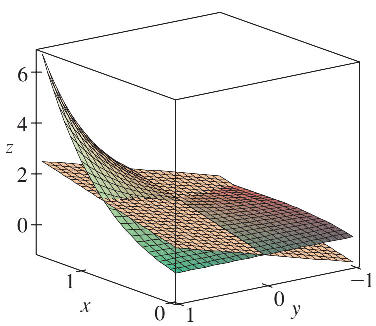


FIGURE 5

SOLUTION The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$

$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable by Theorem 8. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(x - 1) + 1 \cdot y = x + y \end{aligned}$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

so $$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$. ■

EXAMPLE 3 At the beginning of Section 14.3 we discussed the heat index (perceived temperature) I as a function of the actual temperature T and the relative humidity H and gave the following table of values from the National Weather Service.

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find a linear approximation for the heat index $I = f(T, H)$ when T is near 96°F and H is near 70%. Use it to estimate the heat index when the temperature is 97°F and the relative humidity is 72%.

SOLUTION We read from the table that $f(96, 70) = 125$. In Section 14.3 we used the tabular values to estimate that $f_T(96, 70) \approx 3.75$ and $f_H(96, 70) \approx 0.9$. (See pages 952–53.) So the linear approximation is

$$\begin{aligned} f(T, H) &\approx f(96, 70) + f_T(96, 70)(T - 96) + f_H(96, 70)(H - 70) \\ &\approx 125 + 3.75(T - 96) + 0.9(H - 70) \end{aligned}$$

In particular,

$$f(97, 72) \approx 125 + 3.75(1) + 0.9(2) = 130.55$$

Therefore, when $T = 97°\text{F}$ and $H = 72\%$, the heat index is

$$I \approx 131°\text{F}$$

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined as

9 $$dy = f'(x)\,dx$$

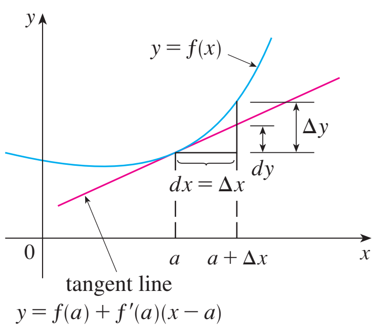


FIGURE 6

(See Section 2.9.) Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

10 $$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

(Compare with Equation 9.) Sometimes the notation df is used in place of dz.

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation (4) can be written as

$$f(x, y) \approx f(a, b) + dz$$

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

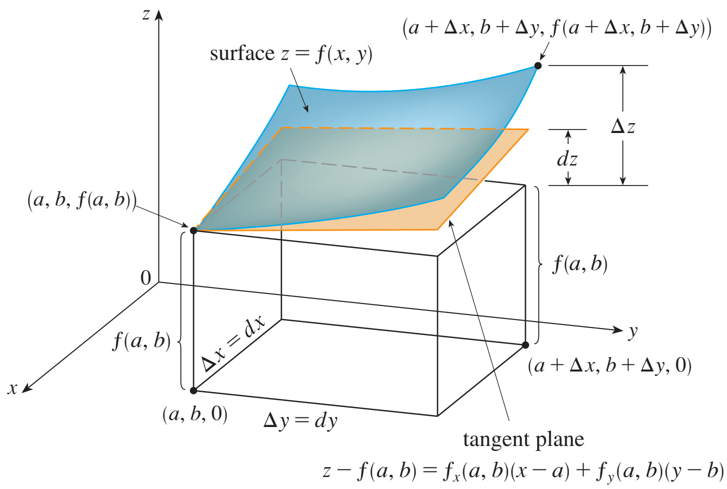


FIGURE 7

EXAMPLE 4
(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

SOLUTION
(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near (2, 3, 13). (See Figure 8.)

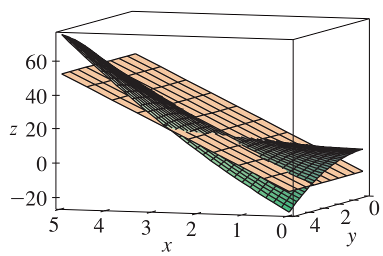


FIGURE 8

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute. ■

EXAMPLE 5 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

SOLUTION The volume V of a cone with base radius r and height h is $V = \pi r^2 h/3$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leqslant 0.1$, $|\Delta h| \leqslant 0.1$. To estimate the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi \text{ cm}^3 \approx 63 \text{ cm}^3$. ■

■ Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

EXAMPLE 6 The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

SOLUTION If the dimensions of the box are x, y, and z, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \leqslant 0.2$, $|\Delta y| \leqslant 0.2$, and $|\Delta z| \leqslant 0.2$. To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box. ■

14.4 EXERCISES

1–6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 2x^2 + y^2 - 5y, \quad (1, 2, -4)$

2. $z = (x + 2)^2 - 2(y - 1)^2 - 5, \quad (2, 3, 3)$

3. $z = e^{x-y}, \quad (2, 2, 1)$

4. $z = x/y^2, \quad (-4, 2, -1)$

5. $z = x \sin(x + y), \quad (-1, 1, 0)$

6. $z = \ln(x - 2y), \quad (3, 1, 0)$

7–8 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. $z = x^2 + xy + 3y^2, \quad (1, 1, 5)$

8. $z = \sqrt{9 + x^2y^2}, \quad (2, 2, 5)$

CAS **9–10** Draw the graph of f and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

9. $f(x, y) = \dfrac{1 + \cos^2(x - y)}{1 + \cos^2(x + y)}, \quad \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

10. $f(x, y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right), \quad (1, 1, 3e^{-0.1})$

11–16 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = 1 + x \ln(xy - 5), \quad (2, 3)$

12. $f(x, y) = \sqrt{xy}, \quad (1, 4)$

13. $f(x, y) = x^2e^y, \quad (1, 0)$

14. $f(x, y) = \dfrac{1 + y}{1 + x}, \quad (1, 3)$

15. $f(x, y) = 4 \arctan(xy), \quad (1, 1)$

16. $f(x, y) = y + \sin(x/y), \quad (0, 3)$

17–18 Verify the linear approximation at (0, 0).

17. $e^x \cos(xy) \approx x + 1$

18. $\dfrac{y - 1}{x + 1} \approx x + y - 1$

19. Given that f is a differentiable function with $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.

20. Find the linear approximation of the function $f(x, y) = 1 - xy \cos \pi y$ at $(1, 1)$ and use it to approximate $f(1.02, 0.97)$. Illustrate by graphing f and the tangent plane.

21. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

22. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table. Use the table to find a linear approximation to the wave height function when v is near 40 knots and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

Duration (hours)

Wind speed (knots)

v \ t	5	10	15	20	30	40	50
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

23. Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94°F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95°F and the relative humidity is 78%.

24. The wind-chill index W is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $W = f(T, v)$. The following table of values is an excerpt from Table 1 in Section 14.1. Use the table to find a linear approximation to the wind-chill index function when T is near $-15°C$ and v is near 50 km/h. Then estimate the wind-chill index when the temperature is $-17°C$ and the wind speed is 55 km/h.

Wind speed (km/h)

Actual temperature (°C)

T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

25–30 Find the differential of the function.

25. $z = e^{-2x} \cos 2\pi t$

26. $u = \sqrt{x^2 + 3y^2}$

27. $m = p^5 q^3$

28. $T = \dfrac{v}{1 + uvw}$

29. $R = \alpha\beta^2 \cos \gamma$

30. $L = xze^{-y^2 - z^2}$

31. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

32. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

33. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

34. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

35. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

36. The wind-chill index is modeled by the function

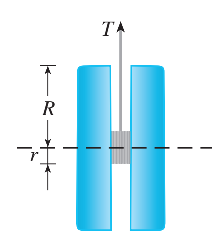
$$W = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

where T is the temperature (in °C) and v is the wind speed (in km/h). The wind speed is measured as 26 km/h, with a possible error of ± 2 km/h, and the temperature is measured as $-11°C$, with a possible error of $\pm 1°C$. Use differentials to estimate the maximum error in the calculated value of W due to the measurement errors in T and v.

37. The tension T in the string of the yo-yo in the figure is

$$T = \frac{mgR}{2r^2 + R^2}$$

where m is the mass of the yo-yo and g is acceleration due to gravity. Use differentials to estimate the change in the tension if R is increased from 3 cm to 3.1 cm and r is increased from 0.7 cm to 0.8 cm. Does the tension increase or decrease?



38. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

39. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

40. A model for the surface area of a human body is given by $S = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

41. In Exercise 14.1.39 and Example 14.3.3, the body mass index of a person was defined as $B(m, h) = m/h^2$, where m is the mass in kilograms and h is the height in meters.

(a) What is the linear approximation of $B(m, h)$ for a child with mass 23 kg and height 1.10 m?

(b) If the child's mass increases by 1 kg and height by 3 cm, use the linear approximation to estimate the new BMI. Compare with the actual new BMI.

42. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle$$
$$\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$$

both lie on S. Find an equation of the tangent plane at P.

43–44 Show that the function is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 7.

43. $f(x, y) = x^2 + y^2$ **44.** $f(x, y) = xy - 5y^2$

45. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b).
Hint: Show that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$$

46. (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 45.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.

APPLIED PROJECT THE SPEEDO LZR RACER

Many technological advances have occurred in sports that have contributed to increased athletic performance. One of the best known is the introduction, in 2008, of the Speedo LZR racer. It was claimed that this full-body swimsuit reduced a swimmer's drag in the water. Figure 1 shows the number of world records broken in men's and women's long-course freestyle swimming events from 1990 to 2011.[1] The dramatic increase in 2008 when the suit was introduced led people to claim that such suits are a form of technological doping. As a result all full-body suits were banned from competition starting in 2010.

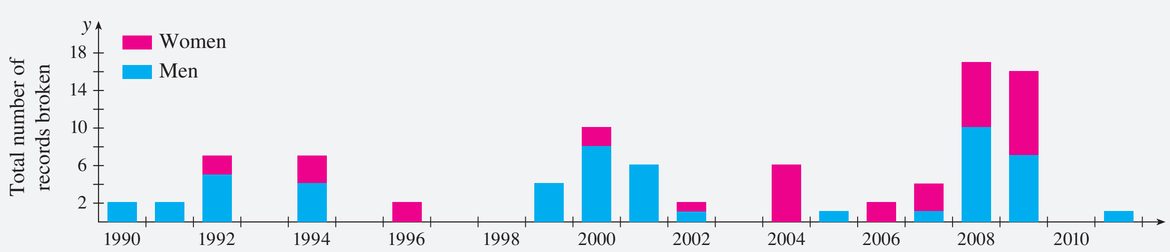
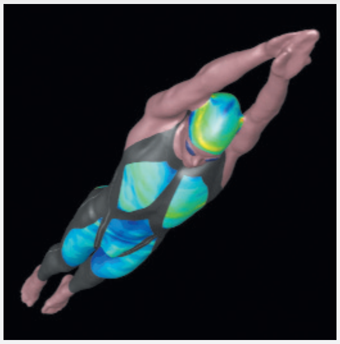


FIGURE 1 Number of world records set in long-course men's and women's freestyle swimming event 1990–2011

It might be surprising that a simple reduction in drag could have such a big effect on performance. We can gain some insight into this using a simple mathematical model.[2]

1. L. Foster et al., "Influence of Full Body Swimsuits on Competitive Performance," *Procedia Engineering* 34 (2012): 712–17.
2. Adapted from http://plus.maths.org/content/swimming.



Courtesy Speedo and ANSYS, Inc.

The speed v of an object being propelled through water is given by

$$v(P, C) = \left(\frac{2P}{kC}\right)^{1/3}$$

where P is the power being used to propel the object, C is the drag coefficient, and k is a positive constant. Athletes can therefore increase their swimming speeds by increasing their power or reducing their drag coefficients. But how effective is each of these?

To compare the effect of increasing power versus reducing drag, we need to somehow compare the two in common units. The most common approach is to determine the percentage change in speed that results from a given percentage change in power and in drag.

If we work with percentages as fractions, then when power is changed by a fraction x (with x corresponding to $100x$ percent), P changes from P to $P + xP$. Likewise, if the drag coefficient is changed by a fraction y, this means that it has changed from C to $C + yC$. Finally, the fractional change in speed resulting from both effects is

$$\boxed{1} \qquad \frac{v(P + xP, C + yC) - v(P, C)}{v(P, C)}$$

1. Expression 1 gives the fractional change in speed that results from a change x in power and a change y in drag. Show that this reduces to the function

$$f(x, y) = \left(\frac{1 + x}{1 + y}\right)^{1/3} - 1$$

Given the context, what is the domain of f?

2. Suppose that the possible changes in power x and drag y are small. Find the linear approximation to the function $f(x, y)$. What does this approximation tell you about the effect of a small increase in power versus a small decrease in drag?

3. Calculate $f_{xx}(x, y)$ and $f_{yy}(x, y)$. Based on the signs of these derivatives, does the linear approximation in Problem 2 result in an overestimate or an underestimate for an increase in power? What about for a decrease in drag? Use your answer to explain why, for changes in power or drag that are not very small, a decrease in drag is more effective.

4. Graph the level curves of $f(x, y)$. Explain how the shapes of these curves relate to your answers to Problems 2 and 3.

14.5 The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f

is differentiable (Definition 14.4.7). Recall that this is the case when f_x and f_y are continuous (Theorem 14.4.8).

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

PROOF A change of Δt in t produces changes of Δx in x and Δy in y. These, in turn, produce a change of Δz in z, and from Definition 14.4.7 we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$. [If the functions ε_1 and ε_2 are not defined at $(0, 0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If we now let $\Delta t \to 0$, then $\Delta x = g(t + \Delta t) - g(t) \to 0$ because g is differentiable and therefore continuous. Similarly, $\Delta y \to 0$. This, in turn, means that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\
&= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right) \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}
\end{aligned}$$

■

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

Notice the similarity to the definition of the differential:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)
\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

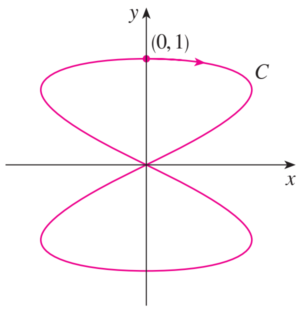


FIGURE 1
The curve $x = \sin 2t$, $y = \cos t$

The derivative in Example 1 can be interpreted as the rate of change of z with respect to t as the point (x, y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$. (See Figure 1.) In particular, when $t = 0$, the point (x, y) is $(0, 1)$ and $dz/dt = 6$ is the rate of increase as we move along the curve C through $(0, 1)$. If, for instance, $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y), then the composite function $z = T(\sin 2t, \cos t)$ represents the temperature at points on C and the derivative dz/dt represents the rate at which the temperature changes along C.

EXAMPLE 2 The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

SOLUTION If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $dT/dt = 0.1$, $V = 100$, $dV/dt = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

the Chain Rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s. ■

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$. Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$
$$= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$
$$= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$ ■

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable. Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

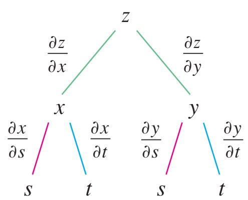


FIGURE 2

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2. We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t. On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$. Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

EXAMPLE 4 Write out the Chain Rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

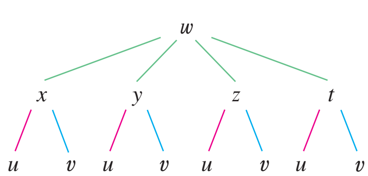


FIGURE 3

SOLUTION We apply Theorem 4 with $n = 4$ and $m = 2$. Figure 3 shows the tree diagram. Although we haven't written the derivatives on the branches, it's understood that if a branch leads from y to u, then the partial derivative for that branch is $\partial y/\partial u$. With the aid of the tree diagram, we can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$ ■

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

SOLUTION With the help of the tree diagram in Figure 4, we have

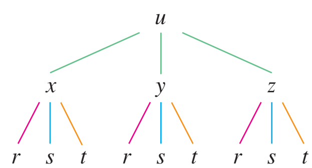


FIGURE 4

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$ ■

EXAMPLE 6 If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

SOLUTION Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$ ■

EXAMPLE 7 If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find (a) $\partial z/\partial r$ and (b) $\partial^2 z/\partial r^2$.

SOLUTION

(a) The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the Product Rule to the expression in part (a), we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

5

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

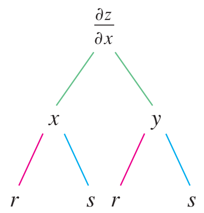


FIGURE 5

But, using the Chain Rule again (see Figure 5), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\,\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into Equation 5 and using the equality of the mixed second-order derivatives, we obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\,\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\,\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\,\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$ ■

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 2.6 and 14.3. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

6
$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: it states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

EXAMPLE 8 Find y' if $x^3 + y^3 = 6xy$.

SOLUTION The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$ ■

The solution to Example 8 should be compared to the one in Example 2.6.2.

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $$\frac{\partial}{\partial x}(x) = 1 \qquad \text{and} \qquad \frac{\partial}{\partial x}(y) = 0$$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7. The formula for $\partial z/\partial y$ is obtained in a similar manner.

7 $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by (7).

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$ ■

The solution to Example 9 should be compared to the one in Example 14.3.5.

14.5 EXERCISES

1–6 Use the Chain Rule to find dz/dt or dw/dt.

1. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

2. $z = \dfrac{x - y}{x + 2y}, \quad x = e^{\pi t}, \quad y = e^{-\pi t}$

3. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

4. $z = \sqrt{1 + xy}, \quad x = \tan t, \quad y = \arctan t$

5. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

6. $w = \ln\sqrt{x^2 + y^2 + z^2}, \quad x = \sin t, \quad y = \cos t, \quad z = \tan t$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = (x - y)^5, \quad x = s^2t, \quad y = st^2$

8. $z = \tan^{-1}(x^2 + y^2), \quad x = s \ln t, \quad y = te^s$

9. $z = \ln(3x + 2y)$, $x = s \sin t$, $y = t \cos s$

10. $z = \sqrt{x}\, e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

11. $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

---

13. Let $p(t) = f(g(t), h(t))$, where f is differentiable, $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4, 5) = 2$, $f_y(4, 5) = 8$. Find $p'(2)$.

14. Let $R(s, t) = G(u(s, t), v(s, t))$, where G, u, and v are differentiable, $u(1, 2) = 5$, $u_s(1, 2) = 4$, $u_t(1, 2) = -3$, $v(1, 2) = 7$, $v_s(1, 2) = 2$, $v_t(1, 2) = 6$, $G_u(5, 7) = 9$, $G_v(5, 7) = -2$. Find $R_s(1, 2)$ and $R_t(1, 2)$.

15. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

16. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

17–20 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

17. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

18. $w = f(x, y, z)$, where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$

19. $T = F(p, q, r)$, where $p = p(x, y, z)$, $q = q(x, y, z)$, $r = r(x, y, z)$

20. $R = F(t, u)$ where $t = t(w, x, y, z)$, $u = u(w, x, y, z)$

21–26 Use the Chain Rule to find the indicated partial derivatives.

21. $z = x^4 + x^2y$, $x = s + 2t - u$, $y = stu^2$;
$\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$

22. $T = \dfrac{v}{2u + v}$, $u = pq\sqrt{r}$, $v = p\sqrt{q}\, r$;
$\dfrac{\partial T}{\partial p}, \dfrac{\partial T}{\partial q}, \dfrac{\partial T}{\partial r}$ when $p = 2$, $q = 1$, $r = 4$

23. $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$;
$\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ when $r = 2$, $\theta = \pi/2$

24. $P = \sqrt{u^2 + v^2 + w^2}$, $u = xe^y$, $v = ye^x$, $w = e^{xy}$;
$\dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y}$ when $x = 0$, $y = 2$

25. $N = \dfrac{p + q}{p + r}$, $p = u + vw$, $q = v + uw$, $r = w + uv$;
$\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$ when $u = 2$, $v = 3$, $w = 4$

26. $u = xe^{ty}$, $x = \alpha^2\beta$, $y = \beta^2\gamma$, $t = \gamma^2\alpha$;
$\dfrac{\partial u}{\partial \alpha}, \dfrac{\partial u}{\partial \beta}, \dfrac{\partial u}{\partial \gamma}$ when $\alpha = -1$, $\beta = 2$, $\gamma = 1$

27–30 Use Equation 6 to find dy/dx.

27. $y\cos x = x^2 + y^2$

28. $\cos(xy) = 1 + \sin y$

29. $\tan^{-1}(x^2y) = x + xy^2$

30. $e^y \sin x = x + xy$

31–34 Use Equations 7 to find $\partial z/\partial x$ and $\partial z/\partial y$.

31. $x^2 + 2y^2 + 3z^2 = 1$

32. $x^2 - y^2 + z^2 - 2z = 4$

33. $e^z = xyz$

34. $yz + x\ln y = z^2$

---

35. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

36. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

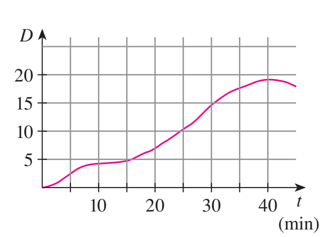
(a) What is the significance of the signs of these partial derivatives?

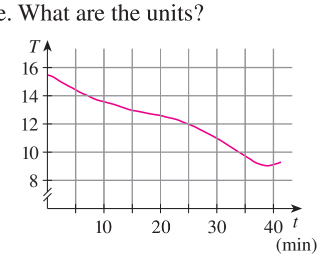
(b) Estimate the current rate of change of wheat production, dW/dt.

37. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?





38. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

39. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.
(a) The volume
(b) The surface area
(c) The length of a diagonal

40. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega$/s.

41. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $PV = 8.31T$ in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

42. A manufacturer has modeled its yearly production function P (the value of its entire production, in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Suppose that when $L = 30$ and $K = 8$, the labor force is decreasing at a rate of 2000 labor hours per year and capital is increasing at a rate of \$500,000 per year. Find the rate of change of production.

43. One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$?

44. A sound with frequency f_s is produced by a source traveling along a line with speed v_s. If an observer is traveling with speed v_o along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s$$

where c is the speed of sound, about 332 m/s. (This is the **Doppler effect**.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/s^2. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/s^2, and sounds its whistle, which has a frequency of 460 Hz. At that instant, what is the perceived frequency that you hear and how fast is it changing?

45–48 Assume that all the given functions are differentiable.

45. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial \theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

46. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]$$

47. If $z = \dfrac{1}{x}[f(x - y) + g(x + y)]$, show that

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial z}{\partial x}\right) = x^2\frac{\partial^2 z}{\partial y^2}$$

48. If $z = \dfrac{1}{y}[f(ax + y) + g(ax - y)]$, show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right)$$

49–54 Assume that all the given functions have continuous second-order partial derivatives.

49. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

[*Hint:* Let $u = x + at$, $v = x - at$.]

50. If $u = f(x, y)$, where $x = e^s\cos t$ and $y = e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$$

51. If $z = f(x, y)$, where $x = r^2 + s^2$ and $y = 2rs$, find $\partial^2 z/\partial r\,\partial s$. (Compare with Example 7.)

52. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (a) $\partial z/\partial r$, (b) $\partial z/\partial\theta$, and (c) $\partial^2 z/\partial r\,\partial\theta$.

53. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

54. Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$.
(a) Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

(b) Find a similar formula for $\partial^2 z/\partial s\,\partial t$.

55. A function f is called **homogeneous of degree n** if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t, where n is a positive integer and f has continuous second-order partial derivatives.

(a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

[*Hint:* Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.]

56. If f is homogeneous of degree n, show that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$$

57. If f is homogeneous of degree n, show that

$$f_x(tx, ty) = t^{n-1}f_x(x, y)$$

58. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

59. Equation 6 is a formula for the derivative dy/dx of a function defined implicitly by an equation $F(x, y) = 0$, provided that F is differentiable and $F_y \neq 0$. Prove that if F has continuous second derivatives, then a formula for the second derivative of y is

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}$$

14.6 Directional Derivatives and the Gradient Vector



FIGURE 1

The weather map in Figure 1 shows a contour map of the temperature function $T(x, y)$ for the states of California and Nevada at 3:00 PM on a day in October. The level curves, or isothermals, join locations with the same temperature. The partial derivative T_x at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno; T_y is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative,* that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

1

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

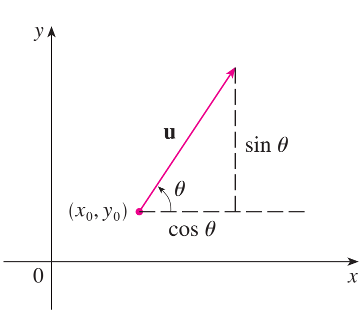


FIGURE 2
A unit vector
$\mathbf{u} = \langle a, b \rangle = \langle \cos u, \sin u \rangle$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of **u** inter-

sects S in a curve C. (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

TEC Visual 14.6A animates Figure 3 by rotating $\mathbf{u}$ and therefore T.

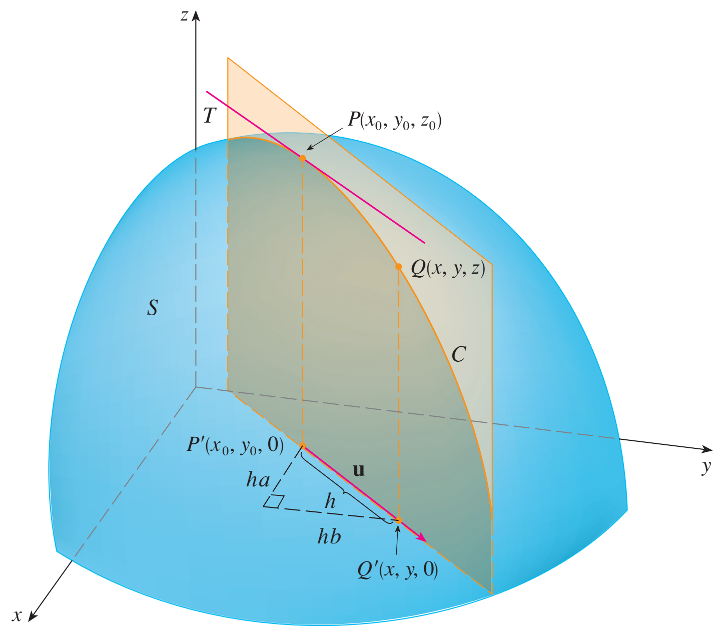


FIGURE 3

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$.

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

EXAMPLE 1 Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

SOLUTION The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression. We start by drawing a line through Reno toward the southeast (see Figure 4).

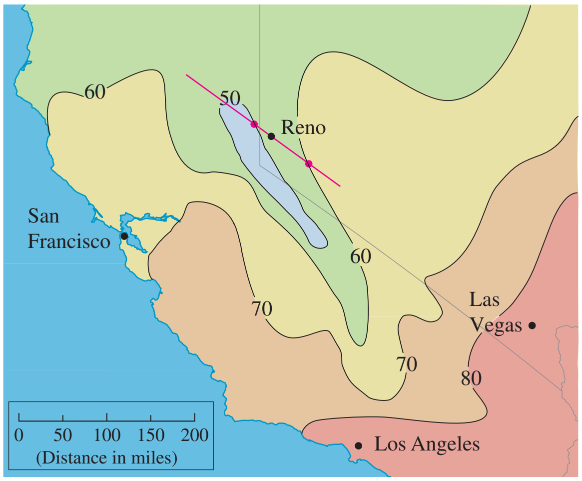


FIGURE 4

We approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$. The temperature at the point southeast of Reno is $T = 60°\text{F}$ and the temperature at the point northwest of Reno is $T = 50°\text{F}$. The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}}T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13°\text{F/mi}$$ ■

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

4
$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$
$$= D_{\mathbf{u}} f(x_0, y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule (Theorem 14.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)\,a + f_y(x, y)\,b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

5 $$g'(0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$ ■

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

6 $$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of $\mathbf{u}$ shown in Figure 5.

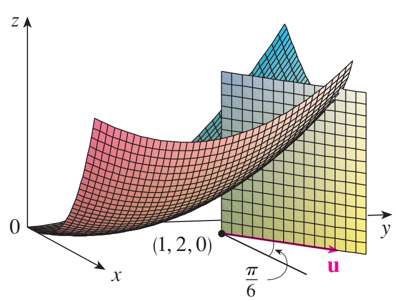


FIGURE 5

SOLUTION Formula 6 gives

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$ ■

The Gradient Vector

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

7 $$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$$

$$= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle$$

$$= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

EXAMPLE 3 If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $$\nabla f(0, 1) = \langle 2, 0 \rangle$$ ■

With this notation for the gradient vector, we can rewrite Equation 7 for the directional derivative of a differentiable function as

9 $$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of a unit vector **u** as the scalar projection of the gradient vector onto **u**.

The gradient vector $\nabla f(2, -1)$ in Example 4 is shown in Figure 6 with initial point $(2, -1)$. Also shown is the vector **v** that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

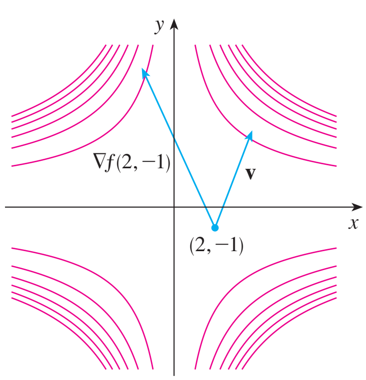


FIGURE 6

EXAMPLE 4 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\,\mathbf{i} + 5\mathbf{j}$.

SOLUTION We first compute the gradient vector at $(2, -1)$:

$$\nabla f(x, y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2, -1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Note that **v** is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}$$

Therefore, by Equation 9, we have

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4\,\mathbf{i} + 8\,\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j} \right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$ ■

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}} f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

11
$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 12.5.1) and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove Theorem 3 can be used to show that

12
$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\, a + f_y(x, y, z)\, b + f_z(x, y, z)\, c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13
$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14
$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

EXAMPLE 5 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION
(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle\end{aligned}$$

(b) At (1, 3, 0) we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$\begin{aligned} D_{\mathbf{u}} f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right) \\ &= 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}} \end{aligned}$$

Maximizing the Directional Derivative

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: in which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

TEC Visual 14.6B provides visual confirmation of Theorem 15.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f.

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}} f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5} \rangle \\ &= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$ ■

At (2, 0) the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through (2, 0). Figure 8 shows the graph of f and the gradient vector.

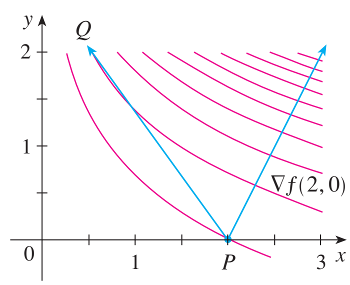


FIGURE 7

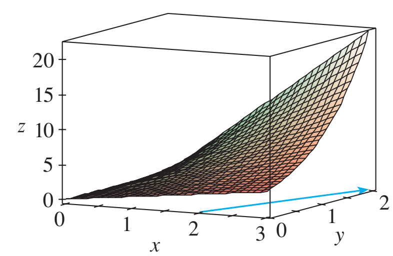


FIGURE 8

EXAMPLE 7 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

SOLUTION The gradient of T is

$$\begin{aligned} \nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\,\mathbf{i} - 2y\,\mathbf{j} - 3z\,\mathbf{k}) \end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \tfrac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \tfrac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \tfrac{5}{8}|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \tfrac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4°\text{C/m}$. ■

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall from Section 13.1 that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

16 $$F(x(t), y(t), z(t)) = k$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

17 $$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

18 $$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

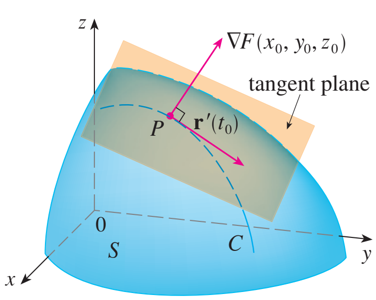


FIGURE 9

Equation 18 says that *the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.* (See Figure 9.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 12.5.7), we can write the equation of this tangent plane as

19 $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 12.5.3, its symmetric equations are

20 $$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$
$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$
$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to Equation 14.4.2. Thus our new, more general, definition of a tangent plane is consistent with the definition that was given for the special case of Section 14.4.

EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

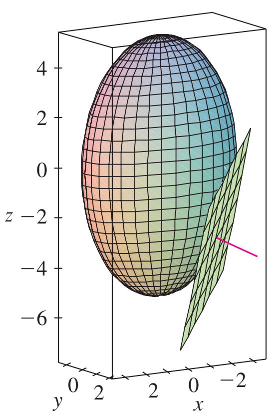


FIGURE 10

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$ ■

Significance of the Gradient Vector

We now summarize the ways in which the gradient vector is significant. We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain. On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f. On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P. (Refer to Figure 9.) These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all. So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain. Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P. Again this is intuitively plausible because the values of f remain constant as we move along the curve. (See Figure 11.)

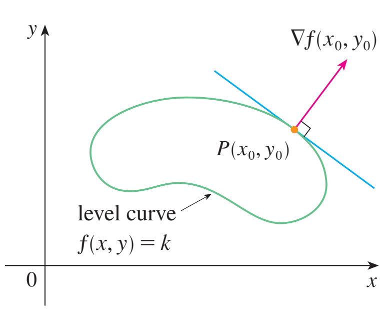


FIGURE 11

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn

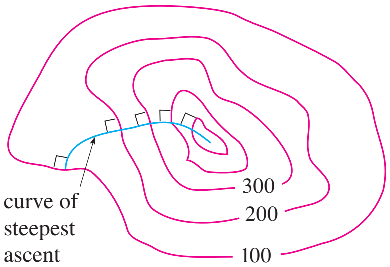


FIGURE 12

as in Figure 12 by making it perpendicular to all of the contour lines. This phenomenon can also be noticed in Figure 14.1.12, where Lonesome Creek follows a curve of steepest descent.

Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

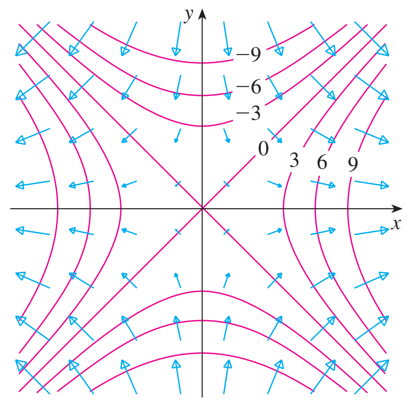
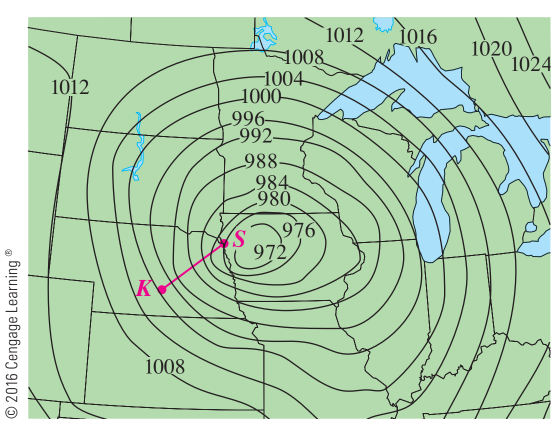


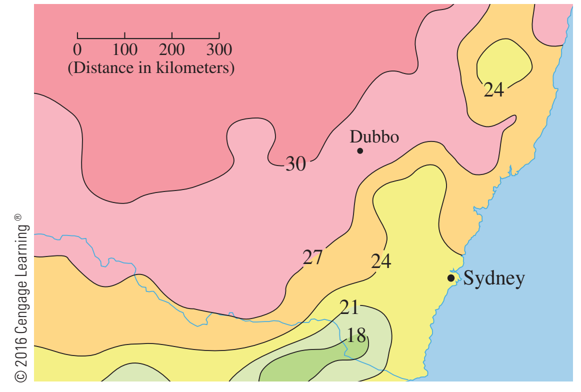
FIGURE 13

14.6 EXERCISES

1. Level curves for barometric pressure (in millibars) are shown for 6:00 AM on a day in November. A deep low with pressure 972 mb is moving over northeast Iowa. The distance along the red line from K (Kearney, Nebraska) to S (Sioux City, Iowa) is 300 km. Estimate the value of the directional derivative of the pressure function at Kearney in the direction of Sioux City. What are the units of the directional derivative?



2. The contour map shows the average maximum temperature for November 2004 (in °C). Estimate the value of the directional derivative of this temperature function at Dubbo, New South Wales, in the direction of Sydney. What are the units?



3. A table of values for the wind-chill index $W = f(T, v)$ is given in Exercise 14.3.3 on page 963. Use the table to estimate the value of $D_{\mathbf{u}} f(-20, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

4–6 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = xy^3 - x^2, \quad (1, 2), \quad \theta = \pi/3$

5. $f(x, y) = y\cos(xy)$, $(0, 1)$, $\theta = \pi/4$

6. $f(x, y) = \sqrt{2x + 3y}$, $(3, 1)$, $\theta = -\pi/6$

7–10
(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = x/y$, $P(2, 1)$, $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

8. $f(x, y) = x^2 \ln y$, $P(3, 1)$, $\mathbf{u} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$

9. $f(x, y, z) = x^2yz - xyz^3$, $P(2, -1, 1)$, $\mathbf{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$

10. $f(x, y, z) = y^2e^{xyz}$, $P(0, 1, -1)$, $\mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = e^x \sin y$, $(0, \pi/3)$, $\mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(1, 2)$, $\mathbf{v} = \langle 3, 5 \rangle$

13. $g(s, t) = s\sqrt{t}$, $(2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$

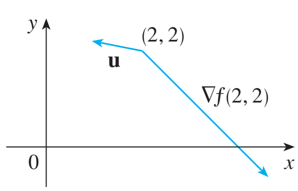
14. $g(u, v) = u^2e^{-v}$, $(3, 0)$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

15. $f(x, y, z) = x^2y + y^2z$, $(1, 2, 3)$, $\mathbf{v} = \langle 2, -1, 2 \rangle$

16. $f(x, y, z) = xy^2 \tan^{-1}z$, $(2, 1, 1)$, $\mathbf{v} = \langle 1, 1, 1 \rangle$

17. $h(r, s, t) = \ln(3r + 6s + 9t)$, $(1, 1, 1)$, $\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

18. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.



19. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

20. Find the directional derivative of $f(x, y, z) = xy^2z^3$ at $P(2, 1, 1)$ in the direction of $Q(0, -3, 5)$.

21–26 Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = 4y\sqrt{x}$, $(4, 1)$

22. $f(s, t) = te^{st}$, $(0, 2)$

23. $f(x, y) = \sin(xy)$, $(1, 0)$

24. $f(x, y, z) = x\ln(yz)$, $\left(1, 2, \frac{1}{2}\right)$

25. $f(x, y, z) = x/(y + z)$, $(8, 1, 3)$

26. $f(p, q, r) = \arctan(pqr)$, $(1, 2, 1)$

27. (a) Show that a differentiable function f decreases most rapidly at $\mathbf{x}$ in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.
(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

28. Find the directions in which the directional derivative of $f(x, y) = x^2 + xy^3$ at the point $(2, 1)$ has the value 2.

29. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

30. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

31. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$.
(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

32. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2 - 3y^2 - 9z^2}$$

where T is measured in °C and x, y, z in meters.
(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
(b) In which direction does the temperature increase fastest at P?
(c) Find the maximum rate of increase at P.

33. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.
(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) In which direction does V change most rapidly at P?
(c) What is the maximum rate of change at P?

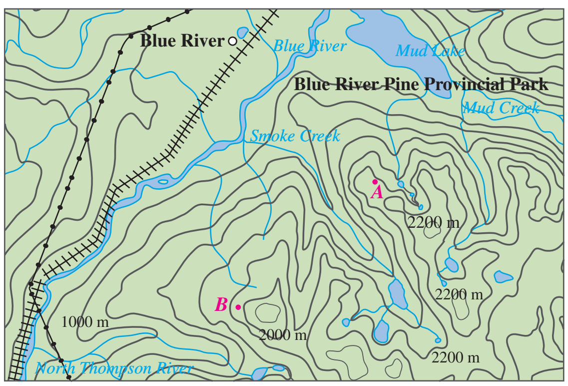
34. Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.
(a) If you walk due south, will you start to ascend or descend? At what rate?

(b) If you walk northwest, will you start to ascend or descend? At what rate?

(c) In which direction is the slope largest? What is the rate of ascent in that direction? At what angle above the horizontal does the path in that direction begin?

35. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is 3 and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\overrightarrow{AD}$.

36. Shown is a topographic map of Blue River Pine Provincial Park in British Columbia. Draw curves of steepest descent from point A (descending to Mud Lake) and from point B.



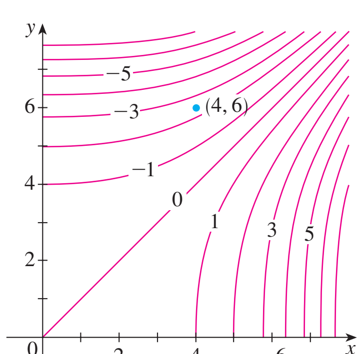
37. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(a) $\nabla(au + bv) = a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$ (d) $\nabla u^n = nu^{n-1}\,\nabla u$

38. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



39. The **second directional derivative** of $f(x, y)$ is

$$D_{\mathbf{u}}^2 f(x, y) = D_{\mathbf{u}}[D_{\mathbf{u}} f(x, y)]$$

If $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, calculate $D_{\mathbf{u}}^2 f(2, 1)$.

40. (a) If $\mathbf{u} = \langle a, b \rangle$ is a unit vector and f has continuous second partial derivatives, show that

$$D_{\mathbf{u}}^2 f = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

(b) Find the second directional derivative of $f(x, y) = xe^{2y}$ in the direction of $\mathbf{v} = \langle 4, 6 \rangle$.

41–46 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

41. $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10$, $(3, 3, 5)$

42. $x = y^2 + z^2 + 1$, $(3, 1, -1)$

43. $xy^2z^3 = 8$, $(2, 2, 1)$

44. $xy + yz + zx = 5$, $(1, 2, 1)$

45. $x + y + z = e^{xyz}$, $(0, 0, 1)$

46. $x^4 + y^4 + z^4 = 3x^2y^2z^2$, $(1, 1, 1)$

47–48 Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

47. $xy + yz + zx = 3$, $(1, 1, 1)$ **48.** $xyz = 6$, $(1, 2, 3)$

---

49. If $f(x, y) = xy$, find the gradient vector $\nabla f(3, 2)$ and use it to find the tangent line to the level curve $f(x, y) = 6$ at the point $(3, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

50. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line, and the gradient vector.

51. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

52. Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0) and express it in a form similar to the one in Exercise 51.

53. Show that the equation of the tangent plane to the elliptic paraboloid $z/c = x^2/a^2 + y^2/b^2$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

54. At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

55. Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

56. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

57. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

58. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

59. Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersect the paraboloid a second time?

60. At what points does the normal line through the point $(1, 2, 1)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 12$ intersect the sphere $x^2 + y^2 + z^2 = 102$?

61. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

62. Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

63. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

64. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

65. Where does the helix $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, t \rangle$ intersect the paraboloid $z = x^2 + y^2$? What is the angle of intersection between the helix and the paraboloid? (This is the angle between the tangent vector to the curve and the tangent plane to the paraboloid.)

66. The helix $\mathbf{r}(t) = \langle \cos(\pi t/2), \sin(\pi t/2), t \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 2$ in two points. Find the angle of intersection at each point.

67. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \quad \text{at } P$$

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

68. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

(b) Graph f near the origin and comment on how the graph confirms part (a).

69. Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two nonparallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

70. Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = \langle x_0, y_0 \rangle$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = 0$$

[*Hint:* Use Definition 14.4.7 directly.]

14.7 Maximum and Minimum Values

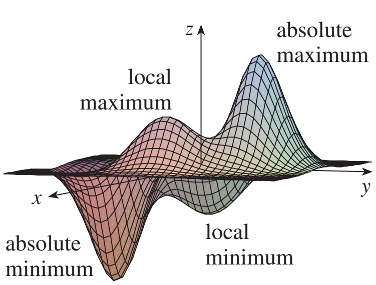


FIGURE 1

As we saw in Chapter 3, one of the main uses of ordinary derivatives is in finding maximum and minimum values (extreme values). In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables. In particular, in Example 6 we will see how to maximize the volume of a box without a lid if we have a fixed amount of cardboard to work with.

Look at the hills and valleys in the graph of f shown in Figure 1. There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the *absolute maximum*. Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the *absolute minimum*.

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem (see Theorem 3.1.4). But $g'(a) = f_x(a, b)$ (see Equation 14.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. ■

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 14.4.2), we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

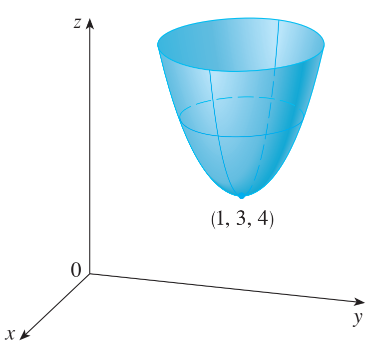


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2. ■

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value. ■

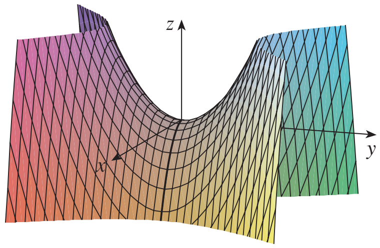


FIGURE 3
$z = y^2 - x^2$



Photo by Stan Wagon, Macalester College

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point. Figure 3 shows how this is possible. The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$, which has a horizontal tangent plane ($z = 0$) at the origin. You can see that $f(0, 0) = 0$ is a maximum in the direction of the x-axis but a minimum in the direction of the y-axis. Near the origin the graph has the shape of a saddle and so $(0, 0)$ is called a *saddle point* of f.

A mountain pass also has the shape of a saddle. As the photograph of the geological formation illustrates, for people hiking in one direction the saddle point is the lowest point on their route, while for those traveling in a different direction the saddle point is the highest point.

We need to be able to determine whether or not a function has an extreme value at a critical point. The following test, which is proved at the end of this section, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

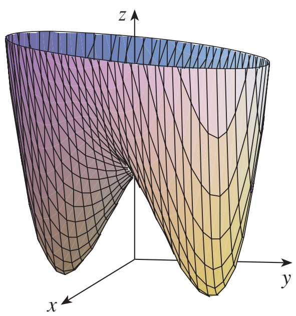


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4. ■

A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

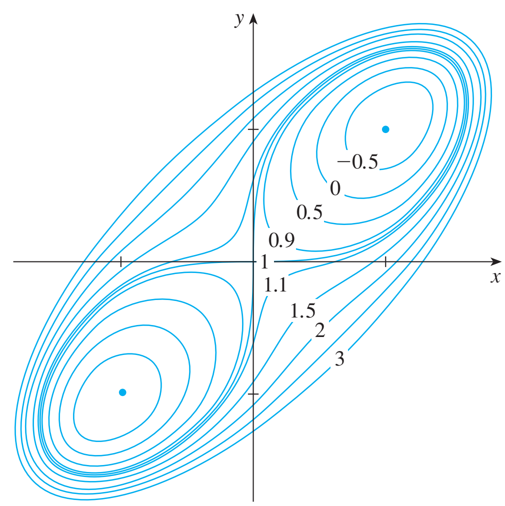


FIGURE 5

TEC In Module 14.7 you can use contour maps to estimate the locations of critical points.

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0 \tag{4}$$

$$5x^2 - 4y - 4y^3 = 0 \tag{5}$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case $(10y - 5 - 2x^2 = 0)$, we get

$$x^2 = 5y - 2.5 \tag{6}$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$4y^3 - 21y + 12.5 = 0 \tag{7}$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

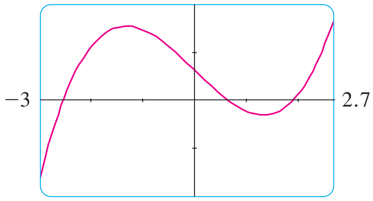


FIGURE 6

as in Figure 6, we see that Equation 7 has three real roots. By zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or solved numerically using a calculator or computer to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	-10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

Figures 7 and 8 give two views of the graph of f and we see that the surface opens downward. [This can also be seen from the expression for $f(x, y)$: the dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.] Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is $f(\pm 2.64, 1.90) \approx 8.50$. In other words, the highest points on the graph of f are $(\pm 2.64, 1.90, 8.50)$.

TEC Visual 14.7 shows several families of surfaces. The surface in Figures 7 and 8 is a member of one of these families.

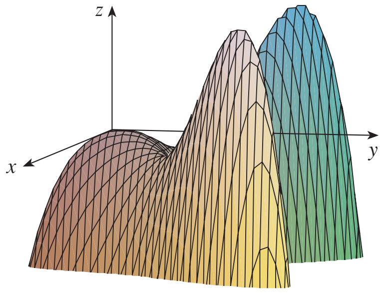


FIGURE 7

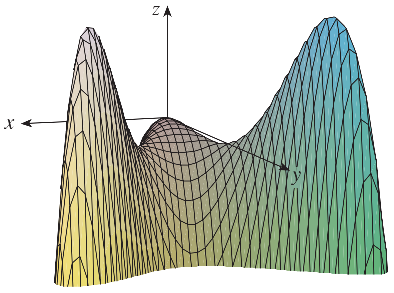


FIGURE 8 ■

The five critical points of the function f in Example 4 are shown in red in the contour map of f in Figure 9.

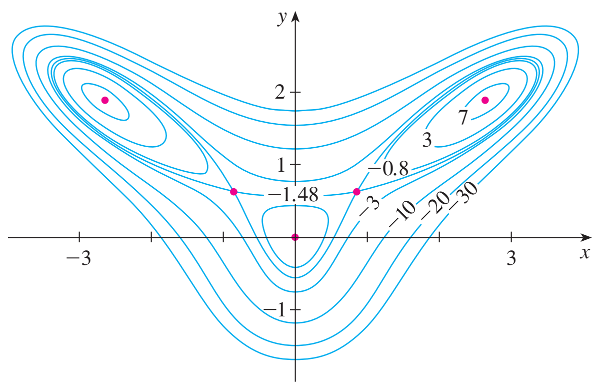


FIGURE 9

EXAMPLE 5 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

Example 5 could also be solved using vectors. Compare with the methods of Section 12.5.

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} = \sqrt{\left(\tfrac{5}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \tfrac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$. ■

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

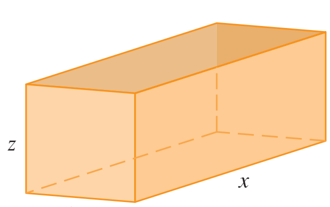


FIGURE 10

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\,\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3. ■

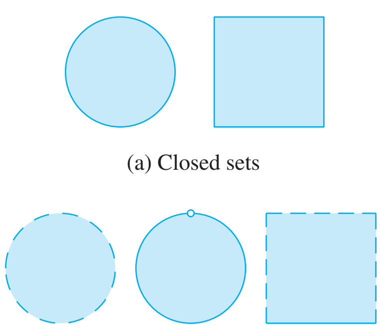
Absolute Maximum and Minimum Values

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. According to the Closed Interval Method in Section 3.1, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b.

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.] For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$$

which consists of all points on or inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)



(a) Closed sets

(b) Sets that are not closed

FIGURE 11

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D. Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

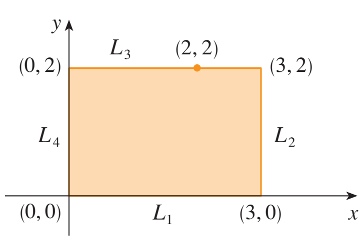


FIGURE 12

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By the methods of Chapter 3, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f. ■

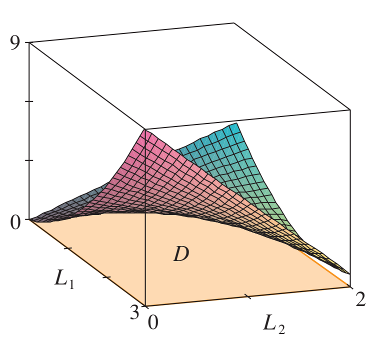


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

We close this section by giving a proof of the first part of the Second Derivatives Test. Part (b) has a similar proof.

PROOF OF THEOREM 3, PART (a) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by Theorem 14.6.3:

$$D_{\mathbf{u}} f = f_x h + f_y k$$

Applying this theorem a second time, we have

$$\begin{aligned} D_{\mathbf{u}}^2 f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)k \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \qquad \text{(by Clairaut's Theorem)} \end{aligned}$$

If we complete the square in this expression, we obtain

10
$$D_{\mathbf{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$. But f_{xx} and $D = f_{xx}f_{yy} - f_{xy}^2$ are continuous functions, so there is a disk B with center (a, b) and radius $\delta > 0$ such that $f_{xx}(x, y) > 0$ and $D(x, y) > 0$ whenever (x, y) is in B. Therefore, by looking at Equation 10, we see that $D_{\mathbf{u}}^2 f(x, y) > 0$ whenever (x, y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a, b, f(a, b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x, y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x, y) \geqslant f(a, b)$ whenever (x, y) is in B. This shows that $f(a, b)$ is a local minimum. ■

14.7 EXERCISES

1. Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

(b) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$

2. Suppose $(0, 2)$ is a critical point of a function g with continuous second derivatives. In each case, what can you say about g?

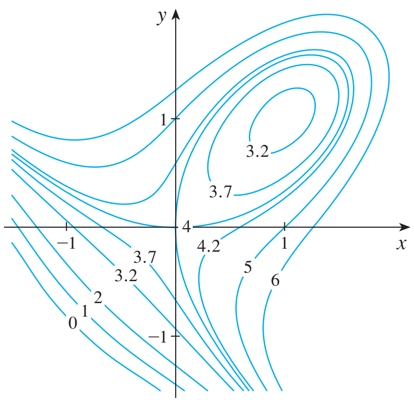
(a) $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 1$

(b) $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 2, \quad g_{yy}(0, 2) = -8$

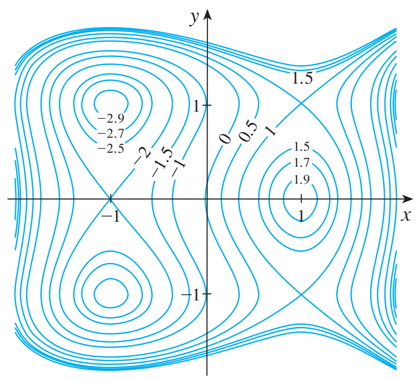
(c) $g_{xx}(0, 2) = 4, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 9$

3–4 Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your reasoning. Then use the Second Derivatives Test to confirm your predictions.

3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–20 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

6. $f(x, y) = xy - 2x - 2y - x^2 - y^2$

7. $f(x, y) = (x - y)(1 - xy)$

8. $f(x, y) = y(e^x - 1)$

9. $f(x, y) = x^2 + y^4 + 2xy$

10. $f(x, y) = 2 - x^4 + 2x^2 - y^2$

11. $f(x, y) = x^3 - 3x + 3xy^2$

12. $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

13. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

14. $f(x, y) = y \cos x$

15. $f(x, y) = e^x \cos y$

16. $f(x, y) = xye^{-(x^2+y^2)/2}$

17. $f(x, y) = xy + e^{-xy}$

18. $f(x, y) = (x^2 + y^2)e^{-x}$

19. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leqslant x \leqslant 7$

20. $f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi$

21. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

22. Show that $f(x, y) = x^2ye^{-x^2-y^2}$ has maximum values at $(\pm 1, 1/\sqrt{2})$ and minimum values at $(\pm 1, -1/\sqrt{2})$. Show also that f has infinitely many other critical points and $D = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

23–26 Use a graph or level curves or both to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

23. $f(x, y) = x^2 + y^2 + x^{-2}y^{-2}$

24. $f(x, y) = (x - y)e^{-x^2-y^2}$

25. $f(x, y) = \sin x + \sin y + \sin(x + y)$,
$0 \leqslant x \leqslant 2\pi, \; 0 \leqslant y \leqslant 2\pi$

26. $f(x, y) = \sin x + \sin y + \cos(x + y)$,
$0 \leqslant x \leqslant \pi/4, \; 0 \leqslant y \leqslant \pi/4$

27–30 Use a graphing device as in Example 4 (or Newton's method or solve numerically using a calculator or computer) to find the critical points of f correct to three decimal places. Then classify the critical points and find the highest or lowest points on the graph, if any.

27. $f(x, y) = x^4 + y^4 - 4x^2y + 2y$

28. $f(x, y) = y^6 - 2y^4 + x^2 - y^2 + y$

29. $f(x, y) = x^4 + y^3 - 3x^2 + y^2 + x - 2y + 1$

30. $f(x, y) = 20e^{-x^2-y^2} \sin 3x \cos 3y, \quad |x| \leqslant 1, \quad |y| \leqslant 1$

31–38 Find the absolute maximum and minimum values of f on the set D.

31. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

32. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$

33. $f(x, y) = x^2 + y^2 + x^2y + 4$,
$D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

34. $f(x, y) = x^2 + xy + y^2 - 6y$,
$D = \{(x, y) \mid -3 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 5\}$

35. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$,
$D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$

36. $f(x, y) = xy^2$, $D = \{(x, y) \mid x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\}$

37. $f(x, y) = 2x^3 + y^4$, $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$

38. $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$

39. For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

40. If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that f has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

41. Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

42. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.

43. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

44. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

45. Find three positive numbers whose sum is 100 and whose product is a maximum.

46. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

47. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

48. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.

49. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

50. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

51. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.

52. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

53. A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

54. A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of 10 units/m^2 per day, the north and south walls at a rate of 8 units/m^2 per day, the floor at a rate of 1 unit/m^2 per day, and the roof at a rate of 5 units/m^2 per day. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly 4000 m^3.

(a) Find and sketch the domain of the heat loss as a function of the lengths of the sides.

(b) Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)

(c) Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

55. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

56. A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$Y(N, P) = kNPe^{-N-P}$$

where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

57. The Shannon index (sometimes called the Shannon-Wiener index or Shannon-Weaver index) is a measure of diversity in an ecosystem. For the case of three species, it is defined as

$$H = -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3$$

where p_i is the proportion of species i in the ecosystem.

(a) Express H as a function of two variables using the fact that $p_1 + p_2 + p_3 = 1$.

(b) What is the domain of H?

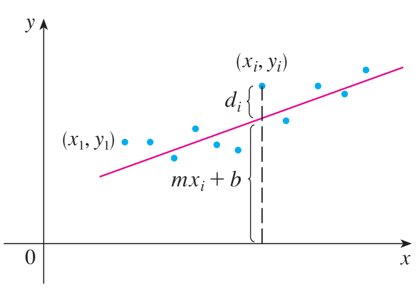
(c) Find the maximum value of H. For what values of p_1, p_2, p_3 does it occur?

58. Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most $\frac{2}{3}$.

59. Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b. The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible (see the figure).



Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least squares** determines m and b so as to minimize $\sum_{i=1}^{n} d_i^2$, the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

and
$$m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

Thus the line is found by solving these two equations in the two unknowns m and b. (See Section 1.2 for a further discussion and applications of the method of least squares.)

60. Find an equation of the plane that passes through the point (1, 2, 3) and cuts off the smallest volume in the first octant.

APPLIED PROJECT DESIGNING A DUMPSTER

For this project we locate a rectangular trash Dumpster in order to study its shape and construction. We then attempt to determine the dimensions of a container of similar design that minimize construction cost.

1. First locate a trash Dumpster in your area. Carefully study and describe all details of its construction, and determine its volume. Include a sketch of the container.

2. While maintaining the general shape and method of construction, determine the dimensions such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:

- The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost \$0.70 per square foot (including any required cuts or bends).
- The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs \$0.90 per square foot.
- Lids cost approximately \$50.00 each, regardless of dimensions.
- Welding costs approximately \$0.18 per foot for material and labor combined.

Give justification of any further assumptions or simplifications made of the details of construction.

3. Describe how any of your assumptions or simplifications may affect the final result.

4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the Dumpster? If so, describe the savings that would result.

DISCOVERY PROJECT QUADRATIC APPROXIMATIONS AND CRITICAL POINTS

The Taylor polynomial approximation to functions of one variable that we discussed in Chapter 11 can be extended to functions of two or more variables. Here we investigate quadratic approximations to functions of two variables and use them to give insight into the Second Derivatives Test for classifying critical points.

In Section 14.4 we discussed the linearization of a function f of two variables at a point (a, b):

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that the graph of L is the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$ and the corresponding linear approximation is $f(x, y) \approx L(x, y)$. The linearization L is also called the **first-degree Taylor polynomial** of f at (a, b).

1. If f has continuous second-order partial derivatives at (a, b), then the **second-degree Taylor polynomial** of f at (a, b) is

$$Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$+ \tfrac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \tfrac{1}{2} f_{yy}(a, b)(y - b)^2$$

and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** to f at (a, b). Verify that Q has the same first- and second-order partial derivatives as f at (a, b).

2. (a) Find the first- and second-degree Taylor polynomials L and Q of $f(x, y) = e^{-x^2-y^2}$ at $(0, 0)$.
(b) Graph f, L, and Q. Comment on how well L and Q approximate f.

3. (a) Find the first- and second-degree Taylor polynomials L and Q for $f(x, y) = xe^y$ at $(1, 0)$.
(b) Compare the values of L, Q, and f at $(0.9, 0.1)$.
(c) Graph f, L, and Q. Comment on how well L and Q approximate f.

4. In this problem we analyze the behavior of the polynomial $f(x, y) = ax^2 + bxy + cy^2$ (without using the Second Derivatives Test) by identifying the graph as a paraboloid.
(a) By completing the square, show that if $a \neq 0$, then

$$f(x, y) = ax^2 + bxy + cy^2 = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) Let $D = 4ac - b^2$. Show that if $D > 0$ and $a > 0$, then f has a local minimum at $(0, 0)$.
(c) Show that if $D > 0$ and $a < 0$, then f has a local maximum at $(0, 0)$.
(d) Show that if $D < 0$, then $(0, 0)$ is a saddle point.

5. (a) Suppose f is any function with continuous second-order partial derivatives such that $f(0, 0) = 0$ and $(0, 0)$ is a critical point of f. Write an expression for the second-degree Taylor polynomial, Q, of f at $(0, 0)$.
(b) What can you conclude about Q from Problem 4?
(c) In view of the quadratic approximation $f(x, y) \approx Q(x, y)$, what does part (b) suggest about f?

14.8 Lagrange Multipliers

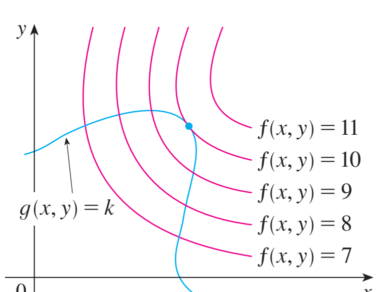


FIGURE 1

TEC Visual 14.8 animates Figure 1 for both level curves and level surfaces.

In Example 14.7.6 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was 12 m^2. In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This

means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know from Section 14.6 that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 14.6.18.) This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

1 $$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

Lagrange multipliers are named after the French-Italian mathematician Joseph-Louis Lagrange (1736–1813). See page 217 for a biographical sketch of Lagrange.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

In deriving Lagrange's method we assumed that $\nabla g \neq \mathbf{0}$. In each of our examples you can check that $\nabla g \neq \mathbf{0}$ at all points where $g(x, y, z) = k$. See Exercise 25 for what can go wrong if $\nabla g = \mathbf{0}$.

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Our first illustration of Lagrange's method is to reconsider the problem given in Example 14.7.6.

EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 14.7.6, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

Another method for solving the system of equations (2–5) is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3),

and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7. ■

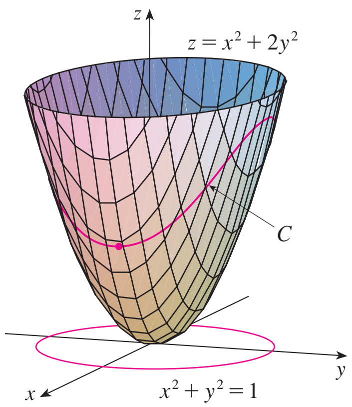


FIGURE 2

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

9 $$2x = 2x\lambda$$

10 $$4y = 2y\lambda$$

11 $$x^2 + y^2 = 1$$

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. In geometric terms, these correspond to the highest and lowest points on the curve C in Figure 2, where C consists of those points on the paraboloid $z = x^2 + 2y^2$ that are directly above the constraint circle $x^2 + y^2 = 1$. ■

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

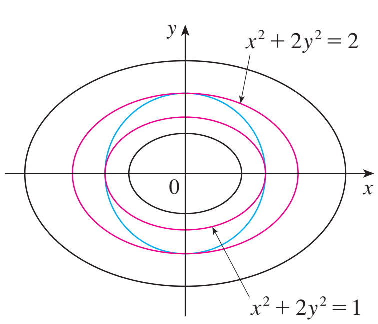


FIGURE 3

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

SOLUTION According to the procedure in (14.7.9), we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$. ■

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

12 $$2(x-3) = 2x\lambda$$

13 $$2(y-1) = 2y\lambda$$

14 $$2(z+1) = 2z\lambda$$

15 $$x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \qquad \text{or} \qquad x(1-\lambda) = 3 \qquad \text{or} \qquad x = \frac{3}{1-\lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1-\lambda} \qquad z = -\frac{1}{1-\lambda}$$

Figure 4 shows the sphere and the nearest point P in Example 4. Can you see how to find the coordinates of P without using calculus?

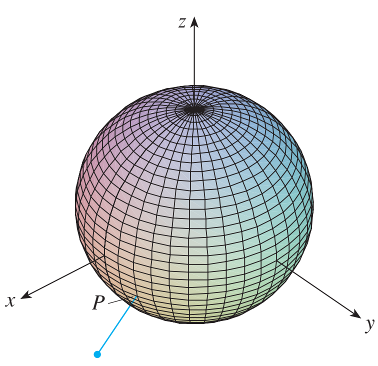


FIGURE 4

Therefore, from (15), we have

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

which gives $(1-\lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \qquad \text{and} \qquad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$. ■

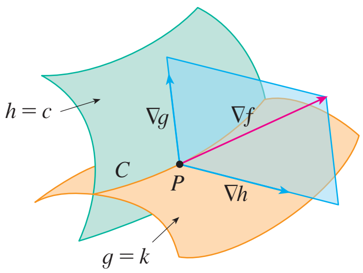


FIGURE 5

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ (called Lagrange multipliers) such that

16 $$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 6). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

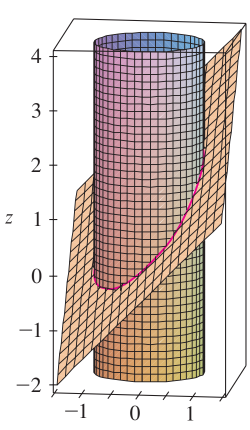


FIGURE 6

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

17 $$1 = \lambda + 2x\mu$$

18 $$2 = -\lambda + 2y\mu$$

19 $$3 = \lambda$$

20 $$x - y + z = 1$$

21 $$x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

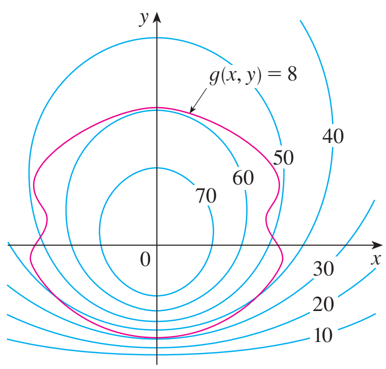
and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$. ■

14.8 EXERCISES

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



2. (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?
(b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3–14 Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 1$

4. $f(x, y) = 3x + y$; $x^2 + y^2 = 10$

5. $f(x, y) = xy$; $4x^2 + y^2 = 8$

6. $f(x, y) = xe^y$; $x^2 + y^2 = 2$

7. $f(x, y, z) = 2x + 2y + z$; $x^2 + y^2 + z^2 = 9$

8. $f(x, y, z) = e^{xyz}$; $2x^2 + y^2 + z^2 = 24$

9. $f(x, y, z) = xy^2z$; $x^2 + y^2 + z^2 = 4$

10. $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1)$; $x^2 + y^2 + z^2 = 12$

11. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$

12. $f(x, y, z) = x^4 + y^4 + z^4$; $x^2 + y^2 + z^2 = 1$

13. $f(x, y, z, t) = x + y + z + t$; $x^2 + y^2 + z^2 + t^2 = 1$

14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$; $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

15. The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$ can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

16. Find the minimum value of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ subject to the constraint $x + 2y + 3z = 10$. Show that f has no maximum value with this constraint.

17–20 Find the extreme values of f subject to both constraints.

17. $f(x, y, z) = x + y + z$; $x^2 + z^2 = 2$, $x + y = 1$

18. $f(x, y, z) = z$; $x^2 + y^2 = z^2$, $x + y + z = 24$

19. $f(x, y, z) = yz + xy$; $xy = 1$, $y^2 + z^2 = 1$

20. $f(x, y, z) = x^2 + y^2 + z^2$; $x - y = 1$, $y^2 - z^2 = 1$

21–23 Find the extreme values of f on the region described by the inequality.

21. $f(x, y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leqslant 9$

22. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$, $x^2 + y^2 \leqslant 16$

23. $f(x, y) = e^{-xy}$, $x^2 + 4y^2 \leqslant 1$

24. Consider the problem of maximizing the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.
(a) Try using Lagrange multipliers to solve the problem.
(b) Does $f(25, 0)$ give a larger value than the one in part (a)?
(c) Solve the problem by graphing the constraint equation and several level curves of f.
(d) Explain why the method of Lagrange multipliers fails to solve the problem.
(e) What is the significance of $f(9, 4)$?

25. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
(a) Try using Lagrange multipliers to solve the problem.
(b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
(c) Explain why Lagrange multipliers fail to find the minimum value in this case.

CAS **26.** (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods.
(b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations numerically. Compare your answers with those in part (a).

27. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 14.1 and 14.3 we discussed how the Cobb-Douglas model $P = bL^{\alpha}K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n, and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

28. Referring to Exercise 27, we now suppose that the production is fixed at $bL^{\alpha}K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?

29. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

30. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s - x)(s - y)(s - z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

31–43 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

31. Exercise 41
32. Exercise 42
33. Exercise 43
34. Exercise 44
35. Exercise 45
36. Exercise 46
37. Exercise 47
38. Exercise 48
39. Exercise 49
40. Exercise 50
41. Exercise 51
42. Exercise 52
43. Exercise 55

44. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

45. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

46. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
(a) Graph the cone and the plane, and observe the elliptical intersection.
(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

CAS **47–48** Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

47. $f(x, y, z) = ye^{x-z}$; $\quad 9x^2 + 4y^2 + 36z^2 = 36,\ xy + yz = 1$

48. $f(x, y, z) = x + y + z$; $\quad x^2 - y^2 = z,\ x^2 + z^2 = 4$

49. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.
(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

50. (a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$.
(b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_j^2}}$$

to show that

$$\sum a_i b_i \leqslant \sqrt{\sum a_j^2}\,\sqrt{\sum b_j^2}$$

for any numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

APPLIED PROJECT ROCKET SCIENCE







Courtesy of Orbital Sciences Corporation

Many rockets, such as the *Pegasus XL* currently used to launch satellites and the *Saturn V* that first put men on the moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln\left(1 - \frac{(1 - S)M_r}{P + M_r}\right)$$

where M_r is the mass of the rocket engine including initial fuel, P is the mass of the payload, S is a *structural factor* determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload), and c is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

1. Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]$$

2. We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity v_f from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions. To simplify, we define variables N_i so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since M is now difficult to express in terms of the N_i's, we wish to use a simpler function that will be minimized at the same place as M. Show that

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1 - S)N_1}{1 - SN_1}$$

$$\frac{M_2 + M_3 + A}{M_3 + A} = \frac{(1 - S)N_2}{1 - SN_2}$$

$$\frac{M_3 + A}{A} = \frac{(1 - S)N_3}{1 - SN_3}$$

and conclude that

$$\frac{M + A}{A} = \frac{(1 - S)^3 N_1 N_2 N_3}{(1 - SN_1)(1 - SN_2)(1 - SN_3)}$$

3. Verify that $\ln((M + A)/A)$ is minimized at the same location as M; use Lagrange multipliers and the results of Problem 2 to find expressions for the values of N_i where the minimum occurs subject to the constraint $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. [*Hint:* Use properties of logarithms to help simplify the expressions.]

4. Find an expression for the minimum value of M as a function of v_f.

5. If we want to put a three-stage rocket into orbit 100 miles above the earth's surface, a final velocity of approximately 17,500 mi/h is required. Suppose that each stage is built with a structural factor $S = 0.2$ and an exhaust speed of $c = 6000$ mi/h.
(a) Find the minimum total mass M of the rocket engines as a function of A.
(b) Find the mass of each individual stage as a function of A. (They are not equally sized!)

6. The same rocket would require a final velocity of approximately 24,700 mi/h in order to escape earth's gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 500-pound probe into deep space.

APPLIED PROJECT HYDRO-TURBINE OPTIMIZATION

At a hydroelectric generating station (once operated by the Katahdin Paper Company) in Millinocket, Maine, water is piped from a dam to the power station. The rate at which the water flows through the pipe varies, depending on external conditions.

The power station has three different hydroelectric turbines, each with a known (and unique) power function that gives the amount of electric power generated as a function of the water flow arriving at the turbine. The incoming water can be apportioned in different volumes to each turbine, so the goal is to determine how to distribute water among the turbines to give the maximum total energy production for any rate of flow.

Using experimental evidence and *Bernoulli's equation*, the following quadratic models were determined for the power output of each turbine, along with the allowable flows of operation:

$$KW_1 = (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5}Q_1^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$KW_2 = (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5}Q_2^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$KW_3 = (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5}Q_3^2)(170 - 1.6 \cdot 10^{-6}Q_T^2)$$

$$250 \leqslant Q_1 \leqslant 1110, \quad 250 \leqslant Q_2 \leqslant 1110, \quad 250 \leqslant Q_3 \leqslant 1225$$

where

Q_i = flow through turbine i in cubic feet per second

KW_i = power generated by turbine i in kilowatts

Q_T = total flow through the station in cubic feet per second

1. If all three turbines are being used, we wish to determine the flow Q_i to each turbine that will give the maximum total energy production. Our limitations are that the flows must sum to the total incoming flow and the given domain restrictions must be observed. Consequently, use Lagrange multipliers to find the values for the individual flows (as functions of Q_T) that maximize the total energy production $KW_1 + KW_2 + KW_3$ subject to the constraints $Q_1 + Q_2 + Q_3 = Q_T$ and the domain restrictions on each Q_i.

2. For which values of Q_T is your result valid?

3. For an incoming flow of 2500 ft^3/s, determine the distribution to the turbines and verify (by trying some nearby distributions) that your result is indeed a maximum.

4. Until now we have assumed that all three turbines are operating; is it possible in some situations that more power could be produced by using only one turbine? Make a graph of the three power functions and use it to help decide if an incoming flow of 1000 ft^3/s should be

distributed to all three turbines or routed to just one. (If you determine that only one turbine should be used, which one would it be?) What if the flow is only 600 ft^3/s?

5. Perhaps for some flow levels it would be advantageous to use two turbines. If the incoming flow is 1500 ft^3/s, which two turbines would you recommend using? Use Lagrange multipliers to determine how the flow should be distributed between the two turbines to maximize the energy produced. For this flow, is using two turbines more efficient than using all three?

6. If the incoming flow is 3400 ft^3/s, what would you recommend to the station management?

14 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) What is a function of two variables?
 (b) Describe three methods for visualizing a function of two variables.

2. What is a function of three variables? How can you visualize such a function?

3. What does

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

mean? How can you show that such a limit does not exist?

4. (a) What does it mean to say that f is continuous at (a, b)?
 (b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.
 (b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret them as rates of change?
 (c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y?

6. What does Clairaut's Theorem say?

7. How do you find a tangent plane to each of the following types of surfaces?
 (a) A graph of a function of two variables, $z = f(x, y)$
 (b) A level surface of a function of three variables, $F(x, y, z) = k$

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

9. (a) What does it mean to say that f is differentiable at (a, b)?
 (b) How do you usually verify that f is differentiable?

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$. How do you interpret it as a rate? How do you interpret it geometrically?
 (b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

14. (a) Define the gradient vector ∇f for a function f of two or three variables.
 (b) Express $D_{\mathbf{u}} f$ in terms of ∇f.
 (c) Explain the geometric significance of the gradient.

15. What do the following statements mean?
 (a) f has a local maximum at (a, b).
 (b) f has an absolute maximum at (a, b).
 (c) f has a local minimum at (a, b).
 (d) f has an absolute minimum at (a, b).
 (e) f has a saddle point at (a, b).

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?
 (b) What is a critical point of f?

17. State the Second Derivatives Test.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?
 (b) State the Extreme Value Theorem for functions of two variables.
 (c) How do you find the values that the Extreme Value Theorem guarantees?

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$

2. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

3. $f_{xy} = \dfrac{\partial^2 f}{\partial x \, \partial y}$

4. $D_{\mathbf{k}} f(x, y, z) = f_z(x, y, z)$

5. If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\lim_{(x, y) \to (a, b)} f(x, y) = L$.

6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

7. If f has a local minimum at (a, b) and f is differentiable at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

8. If f is a function, then

$$\lim_{(x, y) \to (2, 5)} f(x, y) = f(2, 5)$$

9. If $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$.

10. If $(2, 1)$ is a critical point of f and

$$f_{xx}(2, 1) f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$$

then f has a saddle point at $(2, 1)$.

11. If $f(x, y) = \sin x + \sin y$, then $-\sqrt{2} \leqslant D_{\mathbf{u}} f(x, y) \leqslant \sqrt{2}$.

12. If $f(x, y)$ has two local maxima, then f must have a local minimum.

EXERCISES

1–2 Find and sketch the domain of the function.

1. $f(x, y) = \ln(x + y + 1)$

2. $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

3–4 Sketch the graph of the function.

3. $f(x, y) = 1 - y^2$

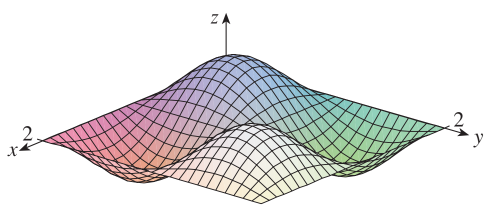
4. $f(x, y) = x^2 + (y - 2)^2$

5–6 Sketch several level curves of the function.

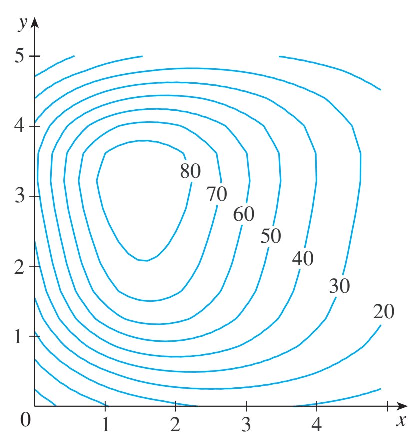
5. $f(x, y) = \sqrt{4x^2 + y^2}$

6. $f(x, y) = e^x + y$

7. Make a rough sketch of a contour map for the function whose graph is shown.



8. The contour map of a function f is shown.
(a) Estimate the value of $f(3, 2)$.
(b) Is $f_x(3, 2)$ positive or negative? Explain.
(c) Which is greater, $f_y(2, 1)$ or $f_y(2, 2)$? Explain.



9–10 Evaluate the limit or show that it does not exist.

9. $\displaystyle\lim_{(x, y) \to (1, 1)} \frac{2xy}{x^2 + 2y^2}$ **10.** $\displaystyle\lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + 2y^2}$

11. A metal plate is situated in the xy-plane and occupies the rectangle $0 \leqslant x \leqslant 10$, $0 \leqslant y \leqslant 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures at equally spaced points were measured and recorded in the table.
(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?

(b) Estimate the value of $D_{\mathbf{u}}T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Interpret your result.

(c) Estimate the value of $T_{xy}(6, 4)$.

x \ y	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

12. Find a linear approximation to the temperature function $T(x, y)$ in Exercise 11 near the point (6, 4). Then use it to estimate the temperature at the point (5, 3.8).

13–17 Find the first partial derivatives.

13. $f(x, y) = (5y^3 + 2x^2y)^8$

14. $g(u, v) = \dfrac{u + 2v}{u^2 + v^2}$

15. $F(\alpha, \beta) = \alpha^2 \ln(\alpha^2 + \beta^2)$

16. $G(x, y, z) = e^{xz}\sin(y/z)$

17. $S(u, v, w) = u \arctan\left(v\sqrt{w}\right)$

18. The speed of sound traveling through ocean water is a function of temperature, salinity, and pressure. It has been modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), S is the salinity (the concentration of salts in parts per thousand, which means the number of grams of dissolved solids per 1000 g of water), and D is the depth below the ocean surface (in meters). Compute $\partial C/\partial T$, $\partial C/\partial S$, and $\partial C/\partial D$ when $T = 10°\text{C}$, $S = 35$ parts per thousand, and $D = 100$ m. Explain the physical significance of these partial derivatives.

19–22 Find all second partial derivatives of f.

19. $f(x, y) = 4x^3 - xy^2$

20. $z = xe^{-2y}$

21. $f(x, y, z) = x^k y^l z^m$

22. $v = r\cos(s + 2t)$

23. If $z = xy + xe^{y/x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = xy + z$.

24. If $z = \sin(x + \sin t)$, show that

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}$$

25–29 Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

25. $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$

26. $z = e^x \cos y$, $(0, 0, 1)$

27. $x^2 + 2y^2 - 3z^2 = 3$, $(2, -1, 1)$

28. $xy + yz + zx = 3$, $(1, 1, 1)$

29. $\sin(xyz) = x + 2y + 3z$, $(2, -1, 0)$

30. Use a computer to graph the surface $z = x^2 + y^4$ and its tangent plane and normal line at (1, 1, 2) on the same screen. Choose the domain and viewpoint so that you get a good view of all three objects.

31. Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

32. Find du if $u = \ln(1 + se^{2t})$.

33. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2, 3, 4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

34. The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

35. If $u = x^2y^3 + z^4$, where $x = p + 3p^2$, $y = pe^p$, and $z = p\sin p$, use the Chain Rule to find du/dp.

36. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\partial v/\partial s$ and $\partial v/\partial t$ when $s = 0$ and $t = 1$.

37. Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

38. Use a tree diagram to write out the Chain Rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$, and $v = v(p, q, r, s)$ are all differentiable functions.

39. If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

40. The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

41. If $z = f(u, v)$, where $u = xy$, $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u\,\partial v} + 2v\frac{\partial z}{\partial v}$$

42. If $\cos(xyz) = 1 + x^2y^2 + z^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

43. Find the gradient of the function $f(x, y, z) = x^2e^{yz^2}$.

44. (a) When is the directional derivative of f a maximum?
(b) When is it a minimum?
(c) When is it 0?
(d) When is it half of its maximum value?

45–46 Find the directional derivative of f at the given point in the indicated direction.

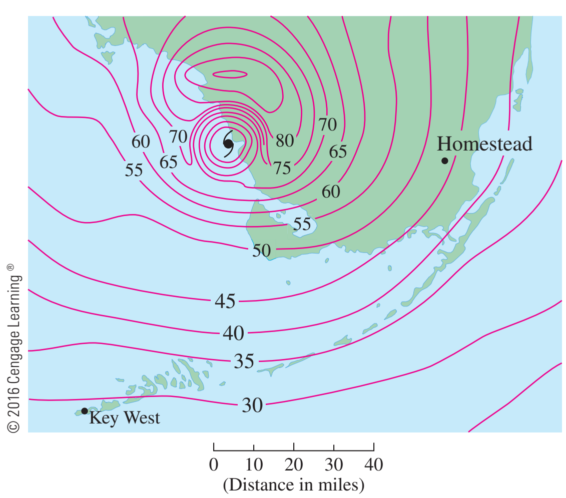
45. $f(x, y) = x^2e^{-y}$, $(-2, 0)$, in the direction toward the point $(2, -3)$

46. $f(x, y, z) = x^2y + x\sqrt{1 + z}$, $(1, 2, 3)$, in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

47. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$. In which direction does it occur?

48. Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

49. The contour map shows wind speed in knots during Hurricane Andrew on August 24, 1992. Use it to estimate the value of the directional derivative of the wind speed at Homestead, Florida, in the direction of the eye of the hurricane.



50. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

51–54 Find the local maximum and minimum values and saddle points of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

51. $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$

52. $f(x, y) = x^3 - 6xy + 8y^3$

53. $f(x, y) = 3xy - x^2y - xy^2$

54. $f(x, y) = (x^2 + y)e^{y/2}$

55–56 Find the absolute maximum and minimum values of f on the set D.

55. $f(x, y) = 4xy^2 - x^2y^2 - xy^3$; D is the closed triangular region in the xy-plane with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$

56. $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$; D is the disk $x^2 + y^2 \leqslant 4$

57. Use a graph or level curves or both to estimate the local maximum and minimum values and saddle points of $f(x, y) = x^3 - 3x + y^4 - 2y^2$. Then use calculus to find these values precisely.

58. Use a graphing calculator or computer (or Newton's method or a computer algebra system) to find the critical points of $f(x, y) = 12 + 10y - 2x^2 - 8xy - y^4$ correct to three decimal places. Then classify the critical points and find the highest point on the graph.

59–62 Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint(s).

59. $f(x, y) = x^2y$; $x^2 + y^2 = 1$

60. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$; $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$

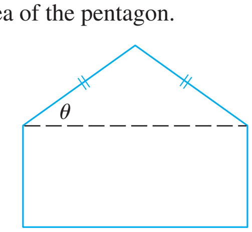
61. $f(x, y, z) = xyz$; $x^2 + y^2 + z^2 = 3$

62. $f(x, y, z) = x^2 + 2y^2 + 3z^2$;
$x + y + z = 1$, $x - y + 2z = 2$

63. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

64. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

65. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the figure. If the pentagon has fixed perimeter P, find the lengths of the sides of the pentagon that maximize the area of the pentagon.



Problems Plus

1. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles.

2. Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests, the concentration of blood (in parts per million) at a point $P(x, y)$ on the surface of seawater is approximated by

$$C(x, y) = e^{-(x^2+2y^2)/10^4}$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

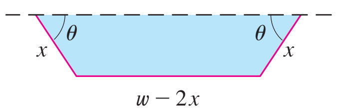
(a) Identify the level curves of the concentration function and sketch several members of this family together with a path that a shark will follow to the source.

(b) Suppose a shark is at the point (x_0, y_0) when it first detects the presence of blood in the water. Find an equation of the shark's path by setting up and solving a differential equation.

3. A long piece of galvanized sheet metal with width w is to be bent into a symmetric form with three straight sides to make a rain gutter. A cross-section is shown in the figure.

(a) Determine the dimensions that allow the maximum possible flow; that is, find the dimensions that give the maximum possible cross-sectional area.

(b) Would it be better to bend the metal into a gutter with a semicircular cross-section?



4. For what values of the number r is the function

$$f(x, y, z) = \begin{cases} \dfrac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}$$

continuous on $\mathbb{R}^3$?

5. Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = xf(y/x)$ intersect in a common point.

6. (a) Newton's method for approximating a root of an equation $f(x) = 0$ (see Section 3.8) can be adapted to approximating a solution of a system of equations $f(x, y) = 0$ and $g(x, y) = 0$. The surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect in a curve that intersects the xy-plane at the point (r, s), which is the solution of the system. If an initial approximation (x_1, y_1) is close to this point, then the tangent planes to the surfaces at (x_1, y_1) intersect in a straight line that intersects the xy-plane in a point (x_2, y_2), which should be closer to (r, s). (Compare with Figure 3.8.2.) Show that

$$x_2 = x_1 - \frac{fg_y - f_y g}{f_x g_y - f_y g_x} \qquad \text{and} \qquad y_2 = y_1 - \frac{f_x g - f g_x}{f_x g_y - f_y g_x}$$

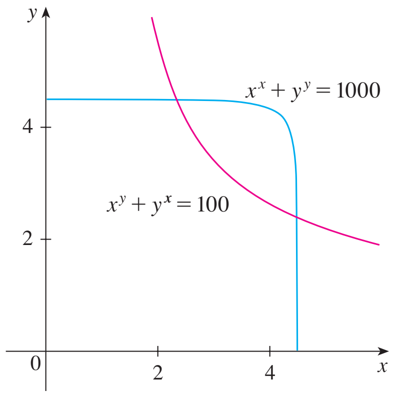
where f, g, and their partial derivatives are evaluated at (x_1, y_1). If we continue this procedure, we obtain successive approximations (x_n, y_n).

(b) It was Thomas Simpson (1710–1761) who formulated Newton's method as we know it today and who extended it to functions of two variables as in part (a). (See the biogra-

phy of Simpson on page 560.) The example that he gave to illustrate the method was to solve the system of equations

$$x^x + y^y = 1000 \qquad x^y + y^x = 100$$

In other words, he found the points of intersection of the curves in the figure. Use the method of part (a) to find the coordinates of the points of intersection correct to six decimal places.



7. If the ellipse $x^2/a^2 + y^2/b^2 = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

8. Show that the maximum value of the function

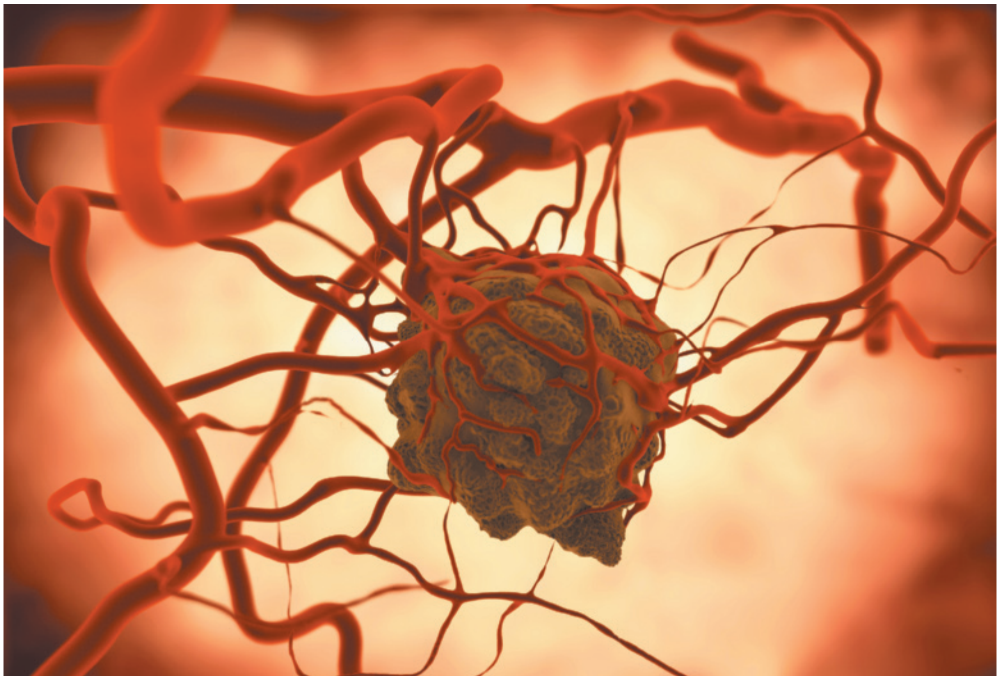
$$f(x, y) = \frac{(ax + by + c)^2}{x^2 + y^2 + 1}$$

is $a^2 + b^2 + c^2$.
Hint: One method for attacking this problem is to use the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$$

(See Exercise 12.3.61.)

15 Multiple Integrals



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Tumors, like the one shown, have been modeled as "bumpy spheres." In Exercise 47 in Section 15.8 you are asked to compute the volume enclosed by such a surface.

IN THIS CHAPTER WE EXTEND the idea of a definite integral to double and triple integrals of functions of two or three variables. These ideas are then used to compute volumes, masses, and centroids of more general regions than we were able to consider in Chapters 5 and 8. We also use double integrals to calculate probabilities when two random variables are involved.

We will see that polar coordinates are useful in computing double integrals over some types of regions. In a similar way, we will introduce two new coordinate systems in three-dimensional space—cylindrical coordinates and spherical coordinates—that greatly simplify the computation of triple integrals over certain commonly occurring solid regions.

15.1 Double Integrals over Rectangles

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1 $$\sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2 $$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x) \, dx$ represents the area under the curve $y = f(x)$ from a to b.

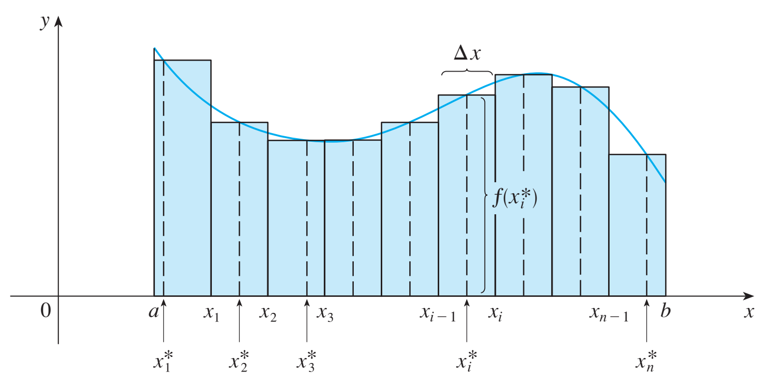


FIGURE 1

Volumes and Double Integrals

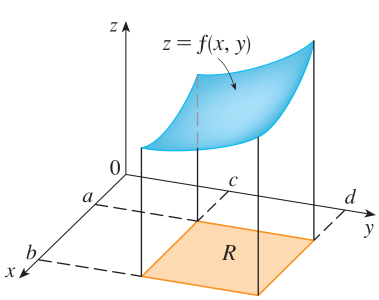


FIGURE 2

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, \ c \leqslant y \leqslant d\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y), \ (x, y) \in R\}$$

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals,

as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

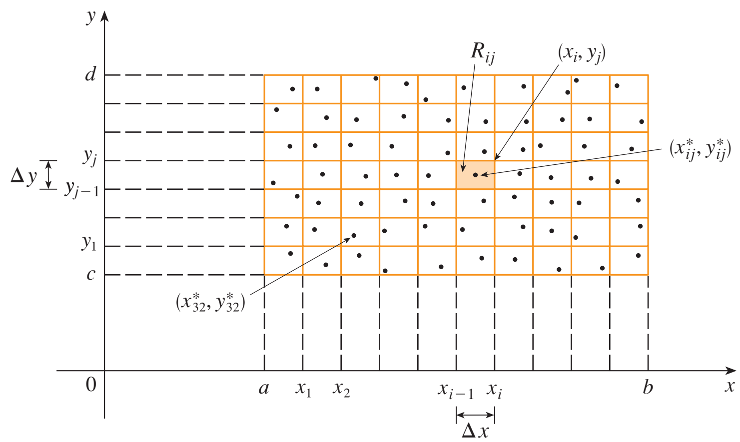


FIGURE 3
Dividing R into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

3
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

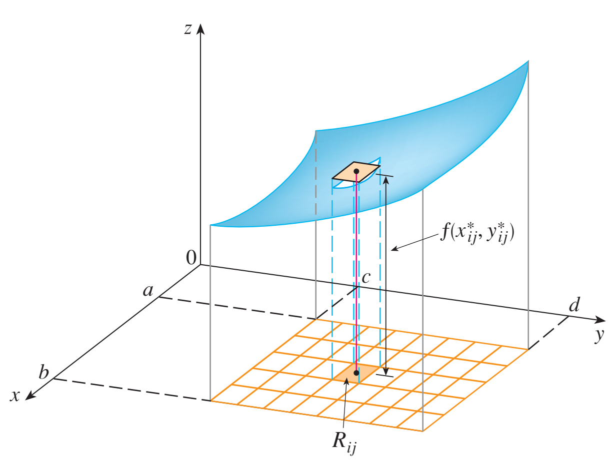


FIGURE 4

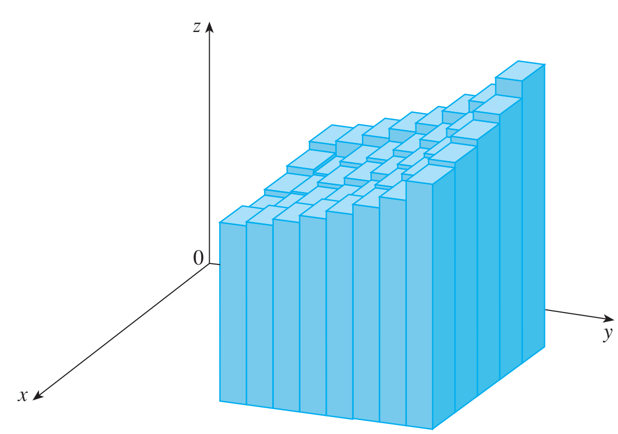


FIGURE 5

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

The meaning of the double limit in Equation 4 is that we can make the double sum as close as we like to the number V [for any choice of (x_{ij}^*, y_{ij}^*) in R_{ij}] by taking m and n sufficiently large.

4
$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

We use the expression in Equation 4 to define the **volume** of the solid S that lies under the graph of f and above the rectangle R. (It can be shown that this definition is consistent with our formula for volume in Section 5.2.)

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations as well—as we will see in Section 15.4—even when f is not a positive function. So we make the following definition.

Notice the similarity between Definition 5 and the definition of a single integral in Equation 2.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y) \, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

if this limit exists.

Although we have defined the double integral by dividing R into equal-sized subrectangles, we could have used subrectangles R_{ij} of unequal size. But then we would have to ensure that all of their dimensions approach 0 in the limiting process.

The precise meaning of the limit in Definition 5 is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x, y) \, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

A function f is called **integrable** if the limit in Definition 5 exists. It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is "not too discontinuous." In particular, if f is bounded on R, [that is, there is a constant M such that $| f(x, y) | \leqslant M$ for all (x, y) in R], and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

6
$$\iint_R f(x, y) \, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \, \Delta A$$

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y) \, dA$$

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f.

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

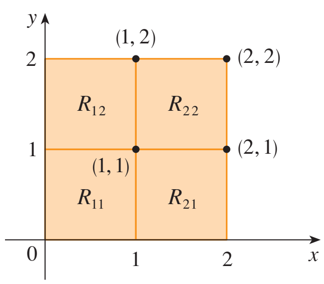


FIGURE 6

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \, \Delta A$$

$$= f(1, 1) \, \Delta A + f(1, 2) \, \Delta A + f(2, 1) \, \Delta A + f(2, 2) \, \Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

This is the volume of the approximating rectangular boxes shown in Figure 7. ■

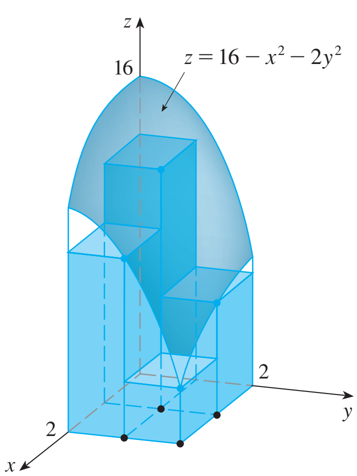
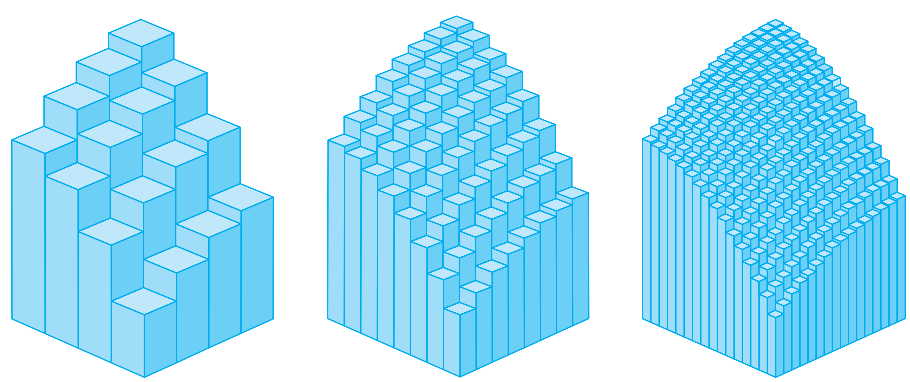


FIGURE 7

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In Example 7 we will be able to show that the exact volume is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

FIGURE 8
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2} \, dA$$

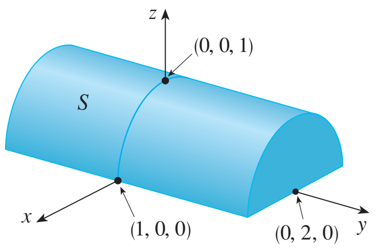


FIGURE 9

SOLUTION It would be very difficult to evaluate this integral directly from Definition 5 but, because $\sqrt{1 - x^2} \geqslant 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geqslant 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. (See Figure 9.) The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

The Midpoint Rule

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_j)$ of R_{ij}. In other words, $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

EXAMPLE 3 Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$.

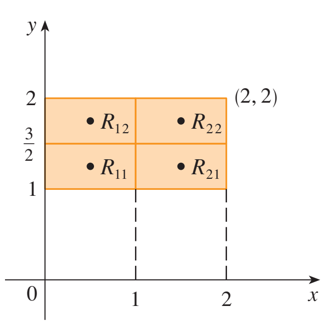


FIGURE 10

SOLUTION In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A \\
&= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A \\
&= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A \\
&= \left(-\tfrac{67}{16}\right)\tfrac{1}{2} + \left(-\tfrac{139}{16}\right)\tfrac{1}{2} + \left(-\tfrac{51}{16}\right)\tfrac{1}{2} + \left(-\tfrac{123}{16}\right)\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875
\end{aligned}$$

Thus we have

$$\iint_R (x - 3y^2)\, dA \approx -11.875$$

Number of subrectangles	Midpoint Rule approximation
1	−11.5000
4	−11.8750
16	−11.9687
64	−11.9922
256	−11.9980
1024	−11.9995

NOTE In Example 5 we will see that the exact value of the double integral in Example 3 is −12. (Remember that the interpretation of a double integral as a volume is valid only when the integrand f is a *positive* function. The integrand in Example 3 is not a positive function, so its integral is not a volume. In Examples 5 and 6 we will discuss how to interpret integrals of functions that are not always positive in terms of volumes.) If we keep dividing each subrectangle in Figure 10 into four smaller ones with similar shape, we get the Midpoint Rule approximations displayed in the chart in the margin. Notice how these approximations approach the exact value of the double integral, −12.

Iterated Integrals

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral, but the Fundamental Theorem of Calculus provides a much easier method. The evaluation of double integrals from first principles is even more difficult, but here we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\,dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\,dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

7 $$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

8 $$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

9 $$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations 8 and 9 we work *from the inside out.*

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\displaystyle\int_1^2 \int_0^3 x^2 y\,dx\,dy$

SOLUTION

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\,dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \left[\int_1^2 x^2 y\,dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2}x^2\,dx = \frac{x^3}{2}\bigg]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Theorem 10) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Theorem 10 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907. But the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

10 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

The proof of Fubini's Theorem is too difficult to include in this book, but we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geqslant 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y) \, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 5, namely,

$$V = \int_a^b A(x) \, dx$$

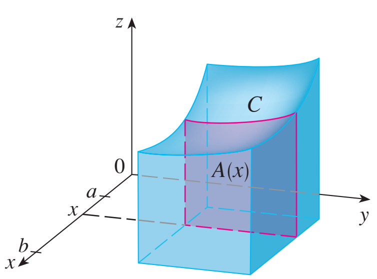


FIGURE 11

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 11 you can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leqslant y \leqslant d$. Therefore

TEC Visual 15.1 illustrates Fubini's Theorem by showing an animation of Figures 11 and 12.

$$A(x) = \int_c^d f(x, y) \, dy$$

and we have

$$\iint_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

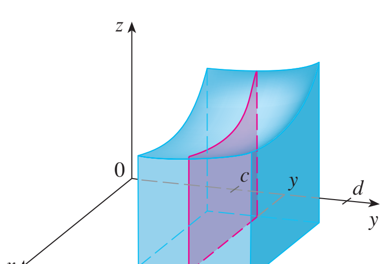


FIGURE 12

A similar argument, using cross-sections perpendicular to the y-axis as in Figure 12, shows that

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

EXAMPLE 5 Evaluate the double integral $\iint_R (x - 3y^2)\,dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3.)

SOLUTION 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx = \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \frac{x^2}{2} - 7x\Big]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12$$ ■

Notice the negative answer in Example 5; nothing is wrong with that. The function f is not a positive function, so its integral doesn't represent a volume. From Figure 13 we see that f is always negative on R, so the value of the integral is the *negative* of the volume that lies *above* the graph of f and *below* R.

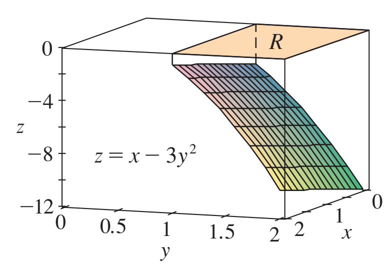


FIGURE 13

EXAMPLE 6 Evaluate $\iint_R y \sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION If we first integrate with respect to x, we get

$$\iint_R y \sin(xy)\,dA = \int_0^\pi \int_1^2 y \sin(xy)\,dx\,dy$$

$$= \int_0^\pi \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$ ■

For a function f that takes on both positive and negative values, $\iint_R f(x, y)\,dA$ is a difference of volumes: $V_1 - V_2$, where V_1 is the volume above R and below the graph of f, and V_2 is the volume below R and above the graph. The fact that the integral in Example 6 is 0 means that these two volumes V_1 and V_2 are equal. (See Figure 14.)

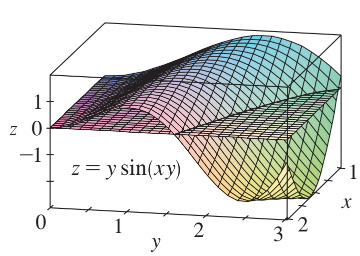


FIGURE 14

NOTE If we reverse the order of integration and first integrate with respect to y in Example 6, we get

$$\iint_R y \sin(xy)\,dA = \int_1^2 \int_0^\pi y \sin(xy)\,dy\,dx$$

but this order of integration is much more difficult than the method given in the example because it involves integration by parts twice. Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

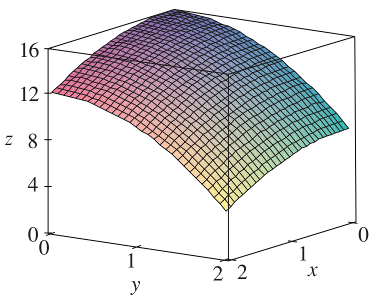


FIGURE 15

SOLUTION We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0, 2] \times [0, 2]$. (See Figure 15.) This solid was considered in Example 1, but we are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned} V &= \iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy \\ &= \int_0^2 \Big[16x - \tfrac{1}{3}x^3 - 2y^2x\Big]_{x=0}^{x=2} dy \\ &= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right) dy = \Big[\tfrac{88}{3}y - \tfrac{4}{3}y^3\Big]_0^2 = 48 \end{aligned}$$

■

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$. Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx\right] dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\, dx\right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\, dx\right)\right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

since $\int_a^b g(x)\, dx$ is a constant. Therefore, in this case the double integral of f can be written as the product of two single integrals:

11 $$\iint_R g(x)\,h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \qquad \text{where } R = [a, b] \times [c, d]$$

EXAMPLE 8 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 11,

$$\begin{aligned} \iint_R \sin x \cos y\, dA &= \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy \\ &= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1 \end{aligned}$$

■

The function $f(x, y) = \sin x \cos y$ in Example 8 is positive on R, so the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 16.

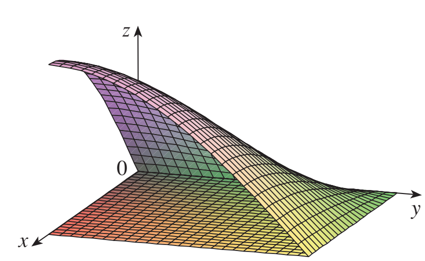


FIGURE 16

Average Value

Recall from Section 5.5 that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In a similar fashion we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$$

where $A(R)$ is the area of R.

If $f(x, y) \geqslant 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\,dA$$

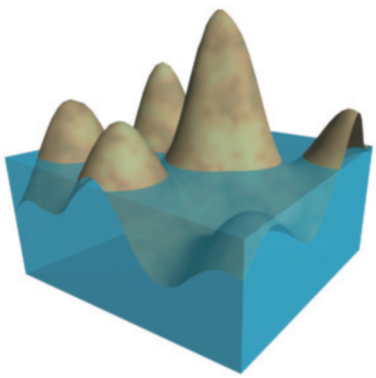


FIGURE 17

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f. [If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 17.]

EXAMPLE 9 The contour map in Figure 18 shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.

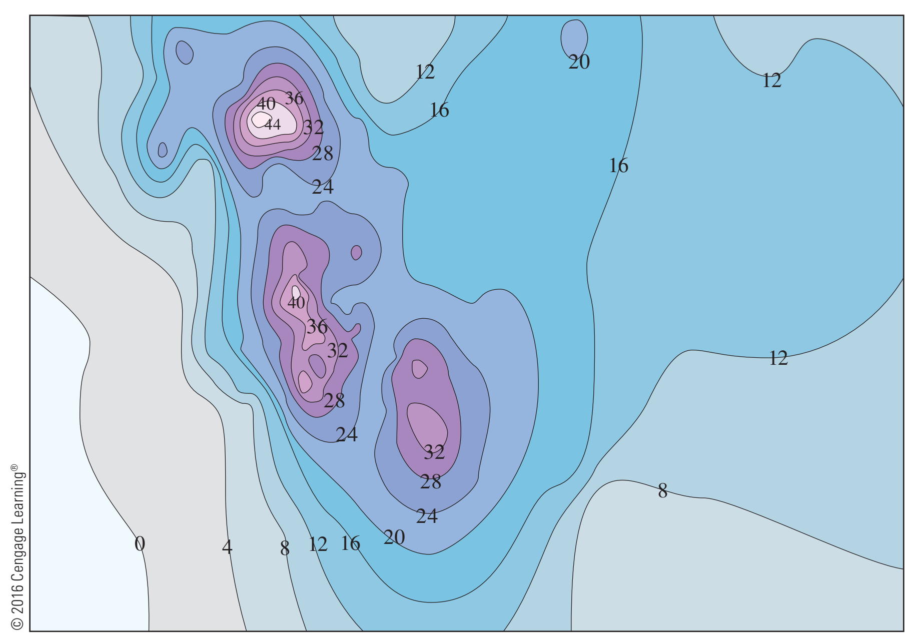


FIGURE 18

SOLUTION Let's place the origin at the southwest corner of the state. Then $0 \leqslant x \leqslant 388$, $0 \leqslant y \leqslant 276$, and $f(x, y)$ is the snowfall, in inches, at a location x miles to the east and y miles to the north of the origin. If R is the rectangle that represents Colorado, then the average snowfall for the state on December 20–21 was

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$$

where $A(R) = 388 \cdot 276$. To estimate the value of this double integral, let's use the Midpoint Rule with $m = n = 4$. In other words, we divide R into 16 subrectangles of equal size, as in Figure 19. The area of each subrectangle is

$$\Delta A = \tfrac{1}{16}(388)(276) = 6693 \text{ mi}^2$$

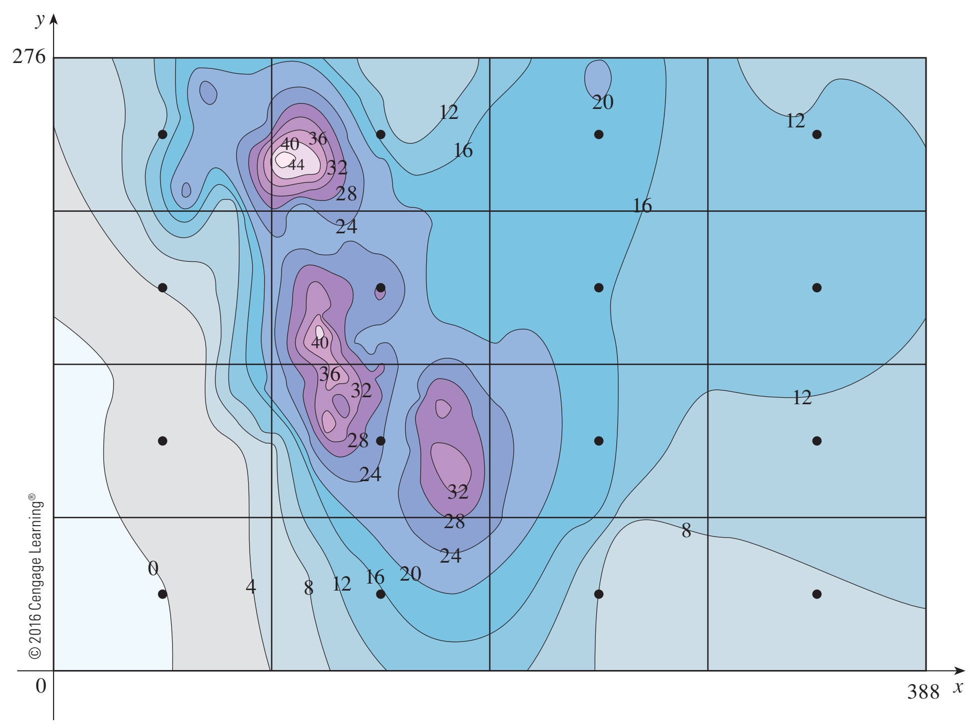


FIGURE 19

Using the contour map to estimate the value of f at the center of each subrectangle, we get

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$\approx \Delta A[0 + 15 + 8 + 7 + 2 + 25 + 18.5 + 11$$

$$+ 4.5 + 28 + 17 + 13.5 + 12 + 15 + 17.5 + 13]$$

$$= (6693)(207)$$

Therefore $$f_{\text{ave}} \approx \frac{(6693)(207)}{(388)(276)} \approx 12.9$$

On December 20–21, 2006, Colorado received an average of approximately 13 inches of snow. ∎

15.1 EXERCISES

1. (a) Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x, y) \mid 0 \leqslant x \leqslant 6, 0 \leqslant y \leqslant 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the upper right corner of each square.

(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

2. If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2$, $n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles.

3. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to be upper right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

4. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2] \times [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the volume in part (a).

5. Let V be the volume of the solid that lies under the graph of $f(x, y) = \sqrt{52 - x^2 - y^2}$ and above the rectangle given by $2 \leqslant x \leqslant 4$, $2 \leqslant y \leqslant 6$. Use the lines $x = 3$ and $y = 4$ to divide R into subrectangles. Let L and U be the Riemann sums computed using lower left corners and upper right corners, respectively. Without calculating the numbers V, L, and U, arrange them in increasing order and explain your reasoning.

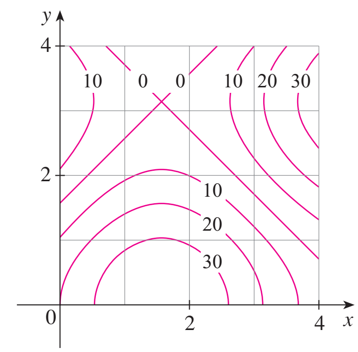
6. A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table. Estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

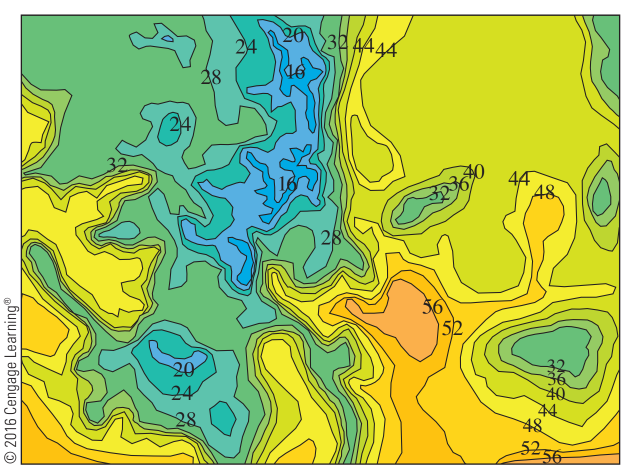
7. A contour map is shown for a function f on the square $R = [0, 4] \times [0, 4]$.

(a) Use the Midpoint Rule with $m = n = 2$ to estimate the value of $\iint_R f(x, y)\, dA$.

(b) Estimate the average value of f.



8. The contour map shows the temperature, in degrees Fahrenheit, at 4:00 PM on February 26, 2007, in Colorado. (The state measures 388 mi west to east and 276 mi south to north.) Use the Midpoint Rule with $m = n = 4$ to estimate the average temperature in Colorado at that time.



9–11 Evaluate the double integral by first identifying it as the volume of a solid.

9. $\iint_R \sqrt{2}\, dA$, $R = \{(x, y) \mid 2 \leqslant x \leqslant 6, -1 \leqslant y \leqslant 5\}$

10. $\iint_R (2x + 1)\, dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 4\}$

11. $\iint_R (4 - 2y)\, dA$, $R = [0, 1] \times [0, 1]$

12. The integral $\iint_R \sqrt{9 - y^2}\, dA$, where $R = [0, 4] \times [0, 2]$, represents the volume of a solid. Sketch the solid.

13–14 Find $\int_0^2 f(x, y)\, dx$ and $\int_0^3 f(x, y)\, dy$

13. $f(x, y) = x + 3x^2y^2$

14. $f(x, y) = y\sqrt{x + 2}$

15–26 Calculate the iterated integral.

15. $\displaystyle\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$

16. $\displaystyle\int_0^1 \int_0^1 (x + y)^2\, dx\, dy$

17. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy$

18. $\int_0^{\pi/6} \int_0^{\pi/2} (\sin x + \sin y)\, dy\, dx$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\, dx\, dy$

20. $\int_1^3 \int_1^5 \frac{\ln y}{xy}\, dy\, dx$

21. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$

22. $\int_0^1 \int_0^2 ye^{x-y}\, dx\, dy$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\, d\phi\, dt$

24. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\, dy\, dx$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\, du\, dv$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\, ds\, dt$

27–34 Calculate the double integral.

27. $\iint_R x \sec^2 y\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant \pi/4\}$

28. $\iint_R (y + xy^{-2})\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

29. $\iint_R \frac{xy^2}{x^2 + 1}\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 1, -3 \leqslant y \leqslant 3\}$

30. $\iint_R \frac{\tan\theta}{\sqrt{1 - t^2}}\, dA, \quad R = \{(\theta, t) \mid 0 \leqslant \theta \leqslant \pi/3, 0 \leqslant t \leqslant \tfrac{1}{2}\}$

31. $\iint_R x \sin(x + y)\, dA, \quad R = [0, \pi/6] \times [0, \pi/3]$

32. $\iint_R \frac{x}{1 + xy}\, dA, \quad R = [0, 1] \times [0, 1]$

33. $\iint_R ye^{-xy}\, dA, \quad R = [0, 2] \times [0, 3]$

34. $\iint_R \frac{1}{1 + x + y}\, dA, \quad R = [1, 3] \times [1, 2]$

35–36 Sketch the solid whose volume is given by the iterated integral.

35. $\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$

36. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\, dy\, dx$

37. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) \mid -1 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1\}$.

38. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

39. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

40. Find the volume of the solid enclosed by the surface $z = x^2 + xy^2$ and the planes $z = 0$, $x = 0$, $x = 5$, and $y = \pm 2$.

41. Find the volume of the solid enclosed by the surface $z = 1 + x^2 ye^y$ and the planes $z = 0$, $x = \pm 1$, $y = 0$, and $y = 1$.

42. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

43. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

44. Graph the solid that lies between the surface $z = 2xy/(x^2 + 1)$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0$, $x = 2$, $y = 0$, and $y = 4$. Then find its volume.

CAS **45.** Use a computer algebra system to find the exact value of the integral $\iint_R x^5 y^3 e^{xy}\, dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

CAS **46.** Graph the solid that lies between the surfaces $z = e^{-x^2} \cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leqslant 1$, $|y| \leqslant 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

47–48 Find the average value of f over the given rectangle.

47. $f(x, y) = x^2 y$, R has vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

48. $f(x, y) = e^y \sqrt{x + e^y}, \quad R = [0, 4] \times [0, 1]$

49–50 Use symmetry to evaluate the double integral.

49. $\iint_R \frac{xy}{1 + x^4}\, dA, \quad R = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

50. $\iint_R (1 + x^2 \sin y + y^2 \sin x)\, dA, \quad R = [-\pi, \pi] \times [-\pi, \pi]$

CAS **51.** Use a CAS to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dx\, dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

52. (a) In what way are the theorems of Fubini and Clairaut similar?

(b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t)\, dt\, ds$$

for $a < x < b$, $c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

1
$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

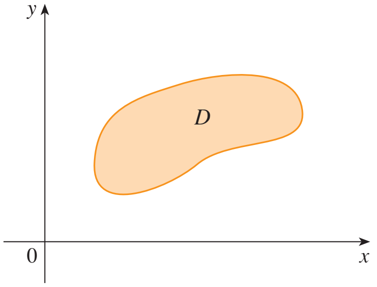


FIGURE 1

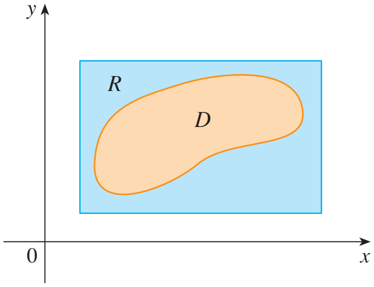


FIGURE 2

If F is integrable over R, then we define the **double integral of f over D** by

2
$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 15.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

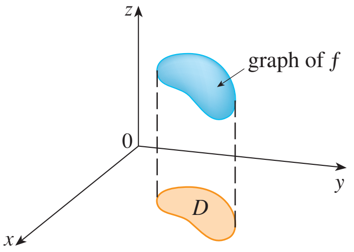


FIGURE 3

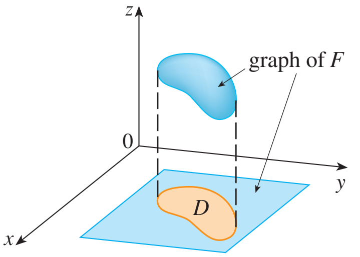


FIGURE 4

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved" (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exists

and therefore $\iint_D f(x, y)\, dA$ exists. In particular, this is the case for the following two types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

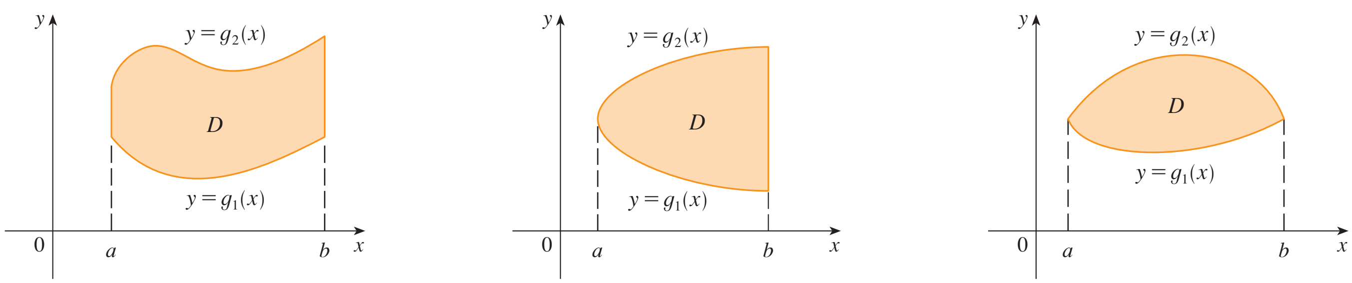


FIGURE 5
Some type I regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

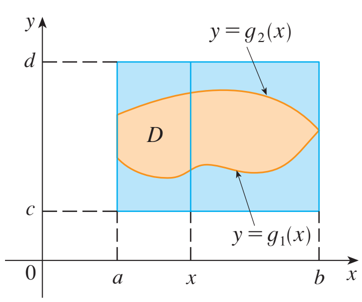


FIGURE 6

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

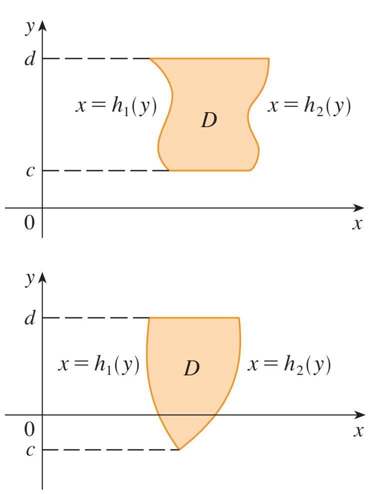


FIGURE 7
Some type II regions

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

4
$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

Using the same methods that were used in establishing (3), we can show that

5
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

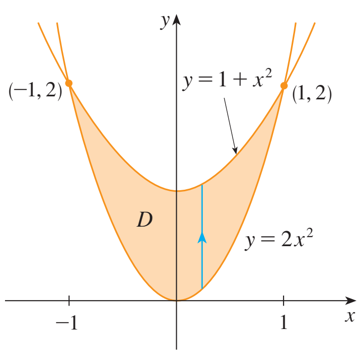


FIGURE 8

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$ ■

NOTE When we set up a double integral as in Example 1, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.

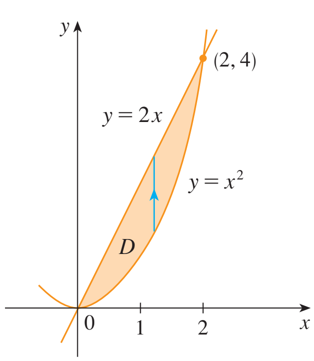


FIGURE 9
D as a type I region

EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

SOLUTION 1 From Figure 9 we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 2x\}$$

Figure 10 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.

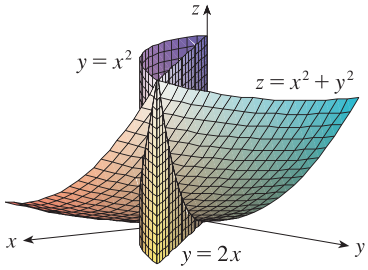


FIGURE 10

Therefore the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\,dy\,dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Big]_0^2 = \frac{216}{35}$$

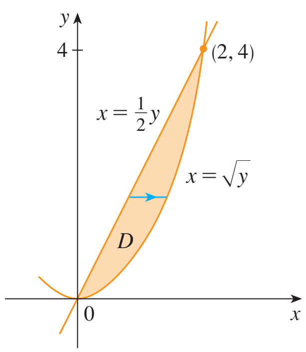


FIGURE 11
D as a type II region

SOLUTION 2 From Figure 11 we see that D can also be written as a type II region:

$$D = \left\{(x, y) \mid 0 \leqslant y \leqslant 4,\ \tfrac{1}{2}y \leqslant x \leqslant \sqrt{y}\right\}$$

Therefore another expression for V is

$$V = \iint_D (x^2 + y^2)\,dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2x \right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy$$

$$= \tfrac{2}{15}y^{5/2} + \tfrac{2}{7}y^{7/2} - \tfrac{13}{96}y^4 \Big]_0^4 = \tfrac{216}{35}$$

■

EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$

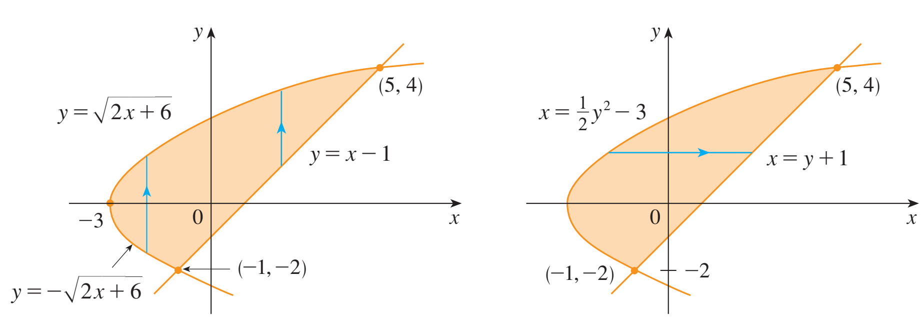


FIGURE 12 (a) D as a type I region (b) D as a type II region

Then (5) gives

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2-3\right)^2\right] dy$$

$$= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method. ■

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION In a question such as this, it's wise to draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies. Figure 13 shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$. Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. (See Figure 14.)

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x/2 \leqslant y \leqslant 1 - x/2\}$$

Therefore

$$V = \iint_D (2 - x - 2y)\,dA$$

$$= \int_0^1\int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx$$

$$= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$= \int_0^1 (x^2 - 2x + 1)\,dx = \frac{x^3}{3} - x^2 + x\Big]_0^1 = \frac{1}{3}$$ ■

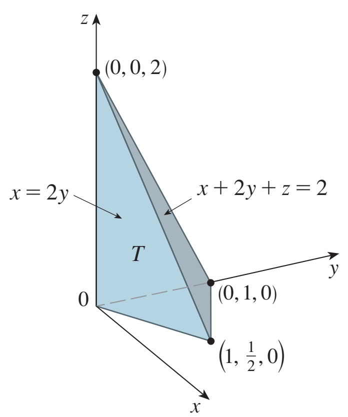


FIGURE 13

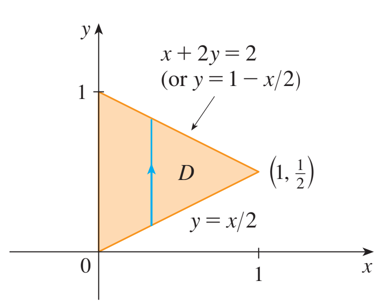


FIGURE 14

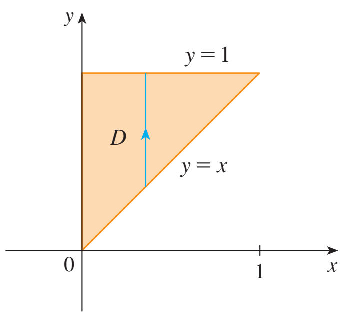


FIGURE 15
D as a type I region

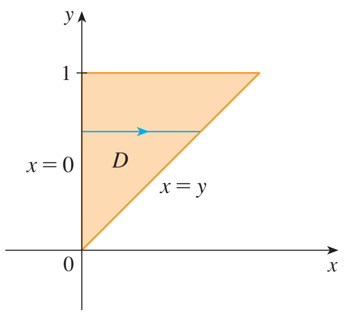


FIGURE 16
D as a type II region

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\,dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\,dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

This enables us to use (5) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x \sin(y^2)\Big]_{x=0}^{x=y} dy \\ &= \int_0^1 y \sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)\end{aligned}$$

Properties of Double Integrals

We assume that all of the following integrals exist. For rectangular regions D the first three properties can be proved in the same manner as in Section 4.2. And then for general regions the properties follow from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA$$

7 $$\iint_D c f(x, y)\,dA = c \iint_D f(x, y)\,dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\,dA \geqslant \iint_D g(x, y)\,dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

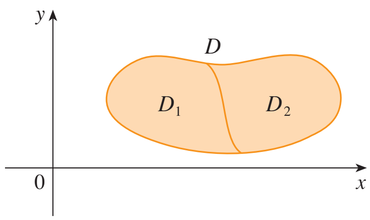
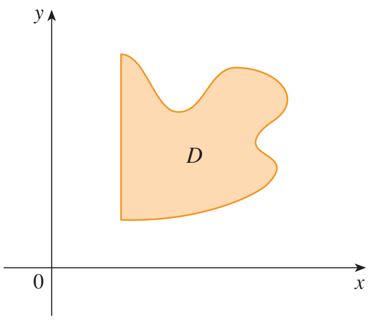


FIGURE 17

9 $$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$$

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 57 and 58.)



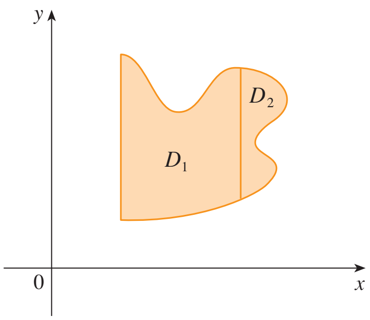


FIGURE 18

(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10
$$\iint_D 1\, dA = A(D)$$

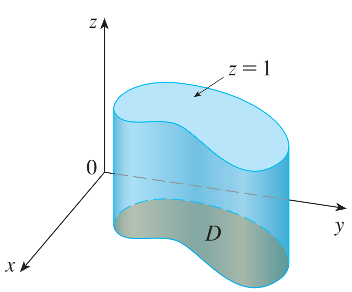


FIGURE 19
Cylinder with base D and height 1

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 63.)

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$ ■

15.2 EXERCISES

1–6 Evaluate the iterated integral.

1. $\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$

2. $\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy$

3. $\int_0^1 \int_0^y x e^{y^3}\, dx\, dy$

4. $\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$

6. $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv$

7–10 Evaluate the double integral.

7. $\iint_D \frac{y}{x^2 + 1}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant \sqrt{x}\}$

8. $\iint_D (2x + y)\, dA, \quad D = \{(x, y) \mid 1 \leqslant y \leqslant 2, y - 1 \leqslant x \leqslant 1\}$

9. $\iint_D e^{-y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant y \leqslant 3, 0 \leqslant x \leqslant y\}$

10. $\iint_D y\sqrt{x^2 - y^2}\, dA, \quad D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant x\}$

11. Draw an example of a region that is
(a) type I but not type II
(b) type II but not type I

12. Draw an example of a region that is
(a) both type I and type II
(b) neither type I nor type II

13–14 Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

13. $\iint_D x\, dA, \quad D$ is enclosed by the lines $y = x, y = 0, x = 1$

14. $\iint_D xy\, dA, \quad D$ is enclosed by the curves $y = x^2, y = 3x$

15–16 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

15. $\iint_D y\, dA, \quad D$ is bounded by $y = x - 2, x = y^2$

16. $\iint_D y^2 e^{xy}\, dA, \quad D$ is bounded by $y = x, y = 4, x = 0$

17–22 Evaluate the double integral.

17. $\iint_D x \cos y\, dA, \quad D$ is bounded by $y = 0,\ y = x^2,\ x = 1$

18. $\iint_D (x^2 + 2y)\, dA, \quad D$ is bounded by $y = x, y = x^3, x \geqslant 0$

19. $\iint_D y^2\, dA,$
D is the triangular region with vertices (0, 1), (1, 2), (4, 1)

20. $\iint_D xy\, dA, \quad D$ is enclosed by the quarter-circle $y = \sqrt{1 - x^2}, x \geqslant 0$, and the axes

21. $\iint_D (2x - y)\, dA,$
D is bounded by the circle with center the origin and radius 2

22. $\iint_D y\, dA, \quad D$ is the triangular region with vertices (0, 0), (1, 1), and (4, 0)

23–32 Find the volume of the given solid.

23. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

24. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

25. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

26. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0, y = 0, z = 0$, and $x + y = 2$

27. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

28. Bounded by the planes $z = x, y = x, x + y = 2$, and $z = 0$

29. Enclosed by the cylinders $z = x^2, y = x^2$ and the planes $z = 0, y = 4$

30. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0, z = 0$ in the first octant

31. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0, z = 0$ in the first octant

32. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

33. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

34. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

35–38 Find the volume of the solid by subtracting two volumes.

35. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

36. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

37. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

38. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

39–40 Sketch the solid whose volume is given by the iterated integral.

39. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

40. $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

CAS **41–44** Use a computer algebra system to find the exact volume of the solid.

41. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

42. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

43. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

44. Enclosed by $z = x^2 + y^2$ and $z = 2y$

45–50 Sketch the region of integration and change the order of integration.

45. $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

46. $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$

47. $\int_0^{\pi/2} \int_0^{\cos x} f(x, y)\, dy\, dx$

48. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$

49. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

50. $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

51–56 Evaluate the integral by reversing the order of integration.

51. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

52. $\int_0^1 \int_{x^2}^1 \sqrt{y}\, \sin y\, dy\, dx$

53. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx$

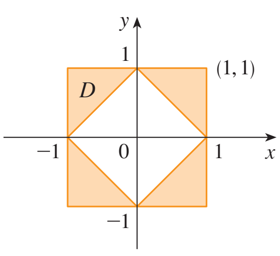
54. $\int_0^2 \int_{y/2}^1 y \cos(x^3 - 1)\, dx\, dy$

55. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

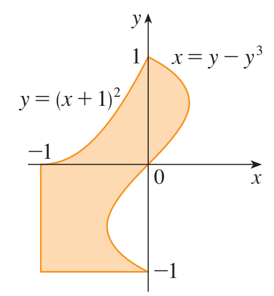
56. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

57–58 Express D as a union of regions of type I or type II and evaluate the integral.

57. $\iint_D x^2\, dA$



58. $\iint_D y\, dA$



59–60 Use Property 11 to estimate the value of the integral.

59. $\iint_S \sqrt{4 - x^2y^2}\, dA, \quad S = \{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

60. $\iint_T \sin^4(x + y)\, dA$, T is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

61–62 Find the averge value of f over the region D.

61. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

62. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

63. Prove Property 11.

64. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

65–69 Use geometry or symmetry, or both, to evaluate the double integral.

65. $\iint_D (x + 2)\, dA$,

$D = \{(x, y) \mid 0 \leqslant y \leqslant \sqrt{9 - x^2}\}$

66. $\iint_D \sqrt{R^2 - x^2 - y^2}\, dA$,

D is the disk with center the origin and radius R

67. $\iint_D (2x + 3y)\, dA$,

D is the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$

68. $\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA$,

$D = \{(x, y) \mid |x| + |y| \leqslant 1\}$

69. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA$,

$D = [-a, a] \times [-b, b]$

CAS **70.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

15.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

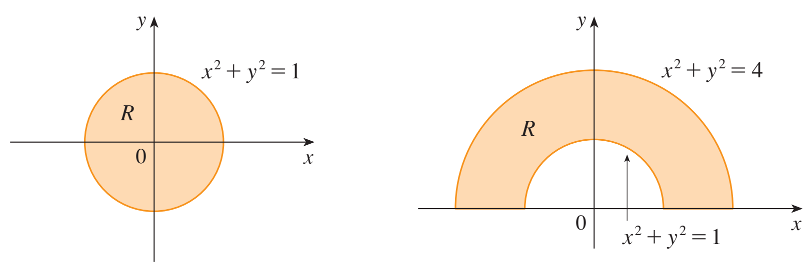


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

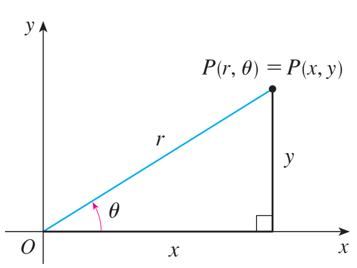


FIGURE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

(See Section 10.3.)

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.

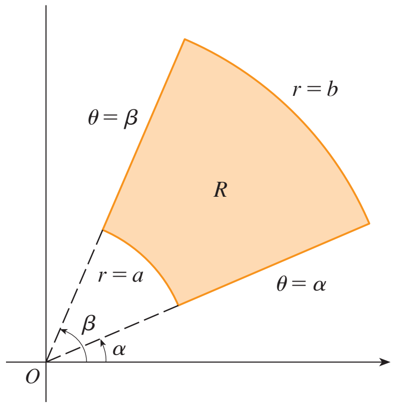


FIGURE 3 Polar rectangle

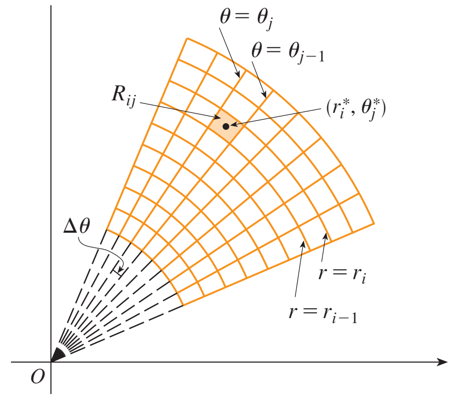


FIGURE 4 Dividing R into polar subrectangles

The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leqslant r \leqslant r_i, \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \tfrac{1}{2}r_i^2\,\Delta\theta - \tfrac{1}{2}r_{i-1}^2\,\Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\,\Delta\theta = r_i^*\,\Delta r\,\Delta\theta\end{aligned}$$

Although we have defined the double integral $\iint_R f(x, y)\,dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so a typical Riemann sum is

1
$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,r_i^*\,\Delta r\,\Delta\theta$$

If we write $g(r, \theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r, \theta)\,dr\,d\theta$$

Therefore we have

$$\begin{aligned}\iint_R f(x, y)\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta = \int_\alpha^\beta \int_a^b g(r, \theta)\,dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\end{aligned}$$

2 Change to Polar Coodinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

The formula in (2) says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\, dr\, d\theta$. Be careful not to forget the additional factor r on the right side of Formula 2. A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\, d\theta$ and dr and therefore has "area" $dA = r\, dr\, d\theta$.

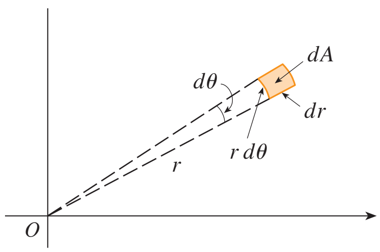


FIGURE 5

EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION The region R can be described as

$$R = \{(x, y) \mid y \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by Formula 2,

$$\begin{aligned}\iint_R (3x + 4y^2)\, dA &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta \\ &= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta \\ &= \int_0^\pi \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta \\ &= \int_0^\pi \Big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\Big]\, d\theta \\ &= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\,\Big]_0^\pi = \frac{15\pi}{2}\end{aligned}$$

Here we use the trigonometric identity

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

See Section 7.2 for advice on integrating trigonometric functions.

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

SOLUTION If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$ [see Figures 6 and 1(a)]. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

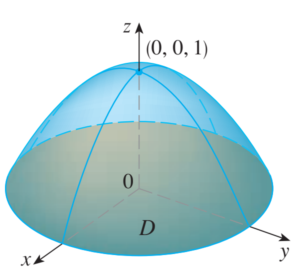


FIGURE 6

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\,dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx$$

which is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\,dx$. ■

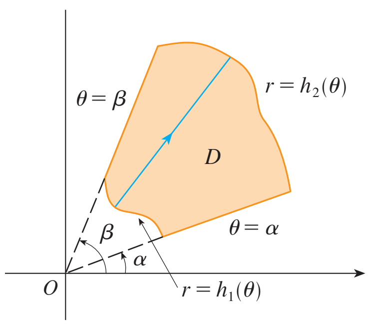


FIGURE 7
$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

What we have done so far can be extended to the more complicated type of region shown in Figure 7. It's similar to the type II rectangular regions considered in Section 15.2. In fact, by combining Formula 2 in this section with Formula 15.2.5, we obtain the following formula.

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\,dA = \int_{\alpha}^{\beta} \int_{0}^{h(\theta)} r\,dr\,d\theta$$

$$= \int_{\alpha}^{\beta} \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta = \int_{\alpha}^{\beta} \tfrac{1}{2}[h(\theta)]^2\,d\theta$$

and this agrees with Formula 10.4.3.

EXAMPLE 3 Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION From the sketch of the curve in Figure 8, we see that a loop is given by the region

$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

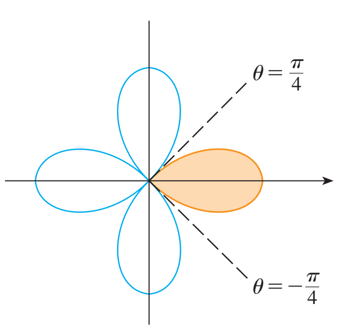


FIGURE 8

So the area is

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_{0}^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \tfrac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\,d\theta = \tfrac{1}{4}\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$ ■

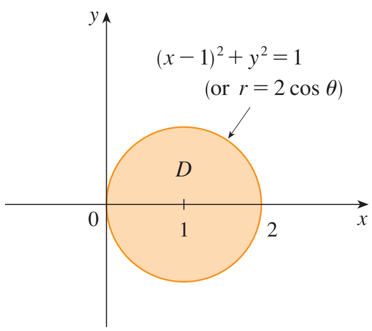


FIGURE 9

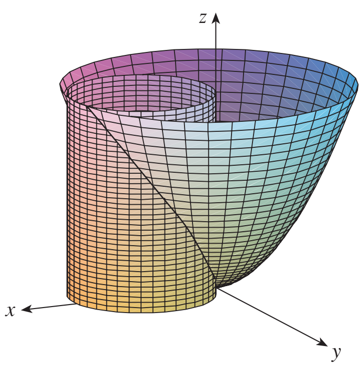


FIGURE 10

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x-1)^2 + y^2 = 1$$

(See Figures 9 and 10.)

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2, 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$V = \iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \cos^4\theta\,d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

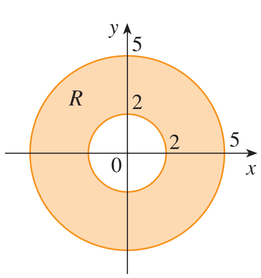
$$= 2\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \tfrac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$ ■

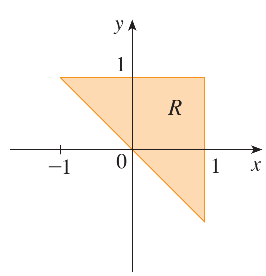
15.3 EXERCISES

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\,dA$ as an iterated integral, where f is an arbitrary continuous function on R.

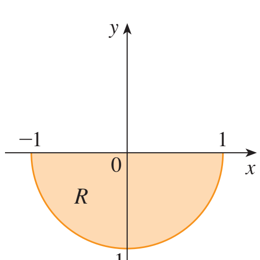
1.



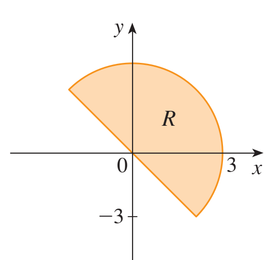
2.



3.



4.



5–6 Sketch the region whose area is given by the integral and evaluate the integral.

5. $\displaystyle\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta$

6. $\displaystyle\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,dr\,d\theta$

7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D x^2 y\,dA$, where D is the top half of the disk with center the origin and radius 5

8. $\iint_R (2x - y)\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$

9. $\iint_R \sin(x^2 + y^2)\,dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3

10. $\displaystyle\iint_R \frac{y^2}{x^2 + y^2}\,dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

11. $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_D \cos\sqrt{x^2+y^2}\,dA$, where D is the disk with center the origin and radius 2

13. $\iint_R \arctan(y/x)\,dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$

14. $\iint_D x\,dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

15–18 Use a double integral to find the area of the region.

15. One loop of the rose $r = \cos 3\theta$

16. The region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

17. The region inside the circle $(x - 1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$

18. The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

19–27 Use polar coordinates to find the volume of the given solid.

19. Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leqslant 25$

20. Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leqslant x^2 + y^2 \leqslant 4$

21. Below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leqslant 1$

22. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

23. A sphere of radius a

24. Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant

25. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

26. Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$

27. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

28. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

29–32 Evaluate the iterated integral by converting to polar coordinates.

29. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$

30. $\displaystyle\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\,dx\,dy$

31. $\displaystyle\int_0^{1/2}\int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\,dx\,dy$

32. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$

33–34 Express the double integral in terms of a single integral with respect to r. Then use your calculator to evaluate the integral correct to four decimal places.

33. $\iint_D e^{(x^2+y^2)^2}\,dA$, where D is the disk with center the origin and radius 1

34. $\iint_D xy\sqrt{1 + x^2 + y^2}\,dA$, where D is the portion of the disk $x^2 + y^2 \leqslant 1$ that lies in the first quadrant

35. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

36. An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler.
(a) If $0 < R \leqslant 100$, what is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?
(b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R.

37. Find the average value of the function $f(x, y) = 1/\sqrt{x^2 + y^2}$ on the annular region $a^2 \leqslant x^2 + y^2 \leqslant b^2$, where $0 < a < b$.

38. Let D be the disk with center the origin and radius a. What is the average distance from points in D to the origin?

39. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

40. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\, x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

41. Use the result of Exercise 40 part (c) to evaluate the following integrals.

(a) $\displaystyle\int_0^{\infty} x^2 e^{-x^2}\, dx$ (b) $\displaystyle\int_0^{\infty} \sqrt{x}\, e^{-x}\, dx$

15.4 Applications of Double Integrals

We have already seen one application of double integrals: computing volumes. Another geometric application is finding areas of surfaces and this will be done in the next section. In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will see that these physical ideas are also important when applied to probability density functions of two random variables.

Density and Mass

In Section 8.3 we were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. But now, equipped with the double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

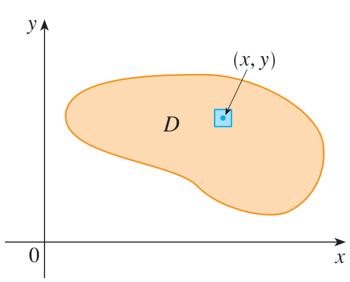


FIGURE 1

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size (as in Figure 2) and consider $\rho(x, y)$ to be 0 outside D. If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of R_{ij}. If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

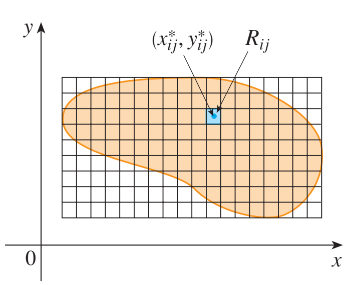


FIGURE 2

If we now increase the number of subrectangles, we obtain the total mass m of the lamina as the limiting value of the approximations:

1
$$m = \lim_{k,\, l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D \rho(x, y)\, dA$$

Physicists also consider other types of density that can be treated in the same manner. For example, if an electric charge is distributed over a region D and the charge density

(in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

2 $$Q = \iint_D \sigma(x, y)\, dA$$

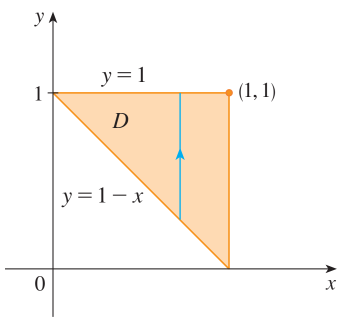


FIGURE 3

EXAMPLE 1 Charge is distributed over the triangular region D in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

SOLUTION From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \left[x \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} [1^2 - (1 - x)^2]\, dx$$

$$= \tfrac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C. ■

Moments and Centers of Mass

In Section 8.3 we found the center of mass of a lamina with constant density; here we consider a lamina with variable density. Suppose the lamina occupies a region D and has density function $\rho(x, y)$. Recall from Chapter 8 that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis. We divide D into small rectangles as in Figure 2. Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\, \Delta A$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\, \Delta A]\, y_{ij}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

3 $$M_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y\, \rho(x, y)\, dA$$

Similarly, the **moment about the *y*-axis** is

4 $$M_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^*\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x\, \rho(x, y)\, dA$$

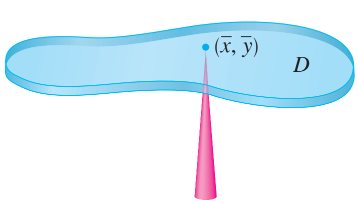


FIGURE 4

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus the lamina balances horizontally when supported at its center of mass (see Figure 4).

5 The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

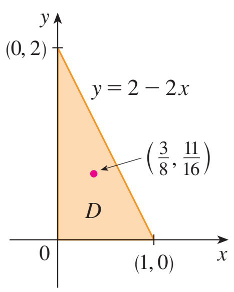


FIGURE 5

SOLUTION The triangle is shown in Figure 5. (Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4 \int_0^1 (1 - x^2)\, dx = 4\left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

Then the formulas in (5) give

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3)\, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4} \int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4} \right]_0^1 = \frac{11}{16}$$

The center of mass is at the point $\left(\tfrac{3}{8}, \tfrac{11}{16}\right)$. ■

EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

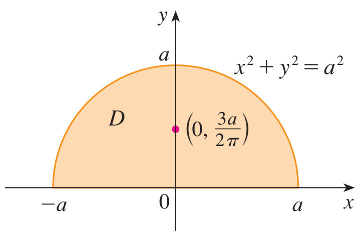


FIGURE 6

SOLUTION Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. (See Figure 6.) Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus the mass of the lamina is

$$\begin{aligned} m &= \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA \\ &= \int_0^\pi \int_0^a (Kr)\, r\, dr\, d\theta = K \int_0^\pi d\theta \int_0^a r^2\, dr \\ &= K\pi \frac{r^3}{3}\bigg]_0^a = \frac{K\pi a^3}{3} \end{aligned}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\begin{aligned} \bar{y} &= \frac{1}{m}\iint_D y\rho(x, y)\, dA = \frac{3}{K\pi a^3}\int_0^\pi \int_0^a r\sin\theta\,(Kr)\, r\, dr\, d\theta \\ &= \frac{3}{\pi a^3}\int_0^\pi \sin\theta\, d\theta \int_0^a r^3\, dr = \frac{3}{\pi a^3}\Big[-\cos\theta\Big]_0^\pi \left[\frac{r^4}{4}\right]_0^a \\ &= \frac{3}{\pi a^3}\frac{2a^4}{4} = \frac{3a}{2\pi} \end{aligned}$$

Compare the location of the center of mass in Example 3 with Example 8.3.4, where we found that the center of mass of a lamina with the same shape but uniform density is located at the point $(0, 4a/(3\pi))$.

Therefore the center of mass is located at the point $(0, 3a/(2\pi))$. ■

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the x-axis**:

6
$$I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

Similarly, the **moment of inertia about the y-axis** is

7
$$I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

8 $$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2)\, \rho(x, y)\, dA$$

Note that $I_0 = I_x + I_y$.

EXAMPLE 4 Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

SOLUTION The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant a$. Let's compute I_0 first:

$$I_0 = \iint_D (x^2 + y^2)\rho\, dA = \rho \int_0^{2\pi} \int_0^a r^2\, r\, dr\, d\theta$$

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3\, dr = 2\pi\rho \left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{2}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem). Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$ ■

In Example 4 notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \tfrac{1}{2}(\rho\pi a^2)a^2 = \tfrac{1}{2}ma^2$$

Thus if we increase the mass or the radius of the disk, we thereby increase the moment of inertia. In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

The **radius of gyration of a lamina about an axis** is the number R such that

9 $$mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis. Equation 9 says that if the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equations

10 $$m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

Thus $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes. (Note the analogy with the center of mass.)

EXAMPLE 5 Find the radius of gyration about the x-axis of the disk in Example 4.

SOLUTION As noted, the mass of the disk is $m = \rho\pi a^2$, so from Equations 10 we have

$$\bar{\bar{y}}^2 = \frac{I_x}{m} = \frac{\frac{1}{4}\pi\rho a^4}{\rho\pi a^2} = \frac{a^2}{4}$$

Therefore the radius of gyration about the x-axis is $\bar{\bar{y}} = \frac{1}{2}a$, which is half the radius of the disk. ■

■ Probability

In Section 8.5 we considered the *probability density function* f of a continuous random variable X. This means that $f(x) \geqslant 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that X lies between a and b is found by integrating f from a to b:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

Now we consider a pair of continuous random variables X and Y, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random. The **joint density function** of X and Y is a function f of two variables such that the probability that (X, Y) lies in a region D is

$$P((X, Y) \in D) = \iint_D f(x, y)\,dA$$

In particular, if the region is a rectangle, the probability that X lies between a and b and Y lies between c and d is

$$P(a \leqslant X \leqslant b,\; c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

(See Figure 7.)

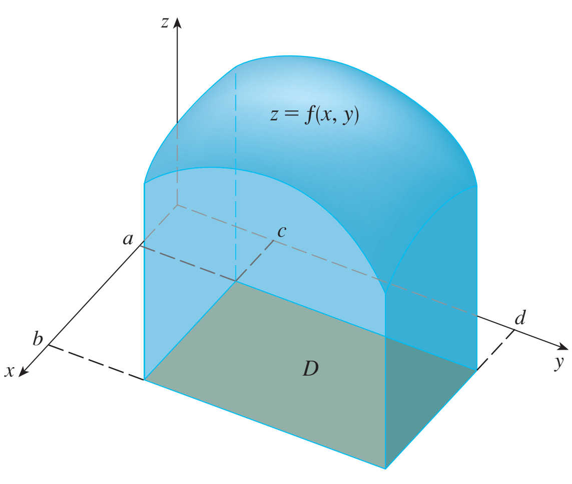


FIGURE 7
The probability that X lies between a and b and Y lies between c and d is the volume that lies above the rectangle $D = [a, b] \times [c, d]$ and below the graph of the joint density function.

Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

As in Exercise 15.3.40, the double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares, and we can write

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

EXAMPLE 6 If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \leqslant x \leqslant 10,\ 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \leqslant 7, Y \geqslant 2)$.

SOLUTION We find the value of C by ensuring that the double integral of f is equal to 1. Because $f(x, y) = 0$ outside the rectangle $[0, 10] \times [0, 10]$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_0^{10} \int_0^{10} C(x + 2y)\, dy\, dx = C \int_0^{10} \Big[xy + y^2 \Big]_{y=0}^{y=10} dx$$

$$= C \int_0^{10} (10x + 100)\, dx = 1500C$$

Therefore $1500C = 1$ and so $C = \frac{1}{1500}$.

Now we can compute the probability that X is at most 7 and Y is at least 2:

$$P(X \leqslant 7, Y \geqslant 2) = \int_{-\infty}^{7} \int_{2}^{\infty} f(x, y)\, dy\, dx = \int_0^7 \int_2^{10} \tfrac{1}{1500}(x + 2y)\, dy\, dx$$

$$= \tfrac{1}{1500} \int_0^7 \Big[xy + y^2 \Big]_{y=2}^{y=10} dx = \tfrac{1}{1500} \int_0^7 (8x + 96)\, dx$$

$$= \tfrac{868}{1500} \approx 0.5787$$

■

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$. Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x) f_2(y)$$

In Section 8.5 we modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1} e^{-t/\mu} & \text{if } t \geqslant 0 \end{cases}$$

where μ is the mean waiting time. In the next example we consider a situation with two independent waiting times.

EXAMPLE 7 The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for this week's film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent,

find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

SOLUTION Assuming that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions, we can write the individual density functions as

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10}e^{-x/10} & \text{if } x \geqslant 0 \end{cases} \qquad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5}e^{-y/5} & \text{if } y \geqslant 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x/10}e^{-y/5} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

We are asked for the probability that $X + Y < 20$:

$$P(X + Y < 20) = P((X, Y) \in D)$$

where D is the triangular region shown in Figure 8. Thus

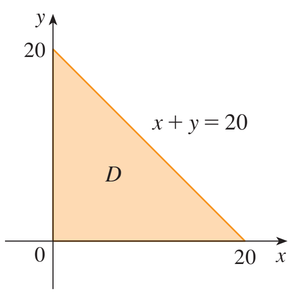


FIGURE 8

$$\begin{aligned}
P(X + Y < 20) &= \iint_D f(x, y)\, dA = \int_0^{20}\int_0^{20-x} \frac{1}{50}e^{-x/10}e^{-y/5}\, dy\, dx \\
&= \frac{1}{50}\int_0^{20} \Big[e^{-x/10}(-5)e^{-y/5}\Big]_{y=0}^{y=20-x} dx \\
&= \frac{1}{10}\int_0^{20} e^{-x/10}\big(1 - e^{(x-20)/5}\big)\, dx \\
&= \frac{1}{10}\int_0^{20} \big(e^{-x/10} - e^{-4}e^{x/10}\big)\, dx \\
&= 1 + e^{-4} - 2e^{-2} \approx 0.7476
\end{aligned}$$

This means that about 75% of the moviegoers wait less than 20 minutes before taking their seats. ■

Expected Values

Recall from Section 8.5 that if X is a random variable with probability density function f, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Now if X and Y are random variables with joint density function f, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of X and Y, to be

11 $$\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Notice how closely the expressions for μ_1 and μ_2 in (11) resemble the moments M_x and M_y of a lamina with density function ρ in Equations 3 and 4. In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function. And because the total "probability mass" is 1, the expressions for $\bar{x}$ and $\bar{y}$ in (5) show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the "center of mass" of the probability distribution.

In the next example we deal with normal distributions. As in Section 8.5, a single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

EXAMPLE 8 A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it. Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

SOLUTION We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$, and $\sigma_1 = \sigma_2 = 0.01$. So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}} e^{-(x-4)^2/0.0002} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}} e^{-(y-6)^2/0.0002}$$

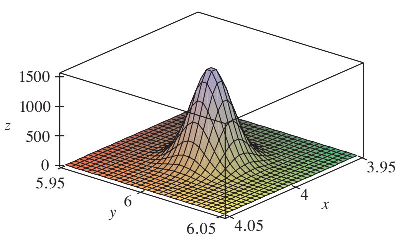


FIGURE 9
Graph of the bivariate normal joint density function in Example 8

Since X and Y are independent, the joint density function is the product:

$$\begin{aligned} f(x, y) &= f_1(x)f_2(y) \\ &= \frac{1}{0.0002\pi} e^{-(x-4)^2/0.0002} e^{-(y-6)^2/0.0002} \\ &= \frac{5000}{\pi} e^{-5000[(x-4)^2+(y-6)^2]} \end{aligned}$$

A graph of this function is shown in Figure 9.

Let's first calculate the probability that both X and Y differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$\begin{aligned} P(3.98 < X < 4.02,\ 5.98 < Y < 6.02) &= \int_{3.98}^{4.02}\int_{5.98}^{6.02} f(x, y)\, dy\, dx \\ &= \frac{5000}{\pi}\int_{3.98}^{4.02}\int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx \\ &\approx 0.91 \end{aligned}$$

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09$$ ■

15.4 EXERCISES

1. Electric charge is distributed over the rectangle $0 \leqslant x \leqslant 5$, $2 \leqslant y \leqslant 5$ so that the charge density at (x, y) is $\sigma(x, y) = 2x + 4y$ (measured in coulombs per square meter). Find the total charge on the rectangle.

2. Electric charge is distributed over the disk $x^2 + y^2 \leqslant 1$ so that the charge density at (x, y) is $\sigma(x, y) = \sqrt{x^2 + y^2}$ (measured in coulombs per square meter). Find the total charge on the disk.

3–10 Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

3. $D = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 1 \leqslant y \leqslant 4\}$; $\rho(x, y) = ky^2$

4. $D = \{(x, y) \mid 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$; $\rho(x, y) = 1 + x^2 + y^2$

5. D is the triangular region with vertices (0, 0), (2, 1), (0, 3); $\rho(x, y) = x + y$

6. D is the triangular region enclosed by the lines $y = 0$, $y = 2x$, and $x + 2y = 1$; $\rho(x, y) = x$

7. D is bounded by $y = 1 - x^2$ and $y = 0$; $\rho(x, y) = ky$

8. D is bounded by $y = x + 2$ and $y = x^2$; $\rho(x, y) = kx^2$

9. D is bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$; $\rho(x, y) = xy$

10. D is enclosed by the curves $y = 0$ and $y = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$; $\rho(x, y) = y$

11. A lamina occupies the part of the disk $x^2 + y^2 \leqslant 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

12. Find the center of mass of the lamina in Exercise 11 if the density at any point is proportional to the square of its distance from the origin.

13. The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$ and $y = \sqrt{4 - x^2}$ together with the portions of the x-axis that join them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

14. Find the center of mass of the lamina in Exercise 13 if the density at any point is inversely proportional to its distance from the origin.

15. Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.

16. A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

17. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 3.

18. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 6.

19. Find the moments of inertia I_x, I_y, I_0 for the lamina of Exercise 15.

20. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\rho(x, y) = 1 + 0.1x$, is it more difficult to rotate the blade about the x-axis or the y-axis?

21–24 A lamina with constant density $\rho(x, y) = \rho$ occupies the given region. Find the moments of inertia I_x and I_y and the radii of gyration $\bar{\bar{x}}$ and $\bar{\bar{y}}$.

21. The rectangle $0 \leqslant x \leqslant b$, $0 \leqslant y \leqslant h$

22. The triangle with vertices $(0, 0)$, $(b, 0)$, and $(0, h)$

23. The part of the disk $x^2 + y^2 \leqslant a^2$ in the first quadrant

24. The region under the curve $y = \sin x$ from $x = 0$ to $x = \pi$

CAS **25–26** Use a computer algebra system to find the mass, center of mass, and moments of inertia of the lamina that occupies the region D and has the given density function.

25. D is enclosed by the right loop of the four-leaved rose $r = \cos 2\theta$; $\rho(x, y) = x^2 + y^2$

26. $D = \{(x, y) \mid 0 \leqslant y \leqslant xe^{-x},\ 0 \leqslant x \leqslant 2\}$; $\rho(x, y) = x^2y^2$

27. The joint density function for a pair of random variables X and Y is

$$f(x, y) = \begin{cases} Cx(1 + y) & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

28. (a) Verify that

$$f(x, y) = \begin{cases} 4xy & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint density function.
(b) If X and Y are random variables whose joint density function is the function f in part (a), find
(i) $P\left(X \geqslant \frac{1}{2}\right)$ (ii) $P\left(X \geqslant \frac{1}{2}, Y \leqslant \frac{1}{2}\right)$
(c) Find the expected values of X and Y.

29. Suppose X and Y are random variables with joint density function

$$f(x, y) = \begin{cases} 0.1e^{-(0.5x+0.2y)} & \text{if } x \geqslant 0,\ y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a joint density function.
(b) Find the following probabilities.
(i) $P(Y \geqslant 1)$ (ii) $P(X \leqslant 2, Y \leqslant 4)$
(c) Find the expected values of X and Y.

30. (a) A lamp has two bulbs, each of a type with average lifetime 1000 hours. Assuming that we can model the probability of failure of a bulb by an exponential density function with mean $\mu = 1000$, find the probability that both of the lamp's bulbs fail within 1000 hours.
(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 1000 hours.

CAS **31.** Suppose that X and Y are independent random variables, where X is normally distributed with mean 45 and standard deviation 0.5 and Y is normally distributed with mean 20 and standard deviation 0.1.

(a) Find $P(40 \leqslant X \leqslant 50, 20 \leqslant Y \leqslant 25)$.

(b) Find $P(4(X - 45)^2 + 100(Y - 20)^2 \leqslant 2)$.

32. Xavier and Yolanda both have classes that end at noon and they agree to meet every day after class. They arrive at the coffee shop independently. Xavier's arrival time is X and Yolanda's arrival time is Y, where X and Y are measured in minutes after noon. The individual density functions are

$$f_1(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f_2(y) = \begin{cases} \frac{1}{50}y & \text{if } 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

(Xavier arrives sometime after noon and is more likely to arrive promptly than late. Yolanda always arrives by 12:10 PM and is more likely to arrive late than promptly.) After Yolanda arrives, she'll wait for up to half an hour for Xavier, but he won't wait for her. Find the probability that they meet.

33. When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 miles in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P) = \tfrac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between points P and A.

(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square mile. Find a double integral that represents the exposure of a person residing at A.

(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

15.5 Surface Area

In Section 16.6 we will deal with areas of more general surfaces, called parametric surfaces, and so this section need not be covered if that later section will be covered.

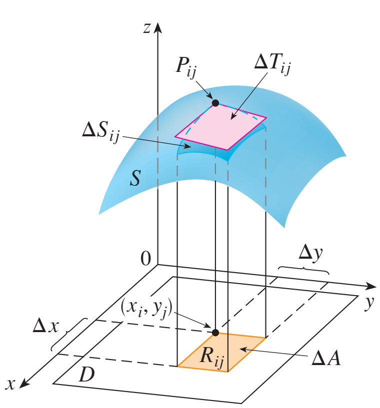


FIGURE 1

In this section we apply double integrals to the problem of computing the area of a surface. In Section 8.2 we found the area of a very special type of surface—a surface of revolution—by the methods of single-variable calculus. Here we compute the area of a surface with equation $z = f(x, y)$, the graph of a function of two variables.

Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geqslant 0$ and the domain D of f is a rectangle. We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x\, \Delta y$. If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it (see Figure 1). The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}. Thus the sum $\Sigma\Sigma\, \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases. Therefore we define the **surface area** of S to be

1
$$A(S) = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

To find a formula that is more convenient than Equation 1 for computational purposes, we let **a** and **b** be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. (See Figure 2.) Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. Recall from Section 14.3 that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of **a** and **b**. Therefore

$$\mathbf{a} = \Delta x\, \mathbf{i} + f_x(x_i, y_j)\, \Delta x\, \mathbf{k}$$

$$\mathbf{b} = \Delta y\, \mathbf{j} + f_y(x_i, y_j)\, \Delta y\, \mathbf{k}$$

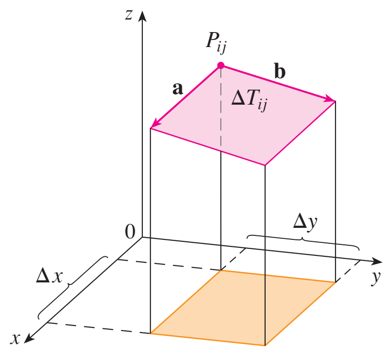


FIGURE 2

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

$$= -f_x(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{j} + \Delta x\,\Delta y\,\mathbf{k}$$

$$= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

Thus $$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\;\Delta A$$

From Definition 1 we then have

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\;\Delta A$$

and by the definition of a double integral we get the following formula.

2 The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\;dA$$

We will verify in Section 16.6 that this formula is consistent with our previous formula for the area of a surface of revolution. If we use the alternative notation for partial derivatives, we can rewrite Formula 2 as follows:

3 $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\;dA$$

Notice the similarity between the surface area formula in Equation 3 and the arc length formula from Section 8.1:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\;dx$$

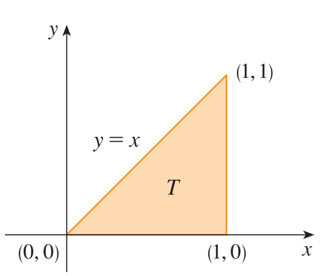


FIGURE 3

EXAMPLE 1 Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices (0, 0), (1, 0), and (1, 1).

SOLUTION The region T is shown in Figure 3 and is described by

$$T = \{(x, y) \mid 0 \leqslant x \leqslant 1,\; 0 \leqslant y \leqslant x\}$$

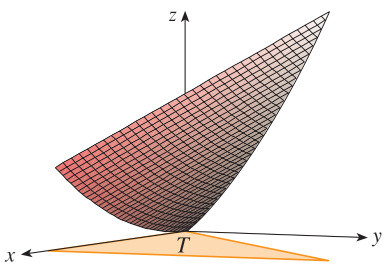


FIGURE 4

Using Formula 2 with $f(x, y) = x^2 + 2y$, we get

$$A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Figure 4 shows the portion of the surface whose area we have just computed. ■

EXAMPLE 2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

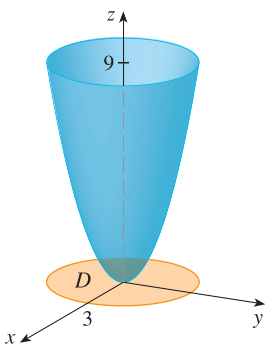


FIGURE 5

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See Figure 5.) Using Formula 3, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \tfrac{1}{8}\sqrt{1 + 4r^2}\,(8r)\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$ ■

15.5 EXERCISES

1–12 Find the area of the surface.

1. The part of the plane $5x + 3y - z + 6 = 0$ that lies above the rectangle $[1, 4] \times [2, 6]$

2. The part of the plane $6x + 4y + 2z = 1$ that lies inside the cylinder $x^2 + y^2 = 25$

3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

4. The part of the surface $2y + 4z - x^2 = 5$ that lies above the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$

5. The part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -2$

6. The part of the cylinder $x^2 + z^2 = 4$ that lies above the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$

7. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

8. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

9. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

10. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$

11. The part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy-plane

12. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$

13–14 Find the area of the surface correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

13. The part of the surface $z = 1/(1 + x^2 + y^2)$ that lies above the disk $x^2 + y^2 \leqslant 1$

14. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

15. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with four squares to estimate the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the square $[0, 1] \times [0, 1]$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

16. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

CAS **17.** Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

CAS **18.** Find the exact area of the surface

$$z = 1 + x + y + x^2 \qquad -2 \leqslant x \leqslant 1 \quad -1 \leqslant y \leqslant 1$$

Illustrate by graphing the surface.

CAS **19.** Find, to four decimal places, the area of the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \leqslant 1$.

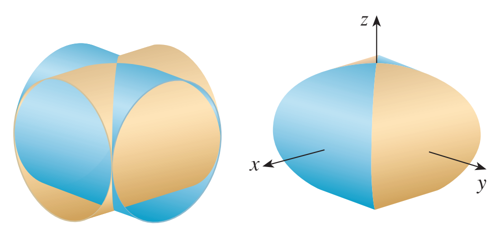
CAS **20.** Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

21. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1}\, A(D)$.

22. If you attempt to use Formula 2 to find the area of the top half of the sphere $x^2 + y^2 + z^2 = a^2$, you have a slight problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle $x^2 + y^2 = a^2$. However, the integral can be computed as the limit of the integral over the disk $x^2 + y^2 \leqslant t^2$ as $t \to a^-$. Use this method to show that the area of a sphere of radius a is $4\pi a^2$.

23. Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$. [*Hint:* Project the surface onto the xz-plane.]

24. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



15.6 Triple Integrals

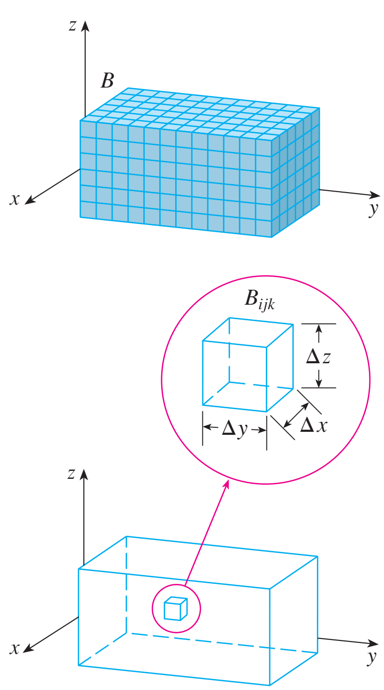


FIGURE 1

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \qquad B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1. Each sub-box has volume $\Delta V = \Delta x\, \Delta y\, \Delta z$.

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral (15.1.5), we define the triple integral as the limit of the triple Riemann sums in (2).

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z. There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

EXAMPLE 1 Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

SOLUTION We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y, and then z, we obtain

$$\begin{aligned}\iiint_B xyz^2\, dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz \\ &= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz \\ &= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4}\bigg]_0^3 = \frac{27}{4}\end{aligned}$$

■

Now we define the **triple integral over a general bounded region E** in three-dimensional space (a solid) by much the same procedure that we used for double integrals (15.2.2). We enclose E in a box B of the type given by Equation 1. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth." The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 15.2).

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

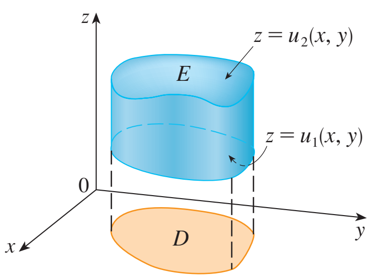


FIGURE 2
A type 1 solid region

5 $$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument that led to (15.2.3), it can be shown that if E is a type 1 region given by Equation 5, then

6 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

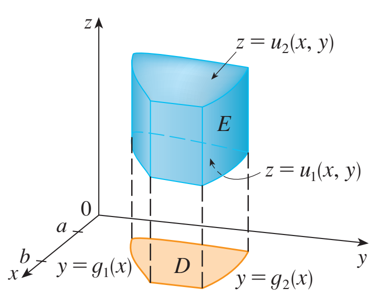


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$$E = \{(x, y, z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

7 $$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

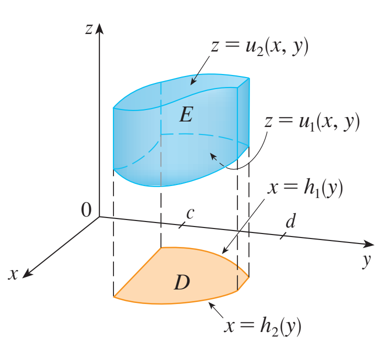


FIGURE 4
A type 1 solid region with a type II projection

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

and Equation 6 becomes

8 $$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$

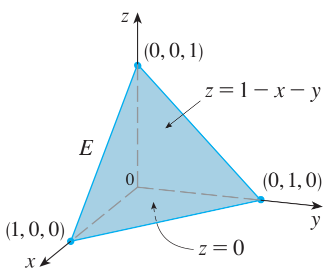


FIGURE 5

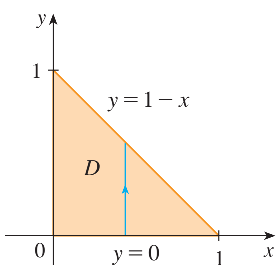


FIGURE 6

EXAMPLE 2 Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

SOLUTION When we set up a triple integral it's wise to draw *two* diagrams: one of the solid region E (see Figure 5) and one of its projection D onto the xy-plane (see Figure 6). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$), so we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

9 $$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

This description of E as a type 1 region enables us to evaluate the integral as follows:

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\,dy\,dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\,dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$ ■

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \leqslant x \leqslant u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7). The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

10 $$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\,dx\right] dA$$

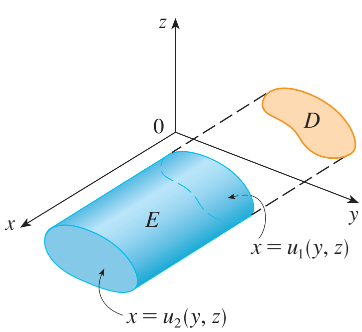


FIGURE 7
A type 2 region

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \leqslant y \leqslant u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8). For this type of region we have

11 $$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\,dy\right] dA$$

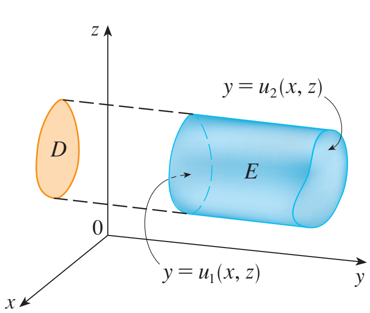


FIGURE 8
A type 3 region

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

EXAMPLE 3 Evaluate $\iiint_E \sqrt{x^2 + z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

SOLUTION The solid E is shown in Figure 9. If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane, which is the parabolic region in Figure 10. (The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$.)

TEC Visual 15.6 illustrates how solid regions (including the one in Figure 9) project onto coordinate planes.

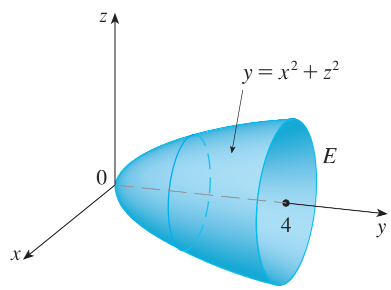


FIGURE 9
Region of integration

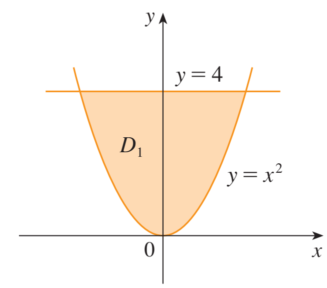


FIGURE 10
Projection onto xy-plane

From $y = x^2 + z^2$ we obtain $z = \pm\sqrt{y - x^2}$, so the lower boundary surface of E is $z = -\sqrt{y - x^2}$ and the upper surface is $z = \sqrt{y - x^2}$. Therefore the description of E as a type 1 region is

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

and so we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

Although this expression is correct, it is extremely difficult to evaluate. So let's instead consider E as a type 3 region. As such, its projection D_3 onto the xz-plane is the disk $x^2 + z^2 \leqslant 4$ shown in Figure 11.

Then the left boundary of E is the paraboloid $y = x^2 + z^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have

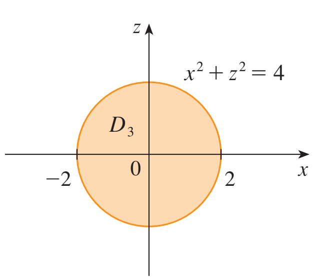


FIGURE 11
Projection onto xz-plane

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy\right] dA = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

Although this integral could be written as

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\, \sqrt{x^2 + z^2}\, dz\, dx$$

it's easier to convert to polar coordinates in the xz-plane: $x = r\cos\theta$, $z = r\sin\theta$. This gives

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$ ■

⊘ The most difficult step in evaluating a triple integral is setting up an expression for the region of integration (such as Equation 9 in Example 2). Remember that the limits of integration in the inner integral contain at most two variables, the limits of integration in the middle integral contain at most one variable, and the limits of integration in the outer integral must be constants.

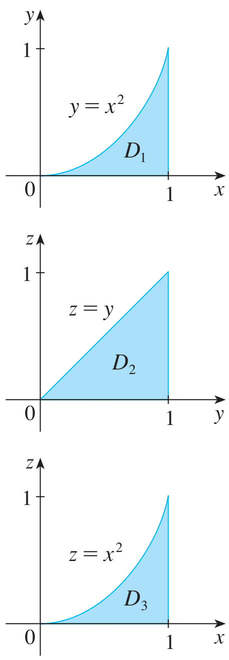


FIGURE 12
Projections of E

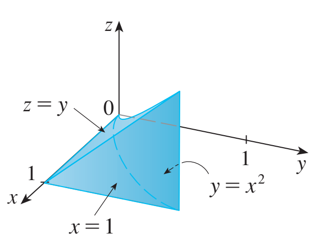


FIGURE 13
The solid E

EXAMPLE 4 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

SOLUTION We can write

$$\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\, dz\, dy\, dx = \iiint_E f(x, y, z)\, dV$$

where $E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2, 0 \leqslant z \leqslant y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

on the xy-plane: $\quad D_1 = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$

$\quad = \{(x, y) \mid 0 \leqslant y \leqslant 1, \sqrt{y} \leqslant x \leqslant 1\}$

on the yz-plane: $\quad D_2 = \{(y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y\}$

on the xz-plane: $\quad D_3 = \{(x, z) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant x^2\}$

From the resulting sketches of the projections in Figure 12 we sketch the solid E in Figure 13. We see that it is the solid enclosed by the planes $z = 0$, $x = 1$, $y = z$ and the parabolic cylinder $y = x^2$ $\left(\text{or } x = \sqrt{y}\right)$.

If we integrate first with respect to x, then z, and then y, we use an alternate description of E:

$$E = \left\{(x, y, z) \mid 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant y, \sqrt{y} \leqslant x \leqslant 1\right\}$$

Thus

$$\iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x, y, z)\, dx\, dz\, dy$$ ■

■ Applications of Triple Integrals

Recall that if $f(x) \geqslant 0$, then the single integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geqslant 0$, then the double integral $\iint_D f(x, y)\, dA$ represents the volume under the surface $z = f(x, y)$ and above D. The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\, dV$, where $f(x, y, z) \geqslant 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.) Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z, and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12
$$V(E) = \iiint_E dV$$

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz\right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

and from Section 15.2 we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

EXAMPLE 5 Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

SOLUTION The tetrahedron T and its projection D onto the xy-plane are shown in Figures 14 and 15. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

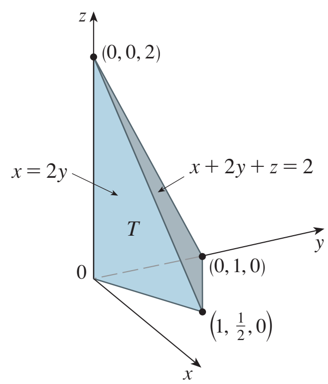


FIGURE 14

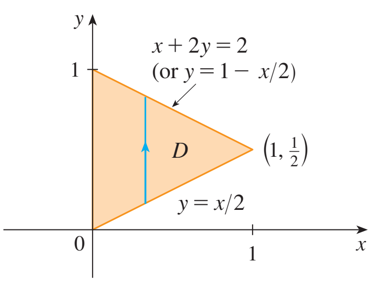


FIGURE 15

Therefore we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

by the same calculation as in Example 15.2.4.

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.) ■

All the applications of double integrals in Section 15.4 can be immediately extended to triple integrals. For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

13 $$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

14 $$M_{yz} = \iiint_E x\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

15 $$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of E. The **moments of inertia** about the three coordinate axes are

16 $$I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\,dV \qquad I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\,dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\,dV$$

As in Section 15.4, the total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\,dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\,dV$$

In particular,

$$P(a \leqslant X \leqslant b,\; c \leqslant Y \leqslant d,\; r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\,dz\,dy\,dx = 1$$

EXAMPLE 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

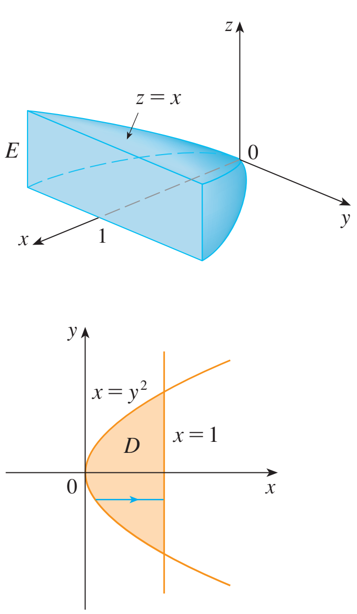


FIGURE 16

SOLUTION The solid E and its projection onto the xy-plane are shown in Figure 16. The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe E as a type 1 region:

$$E = \{(x, y, z) \mid -1 \leqslant y \leqslant 1,\; y^2 \leqslant x \leqslant 1,\; 0 \leqslant z \leqslant x\}$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint_E \rho\,dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\,dz\,dx\,dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x\,dx\,dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy$$

$$= \frac{\rho}{2} \int_{-1}^{1} (1 - y^4)\,dy = \rho \int_0^1 (1 - y^4)\,dy$$

$$= \rho\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that $M_{xz} = 0$ and therefore $\bar{y} = 0$. The other moments are

$$M_{yz} = \iiint_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy = \rho \int_{-1}^{1} \left[\frac{x^3}{3} \right]_{x=y^2}^{x=1} dy$$

$$= \frac{2\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{3} \left[y - \frac{y^7}{7} \right]_0^1 = \frac{4\rho}{7}$$

$$M_{xy} = \iiint_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho\, dz\, dx\, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \left[\frac{z^2}{2} \right]_{z=0}^{z=x} dx\, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2\, dx\, dy$$

$$= \frac{\rho}{3} \int_0^1 (1 - y^6)\, dy = \frac{2\rho}{7}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\tfrac{5}{7}, 0, \tfrac{5}{14} \right)$$ ■

15.6 EXERCISES

1. Evaluate the integral in Example 1, integrating first with respect to y, then z, and then x.

2. Evaluate the integral $\iiint_E (xy + z^2)\, dV$, where

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 3\}$$

using three different orders of integration.

3–8 Evaluate the iterated integral.

3. $\displaystyle\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz$

4. $\displaystyle\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\, dz\, dx\, dy$

5. $\displaystyle\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$

6. $\displaystyle\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y + 1}\, dx\, dz\, dy$

7. $\displaystyle\int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-z^2}} z \sin x\, dy\, dz\, dx$

8. $\displaystyle\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z\, dz\, dy\, dx$

9–18 Evaluate the triple integral.

9. $\iiint_E y\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant x, x - y \leqslant z \leqslant x + y\}$

10. $\iiint_E e^{z/y}\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1, 0 \leqslant z \leqslant xy\}$

11. $\displaystyle\iiint_E \frac{z}{x^2 + z^2}\, dV$, where
$E = \{(x, y, z) \mid 1 \leqslant y \leqslant 4, y \leqslant z \leqslant 4, 0 \leqslant x \leqslant z\}$

12. $\iiint_E \sin y\, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0, 0, 0)$, $(\pi, 0, 0)$, and $(0, \pi, 0)$

13. $\iiint_E 6xy\, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

14. $\iiint_E (x - y)\, dV$, where E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$

15. $\iiint_T y^2\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$

16. $\iiint_T xz\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$

17. $\iiint_E x\, dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

18. $\iiint_E z\, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant

19–22 Use a triple integral to find the volume of the given solid.

19. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

20. The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$

21. The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$

22. The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

23. (a) Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral.

CAS (b) Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to find the exact value of the triple integral in part (a).

24. (a) In the **Midpoint Rule for triple integrals** we use a triple Riemann sum to approximate a triple integral over a box B, where $f(x, y, z)$ is evaluated at the center $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ of the box B_{ijk}. Use the Midpoint Rule to estimate $\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV$, where B is the cube defined by $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant 4$. Divide B into eight cubes of equal size.

CAS (b) Use a computer algebra system to approximate the integral in part (a) correct to the nearest integer. Compare with the answer to part (a).

25–26 Use the Midpoint Rule for triple integrals (Exercise 24) to estimate the value of the integral. Divide B into eight subboxes of equal size.

25. $\iiint_B \cos(xyz)\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1\}$

26. $\iiint_B \sqrt{x}\, e^{xyz}\, dV$, where
$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 2\}$

27–28 Sketch the solid whose volume is given by the iterated integral.

27. $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$

28. $\displaystyle\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$

29–32 Express the integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces.

29. $y = 4 - x^2 - 4z^2$, $y = 0$

30. $y^2 + z^2 = 9$, $x = -2$, $x = 2$

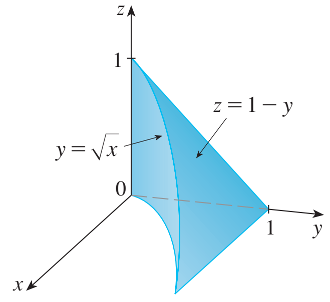
31. $y = x^2$, $z = 0$, $y + 2z = 4$

32. $x = 2$, $y = 2$, $z = 0$, $x + y - 2z = 2$

33. The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

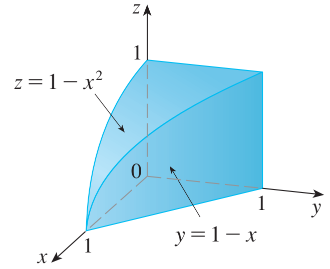
Rewrite this integral as an equivalent iterated integral in the five other orders.



34. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



35–36 Write five other iterated integrals that are equal to the given iterated integral.

35. $\displaystyle\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$

36. $\displaystyle\int_0^1 \int_y^1 \int_0^z f(x, y, z)\, dx\, dz\, dy$

37–38 Evaluate the triple integral using only geometric interpretation and symmetry.

37. $\iiint_C (4 + 5x^2yz^2)\, dV$, where C is the cylindrical region $x^2 + y^2 \leqslant 4$, $-2 \leqslant z \leqslant 2$

38. $\iiint_B (z^3 + \sin y + 3)\, dV$, where B is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

39–42 Find the mass and center of mass of the solid E with the given density function ρ.

39. E lies above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$; $\rho(x, y, z) = 3$

40. E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$; $\rho(x, y, z) = 4$

41. E is the cube given by $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$; $\rho(x, y, z) = x^2 + y^2 + z^2$

42. E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$; $\rho(x, y, z) = y$

43–46 Assume that the solid has constant density k.

43. Find the moments of inertia for a cube with side length L if one vertex is located at the origin and three edges lie along the coordinate axes.

44. Find the moments of inertia for a rectangular brick with dimensions a, b, and c and mass M if the center of the brick is situated at the origin and the edges are parallel to the coordinate axes.

45. Find the moment of inertia about the z-axis of the solid cylinder $x^2 + y^2 \leqslant a^2$, $0 \leqslant z \leqslant h$.

46. Find the moment of inertia about the z-axis of the solid cone $\sqrt{x^2 + y^2} \leqslant z \leqslant h$.

47–48 Set up, but do not evaluate, integral expressions for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

47. The solid of Exercise 21; $\rho(x, y, z) = \sqrt{x^2 + y^2}$

48. The hemisphere $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$; $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

CAS **49.** Let E be the solid in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ with the density function $\rho(x, y, z) = 1 + x + y + z$. Use a computer algebra system to find the exact values of the following quantities for E.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

CAS **50.** If E is the solid of Exercise 18 with density function $\rho(x, y, z) = x^2 + y^2$, find the following quantities, correct to three decimal places.
(a) The mass
(b) The center of mass
(c) The moment of inertia about the z-axis

51. The joint density function for random variables X, Y, and Z is $f(x, y, z) = Cxyz$ if $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 2$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.
(c) Find $P(X + Y + Z \leqslant 1)$.

52. Suppose X, Y, and Z are random variables with joint density function $f(x, y, z) = Ce^{-(0.5x+0.2y+0.1z)}$ if $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, and $f(x, y, z) = 0$ otherwise.
(a) Find the value of the constant C.
(b) Find $P(X \leqslant 1, Y \leqslant 1)$.
(c) Find $P(X \leqslant 1, Y \leqslant 1, Z \leqslant 1)$.

53–54 The **average value** of a function $f(x, y, z)$ over a solid region E is defined to be

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{ave} is the average density of E.

53. Find the average value of the function $f(x, y, z) = xyz$ over the cube with side length L that lies in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

54. Find the average height of the points in the solid hemisphere $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$.

55. (a) Find the region E for which the triple integral

$$\iiint_E (1 - x^2 - 2y^2 - 3z^2)\, dV$$

is a maximum.

CAS (b) Use a computer algebra system to calculate the exact maximum value of the triple integral in part (a).

DISCOVERY PROJECT VOLUMES OF HYPERSPHERES

In this project we find formulas for the volume enclosed by a hypersphere in n-dimensional space.

1. Use a double integral and trigonometric substitution, together with Formula 64 in the Table of Integrals, to find the area of a circle with radius r.

2. Use a triple integral and trigonometric substitution to find the volume of a sphere with radius r.

3. Use a quadruple integral to find the (4-dimensional) volume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$ in $\mathbb{R}^4$. (Use only trigonometric substitution and the reduction formulas for $\int \sin^n x\, dx$ or $\int \cos^n x\, dx$.)

4. Use an n-tuple integral to find the volume enclosed by a hypersphere of radius r in n-dimensional space $\mathbb{R}^n$. [*Hint:* The formulas are different for n even and n odd.]

15.7 Triple Integrals in Cylindrical Coordinates

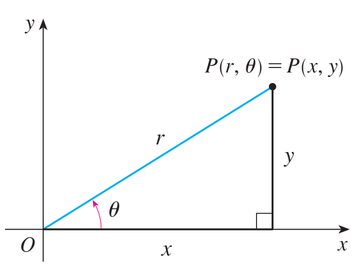


FIGURE 1

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions. (See Section 10.3.) Figure 1 enables us to recall the connection between polar and Cartesian coordinates. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P. (See Figure 2.)

To convert from cylindrical to rectangular coordinates, we use the equations

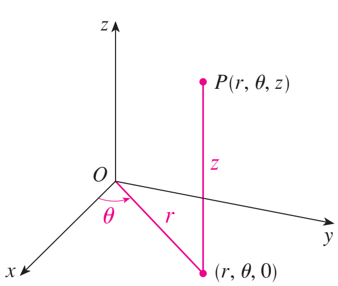


FIGURE 2
The cylindrical coordinates of a point

1
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

2
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

EXAMPLE 1
(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

SOLUTION
(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3. From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

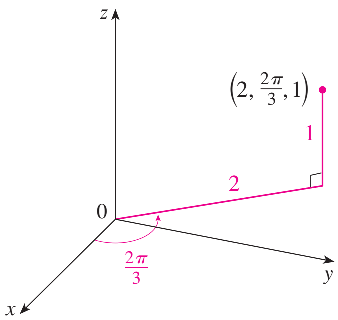


FIGURE 3

So the point is $\left(-1, \sqrt{3}, 1\right)$ in rectangular coordinates.
(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $\left(3\sqrt{2}, 7\pi/4, -7\right)$. Another is $\left(3\sqrt{2}, -\pi/4, -7\right)$. As with polar coordinates, there are infinitely many choices. ■

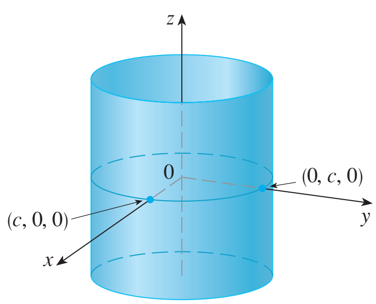


FIGURE 4
$r = c$, a cylinder

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$. (See Figure 4.) This is the reason for the name "cylindrical" coordinates.

EXAMPLE 2 Describe the surface whose equation in cylindrical coordinates is $z = r$.

SOLUTION The equation says that the z-value, or height, of each point on the surface is the same as r, the distance from the point to the z-axis. Because θ doesn't appear, it can vary. So any horizontal trace in the plane $z = k$ $(k > 0)$ is a circle of radius k. These traces suggest that the surface is a cone. This prediction can be confirmed by converting the equation into rectangular coordinates. From the first equation in (2) we have

$$z^2 = r^2 = x^2 + y^2$$

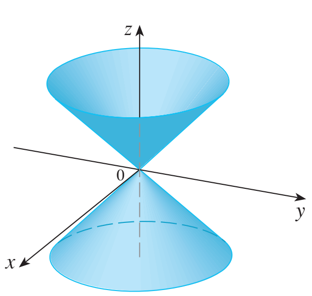


FIGURE 5
$z = r$, a cone

We recognize the equation $z^2 = x^2 + y^2$ (by comparison with Table 1 in Section 12.6) as being a circular cone whose axis is the z-axis (see Figure 5). ■

Evaluating Triple Integrals with Cylindrical Coordinates

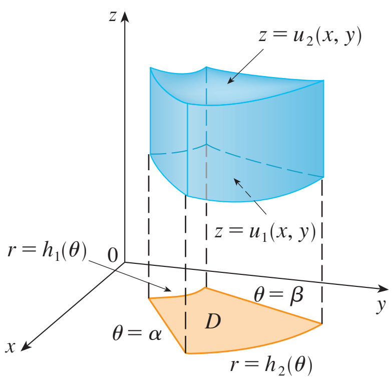


FIGURE 6

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 6). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

We know from Equation 15.6.6 that

3 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining Equation 3 with Equation 15.3.3, we obtain

4 $$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

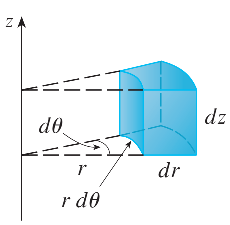


FIGURE 7
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Formula 4 is the **formula for triple integration in cylindrical coordinates**. It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$. (Figure 7 shows how to remember this.) It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

SOLUTION In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

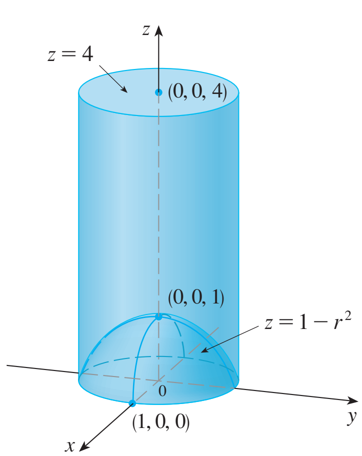


FIGURE 8

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, from Formula 15.6.13, the mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r_2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta = K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$ ■

EXAMPLE 4 Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.

SOLUTION This iterated integral is a triple integral over the solid region

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ -\sqrt{4-x^2} \leqslant y \leqslant \sqrt{4-x^2},\ \sqrt{x^2+y^2} \leqslant z \leqslant 2\right\}$$

and the projection of E onto the xy-plane is the disk $x^2 + y^2 \leqslant 4$. The lower surface of E is the cone $z = \sqrt{x^2+y^2}$ and its upper surface is the plane $z = 2$. (See Figure 9.) This region has a much simpler description in cylindrical coordinates:

$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 2,\ r \leqslant z \leqslant 2\right\}$$

Therefore we have

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx = \iiint_E (x^2+y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)\,dr$$

$$= 2\pi\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 = \tfrac{16}{5}\pi$$

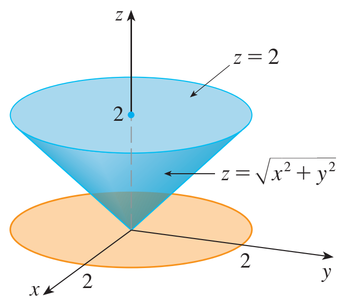


FIGURE 9

15.7 EXERCISES

1–2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(4, \pi/3, -2)$ (b) $(2, -\pi/2, 1)$

2. (a) $(\sqrt{2}, 3\pi/4, 2)$ (b) $(1, 1, 1)$

3–4 Change from rectangular to cylindrical coordinates.

3. (a) $(-1, 1, 1)$ (b) $(-2, 2\sqrt{3}, 3)$

4. (a) $(-\sqrt{2}, \sqrt{2}, 1)$ (b) $(2, 2, 2)$

5–6 Describe in words the surface whose equation is given.

5. $r = 2$ **6.** $\theta = \pi/6$

7–8 Identify the surface whose equation is given.

7. $r^2 + z^2 = 4$ **8.** $r = 2\sin\theta$

9–10 Write the equations in cylindrical coordinates.

9. (a) $x^2 - x + y^2 + z^2 = 1$ (b) $z = x^2 - y^2$

10. (a) $2x^2 + 2y^2 - z^2 = 4$ (b) $2x - y + z = 1$

11–12 Sketch the solid described by the given inequalities.

11. $r^2 \leqslant z \leqslant 8 - r^2$

12. $0 \leqslant \theta \leqslant \pi/2, \quad r \leqslant z \leqslant 2$

13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

14. Use a graphing device to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

15–16 Sketch the solid whose volume is given by the integral and evaluate the integral.

15. $\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^{r^2} r\,dz\,dr\,d\theta$ **16.** $\int_0^2\int_0^{2\pi}\int_0^r r\,dz\,d\theta\,dr$

17–28 Use cylindrical coordinates.

17. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

18. Evaluate $\iiint_E z\,dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

19. Evaluate $\iiint_E (x + y + z)\,dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

20. Evaluate $\iiint_E (x - y)\,dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

21. Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

22. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

23. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

24. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

25. (a) Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.
(b) Find the centroid of E (the center of mass in the case where the density is constant).

26. (a) Find the volume of the solid that the cylinder $r = a\cos\theta$ cuts out of the sphere of radius a centered at the origin.
(b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.

27. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if S has constant density K.

28. Find the mass of a ball B given by $x^2 + y^2 + z^2 \leqslant a^2$ if the density at any point is proportional to its distance from the z-axis.

29–30 Evaluate the integral by changing to cylindrical coordinates.

29. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$

30. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2 + y^2}\,dz\,dy\,dx$

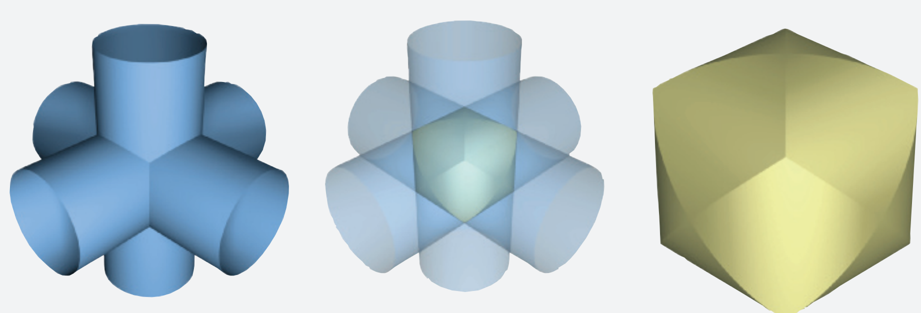
31. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.
(a) Find a definite integral that represents the total work done in forming the mountain.
(b) Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3. How much work was done in forming Mount Fuji if the land was initially at sea level?



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DISCOVERY PROJECT THE INTERSECTION OF THREE CYLINDERS

The figure shows the solid enclosed by three circular cylinders with the same diameter that intersect at right angles. In this project we compute its volume and determine how its shape changes if the cylinders have different diameters.



1. Sketch carefully the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Indicate the positions of the coordinate axes and label the faces with the equations of the corresponding cylinders.

2. Find the volume of the solid in Problem 1.

CAS **3.** Use a computer algebra system to draw the edges of the solid.

4. What happens to the solid in Problem 1 if the radius of the first cylinder is different from 1? Illustrate with a hand-drawn sketch or a computer graph.

5. If the first cylinder is $x^2 + y^2 = a^2$, where $a < 1$, set up, but do not evaluate, a double integral for the volume of the solid. What if $a > 1$?

15.8 Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the *spherical coordinate system*. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

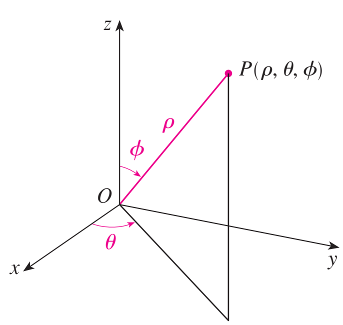


FIGURE 1
The spherical coordinates of a point

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name "spherical" coordinates. The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).

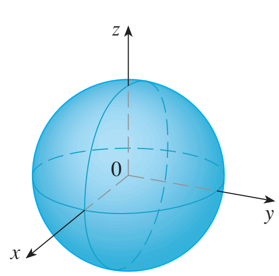


FIGURE 2 $\rho = c$, a sphere

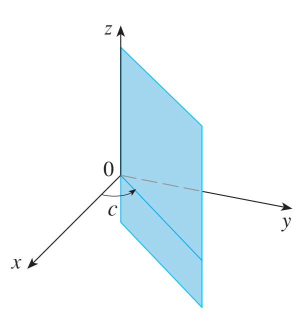


FIGURE 3 $\theta = c$, a half-plane

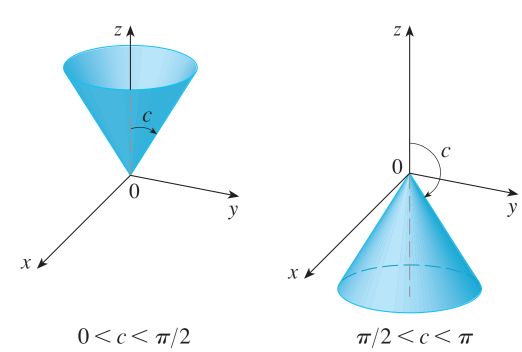


FIGURE 4 $\phi = c$, a half-cone

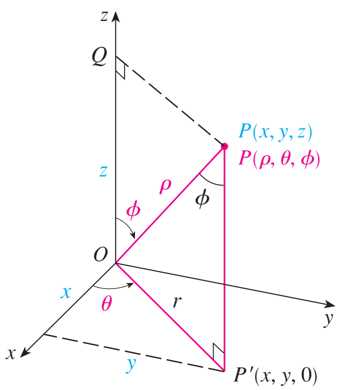


FIGURE 5

The relationship between rectangular and spherical coordinates can be seen from Figure 5. From triangles OPQ and OPP' we have

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

1
$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

Also, the distance formula shows that

2
$$\rho^2 = x^2 + y^2 + z^2$$

We use this equation in converting from rectangular to spherical coordinates.

EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

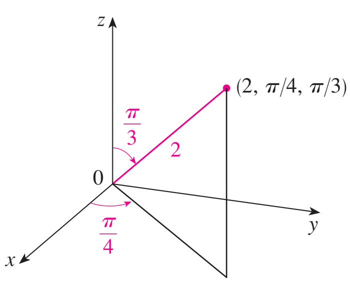


FIGURE 6

SOLUTION We plot the point in Figure 6. From Equations 1 we have

$$x = \rho \sin \phi \cos \theta = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates. ■

EXAMPLE 2 The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

WARNING There is not universal agreement on the notation for spherical coordinates. Most books on physics reverse the meanings of θ and ϕ and use r in place of ρ.

SOLUTION From Equation 2 we have

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

and so Equations 1 give

$$\cos \phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos \theta = \frac{x}{\rho \sin \phi} = 0 \qquad \theta = \frac{\pi}{2}$$

TEC In Module 15.8 you can investigate families of surfaces in cylindrical and spherical coordinates.

(Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$.) Therefore spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$. ■

Evaluating Triple Integrals with Spherical Coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

where $a \geqslant 0$ and $\beta - \alpha \leqslant 2\pi$, and $d - c \leqslant \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result. So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$. Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\,\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\,\Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$). So an approximation to the volume of E_{ijk} is given by

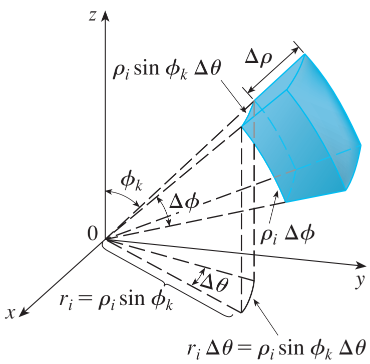


FIGURE 7

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\,\Delta\phi)(\rho_i \sin\phi_k\,\Delta\theta) = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

In fact, it can be shown, with the aid of the Mean Value Theorem (Exercise 49), that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}. Let $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ be the rectangular coordinates of this point. Then

$$\iiint_E f(x, y, z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\,\tilde{\rho}_i^2 \sin\tilde{\phi}_k\,\Delta\rho\Delta\theta\Delta\phi$$

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin\phi\,\cos\theta, \rho \sin\phi\,\sin\theta, \rho \cos\phi)\,\rho^2 \sin\phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint_E f(x, y, z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi\,\cos\theta, \rho \sin\phi\,\sin\theta, \rho \cos\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin\phi\,\cos\theta \qquad y = \rho \sin\phi\,\sin\theta \qquad z = \rho \cos\phi$$

using the appropriate limits of integration, and replacing dV by $\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$. This is illustrated in Figure 8.

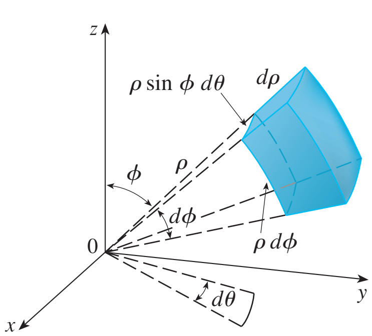


FIGURE 8

Volume element in spherical coordinates: $dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d,\ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

In this case the formula is the same as in (3) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

SOLUTION Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus (3) gives

$$\begin{aligned}\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} d\rho \\ &= \Big[-\cos\phi\Big]_0^{\pi} (2\pi) \Big[\tfrac{1}{3} e^{\rho^3}\Big]_0^1 = \tfrac{4}{3}\pi(e - 1)\end{aligned}$$ ■

NOTE It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

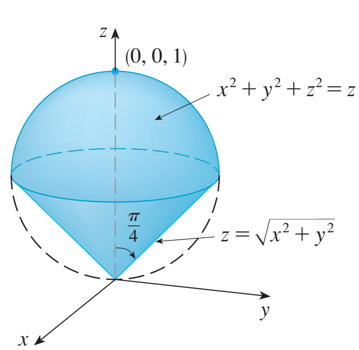


FIGURE 9

Figure 10 gives another look (this time drawn by Maple) at the solid of Example 4.

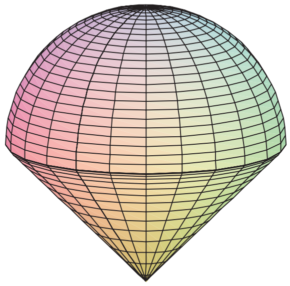


FIGURE 10

SOLUTION Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho\cos\phi \qquad \text{or} \qquad \rho = \cos\phi$$

The equation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\,\cos^2\theta + \rho^2\sin^2\phi\,\sin^2\theta} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi/4,\ 0 \leqslant \rho \leqslant \cos\phi\}$$

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

TEC Visual 15.8 shows an animation of Figure 11.

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\,\cos^3\phi\; d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$$

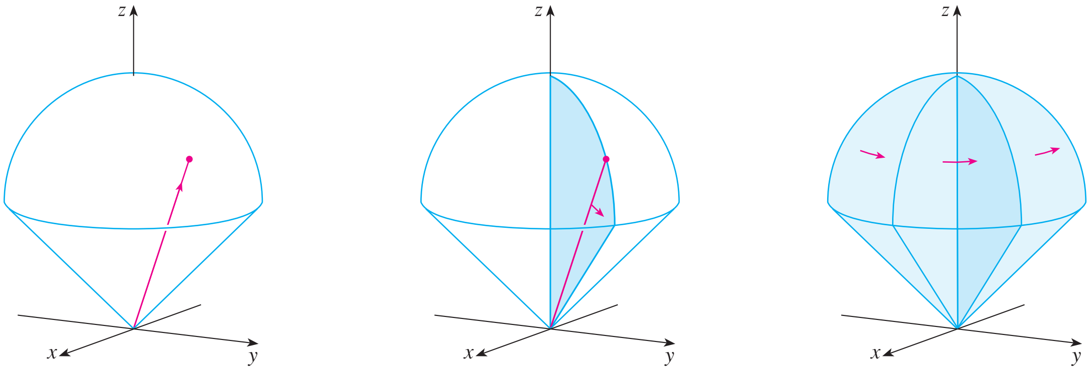


FIGURE 11

ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π. ■

15.8 EXERCISES

1–2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(6, \pi/3, \pi/6)$ (b) $(3, \pi/2, 3\pi/4)$

2. (a) $(2, \pi/2, \pi/2)$ (b) $(4, -\pi/4, \pi/3)$

3–4 Change from rectangular to spherical coordinates.

3. (a) $(0, -2, 0)$ (b) $(-1, 1, -\sqrt{2})$

4. (a) $(1, 0, \sqrt{3})$ (b) $(\sqrt{3}, -1, 2\sqrt{3})$

5–6 Describe in words the surface whose equation is given.

5. $\phi = \pi/3$

6. $\rho^2 - 3\rho + 2 = 0$

7–8 Identify the surface whose equation is given.

7. $\rho \cos \phi = 1$

8. $\rho = \cos \phi$

9–10 Write the equation in spherical coordinates.

9. (a) $x^2 + y^2 + z^2 = 9$ (b) $x^2 - y^2 - z^2 = 1$

10. (a) $z = x^2 + y^2$ (b) $z = x^2 - y^2$

11–14 Sketch the solid described by the given inequalities.

11. $\rho \leqslant 1, \quad 0 \leqslant \phi \leqslant \pi/6, \quad 0 \leqslant \theta \leqslant \pi$

12. $1 \leqslant \rho \leqslant 2, \quad \pi/2 \leqslant \phi \leqslant \pi$

13. $2 \leqslant \rho \leqslant 4, \quad 0 \leqslant \phi \leqslant \pi/3, \quad 0 \leqslant \theta \leqslant \pi$

14. $\rho \leqslant 2, \quad \rho \leqslant \csc \phi$

15. A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

16. (a) Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.
(b) Suppose the ball is cut in half. Write inequalities that describe one of the halves.

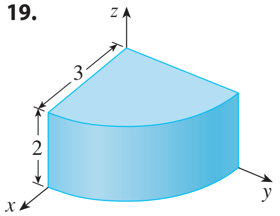
17–18 Sketch the solid whose volume is given by the integral and evaluate the integral.

17. $\displaystyle\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \; d\rho \, d\theta \, d\phi$

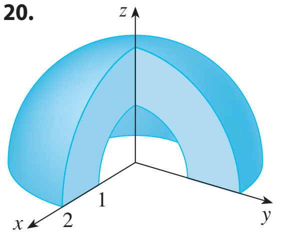
18. $\displaystyle\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

19–20 Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

19.



20.



21–34 Use spherical coordinates.

21. Evaluate $\iiint_B (x^2 + y^2 + z^2)^2 \, dV$, where B is the ball with center the origin and radius 5.

22. Evaluate $\iiint_E y^2 z^2 \, dV$, where E lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$.

23. Evaluate $\iiint_E (x^2 + y^2) \, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

24. Evaluate $\iiint_E y^2 \, dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $y \geqslant 0$.

25. Evaluate $\iiint_E x e^{x^2+y^2+z^2} \, dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

26. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$, where E lies above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

27. Find the volume of the part of the ball $\rho \leqslant a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

28. Find the average distance from a point in a ball of radius a to its center.

29. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4 \cos \phi$.
(b) Find the centroid of the solid in part (a).

30. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

31. (a) Find the centroid of the solid in Example 4. (Assume constant density K.)
(b) Find the moment of inertia about the z-axis for this solid.

32. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
(a) Find the mass of H.
(b) Find the center of mass of H.
(c) Find the moment of inertia of H about its axis.

33. (a) Find the centroid of a solid homogeneous hemisphere of radius a.
(b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

34. Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

35–40 Use cylindrical or spherical coordinates, whichever seems more appropriate.

35. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

36. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.

37. A solid cylinder with constant density has base radius a and height h.
(a) Find the moment of inertia of the cylinder about its axis.
(b) Find the moment of inertia of the cylinder about a diameter of its base.

38. A solid right circular cone with constant density has base radius a and height h.
(a) Find the moment of inertia of the cone about its axis.
(b) Find the moment of inertia of the cone about a diameter of its base.

CAS **39.** Evaluate $\iiint_E z\,dV$, where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$. Use either the Table of Integrals (on Reference Pages 6–10) or a computer algebra system to evaluate the integral.

CAS **40.** (a) Find the volume enclosed by the torus $\rho = \sin\phi$.
(b) Use a computer to draw the torus.

41–43 Evaluate the integral by changing to spherical coordinates.

41. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$

42. $\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3)\,dz\,dx\,dy$

43. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{3/2}\,dz\,dy\,dx$

44. A model for the density δ of the earth's atmosphere near its surface is

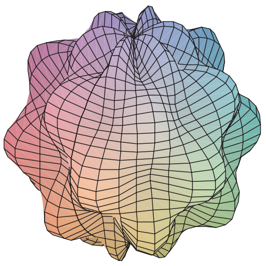
$$\delta = 619.09 - 0.000097\rho$$

where ρ (the distance from the center of the earth) is measured in meters and δ is measured in kilograms per cubic meter. If we take the surface of the earth to be a sphere with radius 6370 km, then this model is a reasonable one for $6.370 \times 10^6 \leqslant \rho \leqslant 6.375 \times 10^6$. Use this model to estimate the mass of the atmosphere between the ground and an altitude of 5 km.

45. Use a graphing device to draw a silo consisting of a cylinder with radius 3 and height 10 surmounted by a hemisphere.

46. The latitude and longitude of a point P in the Northern Hemisphere are related to spherical coordinates ρ, θ, ϕ as follows. We take the origin to be the center of the earth and the positive z-axis to pass through the North Pole. The positive x-axis passes through the point where the prime meridian (the meridian through Greenwich, England) intersects the equator. Then the latitude of P is $\alpha = 90° - \phi°$ and the longitude is $\beta = 360° - \theta°$. Find the great-circle distance from Los Angeles (lat. 34.06° N, long. 118.25° W) to Montréal (lat. 45.50° N, long. 73.60° W). Take the radius of the earth to be 3960 mi. (A *great circle* is the circle of intersection of a sphere and a plane through the center of the sphere.)

CAS **47.** The surfaces $\rho = 1 + \frac{1}{5}\sin m\theta \sin n\phi$ have been used as models for tumors. The "bumpy sphere" with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find the volume it encloses.



48. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

49. (a) Use cylindrical coordinates to show that the volume of the solid bounded above by the sphere $r^2 + z^2 = a^2$ and below by the cone $z = r\cot\phi_0$ (or $\phi = \phi_0$), where $0 < \phi_0 < \pi/2$, is

$$V = \frac{2\pi a^3}{3}(1 - \cos\phi_0)$$

(b) Deduce that the volume of the spherical wedge given by $\rho_1 \leqslant \rho \leqslant \rho_2,\ \theta_1 \leqslant \theta \leqslant \theta_2,\ \phi_1 \leqslant \phi \leqslant \phi_2$ is

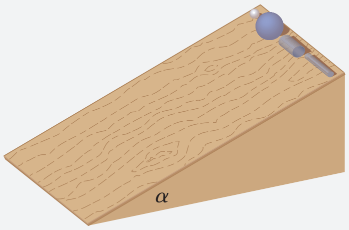
$$\Delta V = \frac{\rho_2^3 - \rho_1^3}{3}(\cos\phi_1 - \cos\phi_2)(\theta_2 - \theta_1)$$

(c) Use the Mean Value Theorem to show that the volume in part (b) can be written as

$$\Delta V = \tilde{\rho}^2 \sin\tilde{\phi}\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

where $\tilde{\rho}$ lies between ρ_1 and ρ_2, $\tilde{\phi}$ lies between ϕ_1 and ϕ_2, $\Delta\rho = \rho_2 - \rho_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta\phi = \phi_2 - \phi_1$.

APPLIED PROJECT ROLLER DERBY



Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question, we consider a ball or cylinder with mass m, radius r, and moment of inertia I (about the axis of rotation). If the vertical drop is h, then the potential energy at the top is mgh. Suppose the object reaches the bottom with velocity v and angular velocity ω, so $v = \omega r$. The kinetic energy at the bottom consists of two parts: $\frac{1}{2}mv^2$ from translation (moving down the slope) and $\frac{1}{2}I\omega^2$ from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

1. Show that

$$v^2 = \frac{2gh}{1 + I^*} \qquad \text{where } I^* = \frac{I}{mr^2}$$

2. If $y(t)$ is the vertical distance traveled at time t, then the same reasoning as used in Problem 1 shows that $v^2 = 2gy/(1 + I^*)$ at any time t. Use this result to show that y satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}}\,(\sin\alpha)\sqrt{y}$$

where α is the angle of inclination of the plane.

3. By solving the differential equation in Problem 2, show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g\sin^2\alpha}}$$

This shows that the object with the smallest value of I^* wins the race.

4. Show that $I^* = \frac{1}{2}$ for a solid cylinder and $I^* = 1$ for a hollow cylinder.

5. Calculate I^* for a partly hollow ball with inner radius a and outer radius r. Express your answer in terms of $b = a/r$. What happens as $a \to 0$ and as $a \to r$?

6. Show that $I^* = \frac{2}{5}$ for a solid ball and $I^* = \frac{2}{3}$ for a hollow ball. Thus the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.

15.9 Change of Variables in Multiple Integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write the Substitution Rule (4.5.5) as

1 $$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

2 $$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (15.3.2) can be written as

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

3 $$x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**. Figure 1 shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the **image of S**, consisting of the images of all points in S.

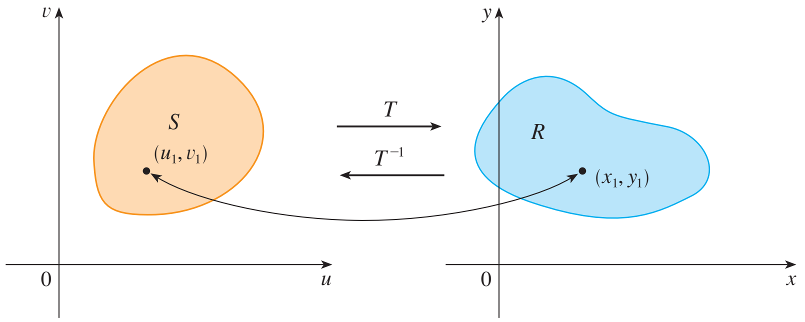


FIGURE 1

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad v = H(x, y)$$

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$

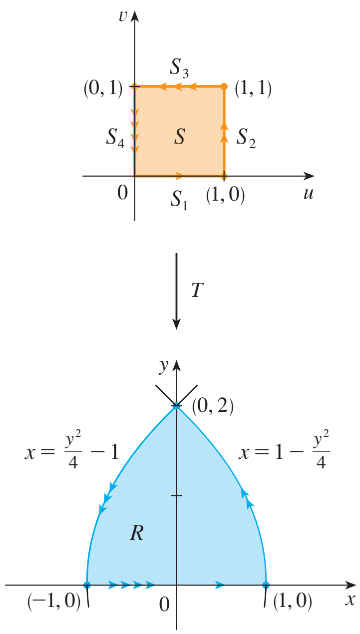


FIGURE 2

$(0 \leqslant u \leqslant 1)$. (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

$$\boxed{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$\boxed{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5. ■

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

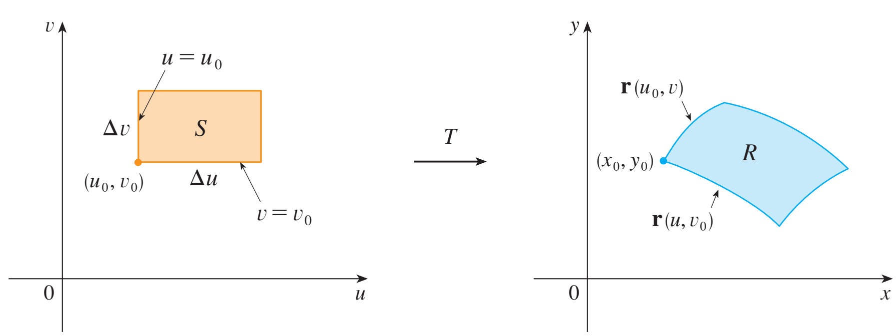


FIGURE 3

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely,

$u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

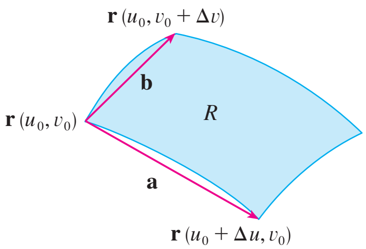


FIGURE 4

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

shown in Figure 4. But

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so $\qquad \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u$

Similarly $\qquad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$

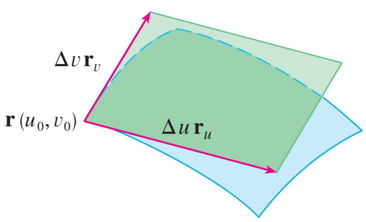


FIGURE 5

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$. (See Figure 5.) Therefore we can approximate the area of R by the area of this parallelogram, which, from Section 12.4, is

6 $$|(\Delta u\,\mathbf{r}_u) \times (\Delta v\,\mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). Although the French mathematician Cauchy first used these special determinants involving partial derivatives, Jacobi developed them into a method for evaluating multiple integrals.

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

8 $$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\,\Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)

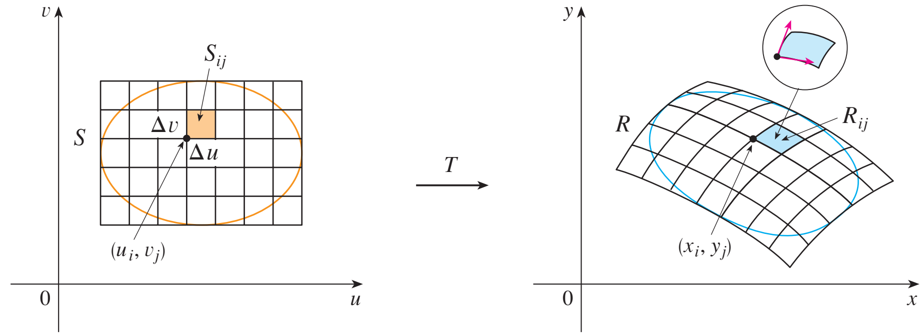


FIGURE 6

Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

The foregoing argument suggests that the following theorem is true. (A full proof is given in books on advanced calculus.)

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Theorem 9 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2. Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is, $|\partial(x, y)/\partial(u, v)|$.

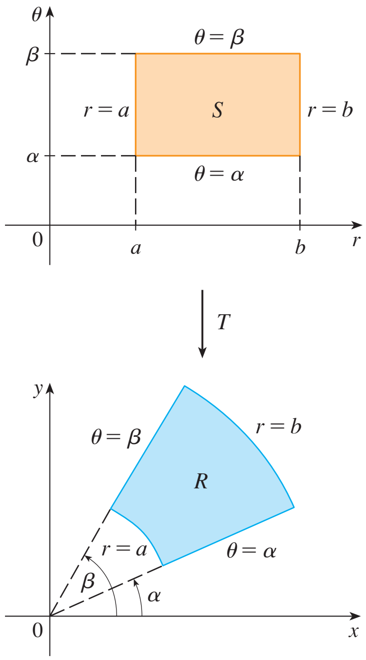


FIGURE 7
The polar coordinate transformation

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7. T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

which is the same as Formula 15.3.2.

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

SOLUTION The region R is pictured in Figure 2 (on page 1094). In Example 1 we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\iint_R y\, dA = \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv$$

$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv = 8 \int_0^1 \left[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v^3 \right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\, dv = \left[v^2 + v^4 \right]_0^1 = 2$$

■

NOTE Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If $f(x, y)$ is difficult to inte-

grate, then the form of $f(x, y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, −2), and (0, −1).

SOLUTION Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

10 $$u = x + y \qquad v = x - y$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

11 $$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices (1, 1), (2, 2), (−2, 2), and (−1, 1) shown in Figure 8. Since

$$S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$$

Theorem 9 gives

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\,dv = \frac{1}{2}\int_1^2 \Big[ve^{u/v}\Big]_{u=-v}^{u=v} dv$$

$$= \frac{1}{2}\int_1^2 (e - e^{-1})v\,dv = \tfrac{3}{4}(e - e^{-1})$$ ■

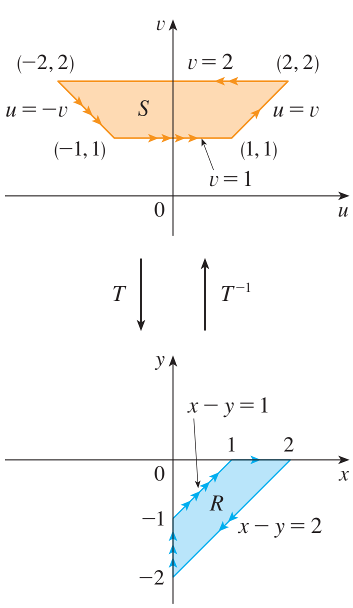


FIGURE 8

Triple Integrals

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

12
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

13
$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

EXAMPLE 4 Use Formula 13 to derive the formula for triple integration in spherical coordinates.

SOLUTION Here the change of variables is given by

$$x = \rho \sin\phi \, \cos\theta \qquad y = \rho \sin\phi \, \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi \, (-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$

$$\quad - \rho \sin\phi \, (\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$

$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = -\rho^2 \sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \left| -\rho^2 \sin\phi \right| = \rho^2 \sin\phi$$

and Formula 13 gives

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

which is equivalent to Formula 15.8.3. ■

15.9 EXERCISES

1–6 Find the Jacobian of the transformation.

1. $x = 2u + v, \quad y = 4u - v$
2. $x = u^2 + uv, \quad y = uv^2$
3. $x = s\cos t, \quad y = s\sin t$
4. $x = pe^q, \quad y = qe^p$
5. $x = uv, \quad y = vw, \quad z = wu$
6. $x = u + vw, \quad y = v + wu, \quad z = w + uv$

7–10 Find the image of the set S under the given transformation.

7. $S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\}$; $x = 2u + 3v$, $y = u - v$
8. S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$
9. S is the triangular region with vertices (0, 0), (1, 1), (0, 1); $x = u^2$, $y = v$
10. S is the disk given by $u^2 + v^2 \leqslant 1$; $x = au$, $y = bv$

11–14 A region R in the xy-plane is given. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u- and v-axes.

11. R is bounded by $y = 2x - 1$, $y = 2x + 1$, $y = 1 - x$, $y = 3 - x$
12. R is the parallelogram with vertices (0, 0), (4, 3), (2, 4), (−2, 1)
13. R lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in the first quadrant
14. R is bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$, $y = 4x$ in the first quadrant

15–20 Use the given transformation to evaluate the integral.

15. $\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices (0, 0), (2, 1), and (1, 2); $x = 2u + v$, $y = u + 2v$
16. $\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices (−1, 3), (1, −3), (3, −1), and (1, 5); $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$
17. $\iint_R x^2\, dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $x = 2u$, $y = 3v$
18. $\iint_R (x^2 - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$; $x = \sqrt{2}\,u - \sqrt{2/3}\,v$, $y = \sqrt{2}\,u + \sqrt{2/3}\,v$
19. $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $x = u/v$, $y = v$
20. $\iint_R y^2\, dA$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$; $u = xy$, $v = xy^2$. Illustrate by using a graphing calculator or computer to draw R.

21. (a) Evaluate $\iiint_E dV$, where E is the solid enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Use the transformation $x = au$, $y = bv$, $z = cw$.
    (b) The earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated by an ellipsoid with $a = b = 6378$ km and $c = 6356$ km. Use part (a) to estimate the volume of the earth.
    (c) If the solid of part (a) has constant density k, find its moment of inertia about the z-axis.
22. An important problem in thermodynamics is to find the work done by an ideal Carnot engine. A cycle consists of alternating expansion and compression of gas in a piston. The work done by the engine is equal to the area of the region R enclosed by two isothermal curves $xy = a$, $xy = b$ and two adiabatic curves $xy^{1.4} = c$, $xy^{1.4} = d$, where $0 < a < b$ and $0 < c < d$. Compute the work done by determining the area of R.

23–27 Evaluate the integral by making an appropriate change of variables.

23. $\displaystyle\iint_R \frac{x - 2y}{3x - y}\, dA$, where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$
24. $\iint_R (x + y)e^{x^2 - y^2}\, dA$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$
25. $\displaystyle\iint_R \cos\left(\frac{y - x}{y + x}\right) dA$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, 2), and (0, 1)
26. $\iint_R \sin(9x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$
27. $\iint_R e^{x+y}\, dA$, where R is given by the inequality $|x| + |y| \leqslant 1$

28. Let f be continuous on $[0, 1]$ and let R be the triangular region with vertices (0, 0), (1, 0), and (0, 1). Show that

$$\iint_R f(x + y)\, dA = \int_0^1 u f(u)\, du$$

15 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.
   (a) Write an expression for a double Riemann sum of f. If $f(x, y) \geqslant 0$, what does the sum represent?
   (b) Write the definition of $\iint_R f(x, y)\, dA$ as a limit.
   (c) What is the geometric interpretation of $\iint_R f(x, y)\, dA$ if $f(x, y) \geqslant 0$? What if f takes on both positive and negative values?
   (d) How do you evaluate $\iint_R f(x, y)\, dA$?
   (e) What does the Midpoint Rule for double integrals say?
   (f) Write an expression for the average value of f.

2. (a) How do you define $\iint_D f(x, y)\, dA$ if D is a bounded region that is not a rectangle?
   (b) What is a type I region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type I region?
   (c) What is a type II region? How do you evaluate $\iint_D f(x, y)\, dA$ if D is a type II region?
   (d) What properties do double integrals have?

3. How do you change from rectangular coordinates to polar coordinates in a double integral? Why would you want to make the change?

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.
   (a) The mass
   (b) The moments about the axes
   (c) The center of mass
   (d) The moments of inertia about the axes and the origin

5. Let f be a joint density function of a pair of continuous random variables X and Y.
   (a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.
   (b) What properties does f possess?
   (c) What are the expected values of X and Y?

6. Write an expression for the area of a surface with equation $z = f(x, y)$, $(x, y) \in D$.

7. (a) Write the definition of the triple integral of f over a rectangular box B.
   (b) How do you evaluate $\iiint_B f(x, y, z)\, dV$?
   (c) How do you define $\iiint_E f(x, y, z)\, dV$ if E is a bounded solid region that is not a box?
   (d) What is a type 1 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?
   (e) What is a type 2 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?
   (f) What is a type 3 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.
   (a) The mass
   (b) The moments about the coordinate planes
   (c) The coordinates of the center of mass
   (d) The moments of inertia about the axes

9. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?
   (b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?
   (c) In what situations would you change to cylindrical or spherical coordinates?

10. (a) If a transformation T is given by $x = g(u, v)$, $y = h(u, v)$, what is the Jacobian of T?
    (b) How do you change variables in a double integral?
    (c) How do you change variables in a triple integral?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\displaystyle\int_{-1}^{2}\int_{0}^{6} x^2 \sin(x - y)\, dx\, dy = \int_{0}^{6}\int_{-1}^{2} x^2 \sin(x - y)\, dy\, dx$

2. $\displaystyle\int_{0}^{1}\int_{0}^{x} \sqrt{x + y^2}\, dy\, dx = \int_{0}^{x}\int_{0}^{1} \sqrt{x + y^2}\, dx\, dy$

3. $\displaystyle\int_{1}^{2}\int_{3}^{4} x^2 e^y\, dy\, dx = \int_{1}^{2} x^2\, dx \int_{3}^{4} e^y\, dy$

4. $\displaystyle\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2} \sin y\, dx\, dy = 0$

5. If f is continuous on $[0, 1]$, then
$$\int_0^1\int_0^1 f(x) f(y)\, dy\, dx = \left[\int_0^1 f(x)\, dx\right]^2$$

6. $\displaystyle\int_{1}^{4}\int_{0}^{1} \left(x^2 + \sqrt{y}\right) \sin(x^2 y^2)\, dx\, dy \leqslant 9$

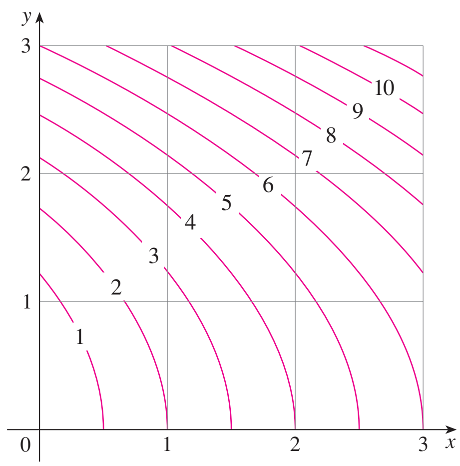
7. If D is the disk given by $x^2 + y^2 \leqslant 4$, then
$$\iint_D \sqrt{4 - x^2 - y^2}\, dA = \tfrac{16}{3}\pi$$

8. The integral $\iiint_E kr^3\, dz\, dr\, d\theta$ represents the moment of inertia about the z-axis of a solid E with constant density k.

9. The integral
$$\int_0^{2\pi}\int_0^2\int_r^2 dz\, dr\, d\theta$$
represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

EXERCISES

1. A contour map is shown for a function f on the square $R = [0, 3] \times [0, 3]$. Use a Riemann sum with nine terms to estimate the value of $\iint_R f(x, y)\, dA$. Take the sample points to be the upper right corners of the squares.



2. Use the Midpoint Rule to estimate the integral in Exercise 1.

3–8 Calculate the iterated integral.

3. $\int_1^2 \int_0^2 (y + 2xe^y)\, dx\, dy$

4. $\int_0^1 \int_0^1 ye^{xy}\, dx\, dy$

5. $\int_0^1 \int_0^x \cos(x^2)\, dy\, dx$

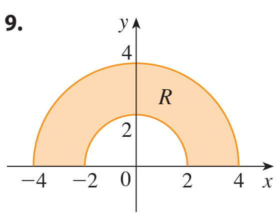
6. $\int_0^1 \int_x^{e^x} 3xy^2\, dy\, dx$

7. $\int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x\, dz\, dy\, dx$

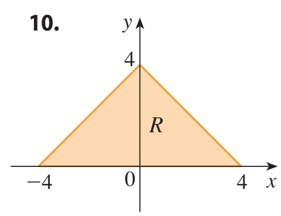
8. $\int_0^1 \int_0^y \int_x^1 6xyz\, dz\, dx\, dy$

9–10 Write $\iint_R f(x, y)\, dA$ as an iterated integral, where R is the region shown and f is an arbitrary continuous function on R.

9.



10.



11. The cylindrical coordinates of a point are $(2\sqrt{3}, \pi/3, 2)$. Find the rectangular and spherical coordinates of the point.

12. The rectangular coordinates of a point are $(2, 2, -1)$. Find the cylindrical and spherical coordinates of the point.

13. The spherical coordinates of a point are $(8, \pi/4, \pi/6)$. Find the rectangular and cylindrical coordinates of the point.

14. Identify the surfaces whose equations are given.
(a) $\theta = \pi/4$ (b) $\phi = \pi/4$

15. Write the equation in cylindrical coordinates and in spherical coordinates.
(a) $x^2 + y^2 + z^2 = 4$ (b) $x^2 + y^2 = 4$

16. Sketch the solid consisting of all points with spherical coordinates (ρ, θ, ϕ) such that $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \phi \leqslant \pi/6$, and $0 \leqslant \rho \leqslant 2\cos\phi$.

17. Describe the region whose area is given by the integral

$$\int_0^{\pi/2} \int_0^{\sin 2\theta} r\, dr\, d\theta$$

18. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

and evaluate the integral.

19–20 Calculate the iterated integral by first reversing the order of integration.

19. $\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$

20. $\int_0^1 \int_{\sqrt{y}}^1 \dfrac{ye^{x^2}}{x^3}\, dx\, dy$

21–34 Calculate the value of the multiple integral.

21. $\iint_R ye^{xy}\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 3\}$

22. $\iint_D xy\, dA$, where $D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ y^2 \leqslant x \leqslant y + 2\}$

23. $\displaystyle\iint_D \frac{y}{1 + x^2}\, dA$,
where D is bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$

24. $\displaystyle\iint_D \frac{1}{1 + x^2}\, dA$, where D is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$

25. $\iint_D y\, dA$, where D is the region in the first quadrant bounded by the parabolas $x = y^2$ and $x = 8 - y^2$

26. $\iint_D y\, dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$

27. $\iint_D (x^2 + y^2)^{3/2}\, dA$, where D is the region in the first quadrant bounded by the lines $y = 0$ and $y = \sqrt{3}\,x$ and the circle $x^2 + y^2 = 9$

28. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$

29. $\iiint_E xy\, dV$, where
$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant x,\ 0 \leqslant z \leqslant x + y\}$

30. $\iiint_T xy\, dV$, where T is the solid tetrahedron with vertices $(0, 0, 0)$, $(\frac{1}{3}, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

31. $\iiint_E y^2z^2\, dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$

32. $\iiint_E z\, dV$, where E is bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant

33. $\iiint_E yz\,dV$, where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$

34. $\iiint_H z^3\sqrt{x^2 + y^2 + z^2}\,dV$, where H is the solid hemisphere that lies above the xy-plane and has center the origin and radius 1

35–40 Find the volume of the given solid.

35. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$

36. Under the surface $z = x^2 y$ and above the triangle in the xy-plane with vertices $(1, 0)$, $(2, 1)$, and $(4, 0)$

37. The solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, and $(2, 2, 0)$

38. Bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$

39. One of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$

40. Above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

41. Consider a lamina that occupies the region D bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant with density function $\rho(x, y) = y$.
(a) Find the mass of the lamina.
(b) Find the center of mass.
(c) Find the moments of inertia and radii of gyration about the x- and y-axes.

42. A lamina occupies the part of the disk $x^2 + y^2 \leqslant a^2$ that lies in the first quadrant.
(a) Find the centroid of the lamina.
(b) Find the center of mass of the lamina if the density function is $\rho(x, y) = xy^2$.

43. (a) Find the centroid of a solid right circular cone with height h and base radius a. (Place the cone so that its base is in the xy-plane with center the origin and its axis along the positive z-axis.)
(b) If the cone has density function $\rho(x, y, z) = \sqrt{x^2 + y^2}$, find the moment of inertia of the cone about its axis (the z-axis).

44. Find the area of the part of the cone $z^2 = a^2(x^2 + y^2)$ between the planes $z = 1$ and $z = 2$.

45. Find the area of the part of the surface $z = x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$.

CAS **46.** Graph the surface $z = x \sin y$, $-3 \leqslant x \leqslant 3$, $-\pi \leqslant y \leqslant \pi$, and find its surface area correct to four decimal places.

47. Use polar coordinates to evaluate

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx$$

48. Use spherical coordinates to evaluate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2 + y^2 + z^2}\,dz\,dx\,dy$$

49. If D is the region bounded by the curves $y = 1 - x^2$ and $y = e^x$, find the approximate value of the integral $\iint_D y^2\,dA$. (Use a graphing device to estimate the points of intersection of the curves.)

CAS **50.** Find the center of mass of the solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ and density function $\rho(x, y, z) = x^2 + y^2 + z^2$.

51. The joint density function for random variables X and Y is

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant C.
(b) Find $P(X \leqslant 2, Y \geqslant 1)$.
(c) Find $P(X + Y \leqslant 1)$.

52. A lamp has three bulbs, each of a type with average lifetime 800 hours. If we model the probability of failure of a bulb by an exponential density function with mean 800, find the probability that all three bulbs fail within a total of 1000 hours.

53. Rewrite the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx$$

as an iterated integral in the order $dx\,dy\,dz$.

54. Give five other iterated integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\,dz\,dx\,dy$$

55. Use the transformation $u = x - y$, $v = x + y$ to evaluate

$$\iint_R \frac{x - y}{x + y}\,dA$$

where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.

56. Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

57. Use the change of variables formula and an appropriate transformation to evaluate $\iint_R xy\,dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.

58. The **Mean Value Theorem for double integrals** says that if f is a continuous function on a plane region D that is of type I or II, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x, y)\,dA = f(x_0, y_0)\,A(D)$$

Use the Extreme Value Theorem (14.7.8) and Property 15.2.11 of integrals to prove this theorem. (Use the proof of the single-variable version in Section 5.5 as a guide.)

59. Suppose that f is continuous on a disk that contains the point (a, b). Let D_r be the closed disk with center (a, b) and radius r. Use the Mean Value Theorem for double integrals (see Exercise 58) to show that

$$\lim_{r \to 0} \frac{1}{\pi r^2} \iint_{D_r} f(x, y)\, dA = f(a, b)$$

60. (a) Evaluate $\displaystyle\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\, dA$, where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0 < r < R$.

(b) For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

(c) Find $\displaystyle\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\, dV$, where E is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.

(d) For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

Problems Plus

1. If $[\![x]\!]$ denotes the greatest integer in x, evaluate the integral

$$\iint_R [\![x + y]\!] \, dA$$

where $R = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 2 \leqslant y \leqslant 5\}$.

2. Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}} \, dy \, dx$$

where $\max\{x^2, y^2\}$ means the larger of the numbers x^2 and y^2.

3. Find the average value of the function $f(x) = \int_x^1 \cos(t^2) \, dt$ on the interval $[0, 1]$.

4. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, $\mathbf{r}$ is the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and E is given by the inequalities $0 \leqslant \mathbf{a} \cdot \mathbf{r} \leqslant \alpha,\ 0 \leqslant \mathbf{b} \cdot \mathbf{r} \leqslant \beta,\ 0 \leqslant \mathbf{c} \cdot \mathbf{r} \leqslant \gamma$, show that

$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r}) \, dV = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}$$

5. The double integral $\displaystyle\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem we ask you to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u - v}{\sqrt{2}} \qquad y = \frac{u + v}{\sqrt{2}}$$

This gives a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the uv-plane.

[*Hint:* If, in evaluating the integral, you encounter either of the expressions $(1 - \sin\theta)/\cos\theta$ or $(\cos\theta)/(1 + \sin\theta)$, you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.]

7. (a) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz} \, dx \, dy \, dz = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

(Nobody has ever been able to find the exact value of the sum of this series.)

(b) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + xyz} \, dx \, dy \, dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

Use this equation to evaluate the triple integral correct to two decimal places.

8. Show that

$$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\,dx = \frac{\pi}{2}\ln \pi$$

by first expressing the integral as an iterated integral.

9. (a) Show that when Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is written in cylindrical coordinates, it becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(b) Show that when Laplace's equation is written in spherical coordinates, it becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot \phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0$$

10. (a) A lamina has constant density ρ and takes the shape of a disk with center the origin and radius R. Use Newton's Law of Gravitation (see Section 13.4) to show that the magnitude of the force of attraction that the lamina exerts on a body with mass m located at the point $(0, 0, d)$ on the positive z-axis is

$$F = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}}\right)$$

[*Hint:* Divide the disk as in Figure 15.3.4 and first compute the vertical component of the force exerted by the polar subrectangle R_{ij}.]

(b) Show that the magnitude of the force of attraction of a lamina with density ρ that occupies an entire plane on an object with mass m located at a distance d from the plane is

$$F = 2\pi Gm\rho$$

Notice that this expression does not depend on d.

11. If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \tfrac{1}{2}\int_0^x (x - t)^2 f(t)\,dt$$

12. Evaluate $\displaystyle\lim_{n\to\infty} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2 + ni + j}}$.

13. The plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad a > 0, \quad b > 0, \quad c > 0$$

cuts the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$$

into two pieces. Find the volume of the smaller piece.

16 Vector Calculus

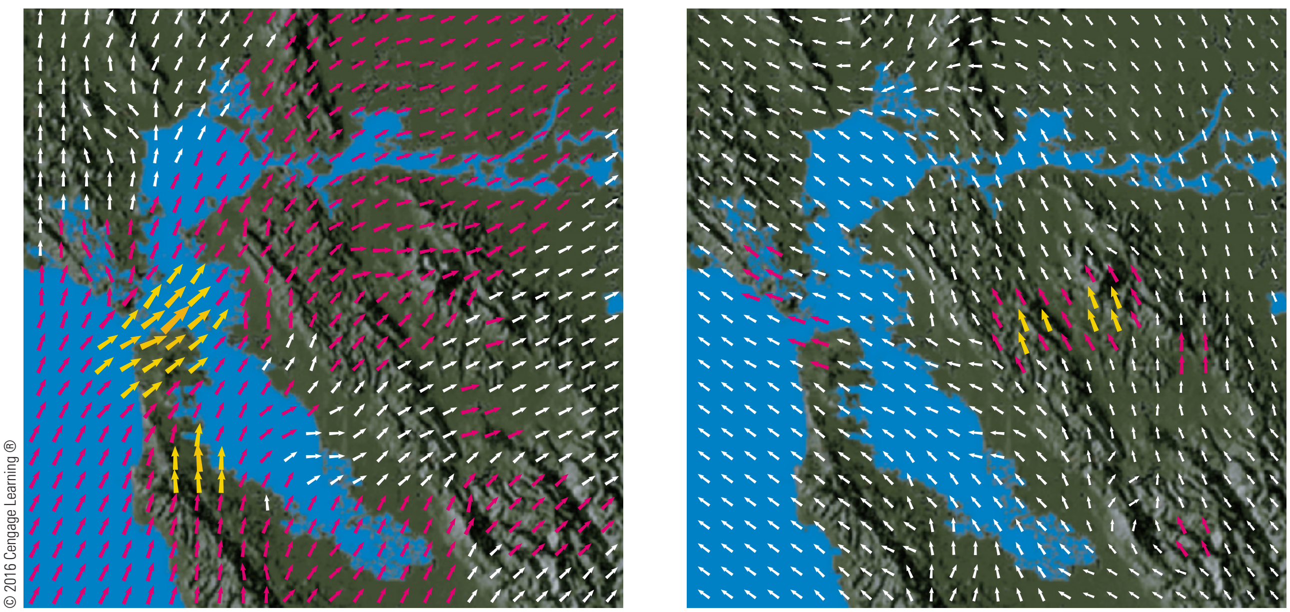

Everett Collection / Glow Images

Parametric surfaces, which are studied in Section 16.6, are frequently used by programmers in creating the sophisticated software used in the development of computer animated films like the *Shrek* series. The software employs parametric and other types of surfaces to create 3D models of the characters and objects in a scene. Color, texture, and lighting is then rendered to bring the scene to life.

IN THIS CHAPTER WE STUDY the calculus of vector fields. (These are functions that assign vectors to points in space.) In particular we define line integrals (which can be used to find the work done by a force field in moving an object along a curve). Then we define surface integrals (which can be used to find the rate of fluid flow across a surface). The connections between these new types of integrals and the single, double, and triple integrals that we have already met are given by the higher-dimensional versions of the Fundamental Theorem of Calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

16.1 Vector Fields

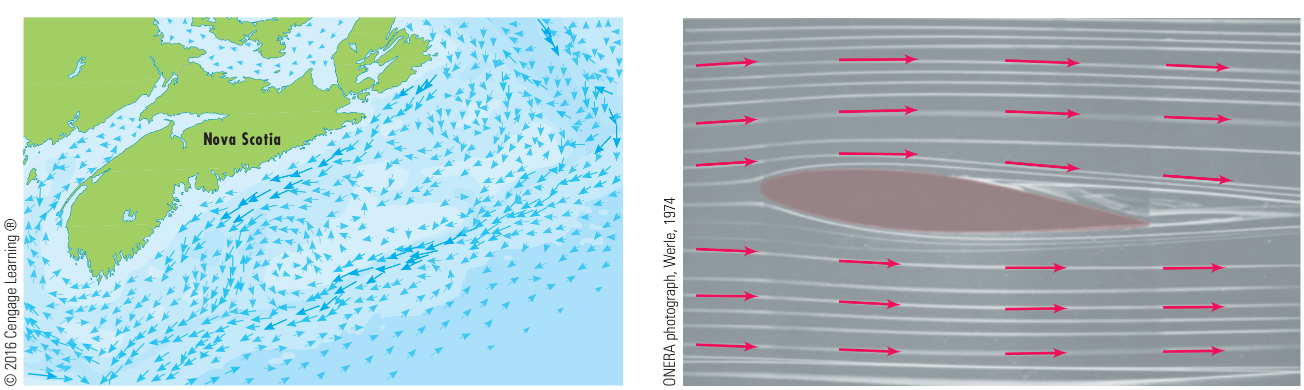
The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area. We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier. Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field*.



(a) 6:00 PM, March 1, 2010 (b) 6:00 AM, March 1, 2010

FIGURE 1 Velocity vector fields showing San Francisco Bay wind patterns

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia (b) Airflow past an inclined airfoil

FIGURE 2
Velocity vector fields

Another type of vector field, called a *force field,* associates a force vector with each point in a region. An example is the gravitational force field that we will look at in Example 4.

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

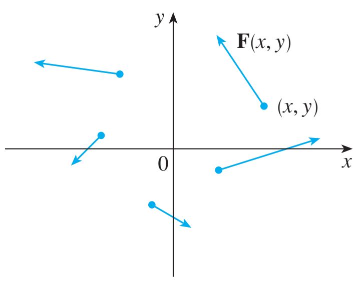


FIGURE 3
Vector field on $\mathbb{R}^2$

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course, it's impossible to do this for all points (x, y), but we can gain a reasonable impression of $\mathbf{F}$ by doing it for a few representative points in D as in Figure 3. Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short, $$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

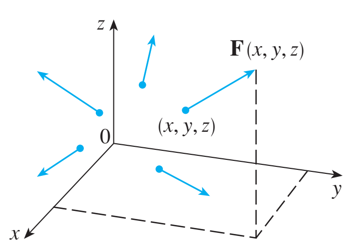


FIGURE 4
Vector field on $\mathbb{R}^3$

A vector field $\mathbf{F}$ on $\mathbb{R}^3$ is pictured in Figure 4. We can express it in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

As with the vector functions in Section 13.1, we can define continuity of vector fields and show that $\mathbf{F}$ is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z\rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$. Then $\mathbf{F}$ becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

SOLUTION Since $\mathbf{F}(1, 0) = \mathbf{j}$, we draw the vector $\mathbf{j} = \langle 0, 1\rangle$ starting at the point $(1, 0)$ in Figure 5. Since $\mathbf{F}(0, 1) = -\mathbf{i}$, we draw the vector $\langle -1, 0\rangle$ with starting point $(0, 1)$. Continuing in this way, we calculate several other representative values of $\mathbf{F}(x, y)$ in the table and draw the corresponding vectors to represent the vector field in Figure 5.

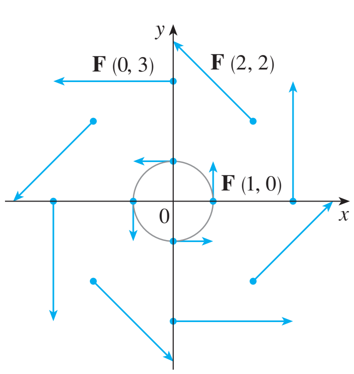


FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, 1\rangle$	$(-1, 0)$	$\langle 0, -1\rangle$
$(2, 2)$	$\langle -2, 2\rangle$	$(-2, -2)$	$\langle 2, -2\rangle$
$(3, 0)$	$\langle 0, 3\rangle$	$(-3, 0)$	$\langle 0, -3\rangle$
$(0, 1)$	$\langle -1, 0\rangle$	$(0, -1)$	$\langle 1, 0\rangle$
$(-2, 2)$	$\langle -2, -2\rangle$	$(2, -2)$	$\langle 2, 2\rangle$
$(0, 3)$	$\langle -3, 0\rangle$	$(0, -3)$	$\langle 3, 0\rangle$

It appears from Figure 5 that each arrow is tangent to a circle with center the origin. To confirm this, we take the dot product of the position vector $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + yx = 0$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$. Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle. ■

Some computer algebra systems are capable of plotting vector fields in two or three dimensions. They give a better impression of the vector field than is possible by hand because the computer can plot a large number of representative vectors. Figure 6 shows a computer plot of the vector field in Example 1; Figures 7 and 8 show two other vector fields. Notice that the computer scales the lengths of the vectors so they are not too long and yet are proportional to their true lengths.

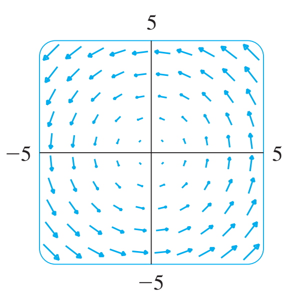


FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

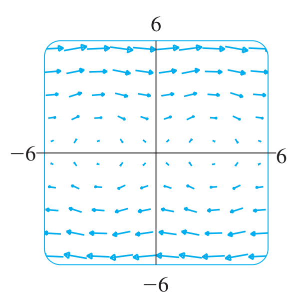


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

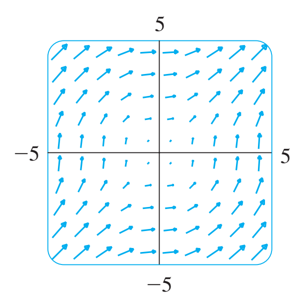


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

EXAMPLE 2 Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

SOLUTION The sketch is shown in Figure 9. Notice that all vectors are vertical and point upward above the xy-plane or downward below it. The magnitude increases with the distance from the xy-plane.

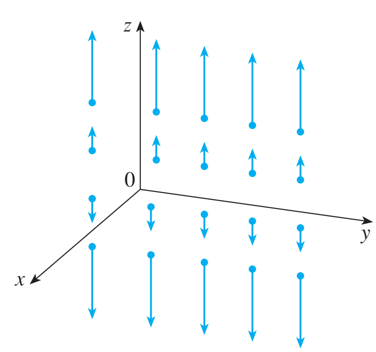


FIGURE 9
$\mathbf{F}(x, y, z) = z\,\mathbf{k}$

■

We were able to draw the vector field in Example 2 by hand because of its particularly simple formula. Most three-dimensional vector fields, however, are virtually impossible

to sketch by hand and so we need to resort to computer software. Examples are shown in Figures 10, 11, and 12. Notice that the vector fields in Figures 10 and 11 have similar formulas, but all the vectors in Figure 11 point in the general direction of the negative y-axis because their y-components are all -2. If the vector field in Figure 12 represents a velocity field, then a particle would be swept upward and would spiral around the z-axis in the clockwise direction as viewed from above.

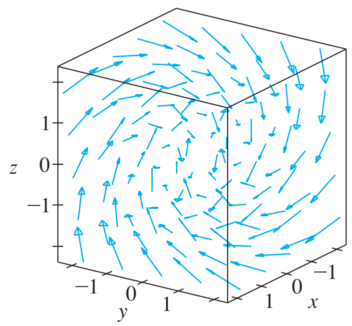


FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

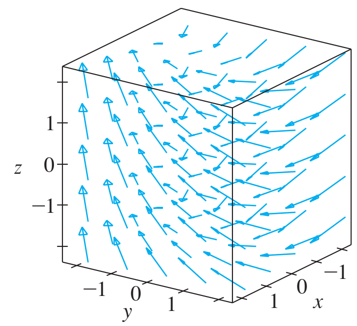


FIGURE 11
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

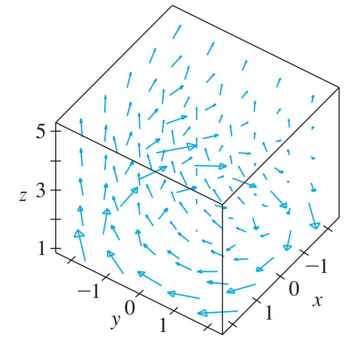


FIGURE 12
$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$

TEC In Visual 16.1 you can rotate the vector fields in Figures 10–12 as well as additional fields.

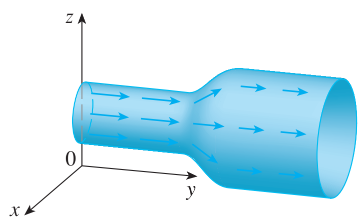


FIGURE 13
Velocity field in fluid flow

EXAMPLE 3 Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z). Then $\mathbf{V}$ assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a **velocity field**. A possible velocity field is illustrated in Figure 13. The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics. For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel. We have seen other examples of velocity fields in Figures 1 and 2. ■

EXAMPLE 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. (This is an example of an inverse square law.) Let's assume that the object with mass M is located at the origin in $\mathbb{R}^3$. (For instance, M could be the mass of the earth and the origin would be at its center.) Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$. The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

[Physicists often use the notation $\mathbf{r}$ instead of $\mathbf{x}$ for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.] The function given by Equation 3 is

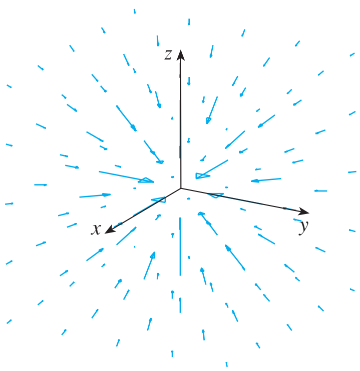


FIGURE 14
Gravitational force field

an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point $\mathbf{x}$ in space.

Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

The gravitational field $\mathbf{F}$ is pictured in Figure 14. ■

EXAMPLE 5 Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z\rangle$ is

4 $$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

where ε is a constant (that depends on the units used). For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive. Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\,\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

Then $\mathbf{E}$ is a vector field on $\mathbb{R}^3$ called the **electric field** of Q. ■

Gradient Fields

If f is a scalar function of two variables, recall from Section 14.6 that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

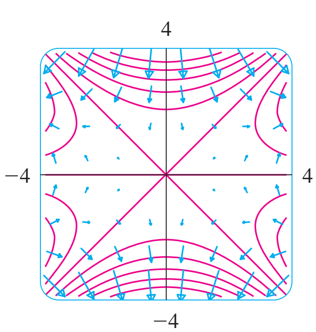


FIGURE 15

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from

Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

16.1 EXERCISES

1–10 Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = 0.3\,\mathbf{i} - 0.4\,\mathbf{j}$

2. $\mathbf{F}(x, y) = \frac{1}{2}x\,\mathbf{i} + y\,\mathbf{j}$

3. $\mathbf{F}(x, y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

5. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

6. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

7. $\mathbf{F}(x, y, z) = \mathbf{i}$

8. $\mathbf{F}(x, y, z) = z\,\mathbf{i}$

9. $\mathbf{F}(x, y, z) = -y\,\mathbf{i}$

10. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

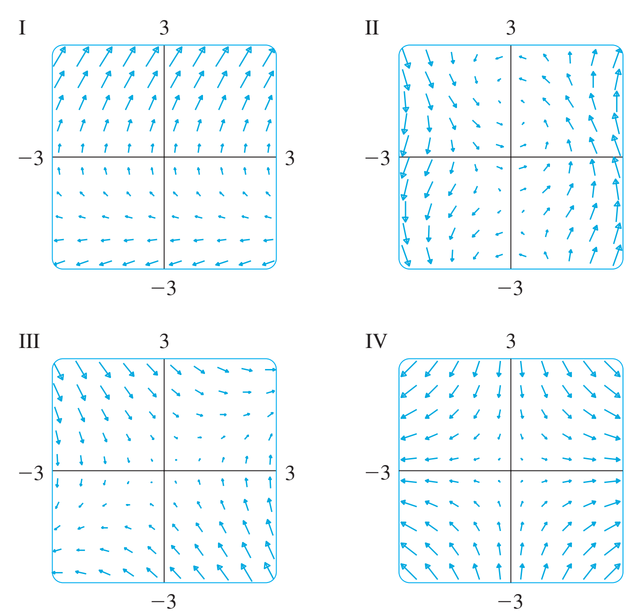
11–14 Match the vector fields $\mathbf{F}$ with the plots labeled I–IV. Give reasons for your choices.

11. $\mathbf{F}(x, y) = \langle x, -y \rangle$

12. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

13. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

14. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$



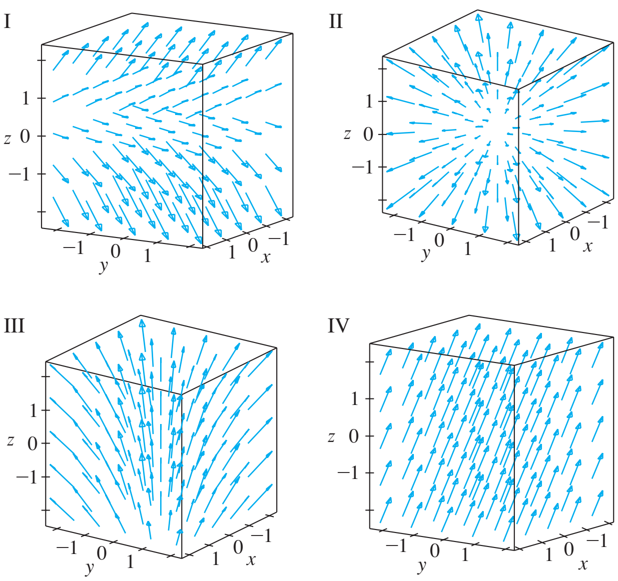
15–18 Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

16. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

18. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$



19. If you have a CAS that plots vector fields (the command is `fieldplot` in Maple and `PlotVectorField` or `VectorPlot` in Mathematica), use it to plot

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

20. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use a CAS to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

21–24 Find the gradient vector field of f.

21. $f(x, y) = y \sin(xy)$

22. $f(s, t) = \sqrt{2s + 3t}$

23. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

24. $f(x, y, z) = x^2 y e^{y/z}$

25–26 Find the gradient vector field ∇f of f and sketch it.

25. $f(x, y) = \frac{1}{2}(x - y)^2$

26. $f(x, y) = \frac{1}{2}(x^2 - y^2)$

27–28 Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

27. $f(x, y) = \ln(1 + x^2 + 2y^2)$

28. $f(x, y) = \cos x - 2 \sin y$

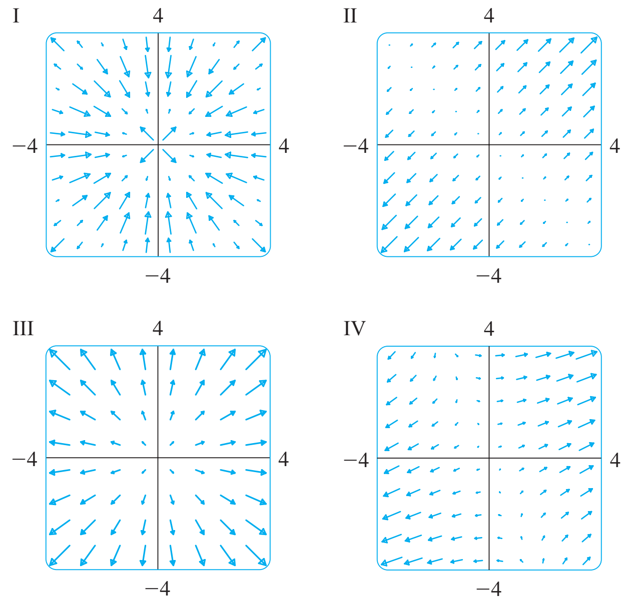
29–32 Match the functions f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

29. $f(x, y) = x^2 + y^2$

30. $f(x, y) = x(x + y)$

31. $f(x, y) = (x + y)^2$

32. $f(x, y) = \sin\sqrt{x^2 + y^2}$



33. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

34. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

35. The **flow lines** (or **streamlines**) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

36. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

We start with a plane curve C given by the parametric equations

1 $$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 13.3.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

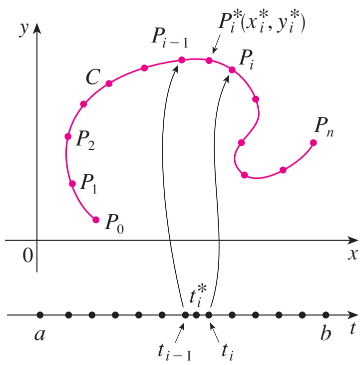


FIGURE 1

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

> **2 Definition** If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is
>
> $$\int_C f(x, y)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$
>
> if this limit exists.

In Section 10.2 we found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

3 $$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

The arc length function s is discussed in Section 13.3.

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t: Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leqslant x \leqslant b$. Formula 3 then becomes

$$\int_C f(x, y)\,ds = \int_a^b f(x, 0)\,dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x, y) \geqslant 0$, $\int_C f(x, y)\,ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

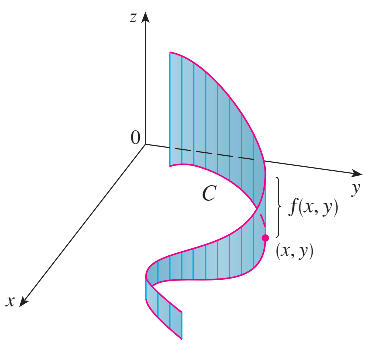


FIGURE 2

EXAMPLE 1 Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

SOLUTION In order to use Formula 3, we first need parametric equations to represent C. Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leqslant t \leqslant \pi$. (See Figure 3.) Therefore Formula 3 gives

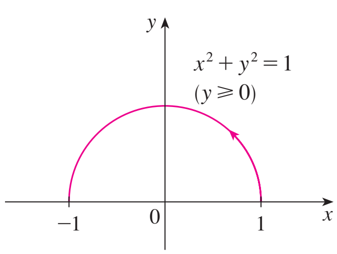


FIGURE 3

$$\int_C (2 + x^2y)\,ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\,dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi$$

$$= 2\pi + \tfrac{2}{3}$$ ■

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

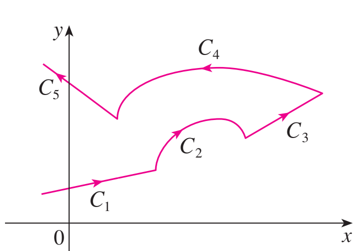


FIGURE 4
A piecewise-smooth curve

$$\int_C f(x, y)\,ds = \int_{C_1} f(x, y)\,ds + \int_{C_2} f(x, y)\,ds + \cdots + \int_{C_n} f(x, y)\,ds$$

EXAMPLE 2 Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

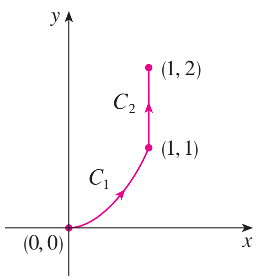


FIGURE 5
$C = C_1 \cup C_2$

SOLUTION The curve C is shown in Figure 5. C_1 is the graph of a function of x, so we can choose x as the parameter and the equations for C_1 become

$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

Therefore

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1 + 4x^2}\,dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

On C_2 we choose y as the parameter, so the equations of C_2 are

$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

and
$$\int_{C_2} 2x\,ds = \int_1^2 2(1)\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = 2$$

Thus
$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5} - 1}{6} + 2$$ ■

Any physical interpretation of a line integral $\int_C f(x, y)\,ds$ depends on the physical interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\,\Delta s_i$ and so the total mass of the wire is approximately $\Sigma\, \rho(x_i^*, y_i^*)\,\Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\,\Delta s_i = \int_C \rho(x, y)\,ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

4
$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x, y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\,\rho(x, y)\,ds$$

Other physical interpretations of line integrals will be discussed later in this chapter.

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant \pi$, and find that $ds = dt$. The linear density is

$$\rho(x, y) = k(1 - y)$$

where k is a constant, and so the mass of the wire is

$$m = \int_C k(1 - y)\, ds = \int_0^\pi k(1 - \sin t)\, dt = k\Big[t + \cos t\Big]_0^\pi = k(\pi - 2)$$

From Equations 4 we have

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x, y)\, ds = \frac{1}{k(\pi - 2)}\int_C y\,k(1 - y)\, ds$$

$$= \frac{1}{\pi - 2}\int_0^\pi (\sin t - \sin^2 t)\, dt = \frac{1}{\pi - 2}\Big[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\Big]_0^\pi$$

$$= \frac{4 - \pi}{2(\pi - 2)}$$

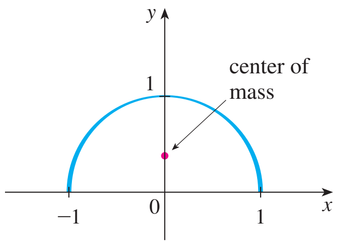


FIGURE 6

By symmetry we see that $\bar{x} = 0$, so the center of mass is

$$\left(0, \frac{4 - \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$$

See Figure 6. ■

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y**:

5
$$\int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

6
$$\int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

7
$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parametrize a line segment, so it's useful to remember that a

vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

8
$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

(See Equation 12.5.4.)

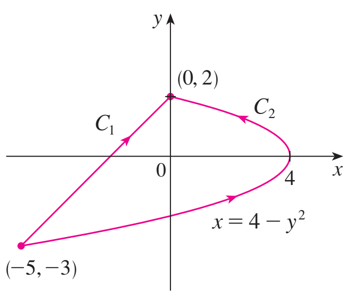


FIGURE 7

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

SOLUTION
(a) A parametric representation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \leqslant t \leqslant 1$$

(Use Equation 8 with $\mathbf{r}_0 = \langle -5, -3 \rangle$ and $\mathbf{r}_1 = \langle 0, 2 \rangle$.) Then $dx = 5\,dt$, $dy = 5\,dt$, and Formulas 7 give

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t - 3)^2(5\,dt) + (5t - 5)(5\,dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\,dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \leqslant y \leqslant 2$$

Then $dx = -2y\,dy$ and by Formulas 7 we have

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^2 y^2(-2y)\,dy + (4 - y^2)\,dy$$

$$= \int_{-3}^2 (-2y^3 - y^2 + 4)\,dy$$

$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^2 = 40\tfrac{5}{6}$$ ■

Notice that we got different answers in parts (a) and (b) of Example 4 even though the two curves had the same endpoints. Thus, in general, the value of a line integral depends not just on the endpoints of the curve but also on the path. (But see Section 16.3 for conditions under which the integral is independent of the path.)

Notice also that the answers in Example 4 depend on the direction, or orientation, of the curve. If $-C_1$ denotes the line segment from $(0, 2)$ to $(-5, -3)$, you can verify, using the parametrization

$$x = -5t \qquad y = 2 - 5t \qquad 0 \leqslant t \leqslant 1$$

that
$$\int_{-C_1} y^2\,dx + x\,dy = \tfrac{5}{6}$$

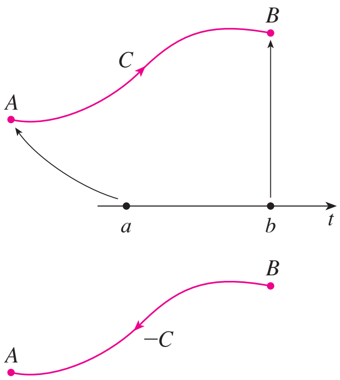


FIGURE 8

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leqslant t \leqslant b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \qquad \int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\,|\,\mathbf{r}'(t)\,|\,dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\,\mathbf{r}'(t)\,|\,dt = L$$

where L is the length of the curve C (see Formula 13.3.3).

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\textbf{10} \qquad \int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

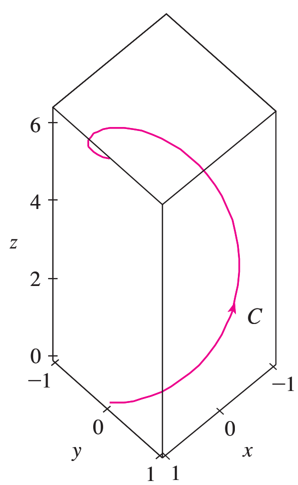


FIGURE 9

EXAMPLE 5 Evaluate $\int_C y \sin z\,ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\int_C y \sin z\,ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{2}\left[t - \tfrac{1}{2}\sin 2t\right]_0^{2\pi} = \sqrt{2}\,\pi$$

■

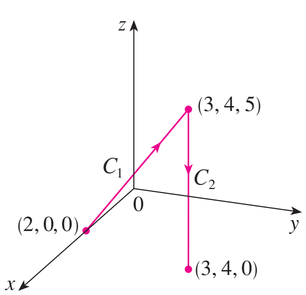


FIGURE 10

EXAMPLE 6 Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2, 0, 0) to (3, 4, 5), followed by the vertical line segment C_2 from (3, 4, 5) to (3, 4, 0).

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2 + t, 4t, 5t\rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \leqslant t \leqslant 1$$

Thus

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)4\,dt + (2 + t)5\,dt$$

$$= \int_0^1 (10 + 29t)\,dt = 10t + 29\frac{t^2}{2}\Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5 - 5t\rangle$$

or $\qquad x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leqslant t \leqslant 1$

Then $dx = 0 = dy$, so

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$

Line Integrals of Vector Fields

Recall from Section 5.4 that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then in Section 12.3 we found that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational field of Example 16.1.4 or the electric force field of Example 16.1.5. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.) We wish to compute the work done by this force in moving a particle along a smooth curve C.

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.) Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*. Thus the work done by the force $\mathbf{F}$ in moving the particle from P_{i-1} to P_i is approximately

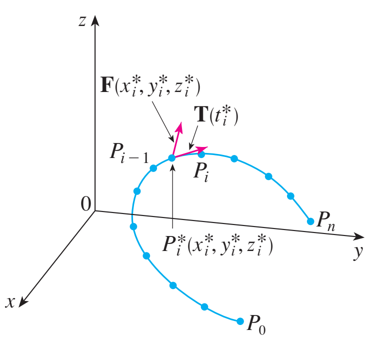


FIGURE 11

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i\, \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)]\,\Delta s_i$$

and the total work done in moving the particle along C is approximately

11 $$\sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)]\,\Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C. Intuitively, we see that these approximations ought to become better as n becomes larger. Therefore we define the **work** W done by the force field $\mathbf{F}$ as the limit of the Riemann sums in (11), namely,

12 $$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,ds = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

> **13 Definition** Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, bear in mind that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for the vector field $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$. Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Figure 12 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.

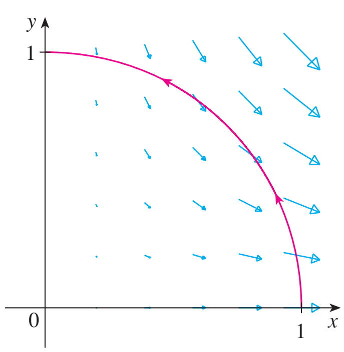


FIGURE 12

EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t\,\sin t\,\mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{\pi/2} (-2\cos^2 t\,\sin t)\, dt$$

$$= 2\left.\frac{\cos^3 t}{3}\right]_0^{\pi/2} = -\frac{2}{3}$$ ■

NOTE Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

Figure 13 shows the twisted cubic C in Example 8 and some typical vectors acting at three points on C.

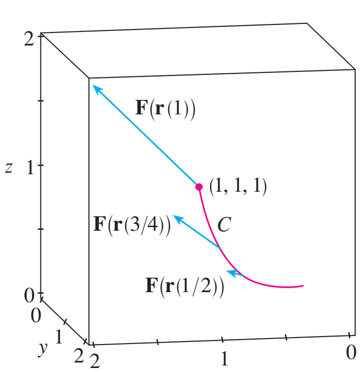


FIGURE 13

EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \leqslant t \leqslant 1$$

SOLUTION We have

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^5\,\mathbf{j} + t^4\,\mathbf{k}$$

Thus
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$
$$= \int_0^1 (t^3 + 5t^6)\, dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big]_0^1 = \frac{27}{28}$$ ■

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$
$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k})\, dt$$
$$= \int_a^b \left[P(x(t), y(t), z(t))\, x'(t) + Q(x(t), y(t), z(t))\, y'(t) + R(x(t), y(t), z(t))\, z'(t)\right] dt$$

But this last integral is precisely the line integral in (10). Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

For example, the integral $\int_C y\, dx + z\, dy + x\, dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

16.2 EXERCISES

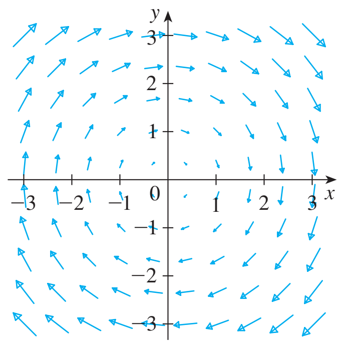
1–16 Evaluate the line integral, where C is the given curve.

1. $\int_C y\, ds$, C: $x = t^2$, $y = 2t$, $0 \leqslant t \leqslant 3$
2. $\int_C (x/y)\, ds$, C: $x = t^3$, $y = t^4$, $1 \leqslant t \leqslant 2$
3. $\int_C xy^4\, ds$, C is the right half of the circle $x^2 + y^2 = 16$
4. $\int_C xe^y\, ds$, C is the line segment from (2, 0) to (5, 4)
5. $\int_C (x^2y + \sin x)\, dy$, C is the arc of the parabola $y = x^2$ from (0, 0) to (π, π^2)
6. $\int_C e^x\, dx$, C is the arc of the curve $x = y^3$ from $(-1, -1)$ to (1, 1)
7. $\int_C (x + 2y)\, dx + x^2\, dy$, C consists of line segments from (0, 0) to (2, 1) and from (2, 1) to (3, 0)
8. $\int_C x^2\, dx + y^2\, dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from (2, 0) to (0, 2) followed by the line segment from (0, 2) to (4, 3)
9. $\int_C x^2y\, ds$, C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant \pi/2$
10. $\int_C y^2z\, ds$, C is the line segment from (3, 1, 2) to (1, 2, 5)
11. $\int_C xe^{yz}\, ds$, C is the line segment from (0, 0, 0) to (1, 2, 3)
12. $\int_C (x^2 + y^2 + z^2)\, ds$, C: $x = t$, $y = \cos 2t$, $z = \sin 2t$, $0 \leqslant t \leqslant 2\pi$
13. $\int_C xye^{yz}\, dy$, C: $x = t$, $y = t^2$, $z = t^3$, $0 \leqslant t \leqslant 1$
14. $\int_C y\, dx + z\, dy + x\, dz$, C: $x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \leqslant t \leqslant 4$
15. $\int_C z^2\, dx + x^2\, dy + y^2\, dz$, C is the line segment from (1, 0, 0) to (4, 1, 2)
16. $\int_C (y + z)\, dx + (x + z)\, dy + (x + y)\, dz$, C consists of line segments from (0, 0, 0) to (1, 0, 1) and from (1, 0, 1) to (0, 1, 2)

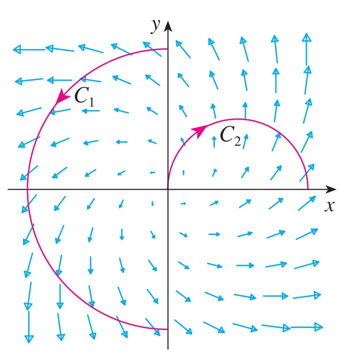
17. Let **F** be the vector field shown in the figure.

(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



18. The figure shows a vector field **F** and two curves C_1 and C_2. Are the line integrals of **F** over C_1 and C_2 positive, negative, or zero? Explain.



19–22 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

19. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} - x^2\,\mathbf{j}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

20. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + xz\,\mathbf{j} + (y + z)\,\mathbf{k}$,
$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 2$

21. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

22. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xy\,\mathbf{k}$,
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi$

23–26 Use a calculator to evaluate the line integral correct to four decimal places.

23. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \sqrt{x + y}\,\mathbf{i} + (y/x)\,\mathbf{j}$ and $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \sin t \cos t\,\mathbf{j}, \ \pi/6 \leqslant t \leqslant \pi/3$

24. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = yze^x\,\mathbf{i} + zxe^y\,\mathbf{j} + xye^z\,\mathbf{k}$ and $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}, \ 0 \leqslant t \leqslant \pi/4$

25. $\int_C xy \arctan z\,ds$, where C has parametric equations $x = t^2,\ y = t^3,\ z = \sqrt{t},\ 1 \leqslant t \leqslant 2$

26. $\int_C z \ln(x + y)\,ds$, where C has parametric equations $x = 1 + 3t,\ y = 2 + t^2,\ z = t^4,\ -1 \leqslant t \leqslant 1$

CAS **27–28** Use a graph of the vector field **F** and the curve C to guess whether the line integral of **F** over C is positive, negative, or zero. Then evaluate the line integral.

27. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$,
C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

28. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$,
C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

29. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j},\ 0 \leqslant t \leqslant 1$.

(b) Illustrate part (a) by using a graphing calculator or computer to graph C and the vectors from the vector field corresponding to $t = 0,\ 1/\sqrt{2}$, and 1 (as in Figure 13).

30. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k},\ -1 \leqslant t \leqslant 1$.

(b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm\frac{1}{2}$ (as in Figure 13).

CAS **31.** Find the exact value of $\int_C x^3y^2z\,ds$, where C is the curve with parametric equations $x = e^{-t}\cos 4t,\ y = e^{-t}\sin 4t,\ z = e^{-t},\ 0 \leqslant t \leqslant 2\pi$.

32. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.

CAS (b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

33. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4,\ x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

34. A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

35. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

(b) Find the center of mass of a wire in the shape of the helix $x = 2 \sin t$, $y = 2 \cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

36. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$, if the density at any point is equal to the square of the distance from the origin.

37. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2 \rho(x, y)\, ds \qquad I_y = \int_C x^2 \rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

38. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 35.

39. Find the work done by the force field

$$\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} \qquad 0 \leqslant t \leqslant 2\pi$$

40. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

41. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

42. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

43. The position of an object with mass m at time t is $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j}$, $0 \leqslant t \leqslant 1$.
(a) What is the force acting on the object at time t?
(b) What is the work done by the force during the time interval $0 \leqslant t \leqslant 1$?

44. An object with mass m moves with position function $\mathbf{r}(t) = a \sin t\,\mathbf{i} + b \cos t\,\mathbf{j} + ct\,\mathbf{k}$, $0 \leqslant t \leqslant \pi/2$. Find the work done on the object during this time period.

45. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

46. Suppose there is a hole in the can of paint in Exercise 45 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?

47. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
(b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?

48. The base of a circular fence with radius 10 m is given by $x = 10 \cos t$, $y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

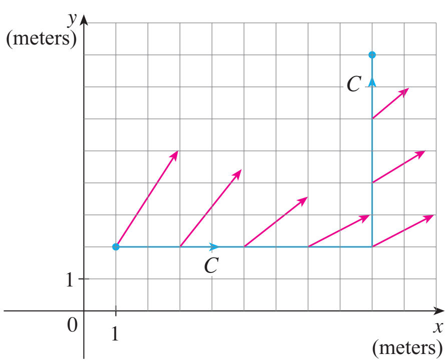
49. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

50. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]$$

51. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.



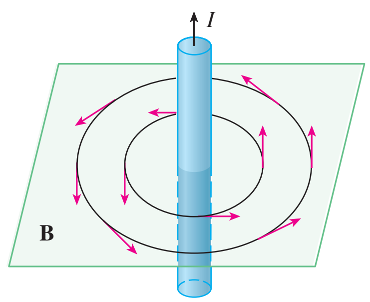
52. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric

current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C, and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



16.3 The Fundamental Theorem for Line Integrals

Recall from Section 4.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

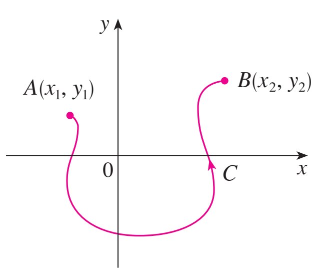
1 $$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$. We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

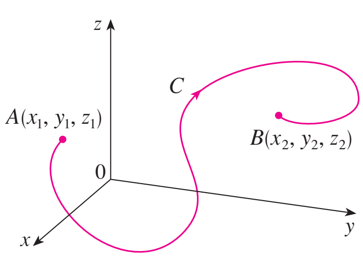
If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



(a)



(b)

FIGURE 1

NOTE Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 2 says that the line integral of ∇f is the net change in f. If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 1(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in Figure 1(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Let's prove Theorem 2 for this case.

PROOF OF THEOREM 2 Using Definition 16.2.13, we have

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt \qquad \text{(by the Chain Rule)}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Although we have proved Theorem 2 for smooth curves, it is also true for piecewise-smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve C. (See Example 16.1.4.)

SOLUTION From Section 16.1 we know that $\mathbf{F}$ is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(2, 2, 0) - f(3, 4, 12)$$

$$= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$ ■

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B. We know from Example 16.2.4 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path.*

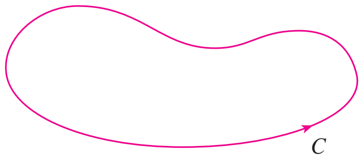


FIGURE 2
A closed curve

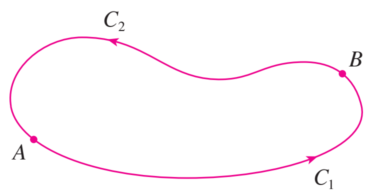


FIGURE 3

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 2.) If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See Figure 3.) Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Thus we have proved the following theorem.

> **3 Theorem** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field $\mathbf{F}$ is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field in Section 16.1) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D.

> **4 Theorem** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

PROOF Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

$$f(x, y) = \int_{(a, b)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$$

for any point (x, y) in D. Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with center (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y). (See Figure 4.) Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a, b)}^{(x_1, y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

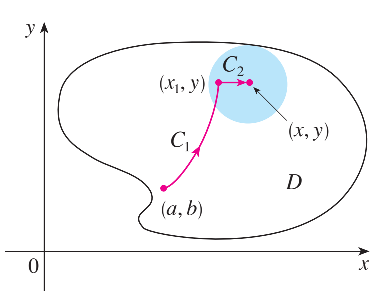


FIGURE 4

If we write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} P\,dx + Q\,dy$$

On C_2, y is constant, so $dy = 0$. Using t as the parameter, where $x_1 \leqslant t \leqslant x$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

by Part 1 of the Fundamental Theorem of Calculus (see Section 4.3). A similar argument, using a vertical line segment (see Figure 5), shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y)$$

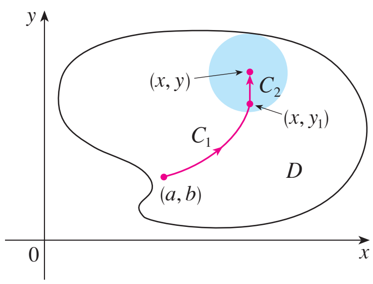


FIGURE 5

Thus
$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f$$

which says that $\mathbf{F}$ is conservative. ■

The question remains: how is it possible to determine whether or not a vector field $\mathbf{F}$ is conservative? Suppose it is known that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is,

$$P = \frac{\partial f}{\partial x} \qquad \text{and} \qquad Q = \frac{\partial f}{\partial y}$$

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

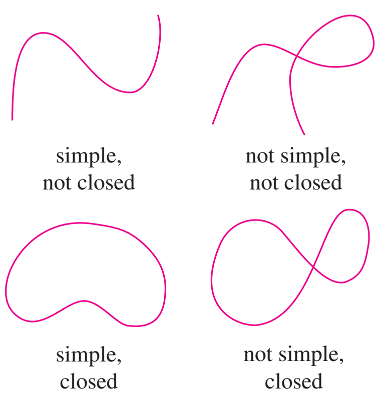


FIGURE 6
Types of curves

5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

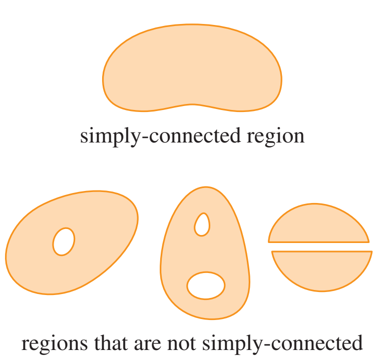


FIGURE 7

The converse of Theorem 5 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative. The proof will be sketched in the next section as a consequence of Green's Theorem.

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

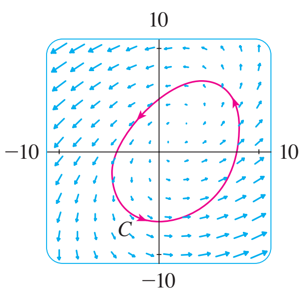


FIGURE 8

Figures 8 and 9 show the vector fields in Examples 2 and 3, respectively. The vectors in Figure 8 that start on the closed curve C all appear to point in roughly the same direction as C. So it looks as if $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$ and therefore $\mathbf{F}$ is not conservative. The calculation in Example 2 confirms this impression. Some of the vectors near the curves C_1 and C_2 in Figure 9 point in approximately the same direction as the curves, whereas others point in the opposite direction. So it appears plausible that line integrals around all closed paths are 0. Example 3 shows that $\mathbf{F}$ is indeed conservative.

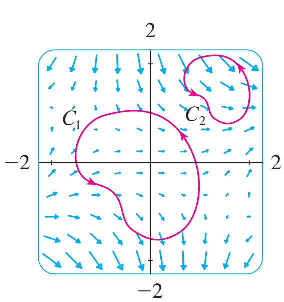


FIGURE 9

EXAMPLE 2 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, $\mathbf{F}$ is not conservative by Theorem 5. ■

EXAMPLE 3 Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.

SOLUTION Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

Also, the domain of $\mathbf{F}$ is the entire plane ($D = \mathbb{R}^2$), which is open and simply-connected. Therefore we can apply Theorem 6 and conclude that $\mathbf{F}$ is conservative. ■

In Example 3, Theorem 6 told us that $\mathbf{F}$ is conservative, but it did not tell us how to find the (potential) function f such that $\mathbf{F} = \nabla f$. The proof of Theorem 4 gives us a clue as to how to find f. We use "partial integration" as in the following example.

EXAMPLE 4
(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.
(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \qquad 0 \leqslant t \leqslant \pi$$

SOLUTION
(a) From Example 3 we know that $\mathbf{F}$ is conservative and so there exists a function f with $\nabla f = \mathbf{F}$, that is,

7 $$f_x(x, y) = 3 + 2xy$$

8 $$f_y(x, y) = x^2 - 3y^2$$

Integrating (7) with respect to x, we obtain

9 $$f(x, y) = 3x + x^2y + g(y)$$

Notice that the constant of integration is a constant with respect to x, that is, a function of y, which we have called $g(y)$. Next we differentiate both sides of (9) with respect to y:

10 $$f_y(x, y) = x^2 + g'(y)$$

Comparing (8) and (10), we see that

$$g'(y) = -3y^2$$

Integrating with respect to y, we have

$$g(y) = -y^3 + K$$

where K is a constant. Putting this in (9), we have

$$f(x, y) = 3x + x^2y - y^3 + K$$

as the desired potential function.

(b) To use Theorem 2 all we have to know are the initial and terminal points of C, namely, $\mathbf{r}(0) = (0, 1)$ and $\mathbf{r}(\pi) = (0, -e^\pi)$. In the expression for $f(x, y)$ in part (a), any value of the constant K will do, so let's choose $K = 0$. Then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

This method is much shorter than the straightforward method for evaluating line integrals that we learned in Section 16.2. ■

A criterion for determining whether or not a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is conservative is given in Section 16.5. Meanwhile, the next example shows that the technique for finding the potential function is much the same as for vector fields on $\mathbb{R}^2$.

EXAMPLE 5 If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

11 $$f_x(x, y, z) = y^2$$

12 $$f_y(x, y, z) = 2xy + e^{3z}$$

13 $$f_z(x, y, z) = 3ye^{3z}$$

Integrating (11) with respect to x, we get

14 $$f(x, y, z) = xy^2 + g(y, z)$$

where $g(y, z)$ is a constant with respect to x. Then differentiating (14) with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with (12) gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite (14) as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with (13), we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$. ■

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion (see Section 13.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt && \text{(Theorem 13.2.3, Formula 4)} \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2\, dt = \frac{m}{2} \Big[|\mathbf{r}'(t)|^2 \Big]_a^b && \text{(Fundamental Theorem of Calculus)} \\ &= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right) \end{aligned}$$

Therefore

15 $$W = \tfrac{1}{2}m|\mathbf{v}(b)|^2 - \tfrac{1}{2}m|\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2}m|\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

16 $$W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

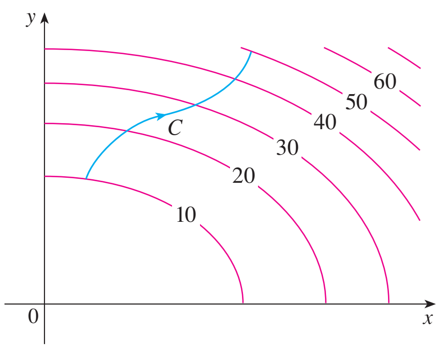
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 EXERCISES

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \qquad y = t^3 + t \qquad 0 \leqslant t \leqslant 1$$

x \ y	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (xy + y^2)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (y^2 - 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$

5. $\mathbf{F}(x, y) = y^2e^{xy}\,\mathbf{i} + (1 + xy)e^{xy}\,\mathbf{j}$

6. $\mathbf{F}(x, y) = ye^x\,\mathbf{i} + (e^x + e^y)\,\mathbf{j}$

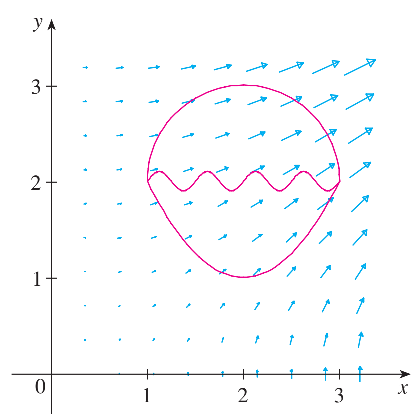
7. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^2 - 2xy^{-3})\,\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (y^2\cos x + \cos y)\,\mathbf{i} + (2y\sin x - x\sin y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (\ln y + y/x)\,\mathbf{i} + (\ln x + x/y)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?



12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

12. $\mathbf{F}(x, y) = (3 + 2xy^2)\,\mathbf{i} + 2x^2y\,\mathbf{j}$,
C is the arc of the hyperbola $y = 1/x$ from (1, 1) to $\left(4, \frac{1}{4}\right)$

13. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, \quad 0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$,
C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \leqslant t \leqslant \pi/2$

15. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from (1, 0, −2) to (4, 6, 3)

16. $\mathbf{F}(x, y, z) = (y^2 z + 2xz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (xy^2 + 2x^2 z)\,\mathbf{k}$,
C: $x = \sqrt{t}$, $y = t + 1$, $z = t^2$, $0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$,
C: $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + (t^2 - 2t)\,\mathbf{k}$,
$0 \leqslant t \leqslant 2$

18. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x \cos y + \cos z)\,\mathbf{j} - y \sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \leqslant t \leqslant \pi/2$

19–20 Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2xe^{-y}\,dx + (2y - x^2 e^{-y})\,dy$,
C is any path from $(1, 0)$ to $(2, 1)$

20. $\int_C \sin y\,dx + (x \cos y - \sin y)\,dy$,
C is any path from $(2, 0)$ to $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field $\mathbf{F}$ to move a particle from one point to another point. You decide to check first whether $\mathbf{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

22. Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?

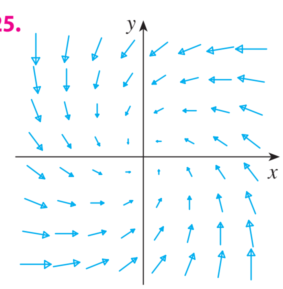
23–24 Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

23. $\mathbf{F}(x, y) = x^3\,\mathbf{i} + y^3\,\mathbf{j}$; $P(1, 0)$, $Q(2, 2)$

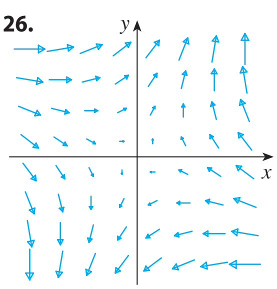
24. $\mathbf{F}(x, y) = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$; $P(1, 1)$, $Q(4, 3)$

25–26 Is the vector field shown in the figure conservative? Explain.

25.



26.



CAS **27.** If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x \cos y)\,\mathbf{j}$, use a plot to guess whether $\mathbf{F}$ is conservative. Then determine whether your guess is correct.

28. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ (b) $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

31–34 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31. $\{(x, y) \mid 0 < y < 3\}$

32. $\{(x, y) \mid 1 < |x| < 2\}$

33. $\{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ y \geqslant 0\}$

34. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

36. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.4. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.5. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\varepsilon = 8.985 \times 10^9$.)

16.4 Green's Theorem

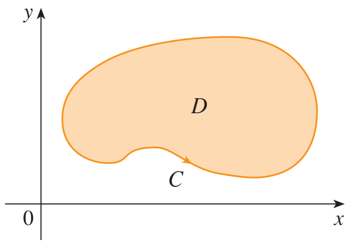


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C. (See Figure 2.)

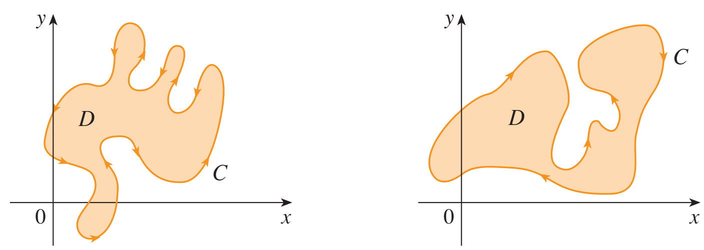


FIGURE 2 (a) Positive orientation (b) Negative orientation

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE The notation

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \ointctrclockwise_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

1
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F', $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F, Q, and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b.)

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both type I and type II (see Section 15.2). Let's call such regions **simple regions**.

George Green

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION Notice that Green's Theorem will be proved if we can show that

2 $$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$$

and

3 $$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

4 $$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,dy\,dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leqslant x \leqslant b$. Thus

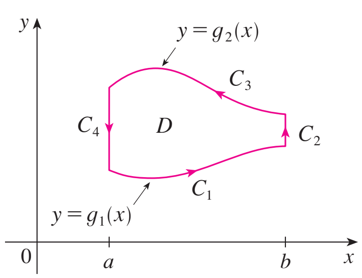


FIGURE 3

$$\int_{C_1} P(x, y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leqslant x \leqslant b$. Therefore

$$\int_{C_3} P(x, y)\,dx = -\int_{-C_3} P(x, y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\,dx = 0 = \int_{C_4} P(x, y)\,dx$$

Hence

$$\int_C P(x, y)\,dx = \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx + \int_{C_3} P(x, y)\,dx + \int_{C_4} P(x, y)\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y)\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 30). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

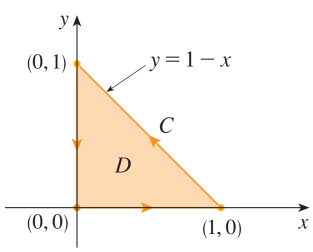


FIGURE 4

EXAMPLE 1 Evaluate $\int_C x^4\, dx + xy\, dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\, dx + xy\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\, dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$ ■

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leqslant 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\, dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi}\int_0^3 (7 - 3)\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi$$ ■

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\, dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\, dx + Q\, dy = 0$$

no matter what values P and Q assume in the region D.

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

5
$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$\begin{aligned} A &= \tfrac{1}{2}\int_C x\,dy - y\,dx \\ &= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \end{aligned}$$

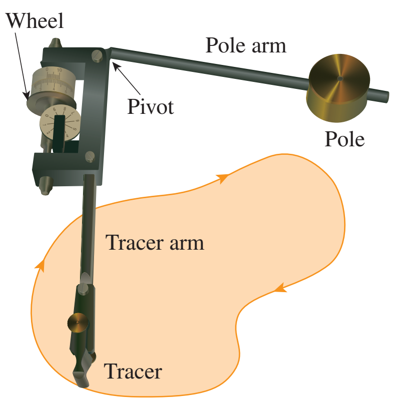


FIGURE 5
A Keuffel and Esser polar planimeter

Formula 5 can be used to explain how planimeters work. A **planimeter** is a mechanical instrument used for measuring the area of a region by tracing its boundary curve. These devices are useful in all the sciences: in biology for measuring the area of leaves or wings, in medicine for measuring the size of cross-sections of organs or tumors, in forestry for estimating the size of forested regions from photographs.

Figure 5 shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region. The explanation as a consequence of Formula 5 can be found in the following articles:

- R. W. Gatterman, "The planimeter as an example of Green's Theorem" *Amer. Math. Monthly,* Vol. 88 (1981), pp. 701–4.
- Tanya Leise, "As the planimeter wheel turns" *College Math. Journal,* Vol. 38 (2007), pp. 24–31.

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

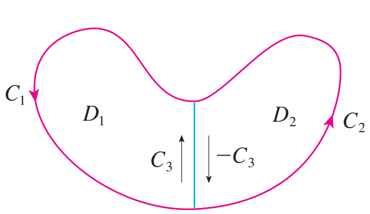


FIGURE 6

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

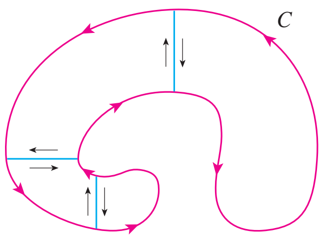


FIGURE 7

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 8). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

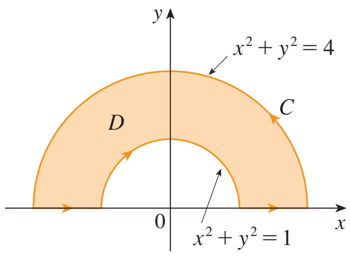


FIGURE 8

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] dA$$

$$= \iint_D y\,dA = \int_0^\pi \int_1^2 (r \sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = \Big[-\cos\theta\Big]_0^\pi \Big[\tfrac{1}{3}r^3\Big]_1^2 = \frac{14}{3}$$ ■

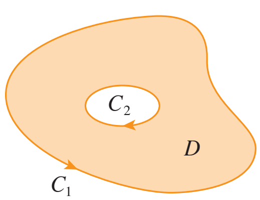


FIGURE 9

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'', we get

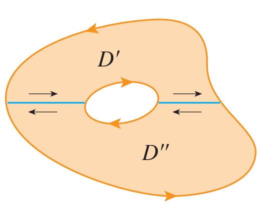


FIGURE 10

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region D.

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

SOLUTION Since C is an *arbitrary* closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle C'

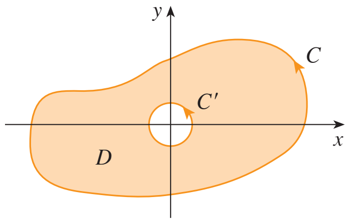


FIGURE 11

with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 11.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2}\right] dA = 0$$

Therefore
$$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is,
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$ ■

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field. ■

16.4 EXERCISES

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C y^2\,dx + x^2 y\,dy$,
C is the rectangle with vertices (0, 0), (5, 0), (5, 4), and (0, 4)

2. $\oint_C y\,dx - x\,dy$,
C is the circle with center the origin and radius 4

3. $\oint_C xy\,dx + x^2 y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x^2y^2\,dx + xy\,dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C ye^x\,dx + 2e^x\,dy$,
C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$

6. $\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$,
C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$

7. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

8. $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$

9. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

10. $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle y\cos x - xy\sin x,\ xy + x\cos x \rangle$,
C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

12. $\mathbf{F}(x, y) = \langle e^{-x} + y^2,\ e^{-y} + x^2 \rangle$,
C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$

13. $\mathbf{F}(x, y) = \langle y - \cos y,\ x\sin y \rangle$,
C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise

14. $\mathbf{F}(x, y) = \left\langle \sqrt{x^2 + 1},\ \tan^{-1}x \right\rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

CAS **15–16** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = x^3y^4$, $Q(x, y) = x^5y^4$,
C consists of the line segment from $(-\pi/2, 0)$ to $(\pi/2, 0)$ followed by the arc of the curve $y = \cos x$ from $(\pi/2, 0)$ to $(-\pi/2, 0)$

16. $P(x, y) = 2x - x^3y^5$, $Q(x, y) = x^3y^8$,
C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

18. A particle starts at the origin, moves along the x-axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$, $x \geqslant 0$, $y \geqslant 0$ to the point $(0, 5)$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \left\langle \sin x,\ \sin y + xy^2 + \tfrac{1}{3}x^3 \right\rangle$.

19. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5\cos t - \cos 5t$, $y = 5\sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a.

24. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3}\oint_C y^3\,dx \qquad I_y = \frac{\rho}{3}\oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 15.4.4.)

27. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

28. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y,\ 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

29. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

30. Complete the proof of the special case of Green's Theorem by proving Equation 3.

31. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

1 $$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \text{curl}\,\mathbf{F}\end{aligned}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

2 $$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

SOLUTION Using Equation 2, we have

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

■

CAS Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\mathbf{0}$.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

PROOF We have

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$$

by Clairaut's Theorem. ■

Notice the similarity to what we know from Section 12.4: $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for every three-dimensional vector $\mathbf{a}$.

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

$$\text{If } \mathbf{F} \text{ is conservative, then curl } \mathbf{F} = \mathbf{0}.$$

Compare this with Exercise 16.3.29.

This gives us a way of verifying that a vector field is not conservative.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\text{curl } \mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by the remarks preceding this example, $\mathbf{F}$ is not conservative. ■

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if $\mathbf{F}$ is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.") Theorem 4 is the three-dimensional version of Theorem 16.3.6. Its proof requires Stokes' Theorem and is sketched at the end of Section 16.8.

4 Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EXAMPLE 3
(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.
(b) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION
(a) We compute the curl of $\mathbf{F}$:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}$$

Since curl $\mathbf{F} = \mathbf{0}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, $\mathbf{F}$ is a conservative vector field by Theorem 4.
(b) The technique for finding f was given in Section 16.3. We have

5 $$f_x(x, y, z) = y^2z^3$$

6 $$f_y(x, y, z) = 2xyz^3$$

7 $$f_z(x, y, z) = 3xy^2z^2$$

Integrating (5) with respect to x, we obtain

8 $$f(x, y, z) = xy^2z^3 + g(y, z)$$

Differentiating (8) with respect to y, we get $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$, so comparison with (6) gives $g_y(y, z) = 0$. Thus $g(y, z) = h(z)$ and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives $h'(z) = 0$. Therefore

$$f(x, y, z) = xy^2z^3 + K$$

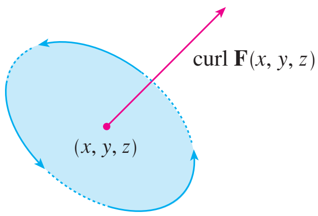


FIGURE 1

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 37. Another occurs when **F** represents the velocity field in fluid flow (see Example 16.1.3). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and **F** is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis. We give a more detailed explanation in Section 16.8 as a consequence of Stokes' Theorem.

Divergence

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9
$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that curl **F** is a vector field but div **F** is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

10
$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div **F**.

SOLUTION By the definition of divergence (Equation 9 or 10) we have

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl}\,\mathbf{F} = 0$$

PROOF Using the definitions of divergence and curl, we have

Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

$$\begin{aligned}
\text{div curl } \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x\, \partial y} - \frac{\partial^2 Q}{\partial x\, \partial z} + \frac{\partial^2 P}{\partial y\, \partial z} - \frac{\partial^2 R}{\partial y\, \partial x} + \frac{\partial^2 Q}{\partial z\, \partial x} - \frac{\partial^2 P}{\partial z\, \partial y} \\
&= 0
\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. ■

EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \text{curl } \mathbf{G}$.

SOLUTION In Example 4 we showed that

$$\text{div } \mathbf{F} = z + xz$$

and therefore div $\mathbf{F} \neq 0$. If it were true that $\mathbf{F} = \text{curl } \mathbf{G}$, then Theorem 11 would give

$$\text{div } \mathbf{F} = \text{div curl } \mathbf{G} = 0$$

which contradicts div $\mathbf{F} \neq 0$. Therefore $\mathbf{F}$ is not the curl of another vector field. ■

The reason for this interpretation of div $\mathbf{F}$ will be explained at the end of Section 16.9 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f. If f is a function of three variables, we have

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and, regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Therefore

$$(\text{curl}\,\mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12 $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\,dA$$

Equation 12 expresses the line integral of the tangential component of $\mathbf{F}$ along C as the double integral of the vertical component of curl $\mathbf{F}$ over the region D enclosed by C. We now derive a similar formula involving the *normal* component of $\mathbf{F}$.

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

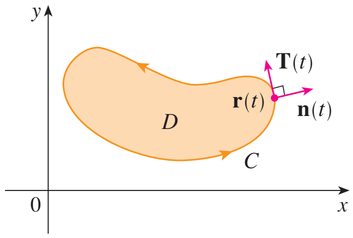


FIGURE 2

(See Figure 2.) Then, from Equation 16.2.3, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t), y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\,x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\,dt$$

$$= \int_a^b P(x(t), y(t))\,y'(t)\,dt - Q(x(t), y(t))\,x'(t)\,dt$$

$$= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

by Green's Theorem. But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

13
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

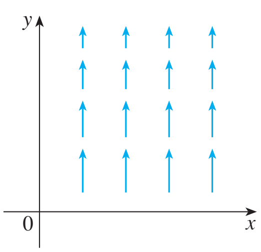
This version says that the line integral of the normal component of **F** along C is equal to the double integral of the divergence of **F** over the region D enclosed by C.

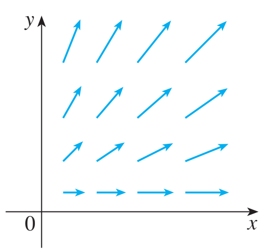
16.5 EXERCISES

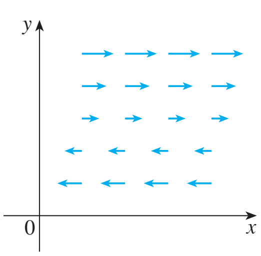
1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xy^2z^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$
2. $\mathbf{F}(x, y, z) = x^3yz^2\,\mathbf{j} + y^4z^3\,\mathbf{k}$
3. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + yze^x\,\mathbf{k}$
4. $\mathbf{F}(x, y, z) = \sin yz\,\mathbf{i} + \sin zx\,\mathbf{j} + \sin xy\,\mathbf{k}$
5. $\mathbf{F}(x, y, z) = \dfrac{\sqrt{x}}{1 + z}\,\mathbf{i} + \dfrac{\sqrt{y}}{1 + x}\,\mathbf{j} + \dfrac{\sqrt{z}}{1 + y}\,\mathbf{k}$
6. $\mathbf{F}(x, y, z) = \ln(2y + 3z)\,\mathbf{i} + \ln(x + 3z)\,\mathbf{j} + \ln(x + 2y)\,\mathbf{k}$
7. $\mathbf{F}(x, y, z) = \langle e^x \sin y,\ e^y \sin z,\ e^z \sin x\rangle$
8. $\mathbf{F}(x, y, z) = \langle \arctan(xy),\ \arctan(yz),\ \arctan(zx)\rangle$

9–11 The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)
(a) Is div **F** positive, negative, or zero? Explain.
(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl **F** point?

9.


10.


11.


12. Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.
(a) curl f (b) grad f
(c) div **F** (d) curl(grad f)
(e) grad **F** (f) grad(div **F**)
(g) div(grad f) (h) grad(div f)
(i) curl(curl **F**) (j) div(div **F**)
(k) (grad f) × (div **F**) (l) div(curl(grad f))

13–18 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

13. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$
14. $\mathbf{F}(x, y, z) = xyz^4\,\mathbf{i} + x^2z^4\,\mathbf{j} + 4x^2yz^3\,\mathbf{k}$
15. $\mathbf{F}(x, y, z) = z\cos y\,\mathbf{i} + xz\sin y\,\mathbf{j} + x\cos y\,\mathbf{k}$
16. $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z\,\mathbf{j} + y\cos z\,\mathbf{k}$
17. $\mathbf{F}(x, y, z) = e^{yz}\,\mathbf{i} + xze^{yz}\,\mathbf{j} + xye^{yz}\,\mathbf{k}$
18. $\mathbf{F}(x, y, z) = e^x\sin yz\,\mathbf{i} + ze^x\cos yz\,\mathbf{j} + ye^x\cos yz\,\mathbf{k}$

19. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x\sin y,\ \cos y,\ z - xy\rangle$? Explain.

20. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x, y, z\rangle$? Explain.

21. Show that any vector field of the form
$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$
where f, g, h are differentiable functions, is irrotational.

22. Show that any vector field of the form
$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$
is incompressible.

23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\,\mathbf{F}(x, y, z)$$

$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$

$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

23. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

24. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

25. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

26. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

27. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

28. $\operatorname{div}(\nabla f\times\nabla g) = 0$

29. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$

30–32 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

30. Verify each identity.
(a) $\nabla\cdot\mathbf{r} = 3$
(b) $\nabla\cdot(r\mathbf{r}) = 4r$
(c) $\nabla^2 r^3 = 12r$

31. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$
(b) $\nabla\times\mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$
(d) $\nabla\ln r = \mathbf{r}/r^2$

32. If $\mathbf{F} = \mathbf{r}/r^p$, find $\operatorname{div}\mathbf{F}$. Is there a value of p for which $\operatorname{div}\mathbf{F} = 0$?

33. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

34. Use Green's first identity (Exercise 33) to prove **Green's second identity**:

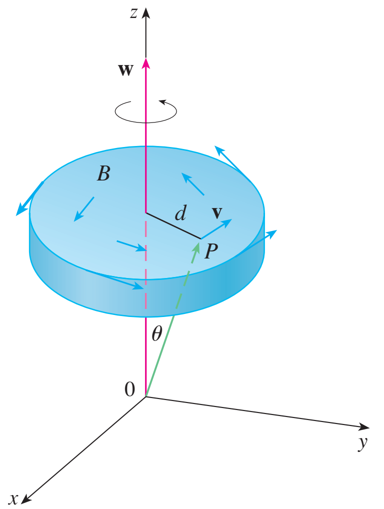
$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

35. Recall from Section 14.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. Use Green's first identity (with the same hypotheses as in Exercise 33) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}g\,ds = 0$. Here $D_{\mathbf{n}}g$ is the normal derivative of g defined in Exercise 33.

36. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2\,dA = 0$. (Assume the same hypotheses as in Exercise 33.)

37. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.
(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.
(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.
(c) Show that $\operatorname{curl}\mathbf{v} = 2\mathbf{w}$.



38. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0 \qquad \operatorname{div}\mathbf{H} = 0$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla\times(\nabla\times\mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$

(b) $\nabla\times(\nabla\times\mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

(c) $\nabla^2\mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 29.]

(d) $\nabla^2\mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$

39. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation curl $\mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \text{curl } \mathbf{G}$ satisfy the equation div $\mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the form $f = \text{div } \mathbf{G}$ must satisfy? Show that the answer to this question is "No" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.
[*Hint:* Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0\rangle$, where $g(x, y, z) = \int_0^x f(t, y, z)\, dt$.]

16.6 Parametric Surfaces and Their Areas

So far we have considered special types of surfaces: cylinders, quadric surfaces, graphs of functions of two variables, and level surfaces of functions of three variables. Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas. Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

1 $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

2 $$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S. Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D. (See Figure 1.)

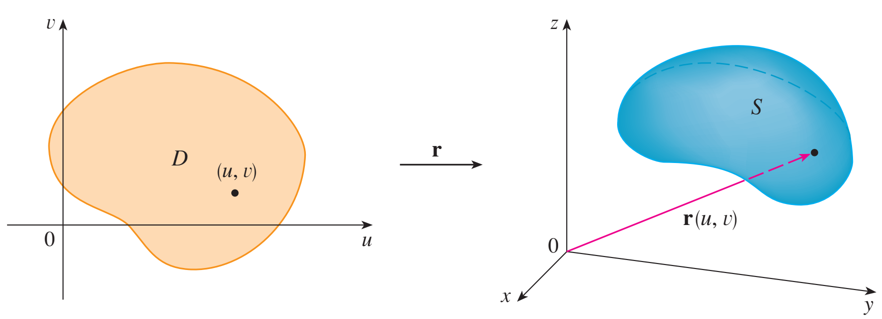


FIGURE 1
A parametric surface

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u\,\mathbf{i} + v\,\mathbf{j} + 2 \sin u\,\mathbf{k}$$

SOLUTION The parametric equations for this surface are

$$x = 2 \cos u \qquad y = v \qquad z = 2 \sin u$$

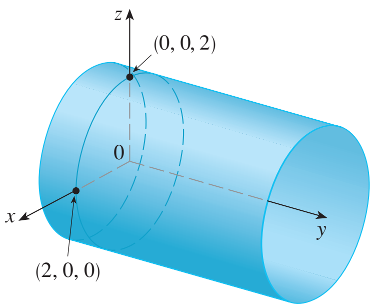


FIGURE 2

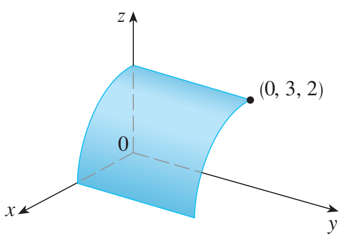


FIGURE 3

TEC Visual 16.6 shows animated versions of Figures 4 and 5, with moving grid curves, for several parametric surfaces.

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

This means that vertical cross-sections parallel to the xz-plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis (see Figure 2). ■

In Example 1 we placed no restrictions on the parameters u and v and so we obtained the entire cylinder. If, for instance, we restrict u and v by writing the parameter domain as

$$0 \leqslant u \leqslant \pi/2 \qquad 0 \leqslant v \leqslant 3$$

then $x \geqslant 0$, $z \geqslant 0$, $0 \leqslant y \leqslant 3$, and we get the quarter-cylinder with length 3 illustrated in Figure 3.

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant. These families correspond to vertical and horizontal lines in the uv-plane. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See Figure 4.)

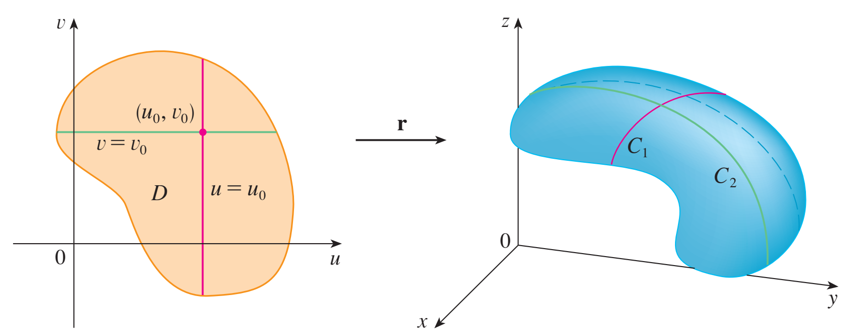


FIGURE 4

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S. We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.) In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves, as we see in the following example.

EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v)\cos u, (2 + \sin v)\sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

SOLUTION We graph the portion of the surface with parameter domain $0 \leqslant u \leqslant 4\pi$, $0 \leqslant v \leqslant 2\pi$ in Figure 5. It has the appearance of a spiral tube. To identify the grid curves, we write the corresponding parametric equations:

$$x = (2 + \sin v)\cos u \qquad y = (2 + \sin v)\sin u \qquad z = u + \cos v$$

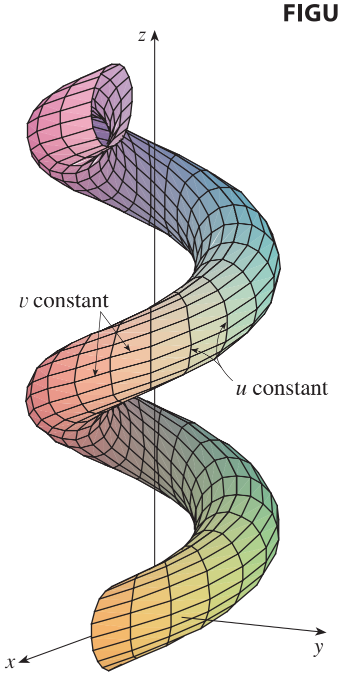


FIGURE 5

If v is constant, then $\sin v$ and $\cos v$ are constant, so the parametric equations resemble those of the helix in Example 13.1.4. Thus the grid curves with v constant are the spiral curves in Figure 5. We deduce that the grid curves with u constant must be the curves

that look like circles in the figure. Further evidence for this assertion is that if u is kept constant, $u = u_0$, then the equation $z = u_0 + \cos v$ shows that the z-values vary from $u_0 - 1$ to $u_0 + 1$. ■

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface. In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface. In the rest of this chapter we will often need to do exactly that.

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

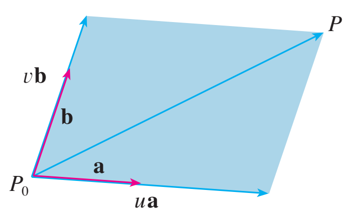


FIGURE 6

SOLUTION If P is any point in the plane, we can get from P_0 to P by moving a certain distance in the direction of $\mathbf{a}$ and another distance in the direction of $\mathbf{b}$. So there are scalars u and v such that $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$. (Figure 6 illustrates how this works, by means of the Parallelogram Law, for the case where u and v are positive. See also Exercise 12.2.46.) If $\mathbf{r}$ is the position vector of P, then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

So the vector equation of the plane can be written as

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

where u and v are real numbers.

If we write $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then we can write the parametric equations of the plane through the point (x_0, y_0, z_0) as follows:

$$x = x_0 + ua_1 + vb_1 \qquad y = y_0 + ua_2 + vb_2 \qquad z = z_0 + ua_3 + vb_3$$ ■

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

SOLUTION The sphere has a simple representation $\rho = a$ in spherical coordinates, so let's choose the angles ϕ and θ in spherical coordinates as the parameters (see Section 15.8). Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates (Equations 15.8.1), we obtain

$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a \cos\phi$$

as the parametric equations of the sphere. The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

We have $0 \leqslant \phi \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$, so the parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$. The grid curves with ϕ constant are the circles of constant latitude (including the equator). The grid curves with θ constant are the meridians (semicircles), which connect the north and south poles (see Figure 7). ■

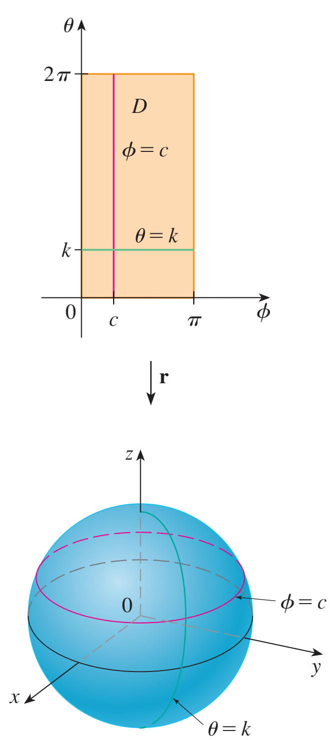


FIGURE 7

NOTE We saw in Example 4 that the grid curves for a sphere are curves of constant latitude or constant longitude. For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude. Describing a point on a parametric surface (like the one in Figure 5) by giving specific values of u and v is like giving the latitude and longitude of a point.

One of the uses of parametric surfaces is in computer graphics. Figure 8 shows the result of trying to graph the sphere $x^2 + y^2 + z^2 = 1$ by solving the equation for z and graphing the top and bottom hemispheres separately. Part of the sphere appears to be missing because of the rectangular grid system used by the computer. The much better picture in Figure 9 was produced by a computer using the parametric equations found in Example 4.

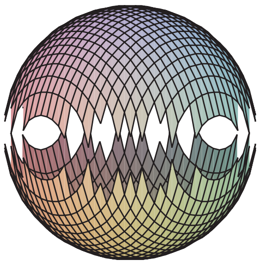


FIGURE 8

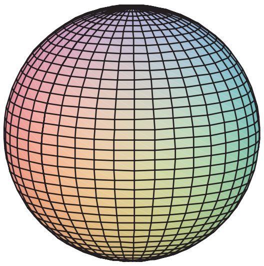


FIGURE 9

EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leqslant z \leqslant 1$$

SOLUTION The cylinder has a simple representation $r = 2$ in cylindrical coordinates, so we choose as parameters θ and z in cylindrical coordinates. Then the parametric equations of the cylinder are

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = z$$

where $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant z \leqslant 1$. ■

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

SOLUTION If we regard x and y as parameters, then the parametric equations are simply

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

and the vector equation is

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + 2y^2)\,\mathbf{k}$$ ■

TEC In Module 16.6 you can investigate several families of parametric surfaces.

In general, a surface given as the graph of a function of x and y, that is, with an equation of the form $z = f(x, y)$, can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equations as

$$x = x \qquad y = y \qquad z = f(x, y)$$

Parametric representations (also called parametrizations) of surfaces are not unique. The next example shows two ways to parametrize a cone.

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

SOLUTION 1 One possible representation is obtained by choosing x and y as parameters:

$$x = x \qquad y = y \qquad z = 2\sqrt{x^2 + y^2}$$

So the vector equation is

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + 2\sqrt{x^2 + y^2}\,\mathbf{k}$$

SOLUTION 2 Another representation results from choosing as parameters the polar coordinates r and θ. A point (x, y, z) on the cone satisfies $x = r\cos\theta$, $y = r\sin\theta$, and

For some purposes the parametric representations in Solutions 1 and 2 are equally good, but Solution 2 might be preferable in certain situations. If we are interested only in the part of the cone that lies below the plane $z = 1$, for instance, all we have to do in Solution 2 is change the parameter domain to

$$0 \leqslant r \leqslant \tfrac{1}{2} \qquad 0 \leqslant \theta \leqslant 2\pi$$

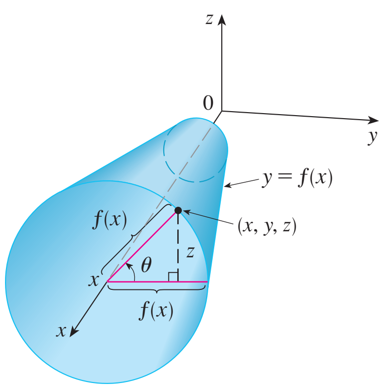


FIGURE 10

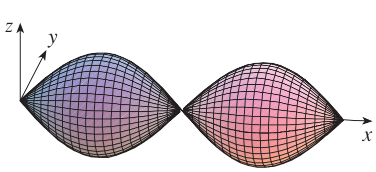


FIGURE 11

$z = 2\sqrt{x^2 + y^2} = 2r$. So a vector equation for the cone is

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2r\,\mathbf{k}$$

where $r \geqslant 0$ and $0 \leqslant \theta \leqslant 2\pi$. ■

Surfaces of Revolution

Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$. Let θ be the angle of rotation as shown in Figure 10. If (x, y, z) is a point on S, then

3 $$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S. The parameter domain is given by $a \leqslant x \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$.

EXAMPLE 8 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leqslant x \leqslant 2\pi$, about the x-axis. Use these equations to graph the surface of revolution.

SOLUTION From Equations 3, the parametric equations are

$$x = x \qquad y = \sin x\cos\theta \qquad z = \sin x\sin\theta$$

and the parameter domain is $0 \leqslant x \leqslant 2\pi$, $0 \leqslant \theta \leqslant 2\pi$. Using a computer to plot these equations and and then rotating the image, we obtain the graph in Figure 11. ■

We can adapt Equations 3 to represent a surface obtained through revolution about the y- or z-axis (see Exercise 30).

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 12.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

4 $$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

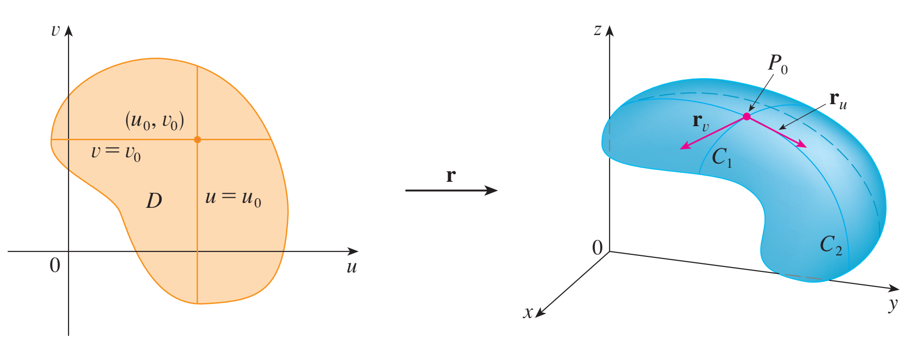


FIGURE 12

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\boxed{5} \qquad \mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth** (it has no "corners"). For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Figure 13 shows the self-intersecting surface in Example 9 and its tangent plane at (1, 1, 3).

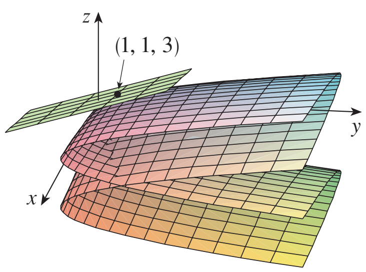


FIGURE 13

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point (1, 1, 3).

SOLUTION We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\,\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\,\mathbf{j} + 2\,\mathbf{k}$$

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

Notice that the point (1, 1, 3) corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$$

Therefore an equation of the tangent plane at (1, 1, 3) is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or

$$x + 2y - 2z + 3 = 0$$ ■

Surface Area

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}. Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. (See Figure 14.)

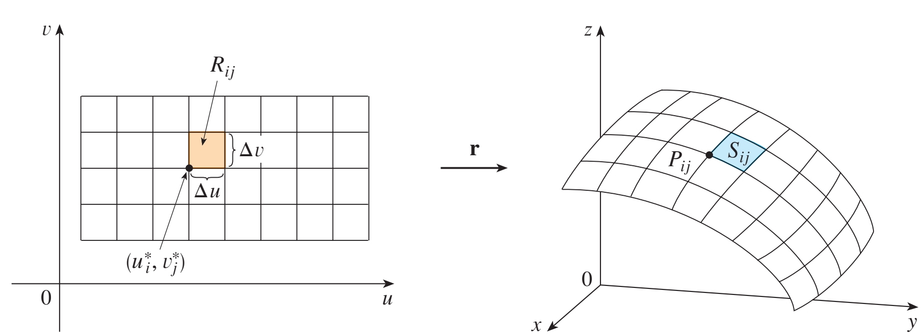


FIGURE 14
The image of the subrectangle R_{ij} is the patch S_{ij}.

The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \qquad \text{and} \qquad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 5 and 4.

Figure 15(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$. This parallelogram is shown in Figure 15(b) and lies in the tangent plane to S at P_{ij}. The area of this parallelogram is

$$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

and so an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$. This motivates the following definition.

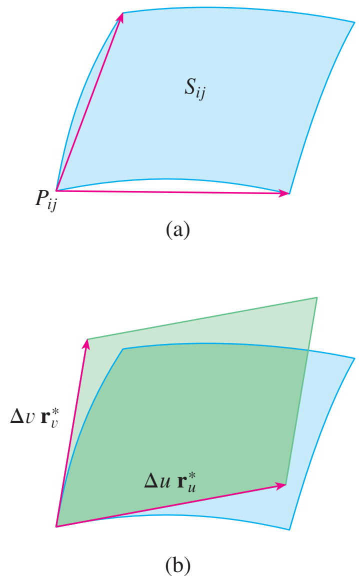


FIGURE 15
Approximating a patch by a parallelogram

6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\, \mathbf{i} + y(u, v)\, \mathbf{j} + z(u, v)\, \mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

EXAMPLE 10 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 we found the parametric representation

$$x = a \sin\phi\, \cos\theta \qquad y = a \sin\phi\, \sin\theta \qquad z = a\cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \theta \leqslant 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\, \cos\theta & a\cos\phi\, \sin\theta & -a\sin\phi \\ -a\sin\phi\, \sin\theta & a\sin\phi\, \cos\theta & 0 \end{vmatrix}$$

$$= a^2 \sin^2\phi\, \cos\theta\, \mathbf{i} + a^2 \sin^2\phi\, \sin\theta\, \mathbf{j} + a^2 \sin\phi\, \cos\phi\, \mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \sin^4\phi \,\cos^2\theta + a^4 \sin^4\phi \,\sin^2\theta + a^4 \sin^2\phi \,\cos^2\phi}$$
$$= \sqrt{a^4 \sin^4\phi + a^4 \sin^2\phi \,\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2 \sin\phi$$

since $\sin\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$. Therefore, by Definition 6, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi \, d\phi\, d\theta$$
$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = a^2(2\pi)2 = 4\pi a^2$$

■

Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

7 $$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Thus we have

8 $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area formula in Definition 6 becomes

9 $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Notice the similarity between the surface area formula in Equation 9 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

from Section 8.1.

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See

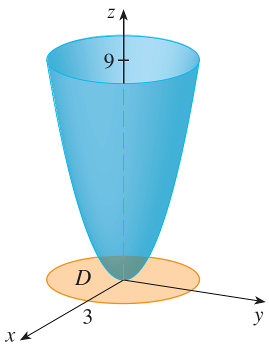


FIGURE 16

Figure 16.) Using Formula 9, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr$$

$$= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$ ■

The question remains whether our definition of surface area (6) is consistent with the surface area formula from single-variable calculus (8.2.4).

We consider the surface S obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$ and f' is continuous. From Equations 3 we know that parametric equations of S are

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta \qquad a \leqslant x \leqslant b \qquad 0 \leqslant \theta \leqslant 2\pi$$

To compute the surface area of S we need the tangent vectors

$$\mathbf{r}_x = \mathbf{i} + f'(x)\cos\theta\,\mathbf{j} + f'(x)\sin\theta\,\mathbf{k}$$

$$\mathbf{r}_\theta = -f(x)\sin\theta\,\mathbf{j} + f(x)\cos\theta\,\mathbf{k}$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= f(x)f'(x)\,\mathbf{i} - f(x)\cos\theta\,\mathbf{j} - f(x)\sin\theta\,\mathbf{k}$$

and so

$$|\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{[f(x)]^2[f'(x)]^2 + [f(x)]^2\cos^2\theta + [f(x)]^2\sin^2\theta}$$

$$= \sqrt{[f(x)]^2[1 + [f'(x)]^2]} = f(x)\sqrt{1 + [f'(x)]^2}$$

because $f(x) \geqslant 0$. Therefore the area of S is

$$A = \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta|\, dA$$

$$= \int_0^{2\pi} \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx\, d\theta$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

This is precisely the formula that was used to define the area of a surface of revolution in single-variable calculus (8.2.4).

16.6 EXERCISES

1–2 Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle u + v, u - 2v, 3 + u - v \rangle$
$P(4, -5, 1)$, $Q(0, 4, 6)$

2. $\mathbf{r}(u, v) = \langle 1 + u - v, u + v^2, u^2 - v^2 \rangle$
$P(1, 2, 1)$, $Q(2, 3, 3)$

3–6 Identify the surface with the given vector equation.

3. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k}$

4. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + u \cos v\,\mathbf{j} + u \sin v\,\mathbf{k}$

5. $\mathbf{r}(s, t) = \langle s \cos t, s \sin t, s \rangle$

6. $\mathbf{r}(s, t) = \langle 3 \cos t, s, \sin t \rangle, \quad -1 \leqslant s \leqslant 1$

7–12 Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have u constant and which have v constant.

7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle$,
$-1 \leqslant u \leqslant 1, \; -1 \leqslant v \leqslant 1$

8. $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle$,
$-2 \leqslant u \leqslant 2, \; -2 \leqslant v \leqslant 2$

9. $\mathbf{r}(u, v) = \langle u^3, u \sin v, u \cos v \rangle$,
$-1 \leqslant u \leqslant 1, \; 0 \leqslant v \leqslant 2\pi$

10. $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle$,
$-\pi \leqslant u \leqslant \pi, \; -\pi \leqslant v \leqslant \pi$

11. $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v$,
$0 \leqslant u \leqslant 2\pi, \; -\pi/2 \leqslant v \leqslant \pi/2$

12. $x = \cos u, \quad y = \sin u \sin v, \quad z = \cos v$,
$0 \leqslant u \leqslant 2\pi, \; 0 \leqslant v \leqslant 2\pi$

13–18 Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have u constant and which have v constant.

13. $\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + v\,\mathbf{k}$

14. $\mathbf{r}(u, v) = uv^2\,\mathbf{i} + u^2v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$

15. $\mathbf{r}(u, v) = (u^3 - u)\,\mathbf{i} + v^2\,\mathbf{j} + u^2\,\mathbf{k}$

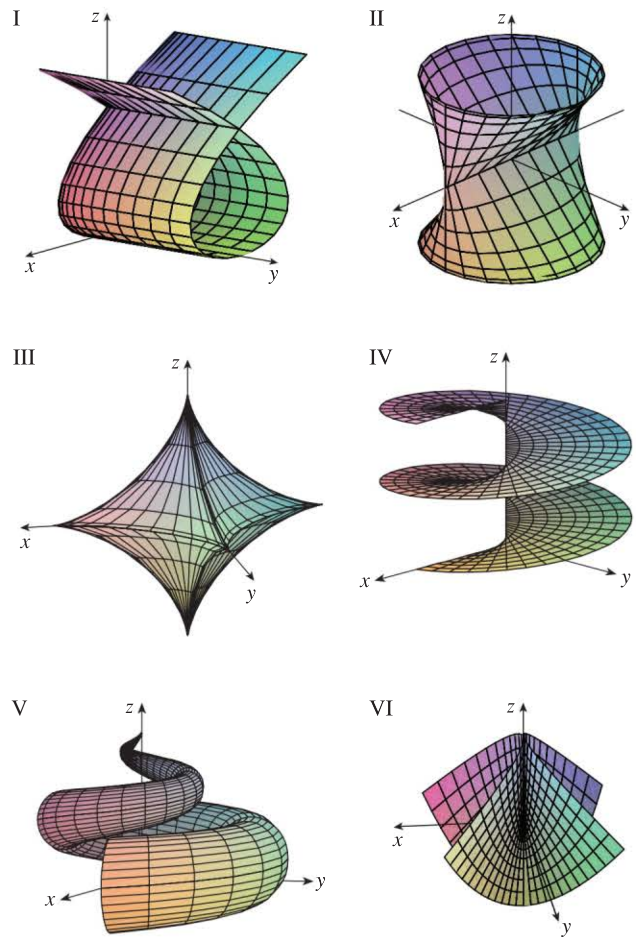
16. $x = (1 - u)(3 + \cos v) \cos 4\pi u$,

$y = (1 - u)(3 + \cos v) \sin 4\pi u$,

$z = 3u + (1 - u) \sin v$

17. $x = \cos^3 u \cos^3 v, \quad y = \sin^3 u \cos^3 v, \quad z = \sin^3 v$

18. $x = \sin u, \quad y = \cos u \sin v, \quad z = \sin v$



19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$

20. The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4 \rangle$ and $\langle -3, 2, 5 \rangle$

21. The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane

22. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

23. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

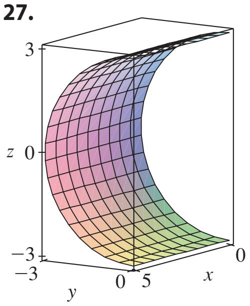
24. The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$

25. The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = 0$ and $z = 3\sqrt{3}$

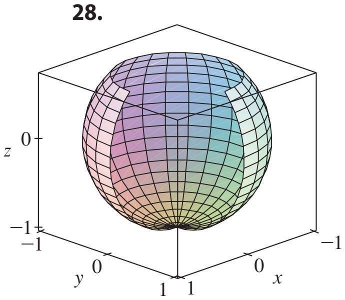
26. The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

27–28 Use a graphing device to produce a graph that looks like the given one.

27.



28.



29. Find parametric equations for the surface obtained by rotating the curve $y = 1/(1 + x^2)$, $-2 \leqslant x \leqslant 2$, about the x-axis and use them to graph the surface.

30. Find parametric equations for the surface obtained by rotating the curve $x = 1/y$, $y \geqslant 1$, about the y-axis and use them to graph the surface.

31. (a) What happens to the spiral tube in Example 2 (see Figure 5) if we replace $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$?
(b) What happens if we replace $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$?

32. The surface with parametric equations

$$x = 2\cos\theta + r\cos(\theta/2)$$

$$y = 2\sin\theta + r\cos(\theta/2)$$

$$z = r\sin(\theta/2)$$

where $-\frac{1}{2} \leqslant r \leqslant \frac{1}{2}$ and $0 \leqslant \theta \leqslant 2\pi$, is called a **Möbius strip**. Graph this surface with several viewpoints. What is unusual about it?

33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

33. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$

34. $x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v; \quad (5, 2, 3)$

35. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}; \quad u = 1, \ v = \pi/3$

36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u\sin v\,\mathbf{j} + \sin v\,\mathbf{k};$
$u = \pi/6, \ v = \pi/6$

CAS **37–38** Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

37. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}; \quad u = 1, \ v = 0$

38. $\mathbf{r}(u, v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}; \quad (-1, -1, -1)$

39–50 Find the area of the surface.

39. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

40. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v\rangle$ that is given by $0 \leqslant u \leqslant 2$, $-1 \leqslant v \leqslant 1$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

42. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$

43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

44. The part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

45. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

46. The part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = 2$, $z = 0$, and $z = 2$

47. The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$

48. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

49. The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2$

50. The part of the sphere $x^2 + y^2 + z^2 = b^2$ that lies inside the cylinder $x^2 + y^2 = a^2$, where $0 < a < b$

51. If the equation of a surface S is $z = f(x, y)$, where $x^2 + y^2 \leqslant R^2$, and you know that $|f_x| \leqslant 1$ and $|f_y| \leqslant 1$, what can you say about $A(S)$?

52–53 Find the area of the surface correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

52. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

53. The part of the surface $z = \ln(x^2 + y^2 + 2)$ that lies above the disk $x^2 + y^2 \leqslant 1$

CAS **54.** Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

55. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with six squares to estimate the area of the surface $z = 1/(1 + x^2 + y^2)$, $0 \leqslant x \leqslant 6$, $0 \leqslant y \leqslant 4$.
CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

CAS **56.** Find the area of the surface with vector equation $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$. State your answer correct to four decimal places.

CAS **57.** Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

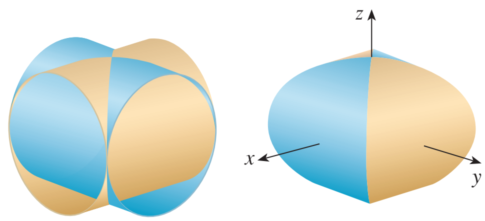
58. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au \cos v$, $y = bu \sin v$, $z = u^2$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$.
(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.
(c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.
CAS (d) For the case $a = 2$, $b = 3$, use a computer algebra system to find the surface area correct to four decimal places.

59. (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$, represent an ellipsoid.
(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.
(c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

60. (a) Show that the parametric equations $x = a \cosh u \cos v$, $y = b \cosh u \sin v$, $z = c \sinh u$, represent a hyperboloid of one sheet.
(b) Use the parametric equations in part (a) to graph the hyperboloid for the case $a = 1$, $b = 2$, $c = 3$.
(c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes $z = -3$ and $z = 3$.

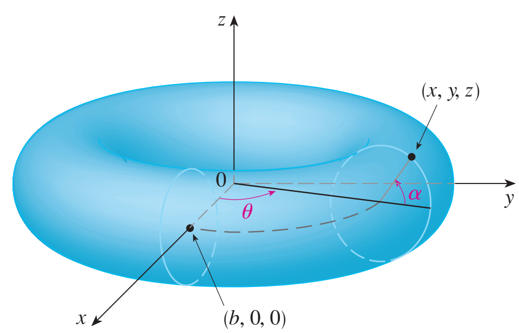
61. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

62. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



63. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

64. (a) Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [*Hint:* Take as parameters the angles θ and α shown in the figure.]
(b) Use the parametric equations found in part (a) to graph the torus for several values of a and b.
(c) Use the parametric representation from part (a) to find the surface area of the torus.



16.7 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S. We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrect-

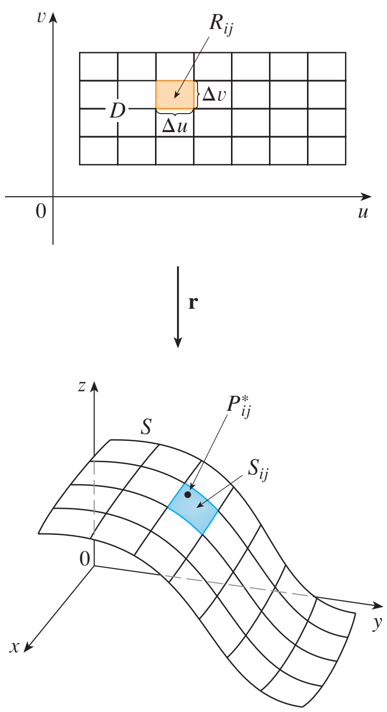


FIGURE 1

angles R_{ij} with dimensions Δu and Δv. Then the surface S is divided into corresponding patches S_{ij} as in Figure 1. We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

1
$$\iint_S f(x, y, z)\, dS = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\, \Delta S_{ij}$$

Notice the analogy with the definition of a line integral (16.2.2) and also the analogy with the definition of a double integral (15.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 16.6 we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

where
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Definition 1, even when D is not a rectangle, that

We assume that the surface is covered only once as (u, v) ranges throughout D. The value of the surface integral does not depend on the parametrization that is used.

2
$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

Observe also that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D. When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2\, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 16.6.4, we use the parametric representation

$$x = \sin\phi\, \cos\theta \qquad y = \sin\phi\, \sin\theta \qquad z = \cos\phi \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

that is, $\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$

As in Example 16.6.10, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

Therefore, by Formula 2,

$$\begin{aligned}\iint_S x^2\,dS &= \iint_D (\sin\phi\,\cos\theta)^2\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA \\ &= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\,\cos^2\theta\,\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi \\ &= \int_0^{2\pi}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta\int_0^{\pi}(\sin\phi - \sin\phi\,\cos^2\phi)\,d\phi \\ &= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi}\Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4\pi}{3}\end{aligned}$$

Here we use the identities

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\phi = 1 - \cos^2\phi$$

Instead, we could use Formulas 64 and 67 in the Table of Integrals.

■

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\,dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x, y, z)\,dS \qquad \bar{y} = \frac{1}{m}\iint_S y\,\rho(x, y, z)\,dS \qquad \bar{z} = \frac{1}{m}\iint_S z\,\rho(x, y, z)\,dS$$

Moments of inertia can also be defined as before (see Exercise 41).

Graphs of Functions

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have $\quad \mathbf{r}_x = \mathbf{i} + \left(\dfrac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\dfrac{\partial g}{\partial y}\right)\mathbf{k}$

Thus

3 $$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

4 $$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

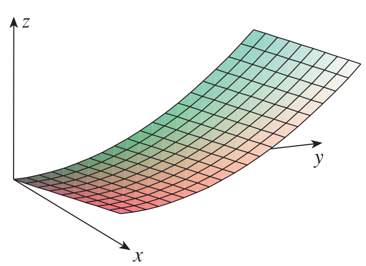


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$. (See Figure 2.)

SOLUTION Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned} \iint_S y\, dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \int_0^1 \int_0^2 y\sqrt{1 + 1 + 4y^2}\, dy\, dx \\ &= \int_0^1 dx\, \sqrt{2} \int_0^2 y\sqrt{1 + 2y^2}\, dy \\ &= \sqrt{2}\left(\tfrac{1}{4}\right)\tfrac{2}{3}(1 + 2y^2)^{3/2}\Big]_0^2 = \frac{13\sqrt{2}}{3} \end{aligned}$$ ■

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

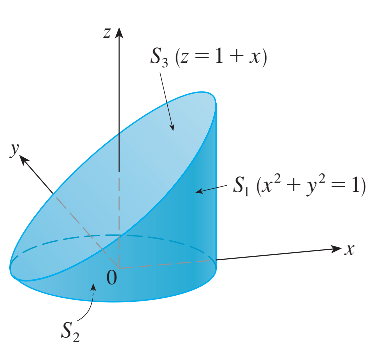


FIGURE 3

EXAMPLE 3 Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leqslant 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S.) For S_1 we use θ and z as parameters (see Example 16.6.5) and write its parametric equations as

$$x = \cos\theta \qquad y = \sin\theta \qquad z = z$$

where

$$0 \leqslant \theta \leqslant 2\pi \qquad \text{and} \qquad 0 \leqslant z \leqslant 1 + x = 1 + \cos\theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

and $$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Thus the surface integral over S_1 is

$$\begin{aligned} \iint_{S_1} z\,dS &= \iint_D z\,|\mathbf{r}_\theta \times \mathbf{r}_z|\,dA \\ &= \int_0^{2\pi}\int_0^{1+\cos\theta} z\,dz\,d\theta = \int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta \\ &= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_{S_2} z\,dS = \iint_{S_2} 0\,dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\begin{aligned} \iint_{S_3} z\,dS &= \iint_D (1+x)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{1+1+0}\;r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_0^1 (r + r^2\cos\theta)\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left(\tfrac{1}{2} + \tfrac{1}{3}\cos\theta\right)d\theta \\ &= \sqrt{2}\left[\frac{\theta}{2} + \frac{\sin\theta}{3}\right]_0^{2\pi} = \sqrt{2}\,\pi \end{aligned}$$

Therefore

$$\begin{aligned} \iint_S z\,dS &= \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS \\ &= \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\tfrac{3}{2} + \sqrt{2}\right)\pi \end{aligned}$$

■

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point P, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 16.6.32.

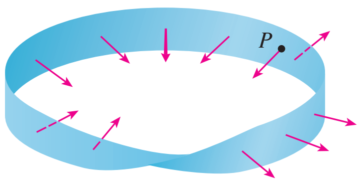


FIGURE 4
A Möbius strip

TEC Visual 16.7 shows a Möbius strip with a normal vector that can be moved along the surface.

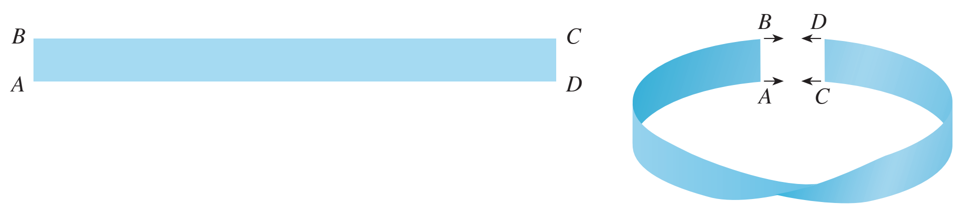


FIGURE 5
Constructing a Möbius strip

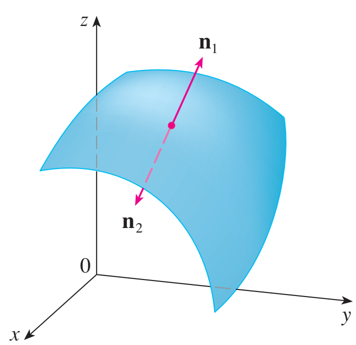


FIGURE 6

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 6.)

If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**. There are two possible orientations for any orientable surface (see Figure 7).

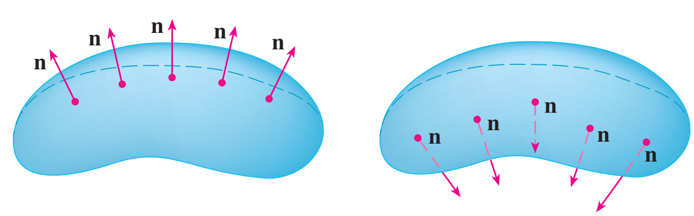


FIGURE 7
The two orientations of an orientable surface

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

5
$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

6
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 16.6.4 we found

the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 16.6.10 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$

and $$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

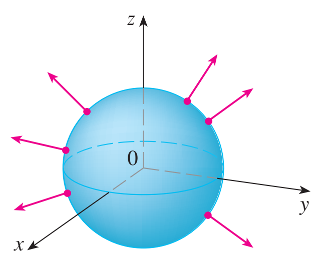


FIGURE 8
Positive orientation

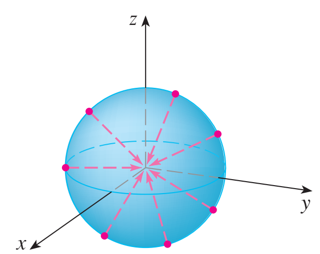


FIGURE 9
Negative orientation

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij}, as in Figure 10 (compare with Figure 1), then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal $\mathbf{n}$ by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

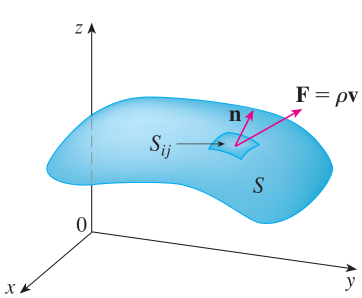


FIGURE 10

where ρ, $\mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector $\mathbf{n}$ is $\rho\mathbf{v} \cdot \mathbf{n}$.) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

7 $$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

A surface integral of this form occurs frequently in physics, even when $\mathbf{F}$ is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of $\mathbf{F}$ over S.

8 Definition If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

In words, Definition 8 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined).

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where D is the parameter domain. Thus we have

Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 16.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

9
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Figure 11 shows the vector field $\mathbf{F}$ in Example 4 at points on the unit sphere.

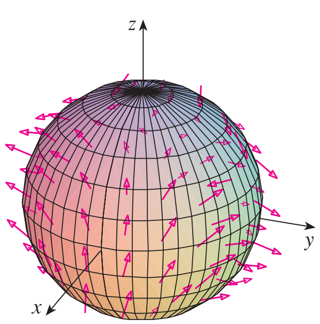


FIGURE 11

SOLUTION As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi\, \cos\theta\, \mathbf{i} + \sin\phi\, \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

Then
$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\, \mathbf{i} + \sin\phi\, \sin\theta\, \mathbf{j} + \sin\phi\, \cos\theta\, \mathbf{k}$$

and, from Example 16.6.10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\!\phi\, \cos\theta\, \mathbf{i} + \sin^2\!\phi\, \sin\theta\, \mathbf{j} + \sin\phi\, \cos\phi\, \mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\, \sin^2\!\phi\, \cos\theta + \sin^3\!\phi\, \sin^2\!\theta + \sin^2\!\phi\, \cos\phi\, \cos\theta$$

and, by Formula 9, the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2 \sin^2\phi \cos\phi \cos\theta + \sin^3\phi \sin^2\theta)\, d\phi\, d\theta$$

$$= 2 \int_0^{\pi} \sin^2\phi \cos\phi\, d\phi \int_0^{2\pi} \cos\theta\, d\theta + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= 0 + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \qquad \left(\text{since } \int_0^{2\pi} \cos\theta\, d\theta = 0\right)$$

$$= \frac{4\pi}{3}$$

by the same calculation as in Example 1. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes

10
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

This formula assumes the upward orientation of S; for a downward orientation we multiply by -1. Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 37 and 38.)

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

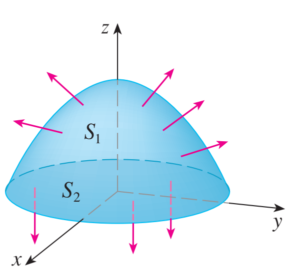


FIGURE 12

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2. (See Figure 12.) Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D being the projection of S_1 onto the xy-plane, namely, the disk $x^2 + y^2 \leqslant 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and
$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

we have

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

$$= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta \sin\theta\right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA = \iint_D 0\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$ ■

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if $\mathbf{E}$ is an electric field (see Example 16.1.5), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of $\mathbf{E}$ through the surface S. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

11 $$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$.) Therefore, if the vector field $\mathbf{F}$ in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\, \nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K \nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and so

$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$. Therefore the rate of heat flow across S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS$$

$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3$$ ■

16.7 EXERCISES

1. Let S be the surface of the box enclosed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Approximate $\iint_S \cos(x + 2y + 3z)\, dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the box S and the points P_{ij}^* to be the centers of the squares.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4$$

Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

5–20 Evaluate the surface integral.

5. $\iint_S (x + y + z)\, dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$

6. $\iint_S xyz\, dS$,
S is the cone with parametric equations $x = u\cos v$, $y = u\sin v$, $z = u$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

7. $\iint_S y\, dS$, S is the helicoid with vector equation $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

8. $\iint_S (x^2 + y^2)\, dS$,
S is the surface with vector equation $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2\rangle$, $u^2 + v^2 \leqslant 1$

9. $\iint_S x^2yz\, dS$,
S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xz\, dS$,
S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant

11. $\iint_S x\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$

12. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. $\iint_S z^2\, dS$,
S is the part of the paraboloid $x = y^2 + z^2$ given by $0 \leqslant x \leqslant 1$

14. $\iint_S y^2z^2\, dS$,
S is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \leqslant y \leqslant 5$

15. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 1$, $0 \leqslant z \leqslant 1$

16. $\iint_S y^2\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2z + y^2z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

18. $\iint_S (x + y + z)\, dS$,
S is the part of the half-cylinder $x^2 + z^2 = 1$, $z \geqslant 0$, that lies between the planes $y = 0$ and $y = 2$

19. $\iint_S xz\, dS$,
S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$

20. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

21–32 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

21. $\mathbf{F}(x, y, z) = ze^{xy}\,\mathbf{i} - 3ze^{xy}\,\mathbf{j} + xy\,\mathbf{k}$,
S is the parallelogram of Exercise 5 with upward orientation

22. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$,
S is the helicoid of Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

24. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation

25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, S is the sphere with radius 1 and center the origin

26. $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented downward

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$,
S is the surface $z = x\sin y$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \pi$, with upward orientation

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm1, \pm1, \pm1)$

30. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

32. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

CAS **33.** Evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$ correct to four decimal places, where S is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

CAS **34.** Find the exact value of $\iint_S xyz\, dS$, where S is the surface $z = x^2y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$.

CAS **35.** Find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

CAS **36.** Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

37. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

41. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 40.

42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k, find (a) the center of mass and (b) the moment of inertia about the z-axis.

43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leqslant z \leqslant 1$.

44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm1, \pm1, \pm1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

49. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere S with center the origin is independent of the radius of S.

16.8 Stokes' Theorem

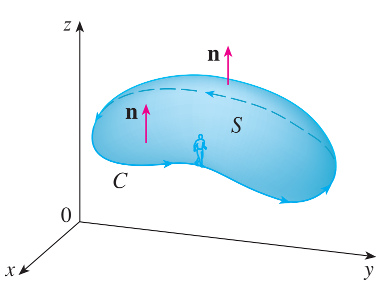


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS$$

George Stokes

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

The positively oriented boundary curve of the oriented surface S is often written as ∂S, so Stokes' Theorem can be expressed as

$$\boxed{1} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl $\mathbf{F}$ is a sort of derivative of $\mathbf{F}$) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S is a graph and $\mathbf{F}$, S, and C are well behaved.

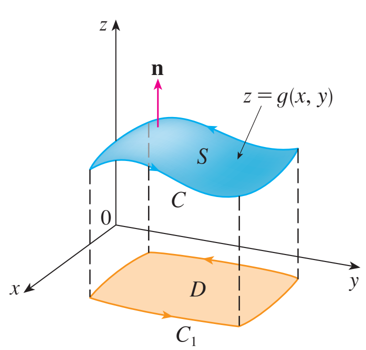


FIGURE 2

PROOF OF A SPECIAL CASE OF STOKES' THEOREM We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. (See Figure 2.) We are also given that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function, we can apply Formula 16.7.10 with $\mathbf{F}$ replaced by curl $\mathbf{F}$. The result is

$$\boxed{2} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt\\ &= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt\\ &= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right]dt\\ &= \int_{C_1}\left(P + R\frac{\partial z}{\partial x}\right)dx + \left(Q + R\frac{\partial z}{\partial y}\right)dy\\ &= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right]dA\end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \Bigg[&\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\,\partial y}\right)\\ &- \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\,\partial x}\right)\Bigg]dA\end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$ ■

EXAMPLE 1 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

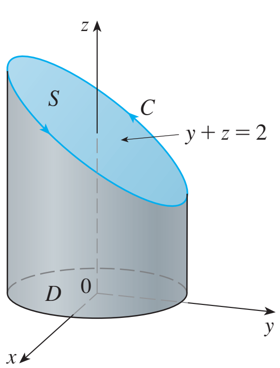


FIGURE 3

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F}\cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y^2 & x & z^2\end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Although there are many surfaces with boundary C, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy-plane is

the disk $x^2 + y^2 \leqslant 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3} \sin \theta\right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$ ■

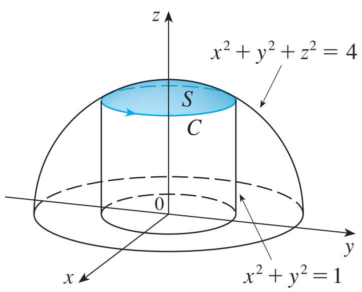


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

so $$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \cos t\,\mathbf{i} + \sqrt{3} \sin t\,\mathbf{j} + \cos t \sin t\,\mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t\right) dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt = 0$$ ■

Note that in Example 2 we computed a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

3 $$\iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector. Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means that the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v} \cdot \mathbf{T}$. Thus $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C. (See Figure 5.)

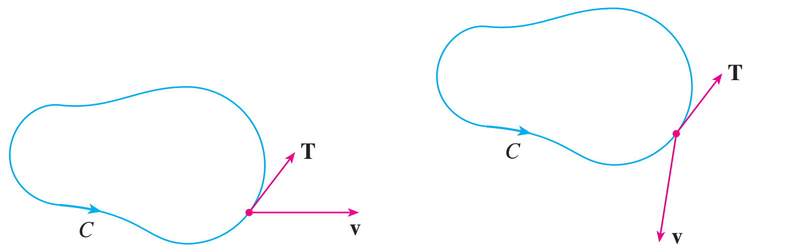


FIGURE 5 (a) $\int_C \mathbf{v} \cdot d\mathbf{r} > 0$, positive circulation (b) $\int_C \mathbf{v} \cdot d\mathbf{r} < 0$, negative circulation

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\operatorname{curl} \mathbf{F})(P) \approx (\operatorname{curl} \mathbf{F})(P_0)$ for all points P on S_a because curl $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \iint_{S_a} \operatorname{curl} \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\, dS$$

$$\approx \iint_{S_a} \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\, dS = \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\pi a^2$$

Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 6; the paddle wheel rotates fastest when its axis is parallel to curl $\mathbf{v}$.

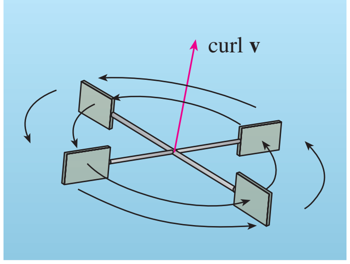


FIGURE 6

This approximation becomes better as $a \to 0$ and we have

4
$$\operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

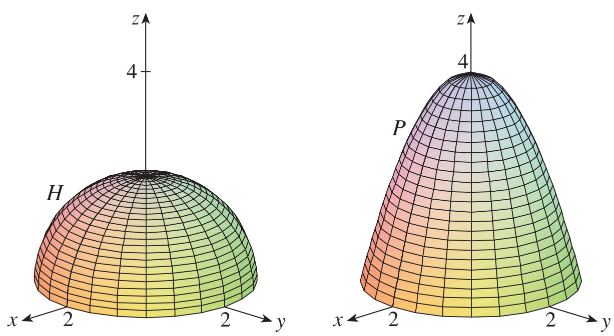
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

16.8 EXERCISES

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_P \text{curl } \mathbf{F} \cdot d\mathbf{S}$$



2–6 Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = x^2 \sin z\, \mathbf{i} + y^2\, \mathbf{j} + xy\, \mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y\, \mathbf{i} + x \cos y\, \mathbf{j} + xz \sin y\, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\, \mathbf{i} + x^2y\, \mathbf{j} + x^2z^2\, \mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz\, \mathbf{i} + xy\, \mathbf{j} + x^2yz\, \mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\, \mathbf{i} + e^{xz}\, \mathbf{j} + x^2z\, \mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2)\, \mathbf{i} + (y + z^2)\, \mathbf{j} + (z + x^2)\, \mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\, \mathbf{j} + \left(xy - \sqrt{z}\right) \mathbf{k}$,
C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

10. $\mathbf{F}(x, y, z) = 2y\, \mathbf{i} + xz\, \mathbf{j} + (x + y)\, \mathbf{k}$, C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2z\, \mathbf{i} + xy^2\, \mathbf{j} + z^2\, \mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2y\, \mathbf{i} + \frac{1}{3}x^3\, \mathbf{j} + xy\, \mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

13–15 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

13. $\mathbf{F}(x, y, z) = -y\, \mathbf{i} + x\, \mathbf{j} - 2\, \mathbf{k}$,
S is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

14. $\mathbf{F}(x, y, z) = -2yz\, \mathbf{i} + y\, \mathbf{j} + 3x\, \mathbf{k}$,
S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

15. $\mathbf{F}(x, y, z) = y\, \mathbf{i} + z\, \mathbf{j} + x\, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\, dx - 2x\, dy + 3y\, dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\, \mathbf{i} + 2xy\, \mathbf{j} + 4y^2\, \mathbf{k}$$

Find the work done.

18. Evaluate

$$\int_C (y + \sin x)\, dx + (z^2 + \cos y)\, dy + x^3\, dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$

(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$

(c) $\int_C (f\,\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$

WRITING PROJECT THREE MEN AND TWO THEOREMS

The photograph shows a stained-glass window at Cambridge University in honor of George Green.



Courtesy of the Masters and Fellows of Gonville and Caius College, Cambridge University, England

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 1137 and 1175.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to His Life and Work* (Philadelphia: Society for Industrial and Applied Mathematics, 2001).

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.

3. I. Grattan-Guinness, "Why Did George Green Write his Essay of 1828 on Electricity and Magnetism?" *Amer. Math. Monthly,* Vol. 102 (1995), pp. 387–96.

4. J. Gray, "There Was a Jolly Miller." *The New Scientist,* Vol. 139 (1993), pp. 24–27.

5. G. E. Hutchinson, *The Enchanted Voyage and Other Studies* (Westport, CT: Greenwood Press, 1978).

6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–80.

7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–85.

8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

1 $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div **F** in this case) over a region to the integral of the original function **F** over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector **n** is directed outward from E.

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of **F** across the boundary surface of E is equal to the triple integral of the divergence of **F** over E.

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

If **n** is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three

equations:

2 $$\iint_S P\,\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial P}{\partial x}\,dV$$

3 $$\iint_S Q\,\mathbf{j}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial Q}{\partial y}\,dV$$

4 $$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\,dz\right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

5 $$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

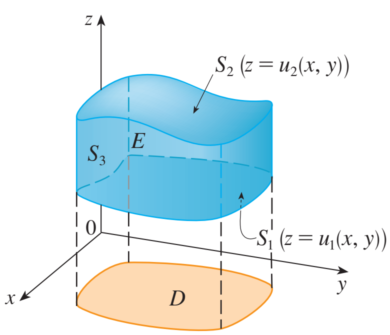


FIGURE 1

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D. (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k}\cdot\mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_3} 0\,dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

6 $$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D R(x, y, u_2(x, y))\,dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

Therefore Equation 6 gives

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leqslant 1$. Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_B \text{div}\,\mathbf{F}\,dV = \iiint_B 1\,dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$ ■

The solution in Example 1 should be compared with the solution in Example 16.7.4.

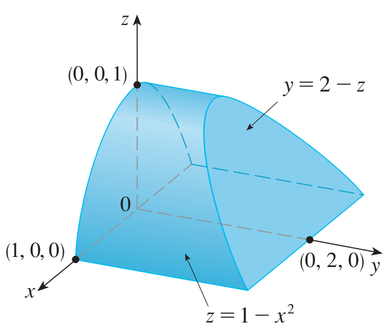


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S.) Furthermore, the divergence of $\mathbf{F}$ is much less complicated than $\mathbf{F}$ itself:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \left\{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2,\ 0 \leqslant y \leqslant 2 - z\right\}$$

Then we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\,dV = \iiint_E 3y\,dV$$

$$= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\,dy\,dz\,dx = 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\,dz\,dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2-z)^3}{3}\right]_0^{1-x^2} dx = -\tfrac{1}{2}\int_{-1}^{1}\left[(x^2+1)^3 - 8\right]dx$$

$$= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\,dx = \frac{184}{35}$$ ■

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 16.4 to extend Green's Theorem.)

For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2. (See Figure 3.) Applying the Divergence Theorem to S, we get

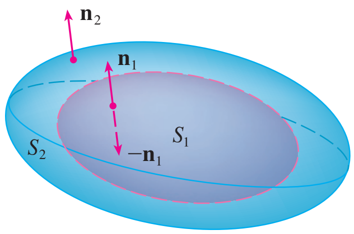


FIGURE 3

7
$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$
$$= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS$$
$$= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

EXAMPLE 3 In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

SOLUTION The difficulty is that we don't have an explicit equation for S_2 because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so we let S_1 be a small sphere with radius a and center the origin. You can verify that $\operatorname{div} \mathbf{E} = 0$. (See Exercise 23.) Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E}\, dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\mathbf{x}/|\mathbf{x}|$. Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\, dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} dS = \frac{\varepsilon Q}{a^2}A(S_1) = \frac{\varepsilon Q}{a^2}4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of $\mathbf{E}$ is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss's Law (Equation 16.7.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.] ■

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P_0)$ for all points P in B_a since $\operatorname{div} \mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0)\, dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

8 $$\operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

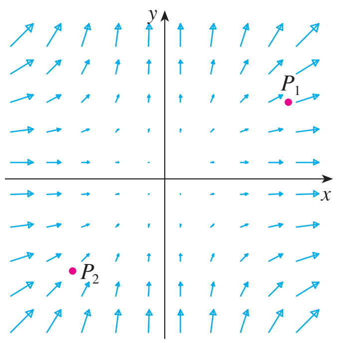


FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

Equation 8 says that $\operatorname{div} \mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.) If $\operatorname{div} \mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If $\operatorname{div} \mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus the net flow is outward near P_1, so $\operatorname{div} \mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so $\operatorname{div} \mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have $\operatorname{div} \mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

16.9 EXERCISES

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2yz\,\mathbf{j} + 4z^2\,\mathbf{k}$,
E is the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

3. $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$,
E is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

5–15 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
S is the sphere with center the origin and radius 2

9. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{i} + (z - e^y)\,\mathbf{j} - xy\,\mathbf{k}$,
S is the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers

11. $\mathbf{F}(x, y, z) = (2x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane

12. $\mathbf{F}(x, y, z) = (xy + 2xz)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j} + (xy - z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = y - 2$ and $z = 0$

13. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

14. $\mathbf{F} = |\mathbf{r}|^2\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, S is the sphere with radius R and center the origin

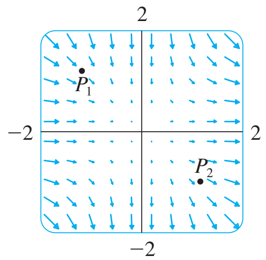
CAS **15.** $\mathbf{F}(x, y, z) = e^y \tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x \sin y\,\mathbf{k}$, S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

CAS **16.** Use a computer algebra system to plot the vector field $\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$ in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

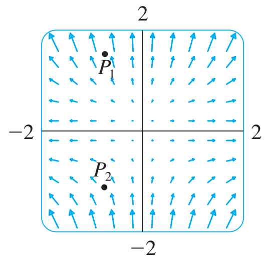
17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = z^2 x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2 z + y^2)\,\mathbf{k}$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

18. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

19. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether div $\mathbf{F}$ is positive or negative at P_1 and at P_2.



20. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.
(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



CAS **21–22** Plot the vector field and guess where div $\mathbf{F} > 0$ and where div $\mathbf{F} < 0$. Then calculate div $\mathbf{F}$ to check your guess.

21. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$

22. $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

23. Verify that div $\mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$.

24. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

25–30 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

25. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where $\mathbf{a}$ is a constant vector

26. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

27. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$

28. $\displaystyle\iint_S D_{\mathbf{n}} f\, dS = \iiint_E \nabla^2 f\, dV$

29. $\displaystyle\iint_S (f \nabla g) \cdot \mathbf{n}\, dS = \iiint_E (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV$

30. $\displaystyle\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f \nabla^2 g - g \nabla^2 f)\, dV$

31. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

32. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid, and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\, dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 31 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' Principle:* The buoyant force on an object equals the weight of the displaced liquid.

16.10 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

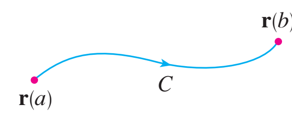
Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



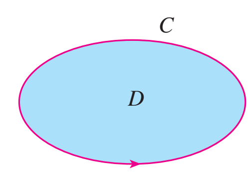
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



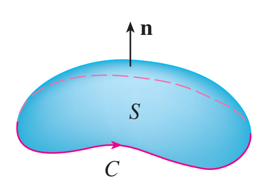
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



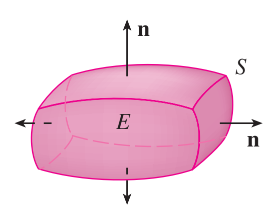
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



16 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. What is a vector field? Give three examples that have physical meaning.

2. (a) What is a conservative vector field?
(b) What is a potential function?

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
(b) How do you evaluate such a line integral?
(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
(e) How do you evaluate these line integrals?

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
(b) If $\mathbf{F}$ is a force field, what does this line integral represent?
(c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

5. State the Fundamental Theorem for Line Integrals.

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

7. State Green's Theorem.

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
(a) Define curl $\mathbf{F}$. (b) Define div $\mathbf{F}$.
(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

11. (a) What is a parametric surface? What are its grid curves?
(b) Write an expression for the area of a parametric surface.
(c) What is the area of a surface given by an equation $z = g(x, y)$?

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.
(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(c) What if S is given by an equation $z = g(x, y)$?
(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

13. (a) What is an oriented surface? Give an example of a non-orientable surface.
(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(d) What if S is given by an equation $z = g(x, y)$?

14. State Stokes' Theorem.

15. State the Divergence Theorem.

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.

2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.

3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$.

4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.

6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$

7. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\operatorname{div} \mathbf{F} = \operatorname{div} \mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

8. The work done by a conservative force field in moving a particle around a closed path is zero.

9. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$$

10. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} \cdot \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$$

11. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

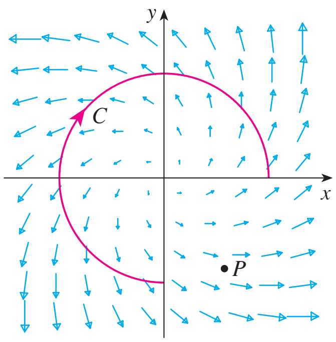
12. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl} \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

13. The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve C is $A = \oint_C y\, dx$.

EXERCISES

1. A vector field $\mathbf{F}$, a curve C, and a point P are shown.
(a) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.
(b) Is div $\mathbf{F}(P)$ positive, negative, or zero? Explain.



2–9 Evaluate the line integral.

2. $\int_C x\, ds$,
C is the arc of the parabola $y = x^2$ from (0, 0) to (1, 1)

3. $\int_C yz \cos x\, ds$,
C: $x = t,\ y = 3\cos t,\ z = 3\sin t,\ 0 \leqslant t \leqslant \pi$

4. $\int_C y\, dx + (x + y^2)\, dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation

5. $\int_C y^3\, dx + x^2\, dy$, C is the arc of the parabola $x = 1 - y^2$ from $(0, -1)$ to $(0, 1)$

6. $\int_C \sqrt{xy}\, dx + e^y\, dy + xz\, dz$,
C is given by $\mathbf{r}(t) = t^4\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

7. $\int_C xy\, dx + y^2\, dy + yz\, dz$,
C is the line segment from $(1, 0, -1)$, to $(3, 4, 2)$

8. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $0 \leqslant t \leqslant \pi$

9. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point (3, 0, 0) to the point $(0, \pi/2, 3)$ along
(a) a straight line
(b) the helix $x = 3\cos t,\ y = t,\ z = 3\sin t$

11–12 Show that $\mathbf{F}$ is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

13–14 Show that $\mathbf{F}$ is conservative and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,
C: $\mathbf{r}(t) = (t + \sin \pi t)\,\mathbf{i} + (2t + \cos \pi t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from (0, 2, 0) to (4, 0, 3)

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\, dx - x^2 y\, dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\, dx + 2xy\, dy$$

where C is the triangle with vertices (0, 0), (1, 0), and (1, 3).

17. Use Green's Theorem to evaluate $\int_C x^2 y\, dx - xy^2\, dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field $\mathbf{G}$ such that

$$\text{curl}\,\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields whose component functions have continuous first partial derivatives, show that

$$\text{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,\text{div}\,\mathbf{G} - \mathbf{G}\,\text{div}\,\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that $\int_C f(x)\, dx + g(y)\, dy = 0$.

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\, dx - f_x\, dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \qquad y = \sin t \qquad z = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(b) Find $\int_C 2xe^{2y}\, dx + (2x^2e^{2y} + 2y\cot z)\, dy - y^2\csc^2 z\, dz$.

25. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices (0, 0), (1, 0), and (1, 2).

26. (a) Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by

$$\mathbf{r}(u, v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k} \qquad 0 \leqslant u \leqslant 3,\ -3 \leqslant v \leqslant 3$$

(b) Use a computer to graph the surface S and the tangent plane found in part (a).

(c) Set up, but do not evaluate, an integral for the surface area of S.

(d) If

$$\mathbf{F}(x, y, z) = \frac{z^2}{1 + x^2}\,\mathbf{i} + \frac{x^2}{1 + y^2}\,\mathbf{j} + \frac{y^2}{1 + z^2}\,\mathbf{k}$$

find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ correct to four decimal places.

27–30 Evaluate the surface integral.

27. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

28. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

29. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

30. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

31. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and S has upward orientation.

32. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

33. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, and C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), oriented counterclockwise as viewed from above.

34. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

35. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

36. Compute the outward flux of

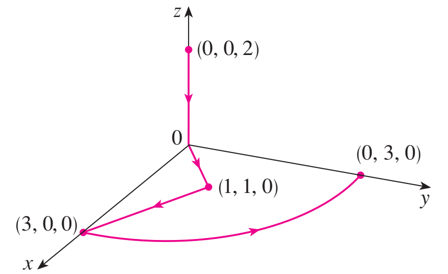
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

37. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}$$

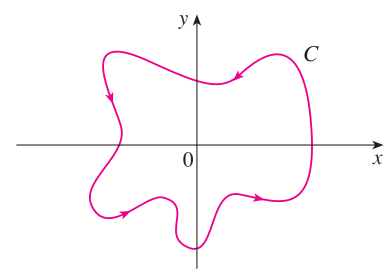
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(0, 3, 0)$ shown in the figure.



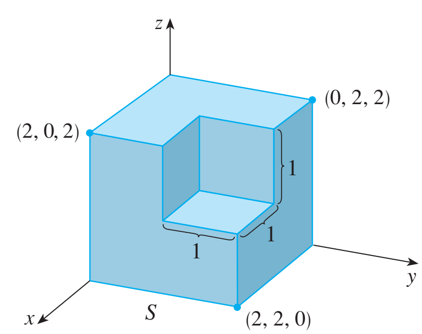
38. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is shown in the figure.



39. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



40. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

41. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

Problems Plus

1. Let S be a smooth parametric surface and let P be a point such that each line that starts at P intersects S at most once. The **solid angle** $\Omega(S)$ subtended by S at P is the set of lines starting at P and passing through S. Let $S(a)$ be the intersection of $\Omega(S)$ with the surface of the sphere with center P and radius a. Then the measure of the solid angle (in *steradians*) is defined to be

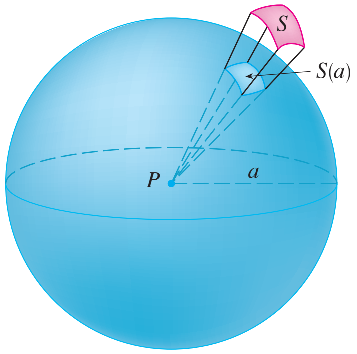
$$|\Omega(S)| = \frac{\text{area of } S(a)}{a^2}$$

Apply the Divergence Theorem to the part of $\Omega(S)$ between $S(a)$ and S to show that

$$|\Omega(S)| = \iint_S \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}\, dS$$

where $\mathbf{r}$ is the radius vector from P to any point on S, $r = |\mathbf{r}|$, and the unit normal vector $\mathbf{n}$ is directed away from P.

This shows that the definition of the measure of a solid angle is independent of the radius a of the sphere. Thus the measure of the solid angle is equal to the area subtended on a *unit* sphere. (Note the analogy with the definition of radian measure.) The total solid angle subtended by a sphere at its center is thus 4π steradians.



2. Find the positively oriented simple closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\, dx - 2x^3\, dy$$

is a maximum.

3. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$ and has positive orientation with respect to $\mathbf{n}$. Show that the plane area enclosed by C is

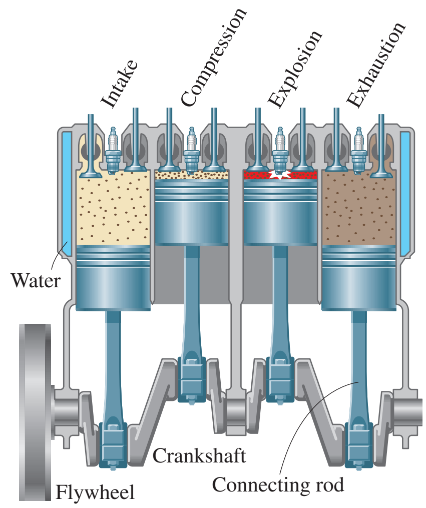
$$\tfrac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

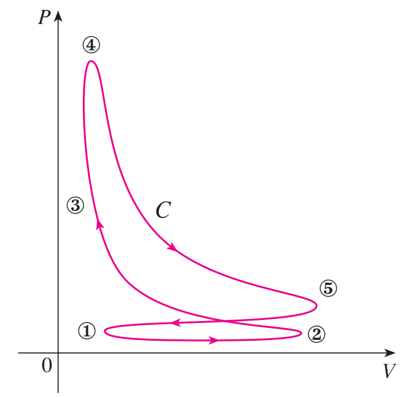
4. Investigate the shape of the surface with parametric equations $x = \sin u$, $y = \sin v$, $z = \sin(u + v)$. Start by graphing the surface from several points of view. Explain the appearance of the graphs by determining the traces in the horizontal planes $z = 0$, $z = \pm 1$, and $z = \pm\frac{1}{2}$.

5. Prove the following identity:

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \operatorname{curl} \mathbf{G} + \mathbf{G} \times \operatorname{curl} \mathbf{F}$$

6. The figure depicts the sequence of events in each cylinder of a four-cylinder internal combustion engine. Each piston moves up and down and is connected by a pivoted arm to a rotating crankshaft. Let $P(t)$ and $V(t)$ be the pressure and volume within a cylinder at time t, where $a \leqslant t \leqslant b$ gives the time required for a complete cycle. The graph shows how P and V vary through one cycle of a four-stroke engine.





During the intake stroke (from ① to ②) a mixture of air and gasoline at atmospheric pressure is drawn into a cylinder through the intake valve as the piston moves downward. Then the piston rapidly compresses the mix with the valves closed in the compression stroke (from ② to ③) during which the pressure rises and the volume decreases. At ③ the sparkplug ignites the fuel, raising the temperature and pressure at almost constant volume to ④. Then, with valves closed, the rapid expansion forces the piston downward during the power stroke (from ④ to ⑤). The exhaust valve opens, temperature and pressure drop, and mechanical energy stored in a rotating flywheel pushes the piston upward, forcing the waste products out of the exhaust valve in the exhaust stroke. The exhaust valve closes and the intake valve opens. We're now back at ① and the cycle starts again.

(a) Show that the work done on the piston during one cycle of a four-stroke engine is $W = \int_C P\,dV$, where C is the curve in the PV-plane shown in the figure.

[*Hint:* Let $x(t)$ be the distance from the piston to the top of the cylinder and note that the force on the piston is $\mathbf{F} = AP(t)\,\mathbf{i}$, where A is the area of the top of the piston. Then $W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leqslant t \leqslant b$. An alternative approach is to work directly with Riemann sums.]

(b) Use Formula 16.4.5 to show that the work is the difference of the areas enclosed by the two loops of C.

17 Second-Order Differential Equations



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The motion of a shock absorber in a motorcycle is described by the differential equations that we solve in Section 17.3.

THE BASIC IDEAS OF DIFFERENTIAL equations were explained in Chapter 9; there we concentrated on first-order equations. In this chapter we study second-order linear differential equations and learn how they can be applied to solve problems concerning the vibrations of springs and the analysis of electric circuits. We will also see how infinite series can be used to solve differential equations.

17.1 Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions. We saw in Section 9.1 that equations of this type arise in the study of the motion of a spring. In Section 17.3 we will further pursue this application as well as the application to electric circuits.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1. Such equations are called **homogeneous** linear equations. Thus the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous** and is discussed in Section 17.2.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

PROOF Since y_1 and y_2 are solutions of Equation 2, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and

$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}
P(x)y'' &+ Q(x)y' + R(x)y \\
&= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&= c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\
&= c_1(0) + c_2(0) = 0
\end{aligned}$$

Thus $y = c_1y_1 + c_2y_2$ is a solution of Equation 2. ■

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2. This means that neither y_1 nor y_2 is a constant multiple of the other. For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it's not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions P, Q, and R are constant functions, that is, if the differential equation has the form

5
$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

6
$$ar^2 + br + c = 0$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

7
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

CASE I $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions of Equation 5. (Note that $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$.) Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

In Figure 1 the graphs of the basic solutions $f(x) = e^{2x}$ and $g(x) = e^{-3x}$ of the differential equation in Example 1 are shown in blue and red, respectively. Some of the other solutions, linear combinations of f and g, are shown in black.

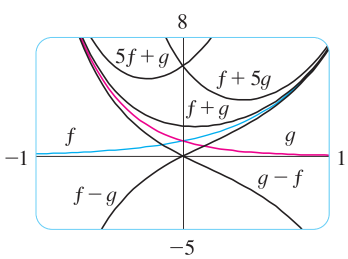


FIGURE 1

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

SOLUTION The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$. Therefore, by (8), the general solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

We could verify that this is indeed a solution by differentiating and substituting into the differential equation. ■

EXAMPLE 2 Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

SOLUTION To solve the auxiliary equation $3r^2 + r - 1 = 0$, we use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{13}}{6}$$

Since the roots are real and distinct, the general solution is

$$y = c_1 e^{(-1+\sqrt{13})x/6} + c_2 e^{(-1-\sqrt{13})x/6}$$ ■

CASE II $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

9 $$r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2 xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

In the first term, $2ar + b = 0$ by Equations 9; in the second term, $ar^2 + br + c = 0$ because r is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

Figure 2 shows the basic solutions $f(x) = e^{-3x/2}$ and $g(x) = xe^{-3x/2}$ in Example 3 and some other members of the family of solutions. Notice that all of them approach 0 as $x \to \infty$.

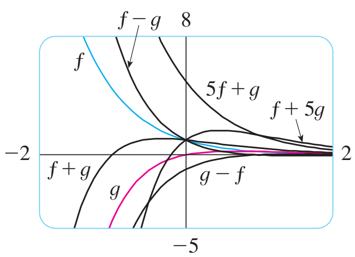


FIGURE 2

EXAMPLE 3 Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By (10) the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$ ■

CASE III $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. (See Appendix H for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix H, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real. We summarize the discussion as follows.

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Figure 3 shows the graphs of the solutions in Example 4, $f(x) = e^{3x}\cos 2x$ and $g(x) = e^{3x}\sin 2x$, together with some linear combinations. All solutions approach 0 as $x \to -\infty$.

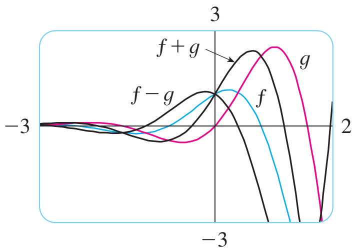


FIGURE 3

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

By (11), the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$ ■

Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants. If P, Q, R, and G are continuous on an interval and $P(x) \neq 0$ there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 and 6 illustrate the technique for solving such a problem.

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

SOLUTION From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

To satisfy the initial conditions we require that

Figure 4 shows the graph of the solution of the initial-value problem in Example 5. Compare with Figure 1.

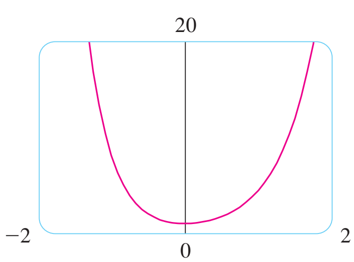


FIGURE 4

12 $$y(0) = c_1 + c_2 = 1$$

13 $$y'(0) = 2c_1 - 3c_2 = 0$$

From (13), we have $c_2 = \frac{2}{3}c_1$ and so (12) gives

$$c_1 + \tfrac{2}{3}c_1 = 1 \qquad c_1 = \tfrac{3}{5} \qquad c_2 = \tfrac{2}{5}$$

Thus the required solution of the initial-value problem is

$$y = \tfrac{3}{5}e^{2x} + \tfrac{2}{5}e^{-3x}$$ ■

EXAMPLE 6 Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

SOLUTION The auxiliary equation is $r^2 + 1 = 0$, or $r^2 = -1$, whose roots are $\pm i$. Thus $\alpha = 0$, $\beta = 1$, and since $e^{0x} = 1$, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Since $$y'(x) = -c_1 \sin x + c_2 \cos x$$

The solution to Example 6 is graphed in Figure 5. It appears to be a shifted sine curve and, indeed, you can verify that another way of writing the solution is

$y = \sqrt{13}\sin(x + \phi)$ where $\tan\phi = \frac{2}{3}$

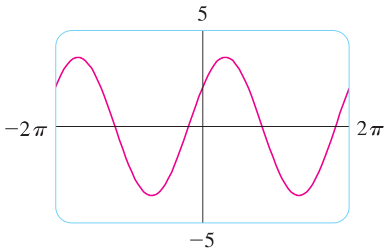


FIGURE 5

the initial conditions become

$$y(0) = c_1 = 2 \qquad y'(0) = c_2 = 3$$

Therefore the solution of the initial-value problem is

$$y(x) = 2\cos x + 3\sin x$$

■

A **boundary-value problem** for Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution. The method is illustrated in Example 7.

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

SOLUTION The auxiliary equation is

$$r^2 + 2r + 1 = 0 \qquad \text{or} \qquad (r + 1)^2 = 0$$

whose only root is $r = -1$. Therefore the general solution is

$$y(x) = c_1e^{-x} + c_2xe^{-x}$$

Figure 6 shows the graph of the solution of the boundary-value problem in Example 7.

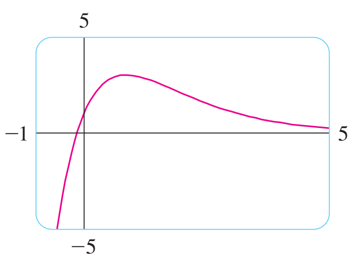


FIGURE 6

The boundary conditions are satisfied if

$$y(0) = c_1 = 1$$

$$y(1) = c_1e^{-1} + c_2e^{-1} = 3$$

The first condition gives $c_1 = 1$, so the second condition becomes

$$e^{-1} + c_2e^{-1} = 3$$

Solving this equation for c_2 by first multiplying through by e, we get

$$1 + c_2 = 3e \qquad \text{so} \qquad c_2 = 3e - 1$$

Thus the solution of the boundary-value problem is

$$y = e^{-x} + (3e - 1)xe^{-x}$$

■

Summary: Solutions of $ay'' + by' + cy = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1e^{r_1x} + c_2e^{r_2x}$
$r_1 = r_2 = r$	$y = c_1e^{rx} + c_2xe^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

17.1 EXERCISES

1–13 Solve the differential equation.

1. $y'' - y' - 6y = 0$

2. $y'' - 6y' + 9y = 0$

3. $y'' + 2y = 0$

4. $y'' + y' - 12y = 0$

5. $4y'' + 4y' + y = 0$

6. $9y'' + 4y = 0$

7. $3y'' = 4y'$

8. $y = y''$

9. $y'' - 4y' + 13y = 0$

10. $3y'' + 4y' - 3y = 0$

11. $2\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - y = 0$

12. $\dfrac{d^2R}{dt^2} + 6\dfrac{dR}{dt} + 34R = 0$

13. $3\dfrac{d^2V}{dt^2} + 4\dfrac{dV}{dt} + 3V = 0$

14–16 Graph the two basic solutions along with several other solutions of the differential equation. What features do the solutions have in common?

14. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

15. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$

16. $2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

17–24 Solve the initial-value problem.

17. $y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$

18. $y'' - 2y' - 3y = 0, \quad y(0) = 2, \quad y'(0) = 2$

19. $9y'' + 12y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

20. $3y'' - 2y' - y = 0, \quad y(0) = 0, \quad y'(0) = -4$

21. $y'' - 6y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = 3$

22. $4y'' - 20y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -3$

23. $y'' - y' - 12y = 0, \quad y(1) = 0, \quad y'(1) = 1$

24. $4y'' + 4y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

25–32 Solve the boundary-value problem, if possible.

25. $y'' + 16y = 0, \quad y(0) = -3, \quad y(\pi/8) = 2$

26. $y'' + 6y' = 0, \quad y(0) = 1, \quad y(1) = 0$

27. $y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y(1) = 0$

28. $y'' - 8y' + 17y = 0, \quad y(0) = 3, \quad y(\pi) = 2$

29. $y'' = y', \quad y(0) = 1, \quad y(1) = 2$

30. $4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y(2) = 0$

31. $y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = 2$

32. $y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = e^{-2\pi}$

33. Let L be a nonzero real number.
(a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.
(b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

34. If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.

35. Consider the boundary-value problem $y'' - 2y' + 2y = 0$, $y(a) = c$, $y(b) = d$.
(a) If this problem has a unique solution, how are a and b related?
(b) If this problem has no solution, how are a, b, c, and d related?
(c) If this problem has infinitely many solutions, how are a, b, c, and d related?

17.2 Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \tag{1}$$

where a, b, and c are constants and G is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \tag{2}$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

3 Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

PROOF We verify that if y is any solution of Equation 1, then $y - y_p$ is a solution of the complementary Equation 2. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0 \end{aligned}$$

This shows that every solution is of the form $y(x) = y_p(x) + y_c(x)$. It is easy to check that every function of this form is a solution. ■

We know from Section 17.1 how to solve the complementary equation. (Recall that the solution is $y_c = c_1y_1 + c_2y_2$, where y_1 and y_2 are linearly independent solutions of Equation 2.) Therefore Theorem 3 says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution y_p. There are two methods for finding a particular solution: The method of undetermined coefficients is straightforward but works only for a restricted class of functions G. The method of variation of parameters works for every function G but is usually more difficult to apply in practice.

The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

EXAMPLE 1 Solve the equation $y'' + y' - 2y = x^2$.

SOLUTION The auxiliary equation of $y'' + y' - 2y = 0$ is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots $r = 1, -2$. So the solution of the complementary equation is

$$y_c = c_1e^x + c_2e^{-2x}$$

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$ so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 1 \qquad 2A - 2B = 0 \qquad 2A + B - 2C = 0$$

The solution of this system of equations is

$$A = -\tfrac{1}{2} \qquad B = -\tfrac{1}{2} \qquad C = -\tfrac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1e^x + c_2e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$ ■

Figure 1 shows four solutions of the differential equation in Example 1 in terms of the particular solution y_p and the functions $f(x) = e^x$ and $g(x) = e^{-2x}$.

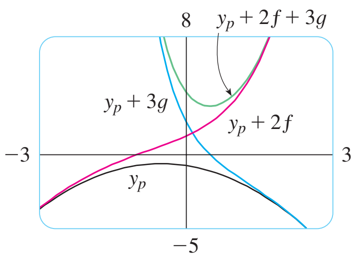


FIGURE 1

If $G(x)$ (the right side of Equation 1) is of the form Ce^{kx}, where C and k are constants, then we take as a trial solution a function of the same form, $y_p(x) = Ae^{kx}$, because the derivatives of e^{kx} are constant multiples of e^{kx}.

EXAMPLE 2 Solve $y'' + 4y = e^{3x}$.

SOLUTION The auxiliary equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$. Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$. Thus a particular solution is

$$y_p(x) = \tfrac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{13}e^{3x}$$ ■

Figure 2 shows solutions of the differential equation in Example 2 in terms of y_p and the functions $f(x) = \cos 2x$ and $g(x) = \sin 2x$. Notice that all solutions approach ∞ as $x \to \infty$ and all solutions (except y_p) resemble sine functions when x is negative.

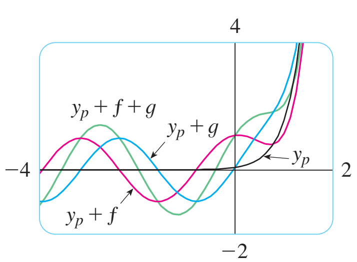


FIGURE 2

If $G(x)$ is either $C \cos kx$ or $C \sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A \cos kx + B \sin kx$$

EXAMPLE 3 Solve $y'' + y' - 2y = \sin x$.

SOLUTION We try a particular solution

$$y_p(x) = A \cos x + B \sin x$$

Then $\qquad y_p' = -A\sin x + B\cos x \qquad y_p'' = -A\cos x - B\sin x$

so substitution in the differential equation gives

$$(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = \sin x$$

or $$(-3A + B)\cos x + (-A - 3B)\sin x = \sin x$$

This is true if

$$-3A + B = 0 \qquad \text{and} \qquad -A - 3B = 1$$

The solution of this system is

$$A = -\tfrac{1}{10} \qquad B = -\tfrac{3}{10}$$

so a particular solution is

$$y_p(x) = -\tfrac{1}{10}\cos x - \tfrac{3}{10}\sin x$$

In Example 1 we determined that the solution of the complementary equation is $y_c = c_1e^x + c_2e^{-2x}$. Thus the general solution of the given equation is

$$y(x) = c_1e^x + c_2e^{-2x} - \tfrac{1}{10}(\cos x + 3\sin x)$$ ■

If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax + B)\cos 3x + (Cx + D)\sin 3x$$

If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition*, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

SOLUTION The auxiliary equation is $r^2 - 4 = 0$ with roots ± 2, so the solution of the complementary equation is $y_c(x) = c_1e^{2x} + c_2e^{-2x}$. For the equation $y'' - 4y = xe^x$ we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then $y_{p_1}' = (Ax + A + B)e^x$, $y_{p_1}'' = (Ax + 2A + B)e^x$, so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or $$(-3Ax + 2A - 3B)e^x = xe^x$$

Thus $-3A = 1$ and $2A - 3B = 0$, so $A = -\frac{1}{3}$, $B = -\frac{2}{9}$, and

$$y_{p_1}(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{9}\right)e^x$$

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C\cos 2x + D\sin 2x$$

Substitution gives

$$-4C\cos 2x - 4D\sin 2x - 4(C\cos 2x + D\sin 2x) = \cos 2x$$

or

$$-8C\cos 2x - 8D\sin 2x = \cos 2x$$

Therefore $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8}\cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1e^{2x} + c_2e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8}\cos 2x$$ ■

In Figure 3 we show the particular solution $y_p = y_{p_1} + y_{p_2}$ of the differential equation in Example 4. The other solutions are given in terms of $f(x) = e^{2x}$ and $g(x) = e^{-2x}$.

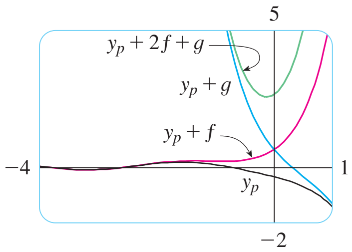


FIGURE 3

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

EXAMPLE 5 Solve $y'' + y = \sin x$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of the complementary equation is

$$y_c(x) = c_1\cos x + c_2\sin x$$

Ordinarily, we would use the trial solution

$$y_p(x) = A\cos x + B\sin x$$

but we observe that it is a solution of the complementary equation, so instead we try

$$y_p(x) = Ax\cos x + Bx\sin x$$

Then

$$y_p'(x) = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$

$$y_p''(x) = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x$$

Substitution in the differential equation gives

$$y_p'' + y_p = -2A\sin x + 2B\cos x = \sin x$$

The graphs of four solutions of the differential equation in Example 5 are shown in Figure 4.

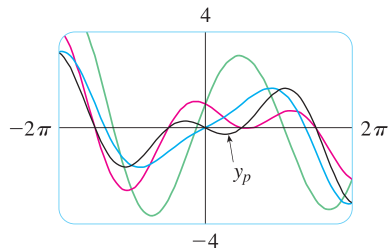


FIGURE 4

so $A = -\frac{1}{2}$, $B = 0$, and

$$y_p(x) = -\tfrac{1}{2}x\cos x$$

The general solution is

$$y(x) = c_1\cos x + c_2\sin x - \tfrac{1}{2}x\cos x$$ ■

We summarize the method of undetermined coefficients as follows:

Summary of the Method of Undetermined Coefficients

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.

SOLUTION Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$. So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A\cos 3x + B\sin 3x)$$

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1\cos 3x + c_2\sin 3x)$$

This means that we have to multiply the suggested trial solution by x. So, instead, we use

$$y_p(x) = xe^{2x}(A\cos 3x + B\sin 3x)$$ ■

The Method of Variation of Parameters

Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

4 $$y(x) = c_1y_1(x) + c_2y_2(x)$$

where y_1 and y_2 are linearly independent solutions. Let's replace the constants (or parameters) c_1 and c_2 in Equation 4 by arbitrary functions $u_1(x)$ and $u_2(x)$. We look for a particu-

lar solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

5
$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

(This method is called **variation of parameters** because we have varied the parameters c_1 and c_2 to make them functions.) Differentiating Equation 5, we get

6
$$y_p' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them. One condition is that y_p is a solution of the differential equation; we can choose the other condition so as to simplify our calculations. In view of the expression in Equation 6, let's impose the condition that

7
$$u_1' y_1 + u_2' y_2 = 0$$

Then
$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Substituting in the differential equation, we get

$$a(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + b(u_1 y_1' + u_2 y_2') + c(u_1 y_1 + u_2 y_2) = G$$

or

8
$$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1' y_1' + u_2' y_2') = G$$

But y_1 and y_2 are solutions of the complementary equation, so

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

and Equation 8 simplifies to

9
$$a(u_1' y_1' + u_2' y_2') = G$$

Equations 7 and 9 form a system of two equations in the unknown functions u_1' and u_2'. After solving this system we may be able to integrate to find u_1 and u_2 and then the particular solution is given by Equation 5.

EXAMPLE 7 Solve the equation $y'' + y = \tan x,\ 0 < x < \pi/2$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of $y'' + y = 0$ is $y(x) = c_1 \sin x + c_2 \cos x$. Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x) \sin x + u_2(x) \cos x$$

Then
$$y_p' = (u_1' \sin x + u_2' \cos x) + (u_1 \cos x - u_2 \sin x)$$

Set

10
$$u_1' \sin x + u_2' \cos x = 0$$

Then $$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For y_p to be a solution we must have

$$y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x \qquad \boxed{11}$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

$$u_1' = \sin x \qquad u_1(x) = -\cos x$$

(We seek a particular solution, so we don't need a constant of integration here.) Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x} u_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

So $$u_2(x) = \sin x - \ln(\sec x + \tan x)$$

(Note that $\sec x + \tan x > 0$ for $0 < x < \pi/2$.) Therefore

$$y_p(x) = -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x$$

$$= -\cos x \ln(\sec x + \tan x)$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$ ■

Figure 5 shows four solutions of the differential equation in Example 7.

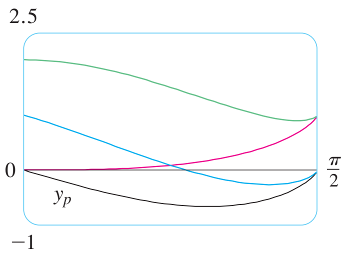


FIGURE 5

17.2 EXERCISES

1–10 Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' + 2y' - 8y = 1 - 2x^2$

2. $y'' - 3y' = \sin 2x$

3. $9y'' + y = e^{2x}$

4. $y'' - 2y' + 2y = x + e^x$

5. $y'' - 4y' + 5y = e^{-x}$

6. $y'' - 4y' + 4y = x - \sin x$

7. $y'' - 2y' + 5y = \sin x, \quad y(0) = 1, \quad y'(0) = 1$

8. $y'' - y = xe^{2x}, \quad y(0) = 0, \quad y'(0) = 1$

9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$

10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11–12 Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $y'' + 3y' + 2y = \cos x$ **12.** $y'' + 4y = e^{-x}$

13–18 Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' - y' - 2y = xe^x \cos x$

14. $y'' + 4y = \cos 4x + \cos 2x$

15. $y'' - 3y' + 2y = e^x + \sin x$

16. $y'' + 3y' - 4y = (x^3 + x)e^x$

17. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$

18. $y'' + 4y = e^{3x} + x \sin 2x$

19–22 Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

19. $4y'' + y = \cos x$

20. $y'' - 2y' - 3y = x + 2$

21. $y'' - 2y' + y = e^{2x}$

22. $y'' - y' = e^x$

23–28 Solve the differential equation using the method of variation of parameters.

23. $y'' + y = \sec^2 x,\ 0 < x < \pi/2$

24. $y'' + y = \sec^3 x,\ 0 < x < \pi/2$

25. $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

26. $y'' + 3y' + 2y = \sin(e^x)$

27. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

28. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^3}$

17.3 Applications of Second-Order Differential Equations

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

Vibrating Springs

We consider the motion of an object with mass m at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

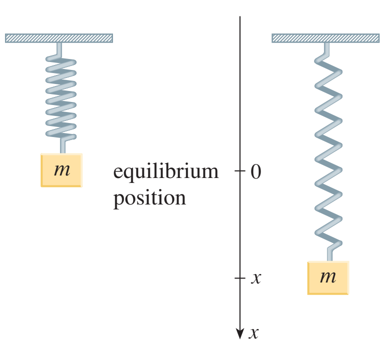


FIGURE 1

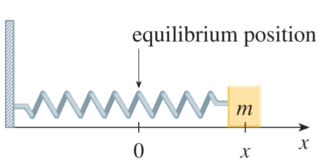


FIGURE 2

In Section 5.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{1} \qquad m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A \cos(\omega t + \delta)$$

where
$$\omega = \sqrt{k/m} \quad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \quad (\delta \text{ is the phase angle})$$

(See Exercise 17.) This type of motion is called **simple harmonic motion**.

EXAMPLE 1 A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$. Using this value of the spring constant k, together with $m = 2$ in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

2 $$x(t) = c_1 \cos 8t + c_2 \sin 8t$$

We are given the initial condition that $x(0) = 0.2$. But, from Equation 2, $x(0) = c_1$. Therefore $c_1 = 0.2$. Differentiating Equation 2, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = 0.2 \cos 8t$$ ■

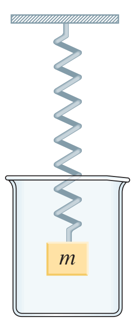


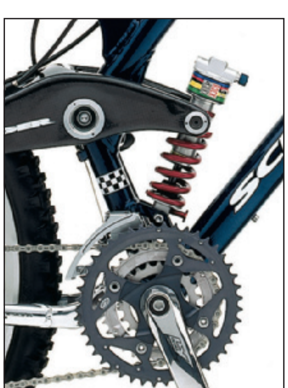
FIGURE 3

Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

$$\text{damping force} = -c\frac{dx}{dt}$$



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where c is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

3 $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Equation 3 is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$\boxed{4} \qquad r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

According to Section 17.1 we need to discuss three cases.

CASE I $c^2 - 4mk > 0$ (overdamping)
In this case r_1 and r_2 are distinct real roots and

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

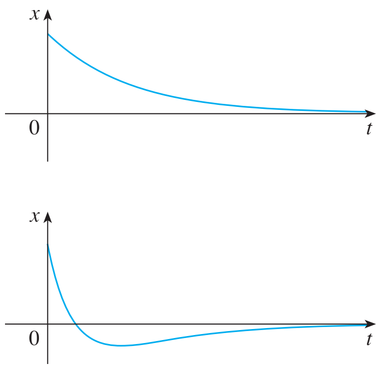


FIGURE 4
Overdamping

CASE II $c^2 - 4mk = 0$ (critical damping)
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2 t)e^{-(c/2m)t}$$

It is similar to Case I, and typical graphs resemble those in Figure 4 (see Exercise 12), but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

CASE III $c^2 - 4mk < 0$ (underdamping)
Here the roots are complex:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$

where
$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

The solution is given by

$$x = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

We see that there are oscillations that are damped by the factor $e^{-(c/2m)t}$. Since $c > 0$ and $m > 0$, we have $-(c/2m) < 0$ so $e^{-(c/2m)t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 5.

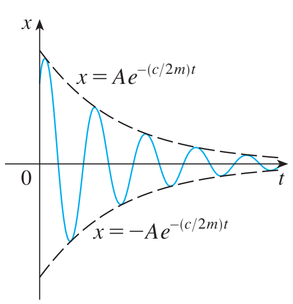


FIGURE 5
Underdamping

EXAMPLE 2 Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

SOLUTION From Example 1, the mass is $m = 2$ and the spring constant is $k = 128$, so the differential equation (3) becomes

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

or

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

The auxiliary equation is $r^2 + 20r + 64 = (r + 4)(r + 16) = 0$ with roots -4 and -16, so the motion is overdamped and the solution is

$$x(t) = c_1e^{-4t} + c_2e^{-16t}$$

Figure 6 shows the graph of the position function for the overdamped motion in Example 2.

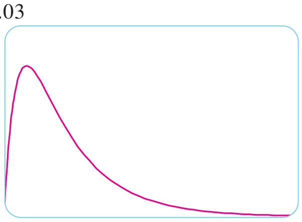


FIGURE 6

We are given that $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating, we get

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

so

$$x'(0) = -4c_1 - 16c_2 = 0.6$$

Since $c_2 = -c_1$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$. Therefore

$$x = 0.05(e^{-4t} - e^{-16t})$$

■

Forced Vibrations

Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} + \text{external force}$$

$$= -kx - c\frac{dx}{dt} + F(t)$$

Thus, instead of the homogeneous equation (3), the motion of the spring is now governed by the following nonhomogeneous differential equation:

5
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

The motion of the spring can be determined by the methods of Section 17.2.

A commonly occurring type of external force is a periodic force function

$$F(t) = F_0 \cos \omega_0 t \qquad \text{where} \quad \omega_0 \neq \omega = \sqrt{k/m}$$

In this case, and in the absence of a damping force ($c = 0$), you are asked in Exercise 9 to use the method of undetermined coefficients to show that

6 $$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

If $\omega_0 = \omega$, then the applied frequency reinforces the natural frequency and the result is vibrations of large amplitude. This is the phenomenon of **resonance** (see Exercise 10).

Electric Circuits

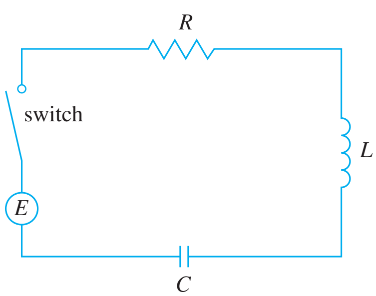


FIGURE 7

In Sections 9.3 and 9.5 we were able to use first-order separable and linear equations to analyze electric circuits that contain a resistor and inductor (see Figure 9.3.5 or Figure 9.5.4) or a resistor and capacitor (see Exercise 9.5.29). Now that we know how to solve second-order linear equations, we are in a position to analyze the circuit shown in Figure 7. It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. As in Section 9.5, it is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = dQ/dt$, this equation becomes

7 $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0$$

and the initial-value problem can be solved by the methods of Section 17.2.

A differential equation for the current can be obtained by differentiating Equation 7 with respect to t and remembering that $I = dQ/dt$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

EXAMPLE 3 Find the charge and current at time t in the circuit of Figure 7 if $R = 40\ \Omega$, $L = 1$ H, $C = 16 \times 10^{-4}$ F, $E(t) = 100\cos 10t$, and the initial charge and current are both 0.

SOLUTION With the given values of L, R, C, and $E(t)$, Equation 7 becomes

8 $$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t$$

The auxiliary equation is $r^2 + 40r + 625 = 0$ with roots

$$r = \frac{-40 \pm \sqrt{-900}}{2} = -20 \pm 15i$$

so the solution of the complementary equation is

$$Q_c(t) = e^{-20t}(c_1\cos 15t + c_2\sin 15t)$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A\cos 10t + B\sin 10t$$

Then $$Q_p'(t) = -10A\sin 10t + 10B\cos 10t$$

$$Q_p''(t) = -100A\cos 10t - 100B\sin 10t$$

Substituting into Equation 8, we have

$$(-100A\cos 10t - 100B\sin 10t) + 40(-10A\sin 10t + 10B\cos 10t) + 625(A\cos 10t + B\sin 10t) = 100\cos 10t$$

or $$(525A + 400B)\cos 10t + (-400A + 525B)\sin 10t = 100\cos 10t$$

Equating coefficients, we have

$$525A + 400B = 100 \qquad \text{or} \qquad 21A + 16B = 4$$

$$-400A + 525B = 0 \qquad \text{or} \qquad -16A + 21B = 0$$

The solution of this system is $A = \frac{84}{697}$ and $B = \frac{64}{697}$, so a particular solution is

$$Q_p(t) = \frac{1}{697}(84\cos 10t + 64\sin 10t)$$

and the general solution is

$$Q(t) = Q_c(t) + Q_p(t)$$
$$= e^{-20t}(c_1 \cos 15t + c_2 \sin 15t) + \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

Imposing the initial condition $Q(0) = 0$, we get

$$Q(0) = c_1 + \tfrac{84}{697} = 0 \qquad c_1 = -\tfrac{84}{697}$$

To impose the other initial condition, we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-20t}[(-20c_1 + 15c_2)\cos 15t + (-15c_1 - 20c_2)\sin 15t]$$
$$+ \tfrac{40}{697}(-21 \sin 10t + 16 \cos 10t)$$

$$I(0) = -20c_1 + 15c_2 + \tfrac{640}{697} = 0 \qquad c_2 = -\tfrac{464}{2091}$$

Thus the formula for the charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63 \cos 15t - 116 \sin 15t) + (21 \cos 10t + 16 \sin 10t)\right]$$

and the expression for the current is

$$I(t) = \tfrac{1}{2091}[e^{-20t}(-1920 \cos 15t + 13{,}060 \sin 15t) + 120(-21 \sin 10t + 16 \cos 10t)]$$

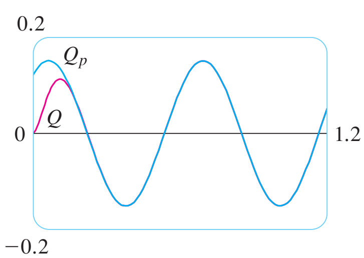


FIGURE 8

NOTE 1 In Example 3 the solution for $Q(t)$ consists of two parts. Since $e^{-20t} \to 0$ as $t \to \infty$ and both $\cos 15t$ and $\sin 15t$ are bounded functions,

$$Q_c(t) = \tfrac{4}{2091}e^{-20t}(-63 \cos 15t - 116 \sin 15t) \to 0 \qquad \text{as } t \to \infty$$

So, for large values of t,

$$Q(t) \approx Q_p(t) = \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

and, for this reason, $Q_p(t)$ is called the **steady state solution**. Figure 8 shows how the graph of the steady state solution compares with the graph of Q in this case.

5 $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

7 $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

NOTE 2 Comparing Equations 5 and 7, we see that mathematically they are identical. This suggests the analogies given in the following chart between physical situations that, at first glance, are very different.

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

We can also transfer other ideas from one situation to the other. For instance, the steady state solution discussed in Note 1 makes sense in the spring system. And the phenomenon of resonance in the spring system can be usefully carried over to electric circuits as electrical resonance.

17.3 EXERCISES

1. A spring has natural length 0.75 m and a 5-kg mass. A force of 25 N is needed to keep the spring stretched to a length of 1 m. If the spring is stretched to a length of 1.1 m and then released with velocity 0, find the position of the mass after t seconds.

2. A spring with an 8-kg mass is kept stretched 0.4 m beyond its natural length by a force of 32 N. The spring starts at its equilibrium position and is given an initial velocity of 1 m/s. Find the position of the mass at any time t.

3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

4. A force of 13 N is needed to keep a spring with a 2-kg mass stretched 0.25 m beyond its natural length. The damping constant of the spring is $c = 8$.
(a) If the mass starts at the equilibrium position with a velocity of 0.5 m/s, find its position at time t.
(b) Graph the position function of the mass.

5. For the spring in Exercise 3, find the mass that would produce critical damping.

6. For the spring in Exercise 4, find the damping constant that would produce critical damping.

7. A spring has a mass of 1 kg and its spring constant is $k = 100$. The spring is released at a point 0.1 m above its equilibrium position. Graph the position function for the following values of the damping constant c: 10, 15, 20, 25, 30. What type of damping occurs in each case?

8. A spring has a mass of 1 kg and its damping constant is $c = 10$. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?

9. Suppose a spring has mass m and spring constant k and let $\omega = \sqrt{k/m}$. Suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by Equation 6.

10. As in Exercise 9, consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

11. Show that if $\omega_0 \neq \omega$, but ω/ω_0 is a rational number, then the motion described by Equation 6 is periodic.

12. Consider a spring subject to a frictional or damping force.
(a) In the critically damped case, the motion is given by $x = c_1 e^{rt} + c_2 t e^{rt}$. Show that the graph of x crosses the t-axis whenever c_1 and c_2 have opposite signs.
(b) In the overdamped case, the motion is given by $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1 > r_2$. Determine a condition on the relative magnitudes of c_1 and c_2 under which the graph of x crosses the t-axis at a positive value of t.

13. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1$ H, a capacitor with $C = 0.002$ F, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

14. A series circuit contains a resistor with $R = 24\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.005$ F, and a 12-V battery. The initial charge is $Q = 0.001$ C and the initial current is 0.
(a) Find the charge and current at time t.
(b) Graph the charge and current functions.

15. The battery in Exercise 13 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$. Find the charge at time t.

16. The battery in Exercise 14 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$.
(a) Find the charge at time t.
(b) Graph the charge function.

17. Verify that the solution to Equation 1 can be written in the form $x(t) = A \cos(\omega t + \delta)$.

18. The figure shows a pendulum with length L and the angle θ from the vertical to the pendulum. It can be shown that θ, as a function of time, satisfies the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration due to gravity. For small values of θ we can use the linear approximation $\sin\theta \approx \theta$ and then the differential equation becomes linear.

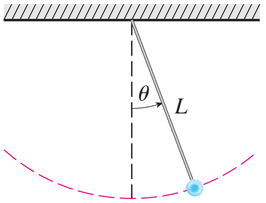
(a) Find the equation of motion of a pendulum with length 1 m if θ is initially 0.2 rad and the initial angular velocity is $d\theta/dt = 1$ rad/s.

(b) What is the maximum angle from the vertical?

(c) What is the period of the pendulum (that is, the time to complete one back-and-forth swing)?

(d) When will the pendulum first be vertical?

(e) What is the angular velocity when the pendulum is vertical?



17.4 Series Solutions

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

$$\boxed{1} \qquad y'' - 2xy' + y = 0$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$. This technique resembles the method of undetermined coefficients discussed in Section 17.2.

Before using power series to solve Equation 1, we illustrate the method on the simpler equation $y'' + y = 0$ in Example 1. It's true that we already know how to solve this equation by the techniques of Section 17.1, but it's easier to understand the power series method when it is applied to this simpler equation.

EXAMPLE 1 Use power series to solve the equation $y'' + y = 0$.

SOLUTION We assume there is a solution of the form

$$\boxed{2} \qquad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\boxed{3} \qquad y'' = 2c_2 + 2 \cdot 3c_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

By writing out the first few terms of (4), you can see that it is the same as (3). To obtain (4), we replaced n by $n + 2$ and began the summation at 0 instead of 2.

$$y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n \tag{4}$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} + c_n]x^n = 0 \tag{5}$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coefficients of x^n in Equation 5 must be 0:

$$(n + 2)(n + 1)c_{n+2} + c_n = 0$$

$$c_{n+2} = -\frac{c_n}{(n + 1)(n + 2)} \qquad n = 0, 1, 2, 3, \ldots \tag{6}$$

Equation 6 is called a *recursion relation.* If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

Put $n = 0$: $\quad c_2 = -\dfrac{c_0}{1 \cdot 2}$

Put $n = 1$: $\quad c_3 = -\dfrac{c_1}{2 \cdot 3}$

Put $n = 2$: $\quad c_4 = -\dfrac{c_2}{3 \cdot 4} = \dfrac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \dfrac{c_0}{4!}$

Put $n = 3$: $\quad c_5 = -\dfrac{c_3}{4 \cdot 5} = \dfrac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \dfrac{c_1}{5!}$

Put $n = 4$: $\quad c_6 = -\dfrac{c_4}{5 \cdot 6} = -\dfrac{c_0}{4!\, 5 \cdot 6} = -\dfrac{c_0}{6!}$

Put $n = 5$: $\quad c_7 = -\dfrac{c_5}{6 \cdot 7} = -\dfrac{c_1}{5!\, 6 \cdot 7} = -\dfrac{c_1}{7!}$

By now we see the pattern:

For the even coefficients, $\quad c_{2n} = (-1)^n \dfrac{c_0}{(2n)!}$

For the odd coefficients, $\quad c_{2n+1} = (-1)^n \dfrac{c_1}{(2n + 1)!}$

Putting these values back into Equation 2, we write the solution as

$$\begin{aligned} y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots \\ &= c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right) \\ &\qquad + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right) \\ &= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Notice that there are two arbitrary constants, c_0 and c_1. ■

NOTE 1 We recognize the series obtained in Example 1 as being the Maclaurin series for $\cos x$ and $\sin x$. (See Equations 11.10.16 and 11.10.15.) Therefore we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions.

EXAMPLE 2 Solve $y'' - 2xy' + y = 0$.

SOLUTION We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then $$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

and $$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

as in Example 1. Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$\sum_{n=1}^{\infty} 2nc_n x^n = \sum_{n=0}^{\infty} 2nc_n x^n$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-1)c_n]x^n = 0$$

This equation is true if the coefficients of x^n are 0:

$$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$$

7 $$c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation 7:

$$\text{Put } n = 0: \qquad c_2 = \frac{-1}{1 \cdot 2} c_0$$

$$\text{Put } n = 1: \qquad c_3 = \frac{1}{2 \cdot 3} c_1$$

$$\text{Put } n = 2: \qquad c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0$$

$$\text{Put } n = 3: \qquad c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1$$

$$\text{Put } n = 4: \qquad c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4!\, 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0$$

$$\text{Put } n = 5: \qquad c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5!\, 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1$$

$$\text{Put } n = 6: \qquad c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$$

$$\text{Put } n = 7: \qquad c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$$

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n - 5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} c_1$$

The solution is

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\ &= c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3 \cdot 7}{6!}x^6 - \frac{3 \cdot 7 \cdot 11}{8!}x^8 - \cdots\right) \\ &\quad + c_1\left(x + \frac{1}{3!}x^3 + \frac{1 \cdot 5}{5!}x^5 + \frac{1 \cdot 5 \cdot 9}{7!}x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!}x^9 + \cdots\right) \end{aligned}$$

or

8
$$\begin{aligned} y &= c_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{(2n)!} x^{2n}\right) \\ &\quad + c_1\left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} x^{2n+1}\right) \end{aligned}$$

■

NOTE 2 In Example 2 we had to *assume* that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

NOTE 3 Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions. The functions

$$y_1(x) = 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!} x^{2n}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

are perfectly good functions but they can't be expressed in terms of familiar functions. We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums $T_0, T_2, T_4, \ldots$ (Taylor polynomials) for $y_1(x)$, and we see how they converge to y_1. In this way we can graph both y_1 and y_2 as in Figure 2.

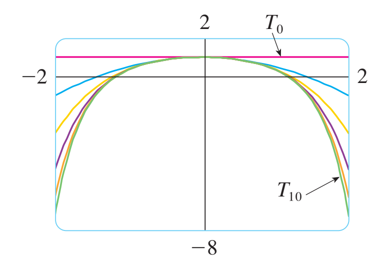


FIGURE 1

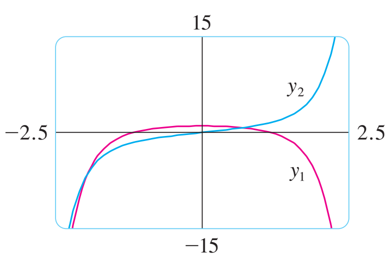


FIGURE 2

NOTE 4 If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

we would observe from Theorem 11.10.5 that

$$c_0 = y(0) = 0 \qquad c_1 = y'(0) = 1$$

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

17.4 EXERCISES

1–11 Use power series to solve the differential equation.

1. $y' - y = 0$

2. $y' = xy$

3. $y' = x^2 y$

4. $(x-3)y' + 2y = 0$

5. $y'' + xy' + y = 0$

6. $y'' = y$

7. $(x-1)y'' + y' = 0$

8. $y'' = xy$

9. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

10. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$

11. $y'' + x^2 y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

12. The solution of the initial-value problem

$$x^2 y'' + xy' + x^2 y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

is called a Bessel function of order 0.

(a) Solve the initial-value problem to find a power series expansion for the Bessel function.

(b) Graph several Taylor polynomials until you reach one that looks like a good approximation to the Bessel function on the interval $[-5, 5]$.

17 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. (a) Write the general form of a second-order homogeneous linear differential equation with constant coefficients.
(b) Write the auxiliary equation.
(c) How do you use the roots of the auxiliary equation to solve the differential equation? Write the form of the solution for each of the three cases that can occur.

2. (a) What is an initial-value problem for a second-order differential equation?
(b) What is a boundary-value problem for such an equation?

3. (a) Write the general form of a second-order nonhomogeneous linear differential equation with constant coefficients.
(b) What is the complementary equation? How does it help solve the original differential equation?
(c) Explain how the method of undetermined coefficients works.
(d) Explain how the method of variation of parameters works.

4. Discuss two applications of second-order linear differential equations.

5. How do you use power series to solve a differential equation?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If y_1 and y_2 are solutions of $y'' + y = 0$, then $y_1 + y_2$ is also a solution of the equation.

2. If y_1 and y_2 are solutions of $y'' + 6y' + 5y = x$, then $c_1 y_1 + c_2 y_2$ is also a solution of the equation.

3. The general solution of $y'' - y = 0$ can be written as

$$y = c_1 \cosh x + c_2 \sinh x$$

4. The equation $y'' - y = e^x$ has a particular solution of the form

$$y_p = Ae^x$$

EXERCISES

1–10 Solve the differential equation.

1. $4y'' - y = 0$

2. $y'' - 2y' + 10y = 0$

3. $y'' + 3y = 0$

4. $y'' + 8y' + 16y = 0$

5. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = e^{2x}$

6. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x^2$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x\cos x$

8. $\dfrac{d^2y}{dx^2} + 4y = \sin 2x$

9. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 1 + e^{-2x}$

10. $\dfrac{d^2y}{dx^2} + y = \csc x, \quad 0 < x < \pi/2$

11–14 Solve the initial-value problem.

11. $y'' + 6y' = 0, \quad y(1) = 3, \quad y'(1) = 12$

12. $y'' - 6y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = 1$

13. $y'' - 5y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

14. $9y'' + y = 3x + e^{-x}, \quad y(0) = 1, \quad y'(0) = 2$

15–16 Solve the boundary-value problem, if possible.

15. $y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -1$

16. $y'' + 4y' + 29y = 0, \quad y(0) = 1, \quad y(\pi) = -e^{-2\pi}$

17. Use power series to solve the initial-value problem

$$y'' + xy' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

18. Use power series to solve the differential equation

$$y'' - xy' - 2y = 0$$

19. A series circuit contains a resistor with $R = 40\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.0025$ F, and a 12-V battery. The initial charge is $Q = 0.01$ C and the initial current is 0. Find the charge at time t.

20. A spring with a mass of 2 kg has damping constant 16, and a force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2.4 m/s.

21. Assume that the earth is a solid sphere of uniform density with mass M and radius $R = 3960$ mi. For a particle of mass m within the earth at a distance r from the earth's center, the gravitational force attracting the particle to the center is

$$F_r = \frac{-GM_r m}{r^2}$$

where G is the gravitational constant and M_r is the mass of the earth within the sphere of radius r.

(a) Show that $F_r = \dfrac{-GMm}{R^3}r$.

(b) Suppose a hole is drilled through the earth along a diameter. Show that if a particle of mass m is dropped from rest at the surface, into the hole, then the distance $y = y(t)$ of the particle from the center of the earth at time t is given by

$$y''(t) = -k^2 y(t)$$

where $k^2 = GM/R^3 = g/R$.

(c) Conclude from part (b) that the particle undergoes simple harmonic motion. Find the period T.

(d) With what speed does the particle pass through the center of the earth?

Appendixes

A Numbers, Inequalities, and Absolute Values

B Coordinate Geometry and Lines

C Graphs of Second-Degree Equations

D Trigonometry

E Sigma Notation

F Proofs of Theorems

G Complex Numbers

H Answers to Odd-Numbered Exercises

A Numbers, Inequalities, and Absolute Values

Calculus is based on the real number system. We start with the **integers**:

$$\ldots, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Then we construct the **rational numbers**, which are ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n} \qquad \text{where } m \text{ and } n \text{ are integers and } n \neq 0$$

Examples are

$$\tfrac{1}{2} \qquad -\tfrac{3}{7} \qquad 46 = \tfrac{46}{1} \qquad 0.17 = \tfrac{17}{100}$$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) Some real numbers, such as $\sqrt{2}$, can't be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that the following are also irrational numbers:

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt[3]{2} \qquad \pi \qquad \sin 1^\circ \qquad \log_{10} 2$$

The set of all real numbers is usually denoted by the symbol $\mathbb{R}$. When we use the word *number* without qualification, we mean "real number."

Every number has a decimal representation. If the number is rational, then the corresponding decimal is repeating. For example,

$$\tfrac{1}{2} = 0.5000\ldots = 0.5\overline{0} \qquad \tfrac{2}{3} = 0.66666\ldots = 0.\overline{6}$$

$$\tfrac{157}{495} = 0.317171717\ldots = 0.3\overline{17} \qquad \tfrac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$$

(The bar indicates that the sequence of digits repeats forever.) On the other hand, if the number is irrational, the decimal is nonrepeating:

$$\sqrt{2} = 1.414213562373095\ldots \qquad \pi = 3.141592653589793\ldots$$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol $\approx$ is read "is approximately equal to." The more decimal places we retain, the better the approximation we get.

The real numbers can be represented by points on a line as in Figure 1. The positive direction (to the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number $-x$ is represented by the point x units to the left of the origin. Thus every real number is represented by a point on the line, and every point P on the line corresponds to exactly one real number. The number associated with the point P is called the **coordinate** of P and the line is then called a

coordinate line, or a **real number line**, or simply a **real line**. Often we identify the point with its coordinate and think of a number as being a point on the real line.

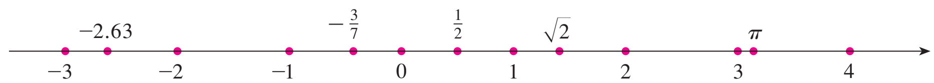


FIGURE 1

The real numbers are ordered. We say a *is less than* b and write $a < b$ if $b - a$ is a positive number. Geometrically this means that a lies to the left of b on the number line. (Equivalently, we say b *is greater than* a and write $b > a$.) The symbol $a \leqslant b$ (or $b \geqslant a$) means that either $a < b$ or $a = b$ and is read "a is less than or equal to b." For instance, the following are true inequalities:

$$7 < 7.4 < 7.5 \qquad -3 > -\pi \qquad \sqrt{2} < 2 \qquad \sqrt{2} \leqslant 2 \qquad 2 \leqslant 2$$

In what follows we need to use *set notation*. A **set** is a collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$. If S and T are sets, then their **union** $S \cup T$ is the set consisting of all elements that are in S or T (or in both S and T). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words, $S \cap T$ is the common part of S and T. The empty set, denoted by $\varnothing$, is the set that contains no element.

Some sets can be described by listing their elements between braces. For instance, the set A consisting of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in *set-builder notation* as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of x such that x is an integer and $0 < x < 7$."

Intervals

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$



FIGURE 2
Open interval (a, b)

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the round brackets ( ) and by the open dots in Figure 2. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$



FIGURE 3
Closed interval $[a, b]$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in Figure 3. It is also possible to include only one endpoint in an interval, as shown in Table 1.

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

This does not mean that ∞ ("infinity") is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

Table 1 lists the nine possible types of intervals. When these intervals are discussed, it is always assumed that $a < b$.

1 Table of Intervals

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Inequalities

When working with inequalities, note the following rules.

2 Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 1 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\frac{4}{6} = -\frac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 2 Solve the inequalities $4 \leqslant 3x - 2 < 13$.

SOLUTION Here the solution set consists of all values of x that satisfy both inequalities. Using the rules given in (2), we see that the following inequalities are equivalent:

$$4 \leqslant 3x - 2 < 13$$

$$6 \leqslant 3x < 15 \qquad \text{(add 2)}$$

$$2 \leqslant x < 5 \qquad \text{(divide by 3)}$$

Therefore the solution set is $[2, 5)$. ■

EXAMPLE 3 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

A visual method for solving Example 3 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 4) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

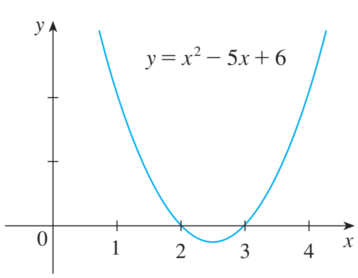


FIGURE 4

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

Another method for obtaining the information in the chart is to use *test values*. For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

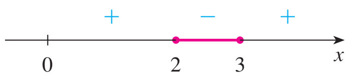


FIGURE 5

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 5. ■

EXAMPLE 4 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x - 1)(x + 4) > 0$$

As in Example 3 we solve the corresponding equation $x(x - 1)(x + 4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign, which we list in the following chart:

Interval	x	$x - 1$	$x + 4$	$x(x - 1)(x + 4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$



FIGURE 6

The solution is illustrated in Figure 6. ■

■ Absolute Value

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0 \qquad |\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

Remember that if a is negative, then $-a$ is positive.

3

$$|a| = a \qquad \text{if } a \geqslant 0$$
$$|a| = -a \qquad \text{if } a < 0$$

EXAMPLE 5 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$ ■

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (3), we then have the equation

4 $$\sqrt{a^2} = |a|$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

5 Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ $(b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

6 Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

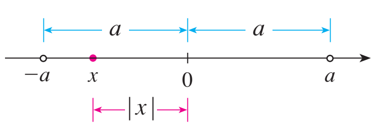


FIGURE 7

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 7 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 8.)

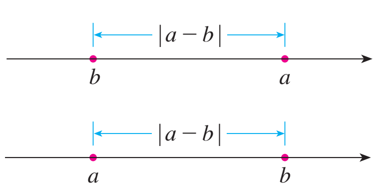


FIGURE 8
Length of a line segment $= |a - b|$

EXAMPLE 6 Solve $|2x - 5| = 3$.

SOLUTION By Property 4 of (6), $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus $x = 4$ or $x = 1$. ■

EXAMPLE 7 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of (6), $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

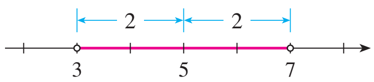


FIGURE 9

SOLUTION 2 Geometrically the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 9 we see that this is the interval $(3, 7)$. ■

EXAMPLE 8 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of (6), $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

Another important property of absolute value, called the Triangle Inequality, is used frequently not only in calculus but throughout mathematics in general.

7 The Triangle Inequality If a and b are any real numbers, then

$$|a + b| \leqslant |a| + |b|$$

Observe that if the numbers a and b are both positive or both negative, then the two sides in the Triangle Inequality are actually equal. But if a and b have opposite signs, the left side involves a subtraction and the right side does not. This makes the Triangle Inequality seem reasonable, but we can prove it as follows.

Notice that

$$-|a| \leqslant a \leqslant |a|$$

is always true because a equals either $|a|$ or $-|a|$. The corresponding statement for b is

$$-|b| \leqslant b \leqslant |b|$$

Adding these inequalities, we get

$$-(|a| + |b|) \leqslant a + b \leqslant |a| + |b|$$

If we now apply Properties 4 and 5 (with x replaced by $a + b$ and a by $|a| + |b|$), we obtain

$$|a + b| \leqslant |a| + |b|$$

which is what we wanted to show.

EXAMPLE 9 If $|x - 4| < 0.1$ and $|y - 7| < 0.2$, use the Triangle Inequality to estimate $|(x + y) - 11|$.

SOLUTION In order to use the given information, we use the Triangle Inequality with $a = x - 4$ and $b = y - 7$:

$$\begin{aligned} |(x + y) - 11| &= |(x - 4) + (y - 7)| \\ &\leqslant |x - 4| + |y - 7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus $$|(x + y) - 11| < 0.3$$ ■

A EXERCISES

1–12 Rewrite the expression without using the absolute-value symbol.

1. $|5 - 23|$

2. $|5| - |-23|$

3. $|-\pi|$

4. $|\pi - 2|$

5. $|\sqrt{5} - 5|$

6. $||-2| - |-3||$

7. $|x - 2|$ if $x < 2$

8. $|x - 2|$ if $x > 2$

9. $|x + 1|$

10. $|2x - 1|$

11. $|x^2 + 1|$

12. $|1 - 2x^2|$

13–38 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

13. $2x + 7 > 3$

14. $3x - 11 < 4$

15. $1 - x \leqslant 2$

16. $4 - 3x \geqslant 6$

17. $2x + 1 < 5x - 8$

18. $1 + 5x > 5 - 3x$

19. $-1 < 2x - 5 < 7$

20. $1 < 3x + 4 \leqslant 16$

21. $0 \leqslant 1 - x < 1$

22. $-5 \leqslant 3 - 2x \leqslant 9$

23. $4x < 2x + 1 \leqslant 3x + 2$

24. $2x - 3 < x + 4 < 3x - 2$

25. $(x - 1)(x - 2) > 0$

26. $(2x + 3)(x - 1) \geqslant 0$

27. $2x^2 + x \leqslant 1$

28. $x^2 < 2x + 8$

29. $x^2 + x + 1 > 0$

30. $x^2 + x > 1$

31. $x^2 < 3$

32. $x^2 \geqslant 5$

33. $x^3 - x^2 \leqslant 0$

34. $(x + 1)(x - 2)(x + 3) \geqslant 0$

35. $x^3 > x$

36. $x^3 + 3x < 4x^2$

37. $\dfrac{1}{x} < 4$

38. $-3 < \dfrac{1}{x} \leqslant 1$

39. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

40. Use the relationship between C and F given in Exercise 39 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

41. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20°C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

42. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

43–46 Solve the equation for x.

43. $|2x| = 3$

44. $|3x + 5| = 1$

45. $|x + 3| = |2x + 1|$

46. $\left|\dfrac{2x - 1}{x + 1}\right| = 3$

47–56 Solve the inequality.

47. $|x| < 3$

48. $|x| \geqslant 3$

49. $|x - 4| < 1$

50. $|x - 6| < 0.1$

51. $|x + 5| \geqslant 2$

52. $|x + 1| \geqslant 3$

53. $|2x - 3| \leqslant 0.4$

54. $|5x - 2| < 6$

55. $1 \leqslant |x| \leqslant 4$

56. $0 < |x - 5| < \frac{1}{2}$

57–58 Solve for x, assuming a, b, and c are positive constants.

57. $a(bx - c) \geqslant bc$

58. $a \leqslant bx + c < 2a$

59–60 Solve for x, assuming a, b, and c are negative constants.

59. $ax + b < c$

60. $\dfrac{ax + b}{c} \leqslant b$

61. Suppose that $|x - 2| < 0.01$ and $|y - 3| < 0.04$. Use the Triangle Inequality to show that $|(x + y) - 5| < 0.05$.

62. Show that if $|x + 3| < \frac{1}{2}$, then $|4x + 13| < 3$.

63. Show that if $a < b$, then $a < \dfrac{a + b}{2} < b$.

64. Use Rule 3 to prove Rule 5 of (2).

65. Prove that $|ab| = |a||b|$. [*Hint:* Use Equation 4.]

66. Prove that $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.

67. Show that if $0 < a < b$, then $a^2 < b^2$.

68. Prove that $|x - y| \geqslant |x| - |y|$. [*Hint:* Use the Triangle Inequality with $a = x - y$ and $b = y$.]

69. Show that the sum, difference, and product of rational numbers are rational numbers.

70. (a) Is the sum of two irrational numbers always an irrational number?
(b) Is the product of two irrational numbers always an irrational number?

B Coordinate Geometry and Lines

Just as the points on a line can be identified with real numbers by assigning them coordinates, as described in Appendix A, so the points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axis.

Any point P in the plane can be located by a unique ordered pair of numbers as follows. Draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates a and b as shown in Figure 1. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** of P; the second number b is called the ***y*-coordinate** of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P(a, b)$. Several points are labeled with their coordinates in Figure 2.

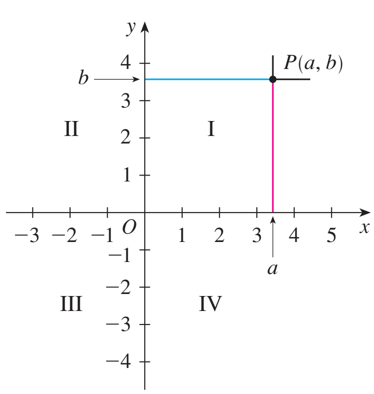


FIGURE 1

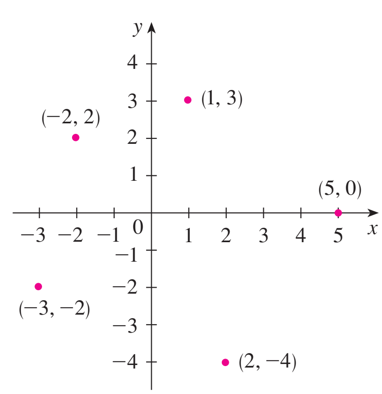


FIGURE 2

By reversing the preceding process we can start with an ordered pair (a, b) and arrive at the corresponding point P. Often we identify the point P with the ordered pair (a, b) and refer to "the point (a, b)." [Although the notation used for an open interval (a, b) is

the same as the notation used for a point (a, b), you will be able to tell from the context which meaning is intended.]

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** in honor of the French mathematician René Descartes (1596–1650), even though another Frenchman, Pierre Fermat (1601–1665), invented the principles of analytic geometry at about the same time as Descartes. The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane** and is denoted by $\mathbb{R}^2$.

The x- and y-axes are called the **coordinate axes** and divide the Cartesian plane into four quadrants, which are labeled I, II, III, and IV in Figure 1. Notice that the first quadrant consists of those points whose x- and y-coordinates are both positive.

EXAMPLE 1 Describe and sketch the regions given by the following sets.

(a) $\{(x, y) \mid x \geqslant 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

SOLUTION

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 3(a).

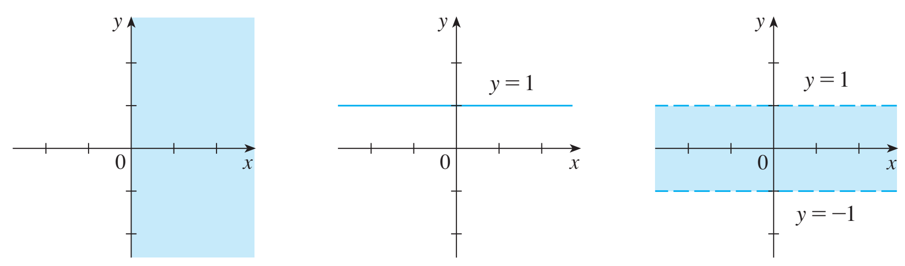


FIGURE 3 (a) $x \geqslant 0$ (b) $y = 1$ (c) $|y| < 1$

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 3(b)].

(c) Recall from Appendix A that

$$|y| < 1 \qquad \text{if and only if} \qquad -1 < y < 1$$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 3(c) to indicate that the points on these lines don't lie in the set.] ■

Recall from Appendix A that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus the distance between points $P_1(x_1, y_1)$ and $P_3(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $P_2(x_2, y_2)$ and $P_3(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.)

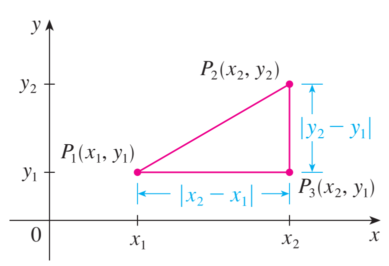


FIGURE 4

To find the distance $|P_1P_2|$ between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 4 is a right triangle, and so by the Pythagorean Theorem we have

$$|P_1P_2| = \sqrt{|P_1P_3|^2 + |P_2P_3|^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1 Distance Formula The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 2 The distance between $(1, -2)$ and $(5, 3)$ is

$$\sqrt{(5 - 1)^2 + [3 - (-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

Lines

We want to find an equation of a given line L; such an equation is satisfied by the coordinates of the points on L and by no other point. To find the equation of L we use its *slope,* which is a measure of the steepness of the line.

2 Definition The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

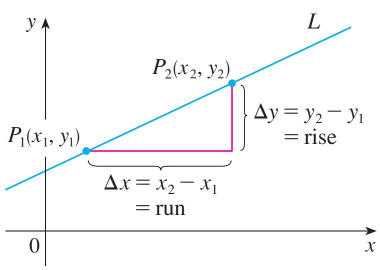


FIGURE 5

Thus the slope of a line is the ratio of the change in y, Δy, to the change in x, Δx. (See Figure 5.) The slope is therefore the rate of change of y with respect to x. The fact that the line is straight means that the rate of change is constant.

Figure 6 shows several lines labeled with their slopes. Notice that lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice also that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0.

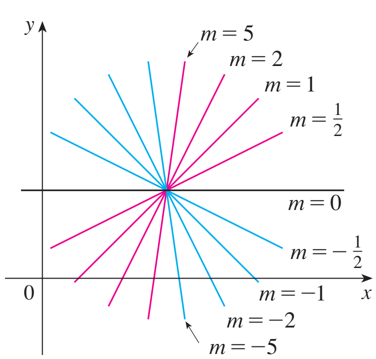


FIGURE 6

Now let's find an equation of the line that passes through a given point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is,

$$\frac{y - y_1}{x - x_1} = m$$

This equation can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

3 Point-Slope Form of the Equation of a Line An equation of the line passing through the point $P_1(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 3 Find an equation of the line through $(1, -7)$ with slope $-\frac{1}{2}$.

SOLUTION Using (3) with $m = -\frac{1}{2}$, $x_1 = 1$, and $y_1 = -7$, we obtain an equation of the line as

$$y + 7 = -\tfrac{1}{2}(x - 1)$$

which we can rewrite as

$$2y + 14 = -x + 1 \qquad \text{or} \qquad x + 2y + 13 = 0$$

■

EXAMPLE 4 Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$.

SOLUTION By Definition 2 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which simplifies to $$3x + 2y = 1$$ ■

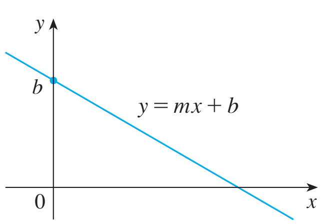


FIGURE 7

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 7.) This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = b$, becomes

$$y - b = m(x - 0)$$

This simplifies as follows.

> **4 Slope-Intercept Form of the Equation of a Line** An equation of the line with slope m and y-intercept b is
>
> $$y = mx + b$$

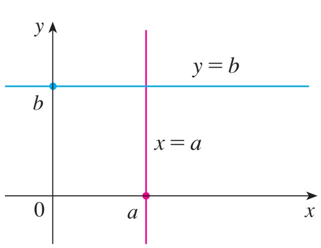


FIGURE 8

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 8). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

Observe that the equation of every line can be written in the form

5 $$Ax + By + C = 0$$

because a vertical line has the equation $x = a$ or $x - a = 0$ $(A = 1, B = 0, C = -a)$ and a nonvertical line has the equation $y = mx + b$ or $-mx + y - b = 0$ $(A = -m, B = 1, C = -b)$. Conversely, if we start with a general first-degree equation, that is, an equation of the form (5), where A, B, and C are constants and A and B are not both 0, then we can show that it is the equation of a line. If $B = 0$, the equation becomes $Ax + C = 0$ or $x = -C/A$, which represents a vertical line with x-intercept $-C/A$. If $B \neq 0$, the

equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and we recognize this as being the slope-intercept form of the equation of a line ($m = -A/B$, $b = -C/B$). Therefore an equation of the form (5) is called a **linear equation** or the **general equation of a line**. For brevity, we often refer to "the line $Ax + By + C = 0$" instead of "the line whose equation is $Ax + By + C = 0$."

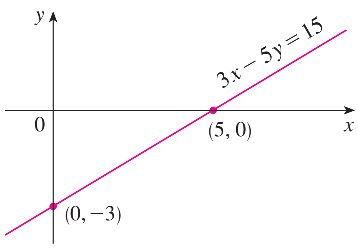


FIGURE 9

EXAMPLE 5 Sketch the graph of the equation $3x - 5y = 15$.

SOLUTION Since the equation is linear, its graph is a line. To draw the graph, we can simply find two points on the line. It's easiest to find the intercepts. Substituting $y = 0$ (the equation of the x-axis) in the given equation, we get $3x = 15$, so $x = 5$ is the x-intercept. Substituting $x = 0$ in the equation, we see that the y-intercept is -3. This allows us to sketch the graph as in Figure 9. ■

EXAMPLE 6 Graph the inequality $x + 2y > 5$.

SOLUTION We are asked to sketch the graph of the set $\{(x, y) \mid x + 2y > 5\}$ and we begin by solving the inequality for y:

$$x + 2y > 5$$
$$2y > -x + 5$$
$$y > -\tfrac{1}{2}x + \tfrac{5}{2}$$

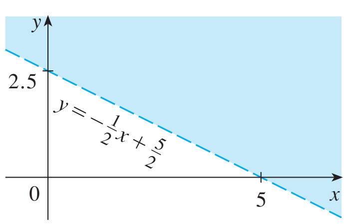


FIGURE 10

Compare this inequality with the equation $y = -\frac{1}{2}x + \frac{5}{2}$, which represents a line with slope $-\frac{1}{2}$ and y-intercept $\frac{5}{2}$. We see that the given graph consists of points whose y-coordinates are *larger* than those on the line $y = -\frac{1}{2}x + \frac{5}{2}$. Thus the graph is the region that lies *above* the line, as illustrated in Figure 10. ■

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved, for instance, in *Precalculus: Mathematics for Calculus,* Seventh Edition, by Stewart, Redlin, and Watson (Belmont, CA, 2016).

6 Parallel and Perpendicular Lines

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

EXAMPLE 7 Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

SOLUTION The given line can be written in the form

$$y = -\tfrac{2}{3}x - \tfrac{5}{6}$$

which is in slope-intercept form with $m = -\frac{2}{3}$. Parallel lines have the same slope, so the required line has slope $-\frac{2}{3}$ and its equation in point-slope form is

$$y - 2 = -\tfrac{2}{3}(x - 5)$$

We can write this equation as $2x + 3y = 16$. ■

EXAMPLE 8 Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

SOLUTION The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular. ■

B EXERCISES

1–6 Find the distance between the points.

1. $(1, 1)$, $(4, 5)$

2. $(1, -3)$, $(5, 7)$

3. $(6, -2)$, $(-1, 3)$

4. $(1, -6)$, $(-1, -3)$

5. $(2, 5)$, $(4, -7)$

6. (a, b), (b, a)

7–10 Find the slope of the line through P and Q.

7. $P(1, 5)$, $Q(4, 11)$

8. $P(-1, 6)$, $Q(4, -3)$

9. $P(-3, 3)$, $Q(-1, -6)$

10. $P(-1, -4)$, $Q(6, 0)$

11. Show that the triangle with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-4, 3)$ is isosceles.

12. (a) Show that the triangle with vertices $A(6, -7)$, $B(11, -3)$, and $C(2, -2)$ is a right triangle using the converse of the Pythagorean Theorem.
(b) Use slopes to show that ABC is a right triangle.
(c) Find the area of the triangle.

13. Show that the points $(-2, 9)$, $(4, 6)$, $(1, 0)$, and $(-5, 3)$ are the vertices of a square.

14. (a) Show that the points $A(-1, 3)$, $B(3, 11)$, and $C(5, 15)$ are collinear (lie on the same line) by showing that $|AB| + |BC| = |AC|$.
(b) Use slopes to show that A, B, and C are collinear.

15. Show that $A(1, 1)$, $B(7, 4)$, $C(5, 10)$, and $D(-1, 7)$ are vertices of a parallelogram.

16. Show that $A(1, 1)$, $B(11, 3)$, $C(10, 8)$, and $D(0, 6)$ are vertices of a rectangle.

17–20 Sketch the graph of the equation.

17. $x = 3$

18. $y = -2$

19. $xy = 0$

20. $|y| = 1$

21–36 Find an equation of the line that satisfies the given conditions.

21. Through $(2, -3)$, slope 6

22. Through $(-1, 4)$, slope -3

23. Through $(1, 7)$, slope $\frac{2}{3}$

24. Through $(-3, -5)$, slope $-\frac{7}{2}$

25. Through $(2, 1)$ and $(1, 6)$

26. Through $(-1, -2)$ and $(4, 3)$

27. Slope 3, y-intercept -2

28. Slope $\frac{2}{5}$, y-intercept 4

29. x-intercept 1, y-intercept -3

30. x-intercept -8, y-intercept 6

31. Through $(4, 5)$, parallel to the x-axis

32. Through $(4, 5)$, parallel to the y-axis

33. Through $(1, -6)$, parallel to the line $x + 2y = 6$

34. y-intercept 6, parallel to the line $2x + 3y + 4 = 0$

35. Through $(-1, -2)$, perpendicular to the line $2x + 5y + 8 = 0$

36. Through $\left(\frac{1}{2}, -\frac{2}{3}\right)$, perpendicular to the line $4x - 8y = 1$

37–42 Find the slope and y-intercept of the line and draw its graph.

37. $x + 3y = 0$

38. $2x - 5y = 0$

39. $y = -2$

40. $2x - 3y + 6 = 0$

41. $3x - 4y = 12$

42. $4x + 5y = 10$

43–52 Sketch the region in the xy-plane.

43. $\{(x, y) \mid x < 0\}$

44. $\{(x, y) \mid y > 0\}$

45. $\{(x, y) \mid xy < 0\}$

46. $\{(x, y) \mid x \geqslant 1 \text{ and } y < 3\}$

47. $\{(x, y) \mid |x| \leqslant 2\}$

48. $\{(x, y) \mid |x| < 3 \text{ and } |y| < 2\}$

49. $\{(x, y) \mid 0 \leqslant y \leqslant 4 \text{ and } x \leqslant 2\}$

50. $\{(x, y) \mid y > 2x - 1\}$

51. $\{(x, y) \mid 1 + x \leqslant y \leqslant 1 - 2x\}$

52. $\{(x, y) \mid -x \leqslant y < \frac{1}{2}(x + 3)\}$

53. Find a point on the y-axis that is equidistant from $(5, -5)$ and $(1, 1)$.

54. Show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

55. Find the midpoint of the line segment joining the given points.
(a) $(1, 3)$ and $(7, 15)$
(b) $(-1, 6)$ and $(8, -12)$

56. Find the lengths of the medians of the triangle with vertices $A(1, 0)$, $B(3, 6)$, and $C(8, 2)$. (A median is a line segment from a vertex to the midpoint of the opposite side.)

57. Show that the lines $2x - y = 4$ and $6x - 2y = 10$ are not parallel and find their point of intersection.

58. Show that the lines $3x - 5y + 19 = 0$ and $10x + 6y - 50 = 0$ are perpendicular and find their point of intersection.

59. Find an equation of the perpendicular bisector of the line segment joining the points $A(1, 4)$ and $B(7, -2)$.

60. (a) Find equations for the sides of the triangle with vertices $P(1, 0)$, $Q(3, 4)$, and $R(-1, 6)$.
(b) Find equations for the medians of this triangle. Where do they intersect?

61. (a) Show that if the x- and y-intercepts of a line are nonzero numbers a and b, then the equation of the line can be put in the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is called the **two-intercept form** of an equation of a line.
(b) Use part (a) to find an equation of the line whose x-intercept is 6 and whose y-intercept is -8.

62. A car leaves Detroit at 2:00 PM, traveling at a constant speed west along I-96. It passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

C Graphs of Second-Degree Equations

In Appendix B we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

The graph of such an equation in x and y is the set of all points (x, y) that satisfy the equation; it gives a visual representation of the equation. Conversely, given a curve in the xy-plane, we may have to find an equation that represents it, that is, an equation satisfied by the coordinates of the points on the curve and by no other point. This is the other half of the basic principle of analytic geometry as formulated by Descartes and Fermat. The idea is that if a geometric curve can be represented by an algebraic equation, then the rules of algebra can be used to analyze the geometric problem.

Circles

As an example of this type of problem, let's find an equation of the circle with radius r and center (h, k). By definition, the circle is the set of all points $P(x, y)$ whose distance

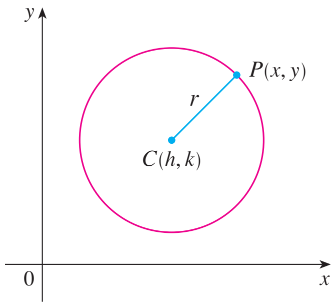


FIGURE 1

from the center $C(h, k)$ is r. (See Figure 1.) Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the desired equation.

1 Equation of a Circle An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

SOLUTION From Equation 1 with $r = 3$, $h = 2$, and $k = -5$, we obtain

$$(x - 2)^2 + (y + 5)^2 = 9$$ ■

EXAMPLE 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants (the squares of half the coefficients of x and y) to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or
$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle (1), we see that $h = -1$, $k = 3$, and $r = \sqrt{3}$, so the given equation represents a circle with center $(-1, 3)$ and radius $\sqrt{3}$. It is sketched in Figure 2.

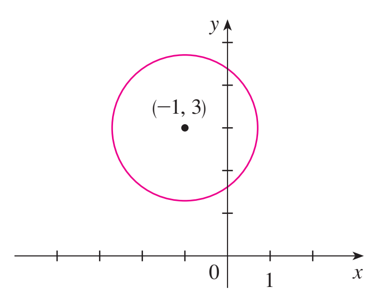


FIGURE 2
$x^2 + y^2 + 2x - 6y + 7 = 0$ ■

Parabolas

The geometric properties of parabolas are reviewed in Section 10.5. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

EXAMPLE 3 Draw the graph of the parabola $y = x^2$.

SOLUTION We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 3.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

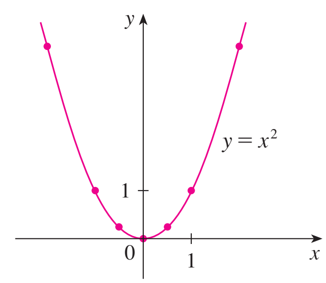


FIGURE 3 ■

Figure 4 shows the graphs of several parabolas with equations of the form $y = ax^2$ for various values of the number a. In each case the *vertex,* the point where the parabola changes direction, is the origin. We see that the parabola $y = ax^2$ opens upward if $a > 0$ and downward if $a < 0$ (as in Figure 5).

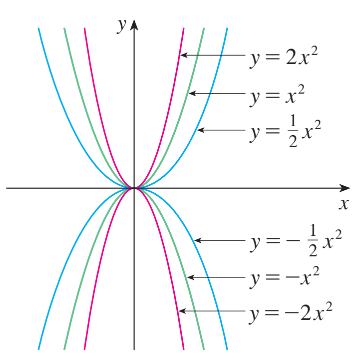
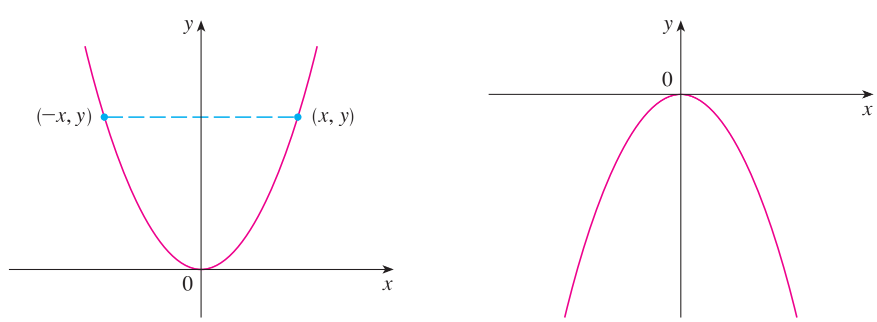


FIGURE 4



(a) $y = ax^2, \ a > 0$

(b) $y = ax^2, \ a < 0$

FIGURE 5

Notice that if (x, y) satisfies $y = ax^2$, then so does $(-x, y)$. This corresponds to the geometric fact that if the right half of the graph is reflected about the y-axis, then the left half of the graph is obtained. We say that the graph is **symmetric with respect to the *y*-axis**.

> The graph of an equation is symmetric with respect to the y-axis if the equation is unchanged when x is replaced by $-x$.

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See

Figure 6.) This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.

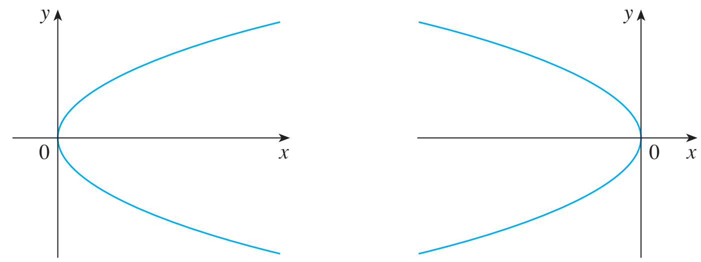


FIGURE 6

(a) $x = ay^2$, $a > 0$ (b) $x = ay^2$, $a < 0$

> The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

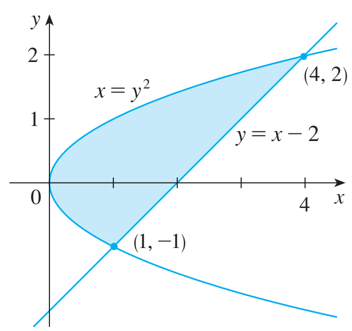


FIGURE 7

EXAMPLE 4 Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$.

SOLUTION First we find the points of intersection by solving the two equations. Substituting $x = y + 2$ into the equation $x = y^2$, we get $y + 2 = y^2$, which gives

$$0 = y^2 - y - 2 = (y - 2)(y + 1)$$

so $y = 2$ or -1. Thus the points of intersection are $(4, 2)$ and $(1, -1)$, and we draw the line $y = x - 2$ passing through these points. We then sketch the parabola $x = y^2$ by referring to Figure 6(a) and having the parabola pass through $(4, 2)$ and $(1, -1)$. The region bounded by $x = y^2$ and $y = x - 2$ means the finite region whose boundaries are these curves. It is sketched in Figure 7. ■

Ellipses

The curve with equation

2 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an **ellipse** in standard position. (Geometric properties of ellipses are discussed in Section 10.5.) Observe that Equation 2 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric with respect to both axes. As a further aid to sketching the ellipse, we find its intercepts.

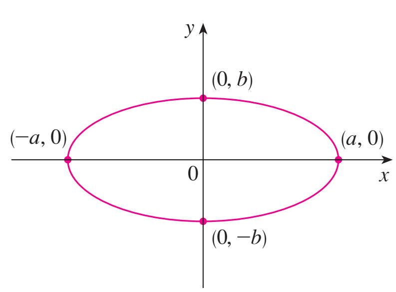


FIGURE 8

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

> The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.
>
> The ***y*-intercepts** are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in its equation.

If we set $y = 0$ in Equation 2, we get $x^2 = a^2$ and so the x-intercepts are $\pm a$. Setting $x = 0$, we get $y^2 = b^2$, so the y-intercepts are $\pm b$. Using this information, together with symmetry, we sketch the ellipse in Figure 8. If $a = b$, the ellipse is a circle with radius a.

EXAMPLE 5 Sketch the graph of $9x^2 + 16y^2 = 144$.

SOLUTION We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse (2), so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4; the y-intercepts are ± 3. The graph is sketched in Figure 9. ■

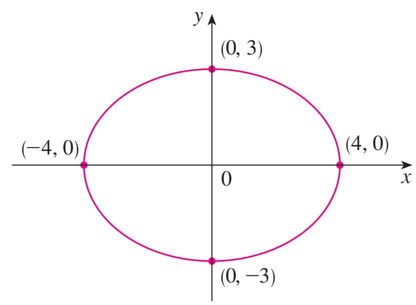


FIGURE 9
$9x^2 + 16y^2 = 144$

Hyperbolas

The curve with equation

3 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is called a **hyperbola** in standard position. Again, Equation 3 is unchanged when x is replaced by $-x$ or y is replaced by $-y$, so the hyperbola is symmetric with respect to both axes. To find the x-intercepts we set $y = 0$ and obtain $x^2 = a^2$ and $x = \pm a$. However, if we put $x = 0$ in Equation 3, we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. In fact, from Equation 3 we obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

which shows that $x^2 \geqslant a^2$ and so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*. It is sketched in Figure 10.

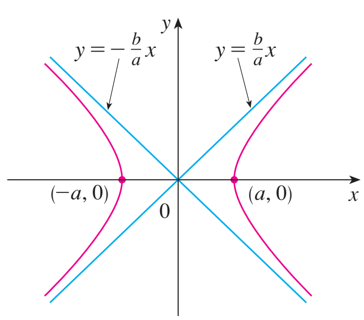


FIGURE 10
The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

In drawing a hyperbola it is useful to draw first its *asymptotes,* which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 10. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in Chapter 1. (See also Exercise 3.5.57.)

By interchanging the roles of x and y we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure 11.

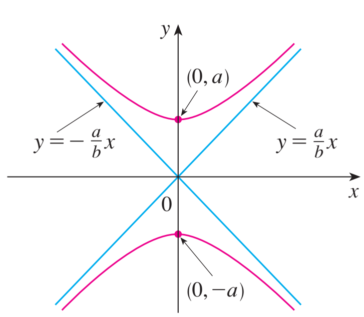


FIGURE 11
The hyperbola $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

EXAMPLE 6 Sketch the curve $9x^2 - 4y^2 = 36$.

SOLUTION Dividing both sides by 36, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola (Equation 3). Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have $b = 3$ and the asymptotes are $y = \pm \frac{3}{2}x$. The hyperbola is sketched in Figure 12.

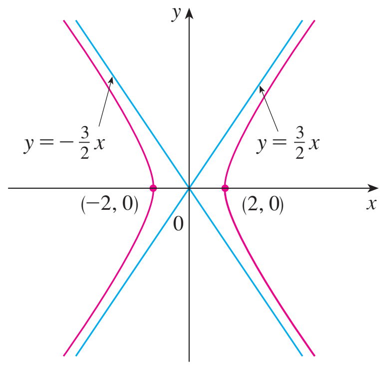
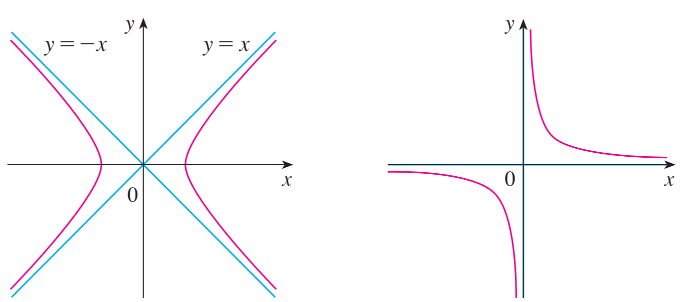


FIGURE 12
The hyperbola $9x^2 - 4y^2 = 36$

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 13(a)]. Its asymptotes are $y = \pm x$, which are perpendicular. If an equilateral hyperbola is rotated by 45°, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 13(b)].



(a) $x^2 - y^2 = a^2$ (b) $xy = k$ $(k > 0)$

FIGURE 13
Equilateral hyperbolas

Shifted Conics

Recall that an equation of the circle with center the origin and radius r is $x^2 + y^2 = r^2$, but if the center is the point (h, k), then the equation of the circle becomes

$$(x - h)^2 + (y - k)^2 = r^2$$

Similarly, if we take the ellipse with equation

4 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and translate it (shift it) so that its center is the point (h, k), then its equation becomes

5
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

(See Figure 14.)

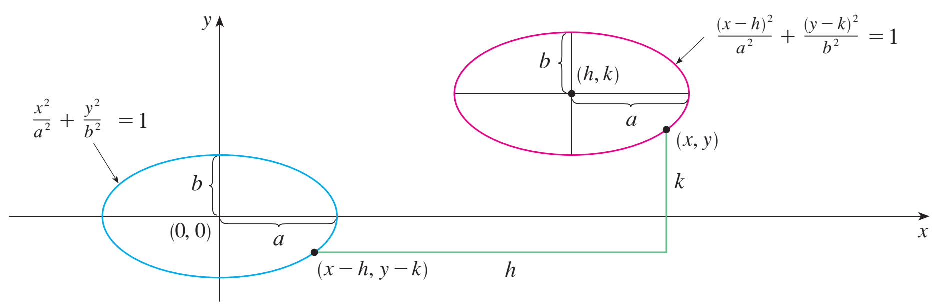


FIGURE 14

Notice that in shifting the ellipse, we replaced x by $x - h$ and y by $y - k$ in Equation 4 to obtain Equation 5. We use the same procedure to shift the parabola $y = ax^2$ so that its vertex (the origin) becomes the point (h, k) as in Figure 15. Replacing x by $x - h$ and y by $y - k$, we see that the new equation is

$$y - k = a(x-h)^2 \qquad \text{or} \qquad y = a(x-h)^2 + k$$

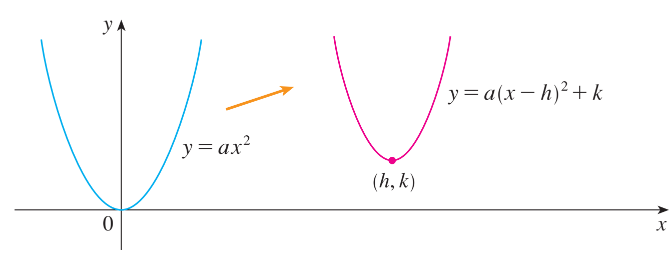


FIGURE 15

EXAMPLE 7 Sketch the graph of the equation $y = 2x^2 - 4x + 1$.

SOLUTION First we complete the square:

$$y = 2(x^2 - 2x) + 1 = 2(x-1)^2 - 1$$

In this form we see that the equation represents the parabola obtained by shifting $y = 2x^2$ so that its vertex is at the point $(1, -1)$. The graph is sketched in Figure 16.

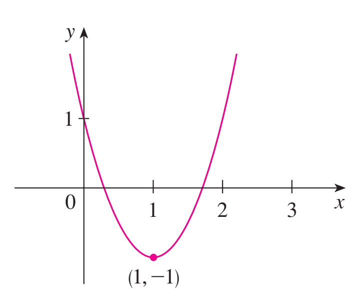


FIGURE 16
$y = 2x^2 - 4x + 1$

■

EXAMPLE 8 Sketch the curve $x = 1 - y^2$.

SOLUTION This time we start with the parabola $x = -y^2$ (as in Figure 6 with $a = -1$) and shift one unit to the right to get the graph of $x = 1 - y^2$. (See Figure 17.)

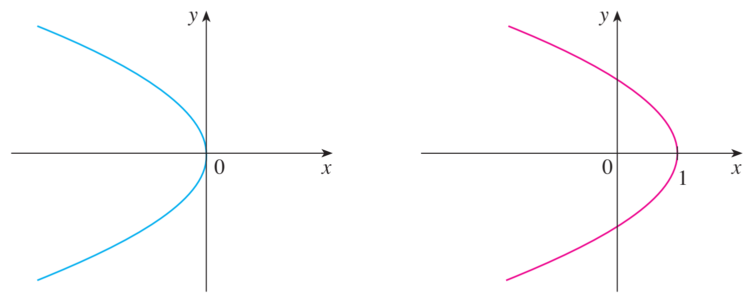


FIGURE 17 (a) $x = -y^2$ (b) $x = 1 - y^2$ ■

C EXERCISES

1–4 Find an equation of a circle that satisfies the given conditions.

1. Center $(3, -1)$, radius 5
2. Center $(-2, -8)$, radius 10
3. Center at the origin, passes through $(4, 7)$
4. Center $(-1, 5)$, passes through $(-4, -6)$

5–9 Show that the equation represents a circle and find the center and radius.

5. $x^2 + y^2 - 4x + 10y + 13 = 0$
6. $x^2 + y^2 + 6y + 2 = 0$
7. $x^2 + y^2 + x = 0$
8. $16x^2 + 16y^2 + 8x + 32y + 1 = 0$
9. $2x^2 + 2y^2 - x + y = 1$

10. Under what condition on the coefficients a, b, and c does the equation $x^2 + y^2 + ax + by + c = 0$ represent a circle? When that condition is satisfied, find the center and radius of the circle.

11–32 Identify the type of curve and sketch the graph. Do not plot points. Just use the standard graphs given in Figures 5, 6, 8, 10, and 11 and shift if necessary.

11. $y = -x^2$
12. $y^2 - x^2 = 1$
13. $x^2 + 4y^2 = 16$
14. $x = -2y^2$
15. $16x^2 - 25y^2 = 400$
16. $25x^2 + 4y^2 = 100$
17. $4x^2 + y^2 = 1$
18. $y = x^2 + 2$
19. $x = y^2 - 1$
20. $9x^2 - 25y^2 = 225$
21. $9y^2 - x^2 = 9$
22. $2x^2 + 5y^2 = 10$
23. $xy = 4$
24. $y = x^2 + 2x$
25. $9(x - 1)^2 + 4(y - 2)^2 = 36$
26. $16x^2 + 9y^2 - 36y = 108$
27. $y = x^2 - 6x + 13$
28. $x^2 - y^2 - 4x + 3 = 0$
29. $x = 4 - y^2$
30. $y^2 - 2x + 6y + 5 = 0$
31. $x^2 + 4y^2 - 6x + 5 = 0$
32. $4x^2 + 9y^2 - 16x + 54y + 61 = 0$

33–34 Sketch the region bounded by the curves.

33. $y = 3x, \quad y = x^2$
34. $y = 4 - x^2, \quad x - 2y = 2$

35. Find an equation of the parabola with vertex $(1, -1)$ that passes through the points $(-1, 3)$ and $(3, 3)$.
36. Find an equation of the ellipse with center at the origin that passes through the points $\left(1, -10\sqrt{2}/3\right)$ and $\left(-2, 5\sqrt{5}/3\right)$.

37–40 Sketch the graph of the set.

37. $\{(x, y) \mid x^2 + y^2 \leqslant 1\}$
38. $\{(x, y) \mid x^2 + y^2 > 4\}$
39. $\{(x, y) \mid y \geqslant x^2 - 1\}$
40. $\{(x, y) \mid x^2 + 4y^2 \leqslant 4\}$

D Trigonometry

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad. Therefore

1 $$\pi \text{ rad} = 180°$$

and

2 $$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.3° \qquad 1° = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

EXAMPLE 1

(a) Find the radian measure of 60°. (b) Express $5\pi/4$ rad in degrees.

SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$ ■

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

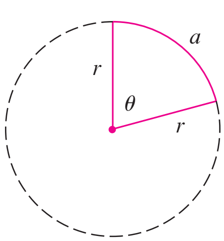


FIGURE 1

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a. Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

3 $$\theta = \frac{a}{r} \qquad a = r\theta$$

Remember that Equations 3 are valid only when θ is measured in radians.

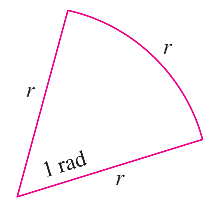


FIGURE 2

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION
(a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \tfrac{6}{5} = 1.2 \text{ rad}$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$ ■

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

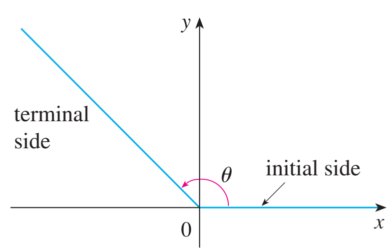


FIGURE 3 $\theta \geqslant 0$

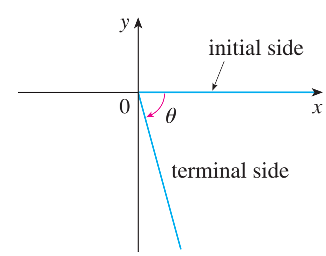


FIGURE 4 $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

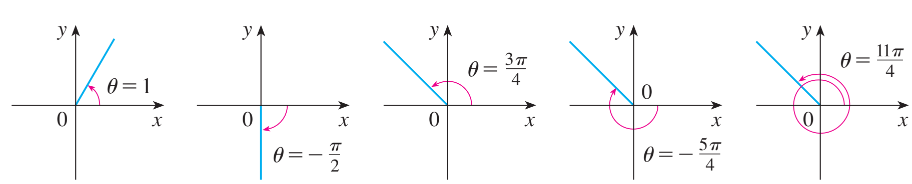


FIGURE 5
Angles in standard position

The Trigonometric Functions

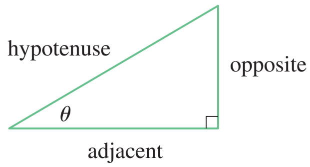


FIGURE 6

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

4

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

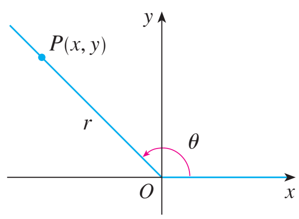


FIGURE 7

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

5

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

If we put $r = 1$ in Definition 5 and draw a unit circle with center the origin and label θ as in Figure 8, then the coordinates of P are $(\cos\theta, \sin\theta)$.

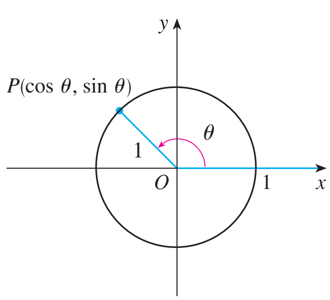


FIGURE 8

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3°$ and, with our calculator in degree mode, we find that

$$\sin 3° \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 9. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

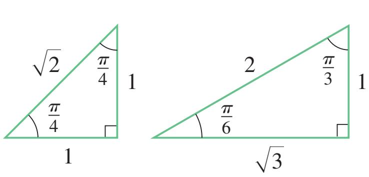


FIGURE 9

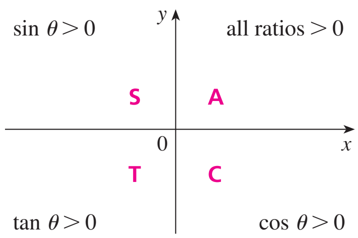


FIGURE 10

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" shown in Figure 10.

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 11 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \qquad y = \sqrt{3} \qquad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$ ■

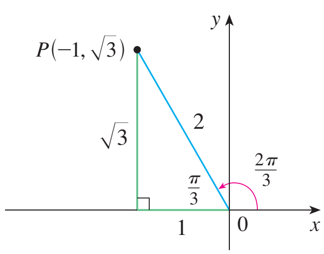


FIGURE 11

The following table gives some values of $\sin\theta$ and $\cos\theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

SOLUTION Since $\cos\theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 12. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$ ■

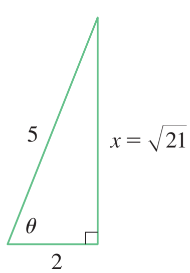


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 13.

SOLUTION From the diagram we see that

$$\tan 40^\circ = \frac{16}{x}$$

Therefore $$x = \frac{16}{\tan 40^\circ} \approx 19.07$$ ■

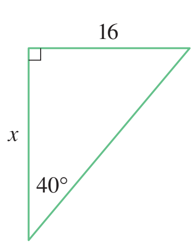


FIGURE 13

Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

6
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

7
$$\sin^2\theta + \cos^2\theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2\theta$ and use Equations 6, we get

8
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2\theta$, we get

9
$$1 + \cot^2\theta = \csc^2\theta$$

The identities

10a
$$\sin(-\theta) = -\sin\theta$$

10b
$$\cos(-\theta) = \cos\theta$$

Odd functions and even functions are discussed in Section 1.1.

show that sine is an odd function and cosine is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

11
$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

These identities show that the sine and cosine functions are periodic with period 2π.

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a $$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

12b $$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 85, 86, and 87.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

13a $$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

13b $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

14a $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

14b $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12), we get the **double-angle formulas**:

15a $$\sin 2x = 2 \sin x \cos x$$

15b $$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

16a $$\cos 2x = 2 \cos^2 x - 1$$

16b $$\cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

17a $$\cos^2 x = \frac{1 + \cos 2x}{2}$$

17b $$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from Equations 12 and 13:

18a $$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

18b $$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

18c $$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of the identities 13–18, remember that they can all be deduced from Equations 12a and 12b.

EXAMPLE 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x\,(1 - 2\cos x) = 0$$

Therefore there are two possibilities:

$$\sin x = 0 \qquad \text{or} \qquad 1 - 2\cos x = 0$$

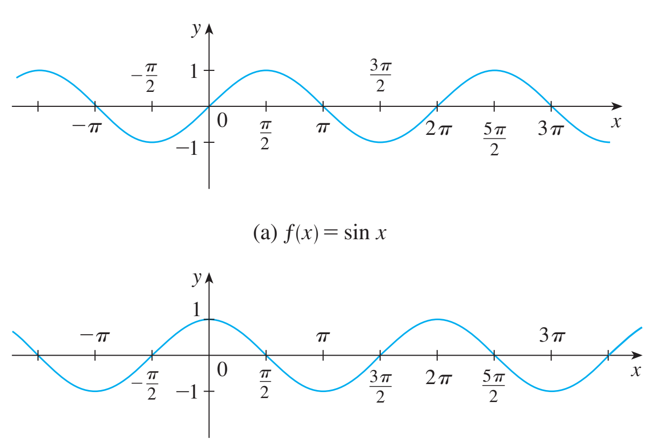
$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \tfrac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The given equation has five solutions: 0, $\pi/3$, π, $5\pi/3$, and 2π. ■

Graphs of the Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 14(a), is obtained by plotting points for $0 \leqslant x \leqslant 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

FIGURE 14

integer multiples of π, that is,

$$\sin x = 0 \qquad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 14(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

The graphs of the remaining four trigonometric functions are shown in Figure 15 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π.

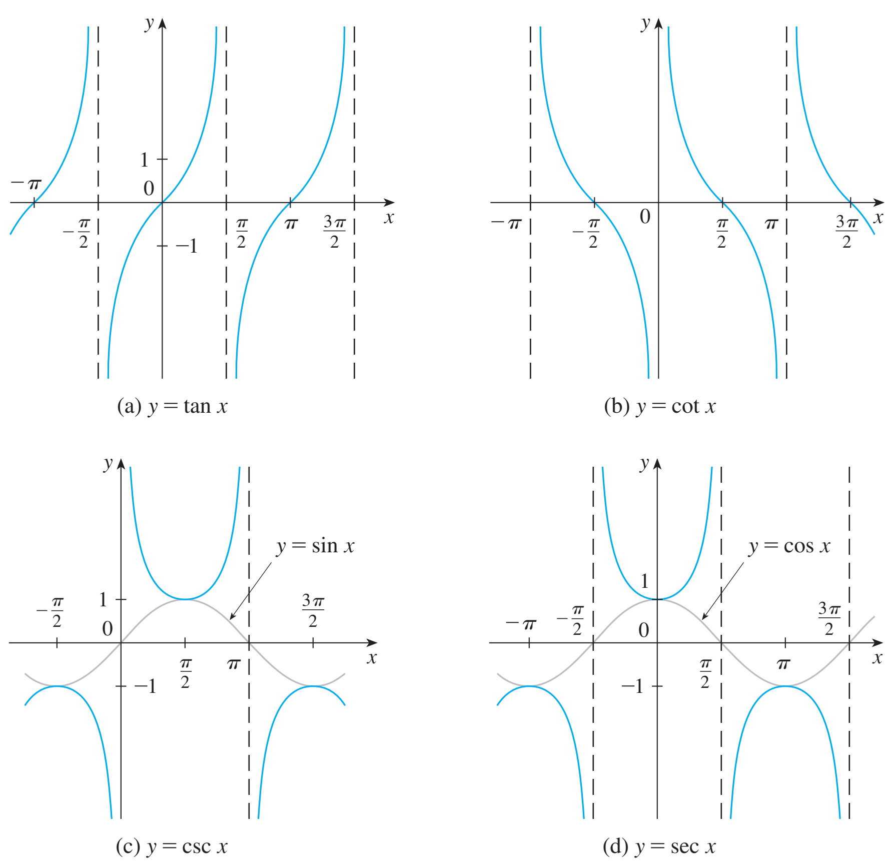


FIGURE 15

D EXERCISES

1–6 Convert from degrees to radians.

1. $210°$ **2.** $300°$ **3.** $9°$

4. $-315°$ **5.** $900°$ **6.** $36°$

7–12 Convert from radians to degrees.

7. 4π **8.** $-\dfrac{7\pi}{2}$ **9.** $\dfrac{5\pi}{12}$

10. $\dfrac{8\pi}{3}$ **11.** $-\dfrac{3\pi}{8}$ **12.** 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of $72°$.

15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17. $315°$ **18.** $-150°$ **19.** $-\dfrac{3\pi}{4}$ rad

20. $\dfrac{7\pi}{3}$ rad **21.** 2 rad **22.** -3 rad

23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23. $\dfrac{3\pi}{4}$ **24.** $\dfrac{4\pi}{3}$ **25.** $\dfrac{9\pi}{2}$

26. -5π **27.** $\dfrac{5\pi}{6}$ **28.** $\dfrac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29. $\sin\theta = \dfrac{3}{5}, \quad 0 < \theta < \dfrac{\pi}{2}$

30. $\tan\alpha = 2, \quad 0 < \alpha < \dfrac{\pi}{2}$

31. $\sec\phi = -1.5, \quad \dfrac{\pi}{2} < \phi < \pi$

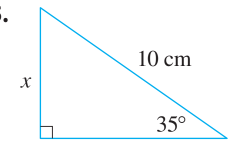
32. $\cos x = -\dfrac{1}{3}, \quad \pi < x < \dfrac{3\pi}{2}$

33. $\cot\beta = 3, \quad \pi < \beta < 2\pi$

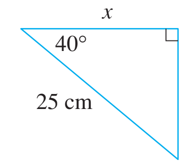
34. $\csc\theta = -\dfrac{4}{3}, \quad \dfrac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled x.

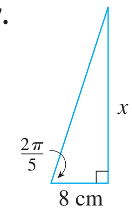
35.



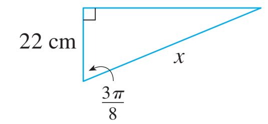
36.



37.



38.



39–41 Prove each equation.

39. (a) Equation 10a (b) Equation 10b

40. (a) Equation 14a (b) Equation 14b

41. (a) Equation 18a (b) Equation 18b
(c) Equation 18c

42–58 Prove the identity.

42. $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

43. $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$ **44.** $\sin(\pi - x) = \sin x$

45. $\sin\theta\,\cot\theta = \cos\theta$ **46.** $(\sin x + \cos x)^2 = 1 + \sin 2x$

47. $\sec y - \cos y = \tan y\,\sin y$

48. $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\,\sin^2\alpha$

49. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

50. $2\csc 2t = \sec t\,\csc t$

51. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

52. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

53. $\sin x\,\sin 2x + \cos x\,\cos 2x = \cos x$

54. $\sin^2 x - \sin^2 y = \sin(x + y)\,\sin(x - y)$

55. $\dfrac{\sin\phi}{1 - \cos\phi} = \csc\phi + \cot\phi$

56. $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x\,\cos y}$

57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58. $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$

59–64 If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x + y)$

60. $\cos(x + y)$

61. $\cos(x - y)$

62. $\sin(x - y)$

63. $\sin 2y$

64. $\cos 2y$

65–72 Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$

66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$

68. $|\tan x| = 1$

69. $\sin 2x = \cos x$

70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$

72. $2 + \cos 2x = 3 \cos x$

73–76 Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leqslant \frac{1}{2}$

74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$

76. $\sin x > \cos x$

77–82 Graph the function by starting with the graphs in Figures 14 and 15 and applying the transformations of Section 1.3 where appropriate.

77. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

78. $y = \tan 2x$

79. $y = \dfrac{1}{3}\tan\left(x - \dfrac{\pi}{2}\right)$

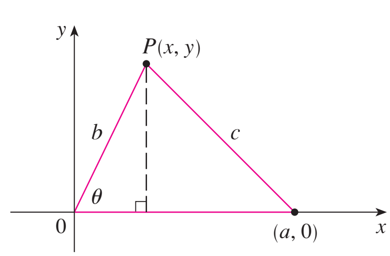
80. $y = 1 + \sec x$

81. $y = |\sin x|$

82. $y = 2 + \sin\left(x + \dfrac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab \cos\theta$$

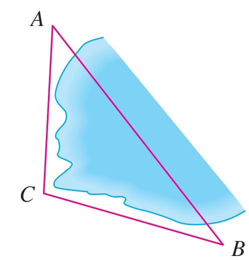


[*Hint:* Introduce a coordinate system so that θ is in standard position, as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.]

84. In order to find the distance $|AB|$ across a small inlet, a point C was located as in the figure and the following measurements were recorded:

$$\angle C = 103° \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

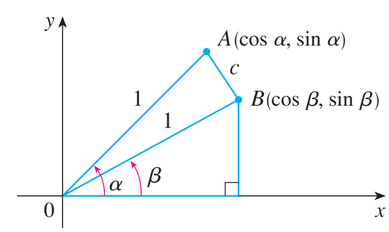
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

to prove the subtraction formula (13a) for the sine function.

88. Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \tfrac{1}{2}ab \sin\theta$$

89. Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle ABC = 107°$$

E Sigma Notation

A convenient way of writing sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.
This tells us to add.
This tells us to start with $i = m$.

$$\sum_{i=m}^{n} a_i$$

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on consecutive integer values beginning with m and ending with n, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$ ■

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$ ■

The following theorem gives three simple rules for working with sigma notation.

2 Theorem If c is any constant (that is, it does not depend on i), then

(a) $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$ (b) $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

PROOF To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned}(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)\\ = (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)\end{aligned}$$

Rule (c) is proved similarly. ■

EXAMPLE 3 Find $\displaystyle\sum_{i=1}^{n} 1$.

SOLUTION
$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$ ■

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 99) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$\begin{aligned}S &= 1 + 2 + 3 + \cdots + (n-1) + n\\ S &= n + (n-1) + (n-2) + \cdots + 2 + 1\end{aligned}$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$ ■

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + [(n+1)^3 - n^3]$$
$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 3S + 3\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Principle of Mathematical Induction
Let S_n be a statement involving the positive integer n. Suppose that
1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

See pages 99 and 100 for a more thorough discussion of mathematical induction.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $\quad 1^2 = \dfrac{1(1+1)(2\cdot 1+1)}{6}$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= (k+1)\frac{k(2k+1) + 6(k+1)}{6}$$
$$= (k+1)\frac{2k^2 + 7k + 6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n. ■

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 4.

3 Theorem Let c be a constant and n a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 6 Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2 - 3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ &= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2 + 2n - 3)}{2}\end{aligned}$$

EXAMPLE 7 Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

The type of calculation in Example 7 arises in Chapter 4 when we compute areas.

SOLUTION

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n}\left[\frac{3}{n^3}i^2 + \frac{3}{n}\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\ &= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4\end{aligned}$$

E EXERCISES

1–10 Write the sum in expanded form.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\sum_{i=4}^{6} 3^i$

4. $\sum_{i=4}^{6} i^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\sum_{k=5}^{8} x^k$

7. $\sum_{i=1}^{n} i^{10}$

8. $\sum_{j=n}^{n+3} j^2$

9. $\sum_{j=0}^{n-1} (-1)^j$

10. $\sum_{i=1}^{n} f(x_i)\, \Delta x_i$

11–20 Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–35 Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i + 2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i + 1)(i + 2)$

34. $\sum_{i=1}^{n} i(i + 1)(i + 2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

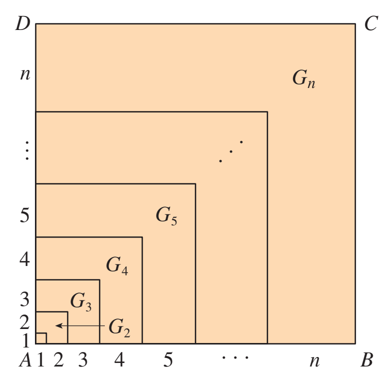
36. Find the number n such that $\sum_{i=1}^{n} i = 78$.

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1 + i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about AD 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths $1, 2, 3, \ldots, n$. Thus the side of the square has length $n(n + 1)/2$ so the area is $[n(n + 1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i - 1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality:

$$\left| \sum_{i=1}^{n} a_i \right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 Find the limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 - 2\left(1 + \frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\sum_{i=1}^{n} \frac{3}{2^{i-1}}$.

49. Evaluate $\sum_{i=1}^{n} (2i + 2^i)$.

50. Evaluate $\sum_{i=1}^{m} \left[\sum_{j=1}^{n} (i + j)\right]$.

F Proofs of Theorems

In this appendix we present proofs of several theorems that are stated in the main body of the text. The sections in which they occur are indicated in the margin.

Section 2.3

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = L + M$

2. $\lim_{x\to a} [f(x) - g(x)] = L - M$

3. $\lim_{x\to a} [cf(x)] = cL$

4. $\lim_{x\to a} [f(x)g(x)] = LM$

5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0$

PROOF OF LAW 4 Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x)g(x) - LM| < \varepsilon$$

In order to get terms that contain $|f(x) - L|$ and $|g(x) - M|$, we add and subtract $Lg(x)$ as follows:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |[f(x) - L]g(x) + L[g(x) - M]| \\ &\leqslant |[f(x) - L]g(x)| + |L[g(x) - M]| \qquad \text{(Triangle Inequality)} \\ &= |f(x) - L||g(x)| + |L||g(x) - M| \end{aligned}$$

We want to make each of these terms less than $\varepsilon/2$.

Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}$$

Also, there is a number $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|g(x) - M| < 1$$

and therefore

$$|g(x)| = |g(x) - M + M| \leqslant |g(x) - M| + |M| < 1 + |M|$$

Since $\lim_{x\to a} f(x) = L$, there is a number $\delta_3 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_3 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x - a| < \delta$, then we have $0 < |x - a| < \delta_1$, $0 < |x - a| < \delta_2$, and $0 < |x - a| < \delta_3$, so we can combine the inequalities to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leqslant |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\varepsilon}{2(1 + |M|)}(1 + |M|) + |L| \frac{\varepsilon}{2(1 + |L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that $\lim_{x\to a} [f(x)g(x)] = LM$. ■

PROOF OF LAW 3 If we take $g(x) = c$ in Law 4, we get

$$\begin{aligned} \lim_{x\to a} [cf(x)] &= \lim_{x\to a} [g(x)f(x)] = \lim_{x\to a} g(x) \cdot \lim_{x\to a} f(x) \\ &= \lim_{x\to a} c \cdot \lim_{x\to a} f(x) \\ &= c \lim_{x\to a} f(x) \qquad \text{(by Law 7)} \end{aligned}$$ ■

PROOF OF LAW 2 Using Law 1 and Law 3 with $c = -1$, we have

$$\begin{aligned} \lim_{x\to a} [f(x) - g(x)] &= \lim_{x\to a} [f(x) + (-1)g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} (-1)g(x) \\ &= \lim_{x\to a} f(x) + (-1) \lim_{x\to a} g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \end{aligned}$$ ■

PROOF OF LAW 5 First let us show that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$$

To do this we must show that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

Observe that
$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|Mg(x)|}$$

We know that we can make the numerator small. But we also need to know that the denominator is not small when x is near a. Since $\lim_{x\to a} g(x) = M$, there is a number $\delta_1 > 0$ such that, whenever $0 < |x - a| < \delta_1$, we have

$$|g(x) - M| < \frac{|M|}{2}$$

and therefore
$$\begin{aligned} |M| &= |M - g(x) + g(x)| \leqslant |M - g(x)| + |g(x)| \\ &< \frac{|M|}{2} + |g(x)| \end{aligned}$$

This shows that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |g(x)| > \frac{|M|}{2}$$

and so, for these values of x,

$$\frac{1}{|Mg(x)|} = \frac{1}{|M||g(x)|} < \frac{1}{|M|} \cdot \frac{2}{|M|} = \frac{2}{M^2}$$

Also, there exists $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{M^2}{2}\varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|Mg(x)|} < \frac{2}{M^2}\frac{M^2}{2}\varepsilon = \varepsilon$$

It follows that $\lim_{x \to a} 1/g(x) = 1/M$. Finally, using Law 4, we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)} \right) = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

■

2 Theorem If $f(x) \leqslant g(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M$$

then $L \leqslant M$.

PROOF We use the method of proof by contradiction. Suppose, if possible, that $L > M$. Law 2 of limits says that

$$\lim_{x \to a} [g(x) - f(x)] = M - L$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |[g(x) - f(x)] - (M - L)| < \varepsilon$$

In particular, taking $\varepsilon = L - M$ (noting that $L - M > 0$ by hypothesis), we have a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |[g(x) - f(x)] - (M - L)| < L - M$$

Since $b \leqslant |b|$ for any number b, we have

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad [g(x) - f(x)] - (M - L) < L - M$$

which simplifies to

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad g(x) < f(x)$$

But this contradicts $f(x) \leqslant g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leqslant M$. ■

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

PROOF Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there is a number $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_1 \quad \text{then} \quad L - \varepsilon < f(x) < L + \varepsilon$$

Since $\lim_{x \to a} h(x) = L$, there is a number $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad |h(x) - L| < \varepsilon$$

that is,

$$\text{if} \quad 0 < |x - a| < \delta_2 \quad \text{then} \quad L - \varepsilon < h(x) < L + \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so

$$L - \varepsilon < f(x) \leqslant g(x) \leqslant h(x) < L + \varepsilon$$

In particular, $$L - \varepsilon < g(x) < L + \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. ■

Section 1.8

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

PROOF Let $\varepsilon > 0$ be given. We want to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(g(x)) - f(b)| < \varepsilon$$

Since f is continuous at b, we have

$$\lim_{y \to b} f(y) = f(b)$$

and so there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |y - b| < \delta_1 \quad \text{then} \quad |f(y) - f(b)| < \varepsilon$$

Since $\lim_{x \to a} g(x) = b$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |g(x) - b| < \delta_1$$

Combining these two statements, we see that whenever $0 < |x - a| < \delta$ we have $|g(x) - b| < \delta_1$, which implies that $|f(g(x)) - f(b)| < \varepsilon$. Therefore we have proved that $\lim_{x \to a} f(g(x)) = f(b)$. ■

Section 2.4

The proof of the following result was promised when we proved that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Theorem If $0 < \theta < \pi/2$, then $\theta \leqslant \tan \theta$.

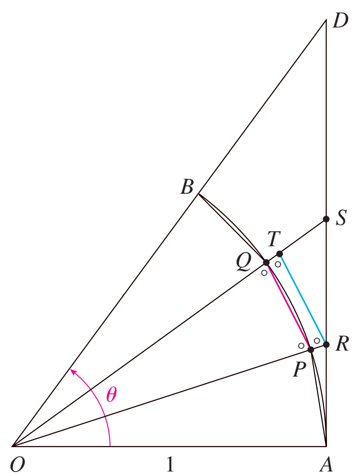


FIGURE 1

PROOF Figure 1 shows a sector of a circle with center O, central angle θ, and radius 1. Then

$$|AD| = |OA| \tan \theta = \tan \theta$$

We approximate the arc AB by an inscribed polygon consisting of n equal line segments and we look at a typical segment PQ. We extend the lines OP and OQ to meet AD in the points R and S. Then we draw $RT \parallel PQ$ as in Figure 2. Observe that

$$\angle RTO = \angle PQO < 90°$$

and so $\angle RTS > 90°$. Therefore we have

$$|PQ| < |RT| < |RS|$$

If we add n such inequalities, we get

$$L_n < |AD| = \tan \theta$$

where L_n is the length of the inscribed polygon. Thus, by Theorem 1.6.2, we have

$$\lim_{n \to \infty} L_n \leqslant \tan \theta$$

But the arc length is defined in Equation 8.1.1 as the limit of the lengths of inscribed polygons, so

$$\theta = \lim_{n \to \infty} L_n \leqslant \tan \theta$$ ■

Section 3.3

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

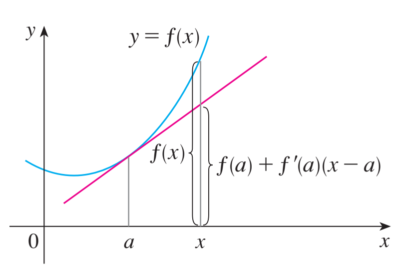


FIGURE 2

PROOF OF (a) Let a be any number in I. We must show that the curve $y = f(x)$ lies above the tangent line at the point $(a, f(a))$. The equation of this tangent is

$$y = f(a) + f'(a)(x - a)$$

So we must show that

$$f(x) > f(a) + f'(a)(x - a)$$

whenever $x \in I$ $(x \neq a)$. (See Figure 2.)

First let us take the case where $x > a$. Applying the Mean Value Theorem to f on the interval $[a, x]$, we get a number c, with $a < c < x$, such that

$$\boxed{1} \qquad f(x) - f(a) = f'(c)(x - a)$$

Since $f'' > 0$ on I, we know from the Increasing/Decreasing Test that f' is increasing on I. Thus, since $a < c$, we have

$$f'(a) < f'(c)$$

and so, multiplying this inequality by the positive number $x - a$, we get

$$\boxed{2} \qquad f'(a)(x - a) < f'(c)(x - a)$$

Now we add $f(a)$ to both sides of this inequality:

$$f(a) + f'(a)(x - a) < f(a) + f'(c)(x - a)$$

But from Equation 1 we have $f(x) = f(a) + f'(c)(x - a)$. So this inequality becomes

$$\boxed{3} \qquad f(x) > f(a) + f'(a)(x - a)$$

which is what we wanted to prove.

For the case where $x < a$ we have $f'(c) < f'(a)$, but multiplication by the negative number $x - a$ reverses the inequality, so we get (2) and (3) as before. ■

Section 6.1

> **6 Theorem** If f is a one-to-one continuous function defined on an interval (a, b), then its inverse function f^{-1} is also continuous.

PROOF First we show that if f is both one-to-one and continuous on (a, b), then it must be either increasing or decreasing on (a, b). If it were neither increasing nor decreasing, then there would exist numbers x_1, x_2, and x_3 in (a, b) with $x_1 < x_2 < x_3$ such that $f(x_2)$ does not lie between $f(x_1)$ and $f(x_3)$. There are two possibilities: either (1) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$ or (2) $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$. (Draw a picture.) In case (1) we apply the Intermediate Value Theorem to the continuous function f to get a number c between x_1 and x_2 such that $f(c) = f(x_3)$. In case (2) the Intermediate Value Theorem gives a number c between x_2 and x_3 such that $f(c) = f(x_1)$. In either case we have contradicted the fact that f is one-to-one.

Let us assume, for the sake of definiteness, that f is increasing on (a, b). We take any number y_0 in the domain of f^{-1} and we let $f^{-1}(y_0) = x_0$; that is, x_0 is the number in (a, b) such that $f(x_0) = y_0$. To show that f^{-1} is continuous at y_0 we take any $\varepsilon > 0$ such that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in the interval (a, b). Since f is increasing, it maps the numbers in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ onto the numbers in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and f^{-1} reverses the correspondence. If we let δ denote the smaller of the numbers $\delta_1 = y_0 - f(x_0 - \varepsilon)$ and $\delta_2 = f(x_0 + \varepsilon) - y_0$, then the interval $(y_0 - \delta, y_0 + \delta)$ is contained in the interval $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$ and so is mapped into the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ by f^{-1}. (See the arrow diagram in Figure 3.) We have therefore found a number $\delta > 0$ such that

$$\text{if} \quad |y - y_0| < \delta \quad \text{then} \quad |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$

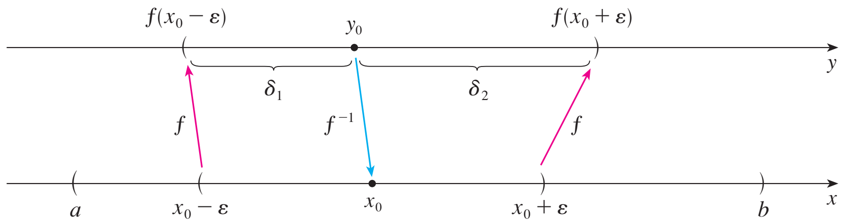


FIGURE 3

This shows that $\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0)$ and so f^{-1} is continuous at any number y_0 in its domain. ■

Section 11.8

In order to prove Theorem 11.8.4, we first need the following results.

Theorem

1. If a power series $\sum c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\sum c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it diverges whenever $|x| > |d|$.

PROOF OF 1 Suppose that $\sum c_n b^n$ converges. Then, by Theorem 11.2.6, we have $\lim_{n \to \infty} c_n b^n = 0$. According to Definition 11.1.2 with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \geqslant N$. Thus, for $n \geqslant N$, we have

$$|c_n x^n| = \left| \frac{c_n b^n x^n}{b^n} \right| = |c_n b^n| \left| \frac{x}{b} \right|^n < \left| \frac{x}{b} \right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n x^n|$ is convergent. Thus the series $\sum c_n x^n$ is absolutely convergent and therefore convergent. ■

PROOF OF 2 Suppose that $\sum c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$. ■

Theorem For a power series $\sum c_n x^n$ there are only three possibilities:

1. The series converges only when $x = 0$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

PROOF Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers b and d such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore the set $S = \{x \mid \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \leqslant |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 11.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n x^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges. ■

4 Theorem For a power series $\Sigma\, c_n(x - a)^n$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

PROOF If we make the change of variable $u = x - a$, then the power series becomes $\Sigma\, c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$. ■

Section 14.3

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

PROOF For small values of h, $h \neq 0$, consider the difference

$$\Delta(h) = [f(a + h, b + h) - f(a + h, b)] - [f(a, b + h) - f(a, b)]$$

Notice that if we let $g(x) = f(x, b + h) - f(x, b)$, then

$$\Delta(h) = g(a + h) - g(a)$$

By the Mean Value Theorem, there is a number c between a and $a + h$ such that

$$g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)]$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b + h$ such that

$$f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a + h, b) - f(a, b)]$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b), we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$. ■

Section 14.4

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

PROOF Let

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

According to (14.4.7), to prove that f is differentiable at (a, b) we have to show that we can write Δz in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Referring to Figure 4, we write

$$\boxed{1} \quad \Delta z = [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

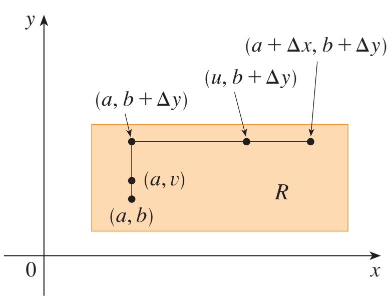


FIGURE 4

Observe that the function of a single variable

$$g(x) = f(x, b + \Delta y)$$

is defined on the interval $[a, a + \Delta x]$ and $g'(x) = f_x(x, b + \Delta y)$. If we apply the Mean Value Theorem to g, we get

$$g(a + \Delta x) - g(a) = g'(u)\,\Delta x$$

where u is some number between a and $a + \Delta x$. In terms of f, this equation becomes

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = f_x(u, b + \Delta y)\,\Delta x$$

This gives us an expression for the first part of the right side of Equation 1. For the second part we let $h(y) = f(a, y)$. Then h is a function of a single variable defined on the interval $[b, b + \Delta y]$ and $h'(y) = f_y(a, y)$. A second application of the Mean Value Theorem then gives

$$h(b + \Delta y) - h(b) = h'(v)\,\Delta y$$

where v is some number between b and $b + \Delta y$. In terms of f, this becomes

$$f(a, b + \Delta y) - f(a, b) = f_y(a, v)\,\Delta y$$

We now substitute these expressions into Equation 1 and obtain

$$\begin{aligned}\Delta z &= f_x(u, b + \Delta y)\,\Delta x + f_y(a, v)\,\Delta y \\ &= f_x(a, b)\,\Delta x + [f_x(u, b + \Delta y) - f_x(a, b)]\,\Delta x + f_y(a, b)\,\Delta y \\ &\qquad + [f_y(a, v) - f_y(a, b)]\,\Delta y \\ &= f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y\end{aligned}$$

where

$$\varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b)$$
$$\varepsilon_2 = f_y(a, v) - f_y(a, b)$$

Since $(u, b + \Delta y) \to (a, b)$ and $(a, v) \to (a, b)$ as $(\Delta x, \Delta y) \to (0, 0)$ and since f_x and f_y are continuous at (a, b), we see that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Therefore f is differentiable at (a, b). ■

G Complex Numbers

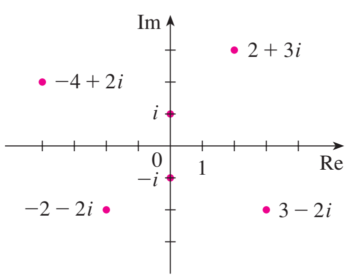


FIGURE 1
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di) \\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i) \\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$ ■

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

EXAMPLE 2 Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely, $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$ ■

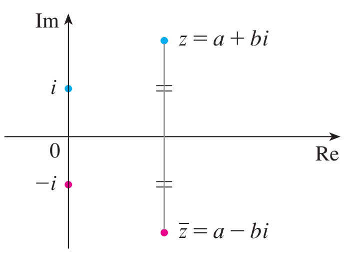


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: $\bar{z}$ is the reflection of z in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

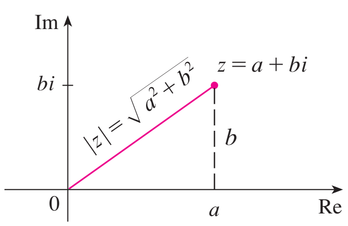


FIGURE 3

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\bar{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$ ■

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b, and c are always complex conjugates. (If z is real, $\bar{z} = z$, so z is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

Polar Form

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

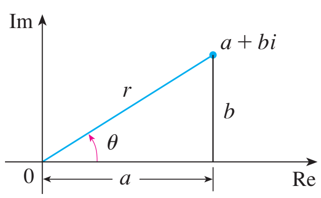


FIGURE 4

as in Figure 4. Therefore we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus we can write any complex number z in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $\qquad r = |z| = \sqrt{a^2 + b^2} \qquad$ and $\qquad \tan\theta = \dfrac{b}{a}$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

EXAMPLE 4 Write the following numbers in polar form.

(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

SOLUTION

(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

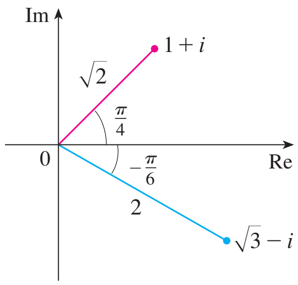


FIGURE 5

(b) Here we have $r = |w| = \sqrt{3+1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since w lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5. ■

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

1

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments*. (See Figure 6.)

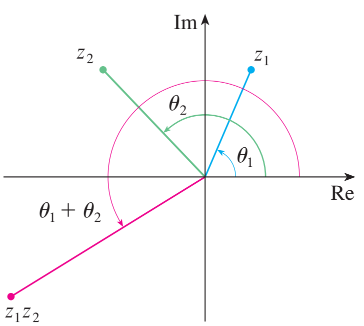


FIGURE 6

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments*.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0$$

In particular, taking $z_1 = 1$ and $z_2 = z$ (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

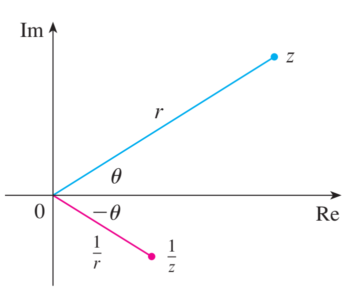


FIGURE 7

$$\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).$$

EXAMPLE 5 Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and
$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$(1 + i)(\sqrt{3} - i) = 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

This is illustrated in Figure 8. ■

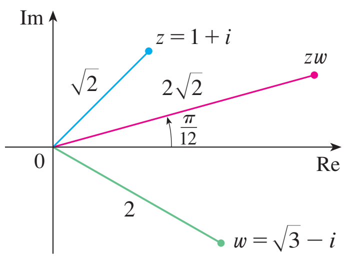


FIGURE 8

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i\sin\theta)$$

then
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and
$$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

2 De Moivre's Theorem If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.*

EXAMPLE 6 Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An nth root of the complex number z is a complex number w such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i\sin\phi) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

The equality of these two complex numbers shows that

$$s^n = r \qquad \text{or} \qquad s = r^{1/n}$$

and $$\cos n\phi = \cos\theta \qquad \text{and} \qquad \sin n\phi = \sin\theta$$

From the fact that sine and cosine have period 2π, it follows that

$$n\phi = \theta + 2k\pi \qquad \text{or} \qquad \phi = \frac{\theta + 2k\pi}{n}$$

Thus $$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n - 1$, we have the following.

3 Roots of a Complex Number Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Notice that each of the nth roots of z has modulus $|w_k| = r^{1/n}$. Thus all the nth roots of z lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive nth root exceeds the argument of the previous root by $2\pi/n$, we see that the nth roots of z are equally spaced on this circle.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos \pi + i \sin \pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$$

$$w_1 = 8^{1/6}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \sqrt{2}\, i$$

$$w_2 = 8^{1/6}\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$$

$$w_3 = 8^{1/6}\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$$

$$w_4 = 8^{1/6}\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right) = -\sqrt{2}\, i$$

$$w_5 = 8^{1/6}\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$$

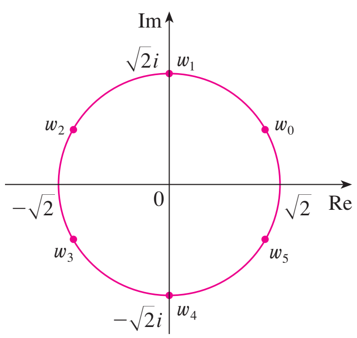


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9. ■

Complex Exponentials

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 11 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (11.10.11) as our guide, we define

4 $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

5 $$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get $$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$$
$$= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots$$
$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$
$$= \cos y + i \sin y$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 11.10.16 and 11.10.15). The result is a famous formula called **Euler's formula**:

6 $$e^{iy} = \cos y + i \sin y$$

Combining Euler's formula with Equation 5, we get

7 $$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

EXAMPLE 8 Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's equation (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

We could write the result of Example 8(a) as

$$e^{i\pi} + 1 = 0$$

This equation relates the five most famous numbers in all of mathematics: 0, 1, e, i, and π.

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$ ■

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos\theta + i \sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

G EXERCISES

1–14 Evaluate the expression and write your answer in the form $a + bi$.

1. $(5 - 6i) + (3 + 2i)$
2. $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right)$
3. $(2 + 5i)(4 - i)$
4. $(1 - 2i)(8 - 3i)$
5. $\overline{12 + 7i}$
6. $\overline{2i\left(\frac{1}{2} - i\right)}$
7. $\dfrac{1 + 4i}{3 + 2i}$
8. $\dfrac{3 + 2i}{1 - 4i}$
9. $\dfrac{1}{1 + i}$
10. $\dfrac{3}{4 - 3i}$
11. i^3
12. i^{100}
13. $\sqrt{-25}$
14. $\sqrt{-3}\sqrt{-12}$

15–17 Find the complex conjugate and the modulus of the number.

15. $12 - 5i$
16. $-1 + 2\sqrt{2}\,i$
17. $-4i$

18. Prove the following properties of complex numbers.
(a) $\overline{z + w} = \bar{z} + \bar{w}$ (b) $\overline{zw} = \bar{z}\,\bar{w}$
(c) $\overline{z^n} = \bar{z}^n$, where n is a positive integer
[*Hint:* Write $z = a + bi$, $w = c + di$.]

19–24 Find all solutions of the equation.

19. $4x^2 + 9 = 0$

20. $x^4 = 1$

21. $x^2 + 2x + 5 = 0$

22. $2x^2 - 2x + 1 = 0$

23. $z^2 + z + 2 = 0$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

25–28 Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$

26. $1 - \sqrt{3}\,i$

27. $3 + 4i$

28. $8i$

29–32 Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i, \quad w = 1 + \sqrt{3}\,i$

30. $z = 4\sqrt{3} - 4i, \quad w = 8i$

31. $z = 2\sqrt{3} - 2i, \quad w = -1 + i$

32. $z = 4\left(\sqrt{3} + i\right), \quad w = -3 - 3i$

33–36 Find the indicated power using De Moivre's Theorem.

33. $(1 + i)^{20}$

34. $\left(1 - \sqrt{3}\,i\right)^5$

35. $\left(2\sqrt{3} + 2i\right)^5$

36. $(1 - i)^8$

37–40 Find the indicated roots. Sketch the roots in the complex plane.

37. The eighth roots of 1

38. The fifth roots of 32

39. The cube roots of i

40. The cube roots of $1 + i$

41–46 Write the number in the form $a + bi$.

41. $e^{i\pi/2}$

42. $e^{2\pi i}$

43. $e^{i\pi/3}$

44. $e^{-i\pi}$

45. $e^{2+i\pi}$

46. $e^{\pi+i}$

47. Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

48. Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

49. If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of u is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

50. (a) If u is a complex-valued function of a real variable, its indefinite integral $\int u(x)\,dx$ is an antiderivative of u. Evaluate

$$\int e^{(1+i)x}\,dx$$

(b) By considering the real and imaginary parts of the integral in part (a), evaluate the real integrals

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$

(c) Compare with the method used in Example 7.1.4.

H Answers to Odd-Numbered Exercises

CHAPTER 1

EXERCISES 1.1 ■ PAGE 19

1. Yes

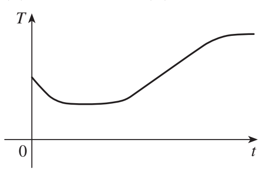
3. (a) 3 (b) -0.2 (c) 0, 3 (d) -0.8
(e) $[-2, 4], [-1, 3]$ (f) $[-2, 1]$

5. $[-85, 115]$ **7.** No

9. Yes, $[-3, 2], [-3, -2) \cup [-1, 3]$

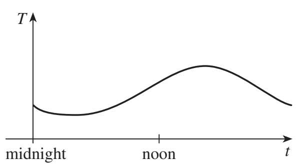
11. (a) 13.8°C (b) 1990 (c) 1910, 2005 (d) $[13.5, 14.5]$

13.

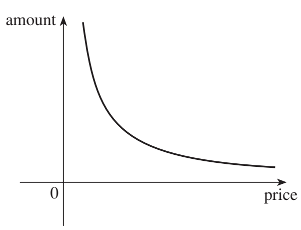


15. (a) 500 MW; 730 MW (b) 4 AM; noon; yes

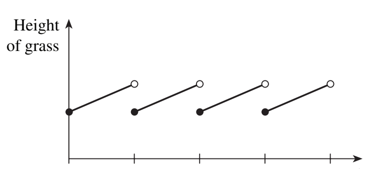
17.



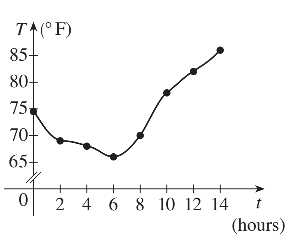
19.



21.



23. (a)



(b) 74°F

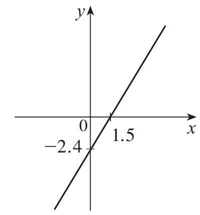
25. 12, 16, $3a^2 - a + 2$, $3a^2 + a + 2$, $3a^2 + 5a + 4$, $6a^2 - 2a + 4$, $12a^2 - 2a + 2$, $3a^4 - a^2 + 2$, $9a^4 - 6a^3 + 13a^2 - 4a + 4$, $3a^2 + 6ah + 3h^2 - a - h + 2$

27. $-3 - h$ **29.** $-1/(ax)$

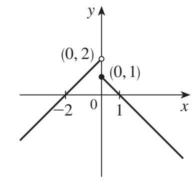
31. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **33.** $(-\infty, \infty)$

35. $(-\infty, 0) \cup (5, \infty)$ **37.** $[0, 4]$

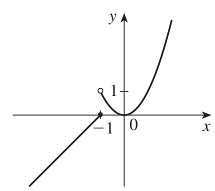
39. $(-\infty, \infty)$



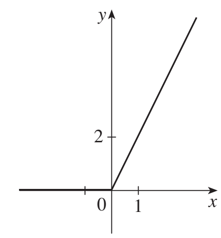
41. $-1, 1, -1$



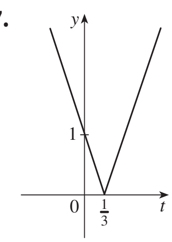
43. $-2, 0, 4$



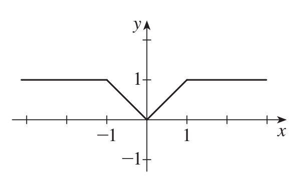
45.



47.



49.



51. $f(x) = \frac{5}{2}x - \frac{11}{2}, 1 \leqslant x \leqslant 5$ **53.** $f(x) = 1 - \sqrt{-x}$

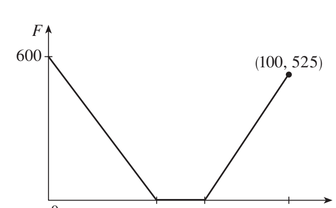
55. $f(x) = \begin{cases} -x + 3 & \text{if } 0 \leqslant x \leqslant 3 \\ 2x - 6 & \text{if } 3 < x \leqslant 5 \end{cases}$

57. $A(L) = 10L - L^2, 0 < L < 10$

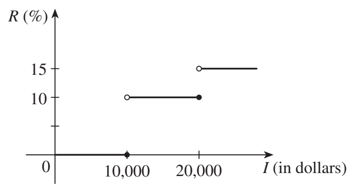
59. $A(x) = \sqrt{3}x^2/4, x > 0$ **61.** $S(x) = x^2 + (8/x), x > 0$

63. $V(x) = 4x^3 - 64x^2 + 240x, 0 < x < 6$

65. $F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \leqslant x < 40 \\ 0 & \text{if } 40 \leqslant x \leqslant 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$

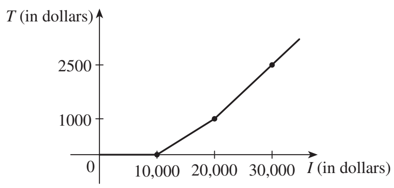


67. (a)



(b) \$400, \$1900

(c)

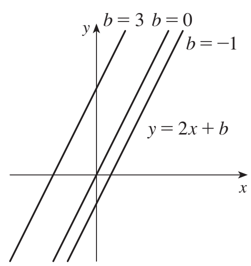


69. f is odd, g is even **71.** (a) $(-5, 3)$ (b) $(-5, -3)$
73. Odd **75.** Neither **77.** Even
79. Even; odd; neither (unless $f = 0$ or $g = 0$)

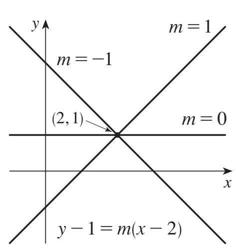
EXERCISES 1.2 ■ PAGE 33

1. (a) Logarithmic (b) Root (c) Rational
(d) Polynomial, degree 2 (e) Exponential (f) Trigonometric
3. (a) h (b) f (c) g
5. $\{x \mid x \neq \pi/2 + 2n\pi, n \text{ an integer}\}$

7. (a) $y = 2x + b$,
where b is the y-intercept.

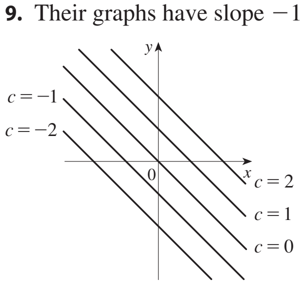


(b) $y = mx + 1 - 2m$,
where m is the slope.



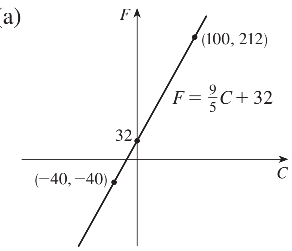
(c) $y = 2x - 3$

9. Their graphs have slope -1.



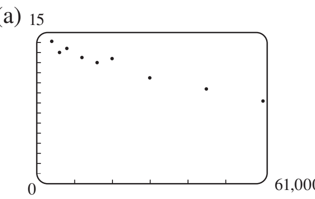
11. $f(x) = -3x(x + 1)(x - 2)$
13. (a) 8.34, change in mg for every 1 year change
(b) 8.34 mg

15. (a)



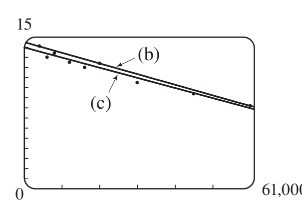
(b) $\frac{9}{5}$, change in °F for every 1°C change; 32, Fahrenheit temperature corresponding to 0°C
17. (a) $T = \frac{1}{6}N + \frac{307}{6}$ (b) $\frac{1}{6}$, change in °F for every chirp per minute change (c) 76°F
19. (a) $P = 0.434d + 15$ (b) 196 ft
21. (a) Cosine (b) Linear

23. (a)



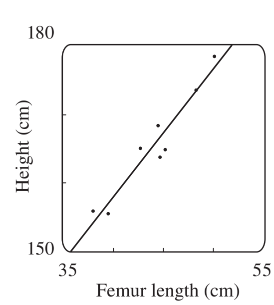
A linear model is appropriate.

(b) $y = -0.000105x + 14.521$



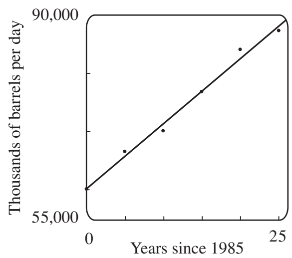
(c) $y = -0.00009979x + 13.951$
(d) About 11.5 per 100 population
(e) About 6% (f) No

25. (a) See graph in part (b).
(b) $y = 1.88074x + 82.64974$



(c) 182.3 cm

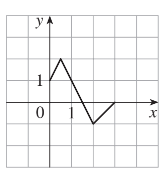
27. (a) A linear model is appropriate. See graph in part (b).
(b) $y = 1116.64x + 60{,}188.33$



(c) In thousands of barrels per day: 79,171 and 90,338
29. Four times as bright
31. (a) $N = 3.1046A^{0.308}$ (b) 18

EXERCISES 1.3 ■ PAGE 42

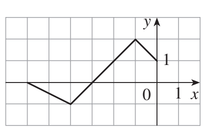
1. (a) $y = f(x) + 3$ (b) $y = f(x) - 3$ (c) $y = f(x - 3)$
(d) $y = f(x + 3)$ (e) $y = -f(x)$ (f) $y = f(-x)$
(g) $y = 3f(x)$ (h) $y = \frac{1}{3}f(x)$
3. (a) 3 (b) 1 (c) 4 (d) 5 (e) 2
5. (a)



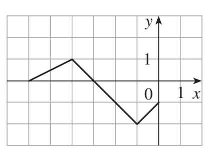
(b)



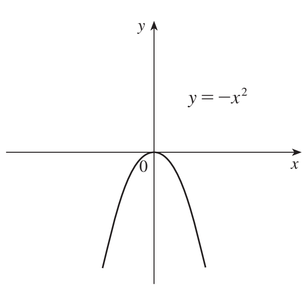
(c)



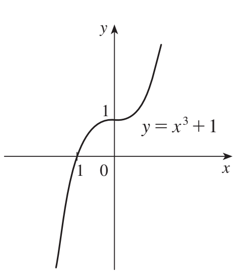
(d)



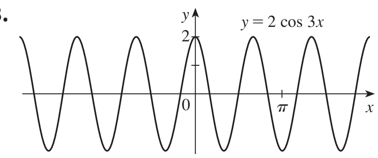
7. $y = -\sqrt{-x^2 - 5x - 4} - 1$
9.



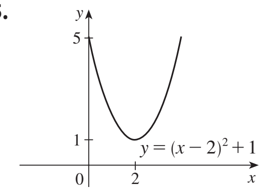
11.



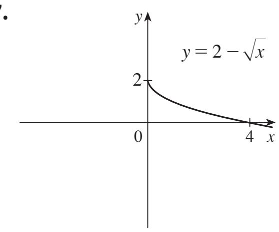
13.



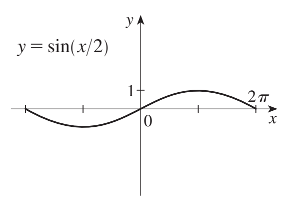
15.



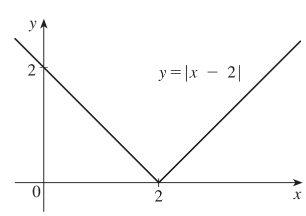
17.



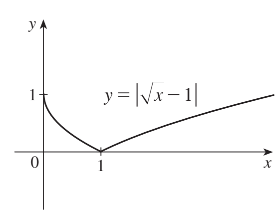
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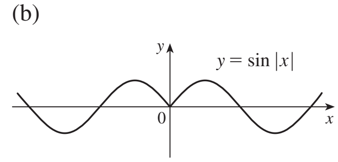
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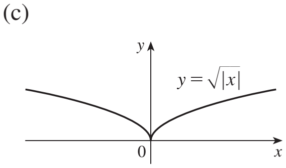
23.



25. $L(t) = 12 + 2\sin\left[\dfrac{2\pi}{365}(t - 80)\right]$
27. $D(t) = 5\cos[(\pi/6)(t - 6.75)] + 7$
29. (a) The portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.
(b)



(c)



31. (a) $(f+g)(x) = x^3 + 5x^2 - 1$, $(-\infty, \infty)$
(b) $(f-g)(x) = x^3 - x^2 + 1$, $(-\infty, \infty)$
(c) $(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2$, $(-\infty, \infty)$
(d) $(f/g)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$, $\left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$

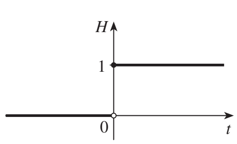
33. (a) $(f \circ g)(x) = 3x^2 + 3x + 5$, $(-\infty, \infty)$
(b) $(g \circ f)(x) = 9x^2 + 33x + 30$, $(-\infty, \infty)$
(c) $(f \circ f)(x) = 9x + 20$, $(-\infty, \infty)$
(d) $(g \circ g)(x) = x^4 + 2x^3 + 2x^2 + x$, $(-\infty, \infty)$

35. (a) $(f \circ g)(x) = \sqrt{4x - 2}$, $\left[\frac{1}{2}, \infty\right)$
(b) $(g \circ f)(x) = 4\sqrt{x+1} - 3$, $[-1, \infty)$
(c) $(f \circ f)(x) = \sqrt{\sqrt{x+1} + 1}$, $[-1, \infty)$
(d) $(g \circ g)(x) = 16x - 15$, $(-\infty, \infty)$

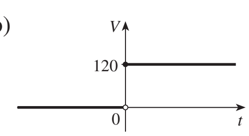
37. (a) $(f \circ g)(x) = \dfrac{2x^2 + 6x + 5}{(x+2)(x+1)}$, $\{x \mid x \neq -2, -1\}$
(b) $(g \circ f)(x) = \dfrac{x^2 + x + 1}{(x+1)^2}$, $\{x \mid x \neq -1, 0\}$
(c) $(f \circ f)(x) = \dfrac{x^4 + 3x^2 + 1}{x(x^2 + 1)}$, $\{x \mid x \neq 0\}$
(d) $(g \circ g)(x) = \dfrac{2x + 3}{3x + 5}$, $\{x \mid x \neq -2, -\frac{5}{3}\}$

39. $(f \circ g \circ h)(x) = 3\sin(x^2) - 2$
41. $(f \circ g \circ h)(x) = \sqrt{x^6 + 4x^3 + 1}$
43. $g(x) = 2x + x^2$, $f(x) = x^4$
45. $g(x) = \sqrt[3]{x}$, $f(x) = x/(1+x)$
47. $g(t) = t^2$, $f(t) = \sec t \tan t$
49. $h(x) = \sqrt{x}$, $g(x) = x - 1$, $f(x) = \sqrt{x}$
51. $h(t) = \cos t$, $g(t) = \sin t$, $f(t) = t^2$
53. (a) 4 (b) 3 (c) 0 (d) Does not exist; $f(6) = 6$ is not in the domain of g. (e) 4 (f) -2
55. (a) $r(t) = 60t$ (b) $(A \circ r)(t) = 3600\pi t^2$; the area of the circle as a function of time
57. (a) $s = \sqrt{d^2 + 36}$ (b) $d = 30t$
(c) $(f \circ g)(t) = \sqrt{900t^2 + 36}$; the distance between the lighthouse and the ship as a function of the time elapsed since noon

59. (a)

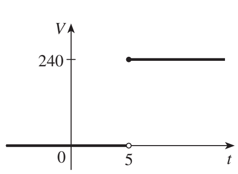


(b)



$V(t) = 120H(t)$

(c)



$V(t) = 240H(t - 5)$

61. Yes; $m_1 m_2$

63. (a) $f(x) = x^2 + 6$ (b) $g(x) = x^2 + x - 1$
65. Yes

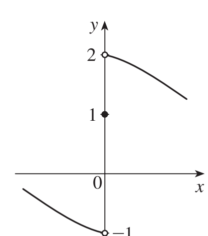
EXERCISES 1.4 ■ PAGE 49

1. (a) $-44.4, -38.8, -27.8, -22.2, -16.\overline{6}$
(b) -33.3 (c) $-33\frac{1}{3}$
3. (a) (i) 2 (ii) 1.111111 (iii) 1.010101 (iv) 1.001001 (v) 0.666667 (vi) 0.909091 (vii) 0.990099 (viii) 0.999001 (b) 1 (c) $y = x - 3$
5. (a) (i) -32 ft/s (ii) -25.6 ft/s (iii) -24.8 ft/s (iv) -24.16 ft/s (b) -24 ft/s
7. (a) (i) 29.3 ft/s (ii) 32.7 ft/s (iii) 45.6 ft/s (iv) 48.75 ft/s (b) 29.7 ft/s
9. (a) 0, 1.7321, -1.0847, -2.7433, 4.3301, -2.8173, 0, -2.1651, -2.6061, -5, 3.4202; no (c) -31.4

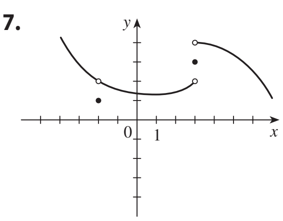
EXERCISES 1.5 ■ PAGE 59

1. Yes
3. (a) $\lim_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
(b) $\lim_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.
5. (a) 2 (b) 1 (c) 4 (d) Does not exist (e) 3
7. (a) -1 (b) -2 (c) Does not exist (d) 2 (e) 0 (f) Does not exist (g) 1 (h) 3
9. (a) $-\infty$ (b) ∞ (c) ∞ (d) $-\infty$ (e) ∞
(f) $x = -7$, $x = -3$, $x = 0$, $x = 6$
11. $\lim\limits_{x \to a} f(x)$ exists for all a except $a = -1$.
13. (a) 1 (b) 0 (c) Does not exist

15.



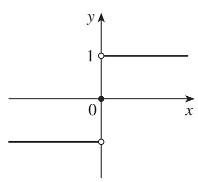
17.



19. $\frac{1}{2}$ **21.** $\frac{1}{2}$ **23.** 1.5 **25.** 1 **27.** (a) -1.5
29. ∞ **31.** ∞ **33.** $-\infty$ **35.** $-\infty$ **37.** $-\infty$
39. ∞ **41.** $-\infty$; ∞
43. (a) 0.998000, 0.638259, 0.358484, 0.158680, 0.038851, 0.008928, 0.001465; 0
(b) 0.000572, -0.000614, -0.000907, -0.000978, -0.000993, -0.001000; -0.001
45. No matter how many times we zoom in toward the origin, the graph appears to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.
47. $x \approx \pm 0.90, \pm 2.24$; $x = \pm\sin^{-1}(\pi/4)$, $\pm(\pi - \sin^{-1}(\pi/4))$
49. (a) 6 (b) Within 0.0649 of 1

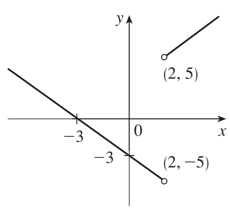
EXERCISES 1.6 ■ PAGE 70

1. (a) -6 (b) -8 (c) 2 (d) -6
(e) Does not exist (f) 0
3. 105 **5.** $\frac{7}{8}$ **7.** 390 **9.** $\frac{3}{2}$ **11.** 4
13. Does not exist **15.** $\frac{6}{5}$ **17.** -10 **19.** $\frac{1}{12}$
21. $\frac{1}{6}$ **23.** $-\frac{1}{9}$ **25.** 1 **27.** $\frac{1}{128}$ **29.** $-\frac{1}{2}$
31. $3x^2$ **33.** (a), (b) $\frac{2}{3}$ **37.** 7 **41.** 6 **43.** -4
45. Does not exist
47. (a)



(b) (i) 1
(ii) -1
(iii) Does not exist
(iv) 1

49. (a) (i) 5 (ii) -5 (b) Does not exist
(c)



51. 7
53. (a) (i) -2 (ii) Does not exist (iii) -3
(b) (i) $n-1$ (ii) n (c) a is not an integer.
59. 8 **65.** 15; -1

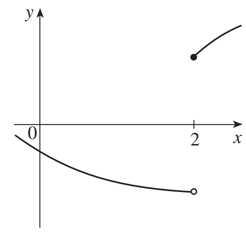
EXERCISES 1.7 ■ PAGE 81

1. 0.1 (or any smaller positive number)
3. 1.44 (or any smaller positive number)
5. 0.0906 (or any smaller positive number)
7. 0.0219 (or any smaller positive number);
0.011 (or any smaller positive number)
9. (a) 0.041 (or any smaller positive number)
(b) $\lim_{x\to 4^+} \dfrac{x^2+4}{\sqrt{x-4}} = \infty$
11. (a) $\sqrt{1000/\pi}$ cm (b) Within approximately 0.0445 cm
(c) Radius; area; $\sqrt{1000/\pi}$; 1000; 5; ≈ 0.0445
13. (a) 0.025 (b) 0.0025
35. (a) 0.093 (b) $\delta = (B^{2/3} - 12)/(6B^{1/3}) - 1$, where
$B = 216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}$
41. Within 0.1

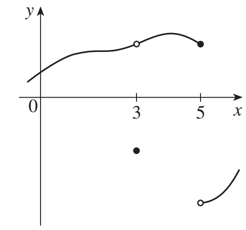
EXERCISES 1.8 ■ PAGE 91

1. $\lim_{x\to 4} f(x) = f(4)$
3. (a) $-4, -2, 2, 4$; $f(-4)$ is not defined and $\lim_{x\to a} f(x)$ does not exist for $a = -2, 2$, and 4
(b) -4, neither; -2, left; 2, right; 4, right

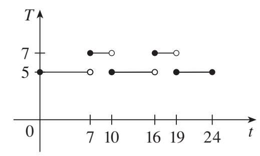
5.



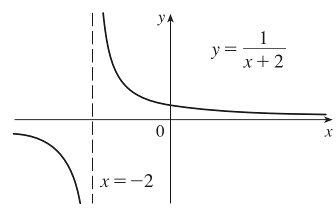
7.



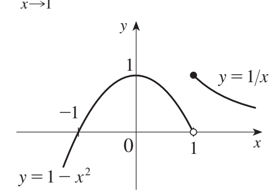
9. (a)



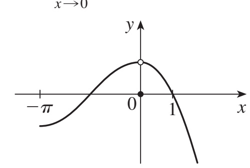
17. $f(-2)$ is undefined.



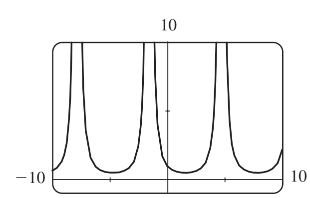
19. $\lim_{x\to 1} f(x)$ does not exist.



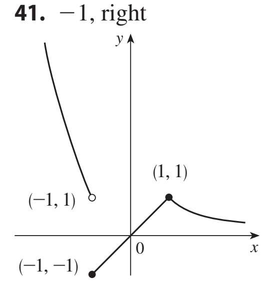
21. $\lim_{x\to 0} f(x) \neq f(0)$



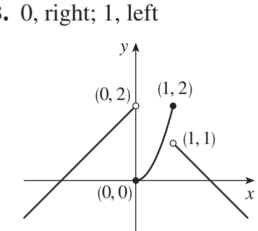
23. Define $f(2) = 3$. **25.** $(-\infty, \infty)$
27. $(-\infty, \sqrt[3]{2}) \cup (\sqrt[3]{2}, \infty)$ **29.** $\mathbb{R}$
31. $(-\infty, -1] \cup (0, \infty)$
33. $x = (-\pi/2) + 2n\pi$, n an integer



35. 8 **37.** $\pi^2/16$
41. -1, right



43. 0, right; 1, left



45. $\frac{2}{3}$ **47.** 4

49. (a) $g(x) = x^3 + x^2 + x + 1$ (b) $g(x) = x^2 + x$

57. (b) (0.86, 0.87) **59.** (b) 1.434 **67.** None

69. Yes

CHAPTER 1 REVIEW ■ PAGE 95

True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True
19. False **21.** True **23.** True **25.** True **27.** False

Exercises

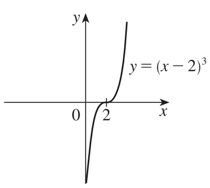
1. (a) 2.7 (b) 2.3, 5.6 (c) $[-6, 6]$ (d) $[-4, 4]$
(e) $[-4, 4]$ (f) Odd; its graph is symmetric about the origin.

3. $2a + h - 2$ **5.** $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$, $(-\infty, 0) \cup (0, \infty)$

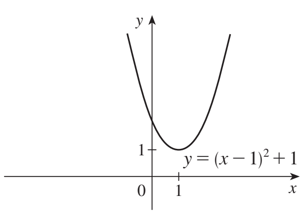
7. $(-\infty, \infty)$, $[0, 2]$

9. (a) Shift the graph 8 units upward.
(b) Shift the graph 8 units to the left.
(c) Stretch the graph vertically by a factor of 2, then shift it 1 unit upward.
(d) Shift the graph 2 units to the right and 2 units downward.
(e) Reflect the graph about the x-axis.
(f) Reflect the graph about the x-axis, then shift 3 units upward.

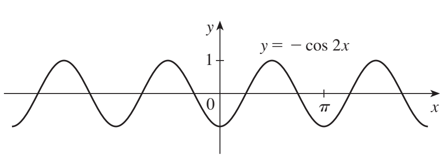
11.



13.



15.



17. (a) Neither (b) Odd (c) Even (d) Neither

19. (a) $(f \circ g)(x) = \sqrt{\sin x}$,
$\{x \mid x \in [2n\pi, \pi + 2n\pi], n \text{ an integer}\}$
(b) $(g \circ f)(x) = \sin \sqrt{x}$, $[0, \infty)$
(c) $(f \circ f)(x) = \sqrt[4]{x}$, $[0, \infty)$
(d) $(g \circ g)(x) = \sin(\sin x)$, $\mathbb{R}$

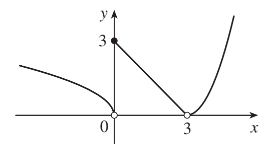
21. $y = 0.2493x - 423.4818$; about 77.6 years

23. (a) (i) 3 (ii) 0 (iii) Does not exist (iv) 2
(v) ∞ (vi) $-\infty$
(b) $x = 0$, $x = 2$ (c) $-3, 0, 2, 4$

25. 1 **27.** $\frac{3}{2}$ **29.** 3 **31.** ∞ **33.** $\frac{4}{7}$ **35.** $-\frac{1}{8}$

37. 0 **39.** 1

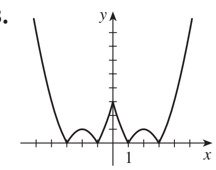
45. (a) (i) 3 (ii) 0 (iii) Does not exist
(iv) 0 (v) 0 (vi) 0
(b) At 0 and 3 (c)



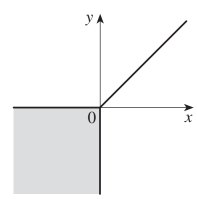
47. $[0, \infty)$ **51.** 0

PRINCIPLES OF PROBLEM SOLVING ■ PAGE 102

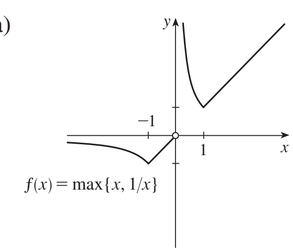
1. $-\frac{7}{3}, 9$ **3.**



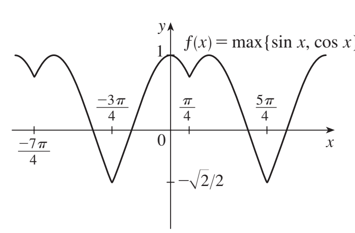
5.



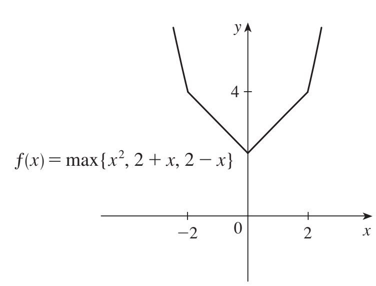
7. (a)



(b)



(c)



9. 40 mi/h **13.** $f_n(x) = x^{2^{n+1}}$ **15.** $\frac{2}{3}$

17. -4 **19.** (a) Does not exist (b) 1

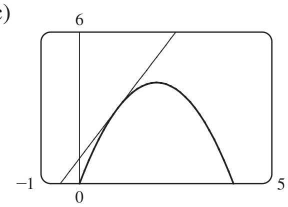
21. $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ **23.** $\frac{3}{4}$ **25.** (b) Yes (c) Yes; no

CHAPTER 2

EXERCISES 2.1 ■ PAGE 113

1. (a) $\dfrac{f(x) - f(3)}{x - 3}$ (b) $\displaystyle\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

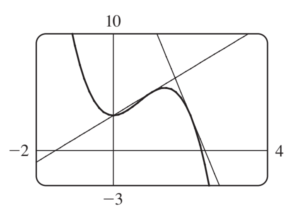
3. (a) 2 (b) $y = 2x + 1$ (c)



5. $y = -8x + 12$ **7.** $y = \frac{1}{2}x + \frac{1}{2}$

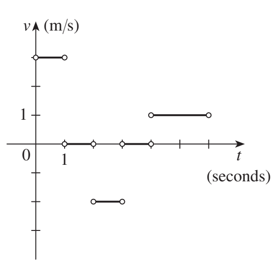
9. (a) $8a - 6a^2$ (b) $y = 2x + 3$, $y = -8x + 19$

(c)



11. (a) Right: $0 < t < 1$ and $4 < t < 6$; left: $2 < t < 3$; standing still: $1 < t < 2$ and $3 < t < 4$

(b)



13. -24 ft/s

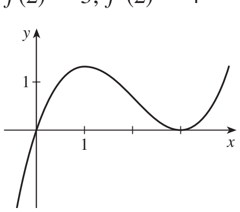
15. $-2/a^3$ m/s; -2 m/s; $-\frac{1}{4}$ m/s; $-\frac{2}{27}$ m/s

17. $g'(0), 0, g'(4), g'(2), g'(-2)$

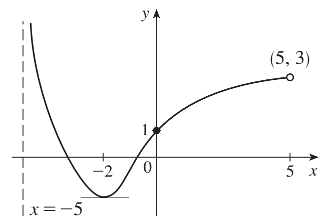
19. (a) 26 (b) No (c) Yes

21. $f(2) = 3$; $f'(2) = 4$

23.

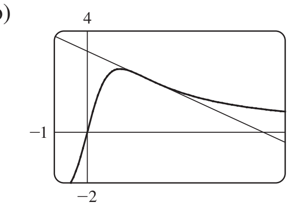


25.



27. 3; $y = 3x - 1$

29. (a) $-\frac{3}{5}$; $y = -\frac{3}{5}x + \frac{16}{5}$ (b)



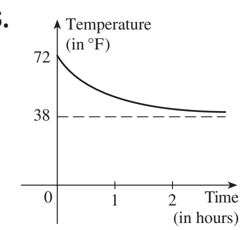
31. $6a - 4$ **33.** $\dfrac{5}{(a + 3)^2}$ **35.** $-\dfrac{1}{\sqrt{1 - 2a}}$

37. $f(x) = \sqrt{x}$, $a = 9$ **39.** $f(x) = x^6$, $a = 2$

41. $f(x) = \cos x$, $a = \pi$ or $f(x) = \cos(\pi + x)$, $a = 0$

43. 32 m/s; 32 m/s

45.



Greater (in magnitude)

47. (a) In (g/dL)/h: (i) -0.015 (ii) -0.012 (iii) -0.012 (iv) -0.011
(b) -0.012 (g/dL)/h; After 2 hours, the BAC is decreasing at a rate of 0.012 (g/dL)/h.

49. (a) 1169.6 thousands of barrels of oil per day per year; oil consumption rose by an average of 1169.6 thousands of barrels of oil per day each year from 1990 to 2005.
(b) 1397.8 thousands of barrels of oil per day per year

51. (a) (i) \$20.25/unit (ii) \$20.05/unit (b) \$20/unit

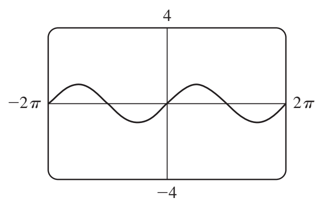
53. (a) The rate at which the cost is changing per ounce of gold produced; dollars per ounce
(b) When the 800th ounce of gold is produced, the cost of production is \$17/oz.
(c) Decrease in the short term; increase in the long term

55. (a) The rate at which daily heating costs change with respect to temperature when the temperature is 58°F; dollars/°F
(b) Negative; If the outside temperature increases, the building should require less heating.

57. (a) The rate at which the oxygen solubility changes with respect to the water temperature; (mg/L)/°C
(b) $S'(16) \approx -0.25$; as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.

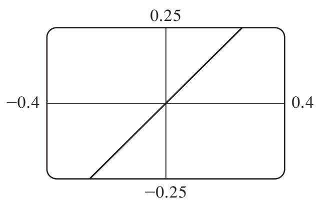
59. Does not exist

61. (a)



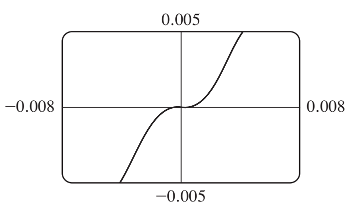
Slope appears to be 1.

(b)



Yes

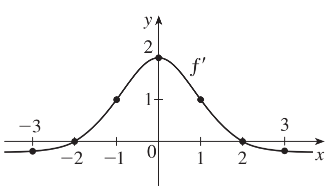
(c)



Yes; 0

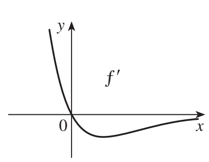
EXERCISES 2.2 ■ PAGE 125

1. (a) −0.2 (b) 0 (c) 1 (d) 2
(e) 1 (f) 0 (g) −0.2

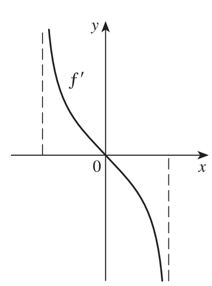


3. (a) II (b) IV (c) I (d) III

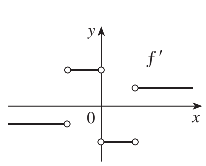
5.



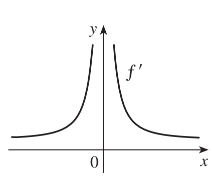
7.



9.

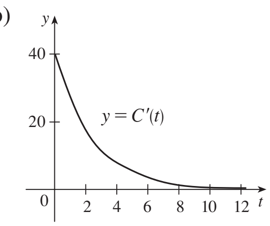


11.



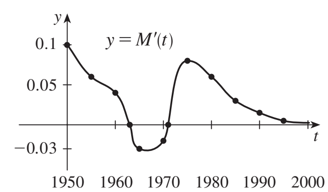
13. (a) The instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours

(b)



The rate of change of percentage of full capacity is decreasing and approaching 0.

15.



1963 to 1971

17. (a) 0, 1, 2, 4 (b) −1, −2, −4 (c) $f'(x) = 2x$
19. $f'(x) = 3$, $\mathbb{R}$, $\mathbb{R}$ **21.** $f'(t) = 5t + 6$, $\mathbb{R}$, $\mathbb{R}$
23. $f'(x) = 2x - 6x^2$, $\mathbb{R}$, $\mathbb{R}$

25. $g'(x) = -\dfrac{1}{2\sqrt{9 - x}}$, $(-\infty, 9]$, $(-\infty, 9)$

27. $G'(t) = \dfrac{-7}{(3 + t)^2}$, $(-\infty, -3) \cup (-3, \infty)$, $(-\infty, -3) \cup (-3, \infty)$

29. $f'(x) = 4x^3$, $\mathbb{R}$, $\mathbb{R}$ **31.** (a) $f'(x) = 4x^3 + 2$

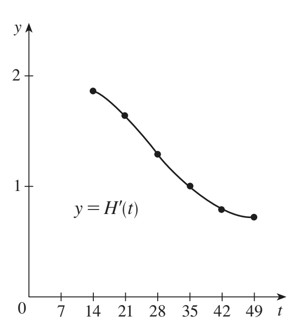
33. (a) The rate at which the unemployment rate is changing, in percent unemployed per year

(b)

t	$U'(t)$	t	$U'(t)$
2003	−0.50	2008	2.35
2004	−0.45	2009	1.90
2005	−0.45	2010	−0.20
2006	−0.25	2011	−0.75
2007	0.60	2012	−0.80

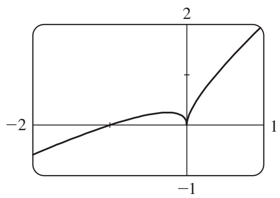
35.

t	14	21	28	35	42	49
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{9}{7}$	1	$\frac{11}{14}$	$\frac{5}{7}$



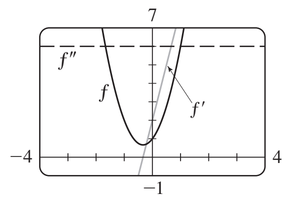
37. (a) The rate at which the percentage of electrical power produced by solar panels is changing, in percentage points per year.
(b) On January 1, 2002, the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

39. -4 (corner); 0 (discontinuity)
41. 1 (not defined); 5 (vertical tangent)
43.

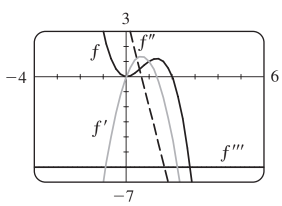


Differentiable at -1; not differentiable at 0

45. $f''(1)$ **47.** $a = f, b = f', c = f''$
49. $a =$ acceleration, $b =$ velocity, $c =$ position
51. $6x + 2$; 6



53.

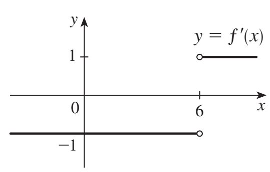


$f'(x) = 4x - 3x^2$,
$f''(x) = 4 - 6x$,
$f'''(x) = -6$,
$f^{(4)}(x) = 0$

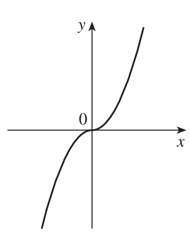
55. (a) $\frac{1}{3}a^{-2/3}$

57. $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

or $f'(x) = \dfrac{x - 6}{|x - 6|}$

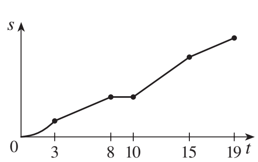


59. (a)

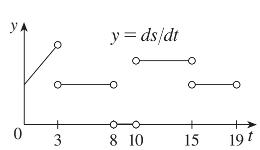


(b) All x
(c) $f'(x) = 2|x|$

63. (a)



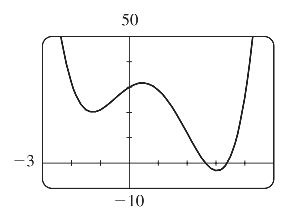
(b)



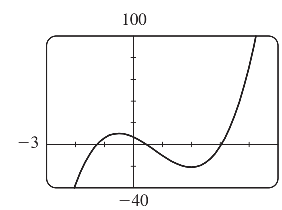
65. $63°$

EXERCISES 2.3 ■ PAGE 140

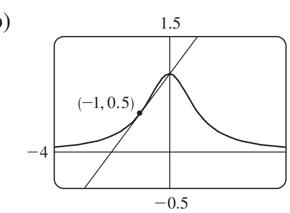
1. $f'(x) = 0$ **3.** $f'(x) = 5.2$ **5.** $f'(t) = 6t^2 - 6t - 4$
7. $g'(x) = 2x - 6x^2$ **9.** $g'(t) = -\frac{3}{2}t^{-7/4}$
11. $F'(r) = -15/r^4$ **13.** $S'(p) = \frac{1}{2}p^{-1/2} - 1$
15. $R'(a) = 18a + 6$ **17.** $y' = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$
19. $G'(q) = -2q^{-2} - 2q^{-3}$ **21.** $u' = -\dfrac{2}{t^3} + \dfrac{3}{t^2\sqrt{t}} - \dfrac{1}{t^2}$
23. $1 - 2x + 6x^2 - 8x^3$ **25.** $f'(x) = 25x^4 + 39x^2 - 6$
27. $F'(y) = 5 + \dfrac{14}{y^2} + \dfrac{9}{y^4}$ **29.** $g'(x) = \dfrac{10}{(3 - 4x)^2}$
31. $y' = \dfrac{x(-x^3 - 3x - 2)}{(x^3 - 1)^2}$ **33.** $y' = \dfrac{t^4 - 8t^3 + 6t^2 + 9}{(t^2 - 4t + 3)^2}$
35. $y' = \dfrac{3 - 2\sqrt{s}}{2s^{5/2}}$ **37.** $f'(t) = \dfrac{-2t - 3}{3t^{2/3}(t - 3)^2}$
39. $F'(x) = 4x + 1 + \dfrac{12}{x^3}$ **41.** $G'(y) = -\dfrac{3ABy^2}{(Ay^3 + B)^2}$
43. $f'(x) = \dfrac{2cx}{(x^2 + c)^2}$
45. $P'(x) = na_nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$
47. $45x^{14} - 15x^2$
49. (a)



(c) $4x^3 - 9x^2 - 12x + 7$



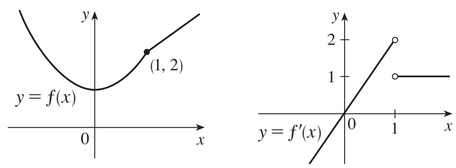
51. $y = \frac{1}{2}x + \frac{1}{2}$
53. (a) $y = \frac{1}{2}x + 1$ (b)



55. $y = \frac{3}{2}x + \frac{1}{2}$, $y = -\frac{2}{3}x + \frac{8}{3}$ **57.** $y = -\frac{1}{2}x + \frac{5}{2}$, $y = 2x$
59. $f'(x) = 0.005x^4 - 0.06x^2$; $f''(x) = 0.02x^3 - 0.12x$
61. $f'(x) = \dfrac{2x^2 + 2x}{(1 + 2x)^2}$; $f''(x) = \dfrac{2}{(1 + 2x)^3}$
63. (a) $v(t) = 3t^2 - 3$, $a(t) = 6t$ (b) 12 m/s^2
(c) $a(1) = 6 \text{ m/s}^2$
65. 1.718; at 12 years, the length of the fish is increasing at a rate of 1.718 in/year.
67. (a) $V = 5.3/P$
(b) -0.00212; instantaneous rate of change of the volume with respect to the pressure at 25°C; m^3/kPa
69. (a) -16 (b) $-\frac{20}{9}$ (c) 20
71. 16 **73.** (a) 0 (b) $-\frac{2}{3}$
75. (a) $y' = xg'(x) + g(x)$ (b) $y' = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$
(c) $y' = \dfrac{xg'(x) - g(x)}{x^2}$

77. $(-2, 21), (1, -6)$ **81.** $y = 3x - 3, y = 3x - 7$
83. $y = -2x + 3$ **85.** $(\pm 2, 4)$
87. (c) $3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$
89. $P(x) = x^2 - x + 3$ **91.** $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$
93. \$1.627 billion/year
95. $\dfrac{0.0021}{(0.015 + [S])^2}$;
the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

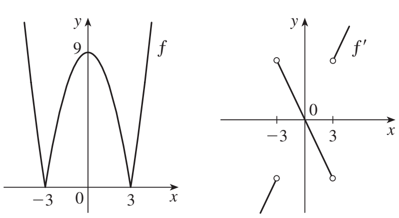
97. No



99. (a) Not differentiable at 3 or −3

$$f'(x) = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

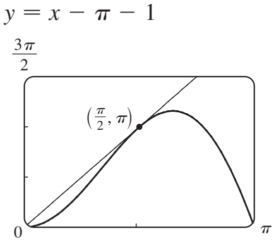
(b)



101. $a = -\frac{1}{2}, b = 2$ **103.** $c = 3\sqrt{4} = 6$
107. 1000 **109.** 3; 1

EXERCISES 2.4 ■ PAGE 150

1. $f'(x) = x^2 \cos x + 2x \sin x$ **3.** $f'(x) = -3 \csc^2 x + 2 \sin x$
5. $y' = \sec\theta\,(\sec^2\theta + \tan^2\theta)$
7. $y' = -c \sin t + t(t \cos t + 2 \sin t)$
9. $y' = \dfrac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$ **11.** $f'(\theta) = \dfrac{1}{1 + \cos\theta}$
13. $y' = \dfrac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$
15. $f'(\theta) = \frac{1}{2} \sin 2\theta + \theta \cos 2\theta$
21. $y = x + 1$ **23.** $y = x - \pi - 1$
25. (a) $y = 2x$ (b)

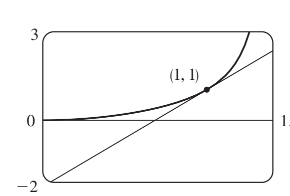


27. (a) $\sec x \tan x - 1$
29. $\theta \cos\theta + \sin\theta$; $2\cos\theta - \theta \sin\theta$
31. (a) $f'(x) = (1 + \tan x)/\sec x$ (b) $f'(x) = \cos x + \sin x$
33. $(2n + 1)\pi \pm \frac{1}{3}\pi$, n an integer
35. (a) $v(t) = 8 \cos t$, $a(t) = -8 \sin t$
(b) $4\sqrt{3}, -4, -4\sqrt{3}$; to the left

37. 5 ft/rad **39.** $\frac{5}{3}$ **41.** 3 **43.** $-\frac{3}{4}$
45. $\frac{1}{2}$ **47.** $-\frac{1}{4}$ **49.** $-\sqrt{2}$ **51.** $-\cos x$
53. $A = -\frac{3}{10}, B = -\frac{1}{10}$
55. (a) $\sec^2 x = \dfrac{1}{\cos^2 x}$ (b) $\sec x \tan x = \dfrac{\sin x}{\cos^2 x}$
(c) $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$
57. 1

EXERCISES 2.5 ■ PAGE 158

1. $\dfrac{4}{3\sqrt[3]{(1 + 4x)^2}}$ **3.** $\pi \sec^2 \pi x$ **5.** $\dfrac{\cos x}{2\sqrt{\sin x}}$
7. $F'(x) = 24x^{11}(5x^3 + 2)^3(5x^3 + 1)$
9. $f'(x) = \dfrac{5}{2\sqrt{5x + 1}}$ **11.** $A'(t) = \dfrac{2(\sin t - \sec^2 t)}{(\cos t + \tan t)^3}$
13. $f'(\theta) = -2\theta \sin(\theta^2)$ **15.** $h'(v) = \dfrac{5v^2 + 3}{3(\sqrt[3]{1 + v^2})^2}$
17. $f'(x) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7)$
19. $h'(t) = \frac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1)$
21. $g'(u) = \dfrac{48u^2(u^3 - 1)^7}{(u^3 + 1)^9}$ **23.** $y' = \dfrac{1}{2\sqrt{x}\,(x + 1)^{3/2}}$
25. $h'(\theta) = \theta \sec^2(\theta^2 \sin\theta)(\theta \cos\theta + 2 \sin\theta)$
27. $y' = -\frac{1}{2}\sqrt{1 + \sin x}$
29. $H'(r) = \dfrac{2(r^2 - 1)^2(r^2 + 3r + 5)}{(2r + 1)^6}$
31. $y' = -4 \sin(\sec 4x) \sec 4x \tan 4x$
33. $y' = \left(x \cos\sqrt{1 + x^2}\right)/\sqrt{1 + x^2}$
35. $y' = \dfrac{16 \sin 2x(1 - \cos 2x)^3}{(1 + \cos 2x)^5}$
37. $y' = -2 \cos\theta \cot(\sin\theta) \csc^2(\sin\theta)$
39. $f'(t) = -\sin t \sec^2(\sec(\cos t)) \sec(\cos t) \tan(\cos t)$
41. $y' = \dfrac{1 + 1/(2\sqrt{x})}{2\sqrt{x + \sqrt{x}}}$
43. $g'(x) = p(2r \sin rx + n)^{p-1}(2r^2 \cos rx)$
45. $y' = \dfrac{-\pi \cos(\tan \pi x) \sec^2(\pi x) \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$
47. $y' = -3 \cos 3\theta \sin(\sin 3\theta)$;
$y'' = -9 \cos^2(3\theta) \cos(\sin 3\theta) + 9(\sin 3\theta) \sin(\sin 3\theta)$
49. $y' = \dfrac{-\sec t \tan t}{2\sqrt{1 - \sec t}}$;
$y'' = \dfrac{\sec t\,(3 \sec^3 t - 4 \sec^2 t - \sec t + 2)}{4(1 - \sec t)^{3/2}}$
51. $y = 18x + 1$ **53.** $y = -x + \pi$
55. (a) $y = \pi x - \pi + 1$ (b)



57. (a) $f'(x) = \dfrac{2 - 2x^2}{\sqrt{2 - x^2}}$

59. $((\pi/2) + 2n\pi, 3)$, $((3\pi/2) + 2n\pi, -1)$, n an integer

61. 24 **63.** (a) 30 (b) 36

65. (a) $\frac{3}{4}$ (b) Does not exist (c) -2 **67.** $-\frac{1}{6}\sqrt{2}$

69. 120 **71.** 96 **73.** $2^{103} \sin 2x$

75. $v(t) = \frac{5}{2}\pi \cos(10\pi t)$ cm/s

77. (a) $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$ (b) 0.16

79. dv/dt is the rate of change of velocity with respect to time; dv/ds is the rate of change of velocity with respect to displacement

81. (b) The factored form **85.** (b) $-n \cos^{n-1}x \sin[(n + 1)x]$

EXERCISES 2.6 ■ PAGE 166

1. (a) $y' = 9x/y$ (b) $y = \pm\sqrt{9x^2 - 1}$, $y' = \pm 9x/\sqrt{9x^2 - 1}$

3. (a) $y' = -\sqrt{y}/\sqrt{x}$ (b) $y = (1 - \sqrt{x})^2$, $y' = 1 - 1/\sqrt{x}$

5. $y' = \dfrac{2y - x}{y - 2x}$ **7.** $y' = -\dfrac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}$

9. $y' = \dfrac{x(x + 2y)}{2x^2y + 4xy^2 + 2y^3 + x^2}$ **11.** $y' = \dfrac{2x + y \sin x}{\cos x - 2y}$

13. $y' = \dfrac{1 - 8x^3\sqrt{x + y}}{8y^3\sqrt{x + y} - 1}$ **15.** $y' = \dfrac{y \sec^2(x/y) - y^2}{y^2 + x \sec^2(x/y)}$

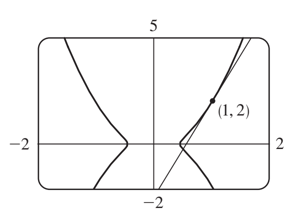
17. $y' = \dfrac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

19. $y' = -\dfrac{y \cos(xy) + \sin(x + y)}{x \cos(xy) + \sin(x + y)}$ **21.** $-\frac{16}{13}$

23. $\dfrac{dx}{dy} = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$ **25.** $y = \frac{1}{2}x$

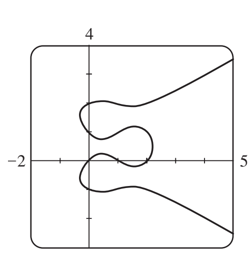
27. $y = \frac{3}{4}x - \frac{1}{2}$ **29.** $y = x + \frac{1}{2}$ **31.** $y = -\frac{9}{13}x + \frac{40}{13}$

33. (a) $y = \frac{9}{2}x - \frac{5}{2}$ (b)



35. $-1/(4y^3)$ **37.** $\dfrac{\cos^2y \cos x + \sin^2x \sin y}{\cos^3y}$ **39.** 0

41. (a)

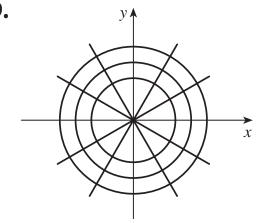


Eight; $x \approx 0.42, 1.58$

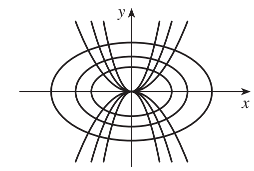
(b) $y = -x + 1$, $y = \frac{1}{3}x + 2$ (c) $1 \mp \frac{1}{3}\sqrt{3}$

43. $(\pm\frac{5}{4}\sqrt{3}, \pm\frac{5}{4})$ **45.** $(x_0x/a^2) - (y_0y/b^2) = 1$

49.



51.



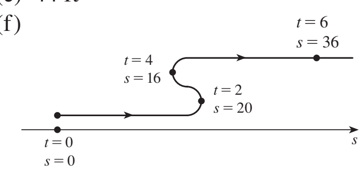
55. (a) $\dfrac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$ (b) ≈ -4.04 L/atm

57. $(\pm\sqrt{3}, 0)$ **59.** $(-1, -1)$, $(1, 1)$

61. (a) 0 (b) $-\frac{1}{2}$

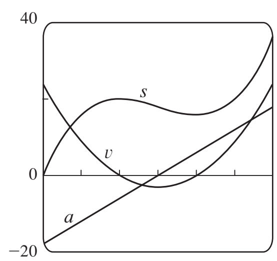
EXERCISES 2.7 ■ PAGE 178

1. (a) $3t^2 - 18t + 24$ (b) 9 ft/s (c) $t = 2$ s, 4 s
(d) $0 \leqslant t < 2$ or $t > 4$
(e) 44 ft
(f)



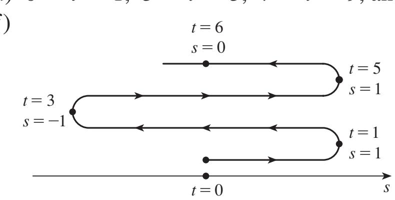
(g) $6t - 18$; -12 ft/s^2

(h)



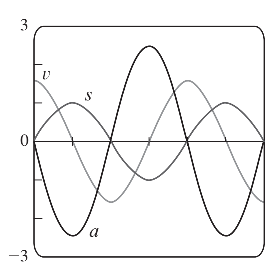
(i) Speeding up when $2 < t < 3$ and $4 < t \leqslant 6$; slowing down when $0 \leqslant t < 2$ and $3 < t < 4$

3. (a) $(\pi/2) \cos(\pi t/2)$ (b) 0 ft/s
(c) $t = 2n + 1$, t a nonnegative integer
(d) $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on (e) 6 ft
(f)



(g) $(-\pi^2/4) \sin(\pi t/2)$; $-\pi^2/4$ ft/s^2

(h)



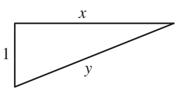
(i) Speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$; slowing down when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$

5. (a) Speeding up when $0 < t < 1$ or $2 < t < 3$; slowing down when $1 < t < 2$
(b) Speeding up when $1 < t < 2$ or $3 < t < 4$; slowing down when $0 < t < 1$ or $2 < t < 3$

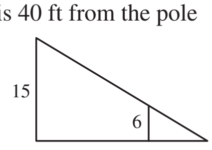
7. (a) 4.9 m/s; −14.7 m/s (b) After 2.5 s (c) $32\frac{5}{8}$ m
(d) ≈5.08 s (e) ≈−25.3 m/s
9. (a) 7.56 m/s (b) ≈6.24 m/s; ≈−6.24 m/s
11. (a) 30 mm^2/mm; the rate at which the area is increasing with respect to side length as x reaches 15 mm
(b) $\Delta A \approx 2x\,\Delta x$
13. (a) (i) 5π (ii) 4.5π (iii) 4.1π
(b) 4π (c) $\Delta A \approx 2\pi r\,\Delta r$
15. (a) 8π ft^2/ft (b) 16π ft^2/ft (c) 24π ft^2/ft
The rate increases as the radius increases.
17. (a) 6 kg/m (b) 12 kg/m (c) 18 kg/m
At the right end; at the left end
19. (a) 4.75 A (b) 5 A; $t = \frac{2}{3}$ s
23. (a) $dV/dP = -C/P^2$ (b) At the beginning
25. (a) 16 million/year; 78.5 million/year
(b) $P(t) = at^3 + bt^2 + ct + d$, where $a \approx -0.0002849$, $b \approx 0.5224331$, $c \approx -6.395641$, $d \approx 1720.586$
(c) $P'(t) = 3at^2 + 2bt + c$
(d) 14.16 million/year (smaller); 71.72 million/year (smaller)
(e) $P'(85) \approx 76.24$ million/year
27. (a) 0.926 cm/s; 0.694 cm/s; 0
(b) 0; −92.6 (cm/s)/cm; −185.2 (cm/s)/cm
(c) At the center; at the edge
29. (a) $C'(x) = 3 + 0.02x + 0.0006x^2$
(b) \$11/pair; the rate at which the cost is changing as the 100th pair of jeans is being produced; the cost of the 101st pair
(c) \$11.07
31. (a) $[xp'(x) - p(x)]/x^2$; the average productivity increases as new workers are added.
33. −0.2436 K/min
35. (a) 0 and 0 (b) $C = 0$
(c) (0, 0), (500, 50); it is possible for the species to coexist.

EXERCISES 2.8 ■ PAGE 185

1. $dV/dt = 3x^2\,dx/dt$ **3.** 48 cm^2/s **5.** $3/(25\pi)$ m/min
7. 128π cm^2/min **9.** (a) 1 (b) 25 **11.** −18
13. (a) The plane's altitude is 1 mi and its speed is 500 mi/h.
(b) The rate at which the distance from the plane to the station is increasing when the plane is 2 mi from the station
(c)



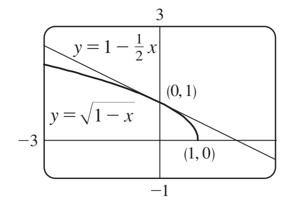
(d) $y^2 = x^2 + 1$
(e) $250\sqrt{3}$ mi/h
15. (a) The height of the pole (15 ft), the height of the man (6 ft), and the speed of the man (5 ft/s)
(b) The rate at which the tip of the man's shadow is moving when he is 40 ft from the pole
(c)



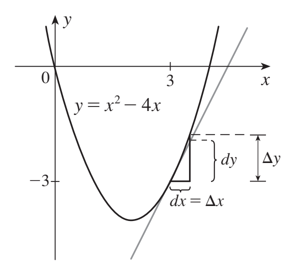
(d) $\dfrac{15}{6} = \dfrac{x + y}{y}$ (e) $\frac{25}{3}$ ft/s
17. 65 mi/h **19.** $837/\sqrt{8674} \approx 8.99$ ft/s
21. −1.6 cm/min **23.** $\frac{720}{13} \approx 55.4$ km/h
25. $(10{,}000 + 800{,}000\pi/9) \approx 2.89 \times 10^5$ cm^3/min
27. $\frac{10}{3}$ cm/min **29.** $6/(5\pi) \approx 0.38$ ft/min
31. $150\sqrt{3}$ cm^2/min **33.** 5 m **35.** πr^2 cm^2/h
37. 80 cm^3/min **39.** $\frac{107}{810} \approx 0.132$ Ω/s
41. $\sqrt{7}\,\pi/21 \approx 0.396$ m/min
43. (a) 360 ft/s (b) 0.096 rad/s
45. $\frac{10}{9}\pi$ km/min **47.** $1650/\sqrt{31} \approx 296$ km/h
49. $\frac{7}{4}\sqrt{15} \approx 6.78$ m/s

EXERCISES 2.9 ■ PAGE 192

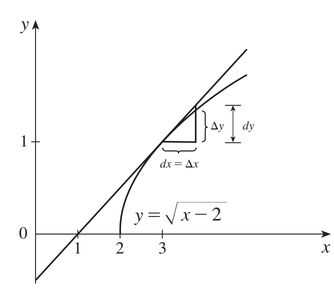
1. $L(x) = 16x + 23$ **3.** $L(x) = \frac{1}{4}x + 1$
5. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$;
$\sqrt{0.9} \approx 0.95$,
$\sqrt{0.99} \approx 0.995$



7. $-0.368 < x < 0.677$ **9.** $-0.045 < x < 0.055$
11. (a) $dy = -\dfrac{4x}{(x^2 - 3)^3}\,dx$ (b) $dy = -\dfrac{2t^3}{\sqrt{1 - t^4}}\,dt$
13. (a) $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$ (b) $dy = \dfrac{-4v}{(1 + v^2)^2}\,dv$
15. (a) $dy = \sec^2 x\,dx$ (b) −0.2
17. (a) $dy = \dfrac{x}{\sqrt{3 + x^2}}\,dx$ (b) −0.05
19. $\Delta y = 1.25$, $dy = 1$



21. $\Delta y \approx 0.34$, $dy = 0.4$



23. 15.968 **25.** $10.00\overline{3}$ **27.** $\pi/90 \approx 0.0349$
31. (a) 270 cm^3, 0.01, 1% (b) 36 cm^2, $0.00\overline{6}$, $0.\overline{6}$%
33. (a) $84/\pi \approx 27$ cm^2; $\frac{1}{84} \approx 0.012 = 1.2\%$
(b) $1764/\pi^2 \approx 179$ cm^3; $\frac{1}{56} \approx 0.018 = 1.8\%$
35. (a) $2\pi rh\,\Delta r$ (b) $\pi(\Delta r)^2 h$
41. (a) 4.8, 5.2 (b) Too large

CHAPTER 2 REVIEW ■ PAGE 196

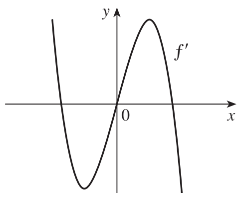
True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. False **13.** True **15.** True

Exercises

1. (a) (i) 3 m/s (ii) 2.75 m/s (iii) 2.625 m/s (iv) 2.525 m/s (b) 2.5 m/s

3.



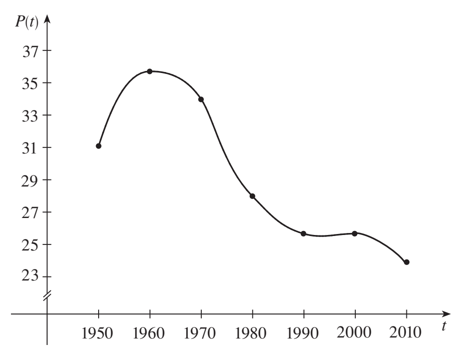
5. $a = f,\ c = f',\ b = f''$

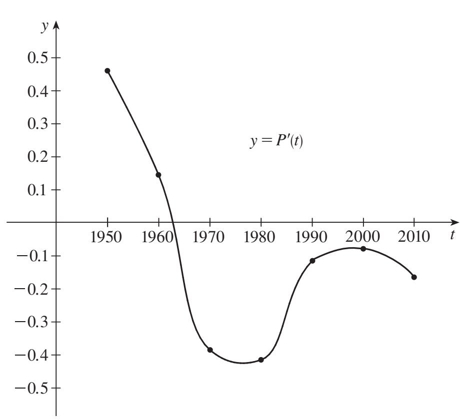
7. (a) The rate at which the cost changes with respect to the interest rate; dollars/(percent per year)
(b) As the interest rate increases past 10%, the cost is increasing at a rate of \$1200/(percent per year).
(c) Always positive

9. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/year).
(b)

t	$P'(t)$
1950	0.460
1960	0.145
1970	−0.385
1980	−0.415
1990	−0.115
2000	−0.085
2010	−0.170

(c)





(d) By obtaining data for the mid-decade years

11. $f'(x) = 3x^2 + 5$ **13.** $4x^7(x+1)^3(3x+2)$

15. $\frac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$ **17.** $x(\pi x \cos \pi x + 2 \sin \pi x)$

19. $\dfrac{8t^3}{(t^4+1)^2}$ **21.** $-\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

23. $\dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$ **25.** $\dfrac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1+\tan 2\theta)^2}$

27. $-(x-1)^{-2}$ **29.** $\dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

31. $-6x\csc^2(3x^2+5)$ **33.** $\dfrac{\cos\sqrt{x} - \sqrt{x}\sin\sqrt{x}}{2\sqrt{x}}$

35. $2\cos\theta\,\tan(\sin\theta)\sec^2(\sin\theta)$

37. $\frac{1}{5}(x\tan x)^{-4/5}(\tan x + x\sec^2 x)$

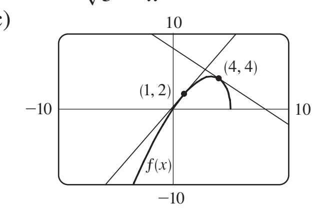
39. $\cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\dfrac{3x^2}{2\sqrt{1+x^3}}$

41. $-\frac{4}{27}$ **43.** $-5x^4/y^{11}$ **45.** 1

47. $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$

49. $y = 2x + 1,\ y = -\frac{1}{2}x + 1$

51. (a) $\dfrac{10 - 3x}{2\sqrt{5-x}}$ (b) $y = \frac{7}{4}x + \frac{1}{4},\ y = -x + 8$
(c)



53. $(\pi/4, \sqrt{2}),\ (5\pi/4, -\sqrt{2})$ **55.** $y = -\frac{2}{3}x^2 + \frac{14}{3}x$

59. (a) 4 (b) 6 (c) $\frac{7}{9}$ (d) 12

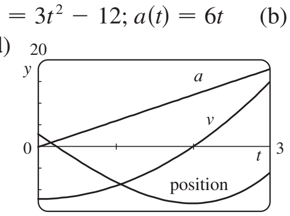
61. $2xg(x) + x^2g'(x)$ **63.** $2g(x)g'(x)$

65. $g'(g(x))g'(x)$ **67.** $g'(\sin x)\cdot\cos x$

69. $\dfrac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2}$

71. $f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$

73. (a) $v(t) = 3t^2 - 12;\ a(t) = 6t$ (b) $t > 2;\ 0 \leqslant t < 2$
(c) 23 (d)



(e) $t > 2;\ 0 < t < 2$

75. 4 kg/m **77.** $\frac{4}{3}$ cm²/min **79.** 13 ft/s **81.** 400 ft/h

83. (a) $L(x) = 1 + x;\ \sqrt[3]{1+3x} \approx 1 + x;\ \sqrt[3]{1.03} \approx 1.01$
(b) $-0.23 < x < 0.40$

85. $12 + \frac{3}{2}\pi \approx 16.7\ \text{cm}^2$ **87.** $\left[\dfrac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{32}$

89. $\frac{1}{4}$ **91.** $\frac{1}{8}x^2$

PROBLEMS PLUS ■ PAGE 200

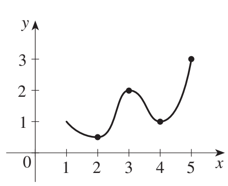
1. $\left(\pm\sqrt{3}/2, \frac{1}{4}\right)$ **5.** $3\sqrt{2}$ **9.** $\left(0, \frac{5}{4}\right)$

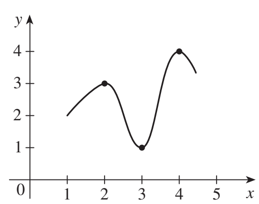
11. 3 lines; $(0, 2)$, $\left(\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$, $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$
13. (a) $4\pi\sqrt{3}/\sqrt{11}$ rad/s (b) $40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm
(c) $-480\pi \sin\theta\left(1 + \cos\theta/\sqrt{8 + \cos^2\theta}\right)$ cm/s
15. $x_T \in (3, \infty)$, $y_T \in (2, \infty)$, $x_N \in \left(0, \frac{5}{3}\right)$, $y_N \in \left(-\frac{5}{2}, 0\right)$
17. (b) (i) 53° (or 127°) (ii) 63° (or 117°)
19. R approaches the midpoint of the radius AO.
21. $-\sin a$ **23.** $(1, -2)$, $(-1, 0)$
25. $\sqrt{29}/58$ **27.** $2 + \frac{375}{128}\pi \approx 11.204$ cm³/min

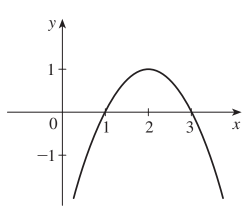
CHAPTER 3

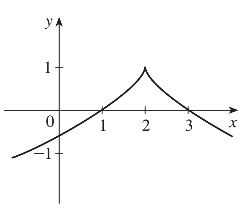
EXERCISES 3.1 ■ PAGE 211

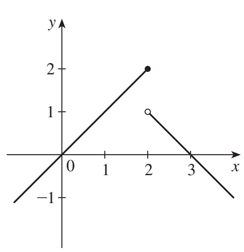
Abbreviations: abs, absolute; loc, local; max, maximum; min, minimum
1. Abs min: smallest function value on the entire domain of the function; loc min at c: smallest function value when x is near c
3. Abs max at s, abs min at r, loc max at c, loc min at b and r, neither a max nor a min at a and d
5. Abs max $f(4) = 5$, loc max $f(4) = 5$ and $f(6) = 4$, loc min $f(2) = 2$ and $f(1) = f(5) = 3$

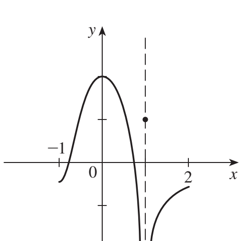
7.


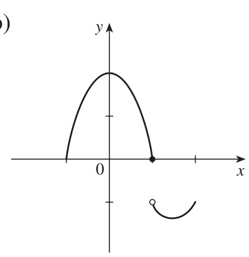
9.


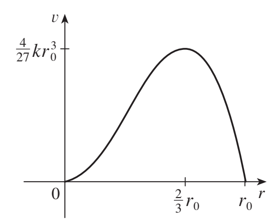
11. (a)


(b)


(c)


13. (a)


(b)


15. Abs max $f(3) = 4$ **17.** Abs max $f(1) = 1$
19. Abs min $f(0) = 0$
21. Abs max $f(\pi/2) = 1$; abs min $f(-\pi/2) = -1$
23. Abs min $f(-1) = 1$; loc min $f(-1) = 1$
25. Abs max $f(0) = 1$
27. Abs min $f(1) = -1$; loc min $f(0) = 0$
29. $\frac{1}{3}$ **31.** $-2, 3$ **33.** 0 **35.** 0, 2 **37.** $0, \frac{4}{9}$
39. $0, \frac{8}{7}, 4$ **41.** $n\pi$ (n an integer) **43.** 10
45. $f(2) = 16$, $f(5) = 7$ **47.** $f(-1) = 8$, $f(2) = -19$
49. $f(-2) = 33$, $f(2) = -31$ **51.** $f(0.2) = 5.2$, $f(1) = 2$
53. $f(4) = 4 - \sqrt[3]{4}$, $f\left(\sqrt{3}/9\right) = -2\sqrt{3}/9$
55. $f(\pi/6) = \frac{3}{2}\sqrt{3}$, $f(\pi/2) = 0$
57. $f\left(\dfrac{a}{a + b}\right) = \dfrac{a^a b^b}{(a + b)^{a+b}}$
59. (a) 2.19, 1.81 (b) $\frac{6}{25}\sqrt{\frac{3}{5}} + 2$, $-\frac{6}{25}\sqrt{\frac{3}{5}} + 2$
61. (a) 0.32, 0.00 (b) $\frac{3}{16}\sqrt{3}, 0$ **63.** ≈ 3.9665°C
65. About 4.1 months after January 1
67. (a) $r = \frac{2}{3}r_0$ (b) $v = \frac{4}{27}kr_0^3$
(c)


EXERCISES 3.2 ■ PAGE 219

1. 1, 5
3. (a) g is continuous on $[0, 8]$ and differentiable on $(0, 8)$.
(b) 2.2, 6.4 (c) 3.7, 5.5
5. 1 **7.** π
9. f is not differentiable on $(-1, 1)$ **11.** 1
13. $\sqrt{3}/9$ **15.** 1; yes **17.** f is not continous at 3
25. 16 **27.** No **33.** No

EXERCISES 3.3 ■ PAGE 227

Abbreviations: inc, increasing; dec, decreasing; CD, concave downward; CU, concave upward; HA, horizontal asymptote; VA, vertical asymptote; IP, inflection point(s)
1. (a) (1, 3), (4, 6) (b) (0, 1), (3, 4) (c) (0, 2)
(d) (2, 4), (4, 6) (e) (2, 3)
3. (a) I/D Test (b) Concavity Test
(c) Find points at which the concavity changes.
5. (a) Inc on (1, 5); dec on (0, 1) and (5, 6)
(b) Loc max at $x = 5$, loc min at $x = 1$
7. (a) 3, 5 (b) 2, 4, 6 (c) 1, 7
9. (a) Inc on $(-\infty, -1)$, $(3, \infty)$; dec on $(-1, 3)$
(b) Loc max $f(-1) = 9$; loc min $f(3) = -23$
(c) CU on $(1, \infty)$, CD on $(-\infty, 1)$; IP $(1, -7)$
11. (a) Inc on $(-1, 0)$, $(1, \infty)$; dec on $(-\infty, -1)$, $(0, 1)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 1) = 2$
(c) CU on $\left(-\infty, -\sqrt{3}/3\right)$, $\left(\sqrt{3}/3, \infty\right)$;
CD on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$; IP $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$

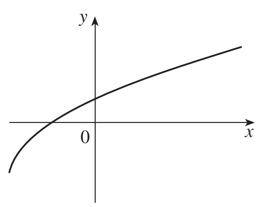
13. (a) Inc on $(0, \pi/4)$, $(5\pi/4, 2\pi)$; dec on $(\pi/4, 5\pi/4)$
(b) Loc max $f(\pi/4) = \sqrt{2}$; loc min $f(5\pi/4) = -\sqrt{2}$
(c) CU on $(3\pi/4, 7\pi/4)$; CD on $(0, 3\pi/4)$, $(7\pi/4, 2\pi)$;
IP $(3\pi/4, 0)$, $(7\pi/4, 0)$

15. Loc max $f(1) = 2$; loc min $f(0) = 1$

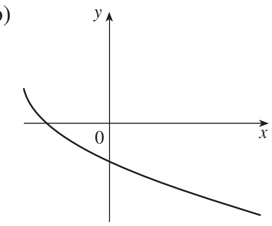
17. Loc min $f\left(\frac{1}{16}\right) = -\frac{1}{4}$

19. (a) f has a local maximum at 2.
(b) f has a horizontal tangent at 6.

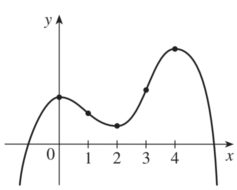
21. (a)



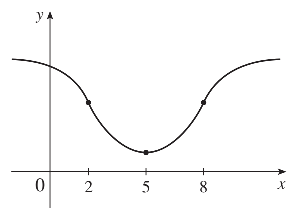
(b)



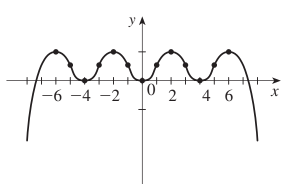
23.



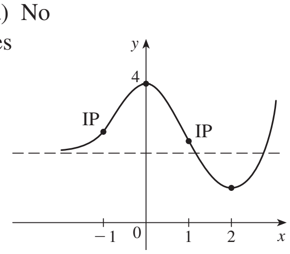
25.



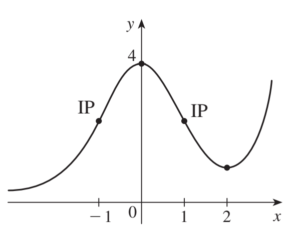
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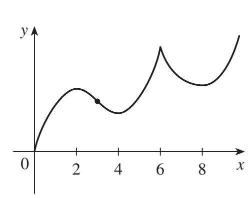
29. (a) No
(b) Yes



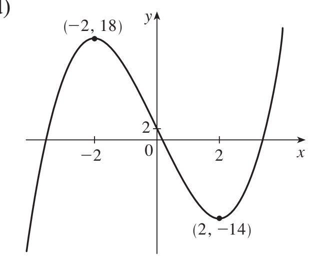
(c)



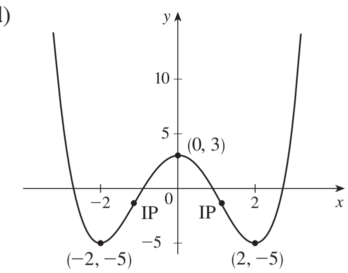
31. (a) Inc on $(0, 2)$, $(4, 6)$, $(8, \infty)$;
dec on $(2, 4)$, $(6, 8)$
(b) Loc max at $x = 2, 6$;
loc min at $x = 4, 8$
(c) CU on $(3, 6)$, $(6, \infty)$;
CD on $(0, 3)$ (d) 3
(e) See graph at right.



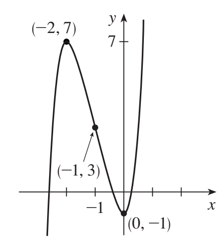
33. (a) Inc on $(-\infty, -2)$, $(2, \infty)$; dec on $(-2, 2)$
(b) Loc max $f(-2) = 18$; loc min $f(2) = -14$
(c) CU on $(0, \infty)$, CD on $(-\infty, 0)$; IP $(0, 2)$
(d)



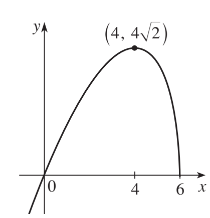
35. (a) Inc on $(-2, 0)$, $(2, \infty)$; dec on $(-\infty, -2)$, $(0, 2)$
(b) Loc max $f(0) = 3$; loc min $f(\pm 2) = -5$
(c) CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$, $\left(\frac{2}{\sqrt{3}}, \infty\right)$; CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$;
IPs $\left(\pm\frac{2}{\sqrt{3}}, -\frac{13}{9}\right)$
(d)



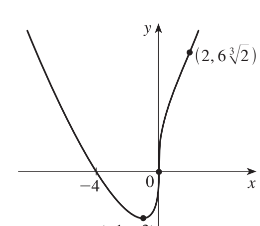
37. (a) Inc on $(-\infty, -2)$, $(0, \infty)$;
dec on $(-2, 0)$
(b) Loc max $h(-2) = 7$;
loc min $h(0) = -1$
(c) CU on $(-1, \infty)$;
CD on $(-\infty, -1)$; IP $(-1, 3)$
(d) See graph at right.



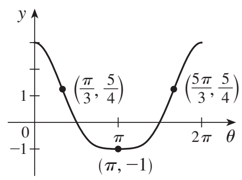
39. (a) Inc on $(-\infty, 4)$;
dec on $(4, 6)$
(b) Loc max $F(4) = 4\sqrt{2}$
(c) CD on $(-\infty, 6)$; No IP
(d) See graph at right.



41. (a) Inc on $(-1, \infty)$;
dec on $(-\infty, -1)$
(b) Loc min $C(-1) = -3$
(c) CU on $(-\infty, 0)$, $(2, \infty)$;
CD on $(0, 2)$;
IPs $(0, 0)$, $\left(2, 6\sqrt[3]{2}\right)$
(d) See graph at right.



43. (a) Inc on $(\pi, 2\pi)$; dec on $(0, \pi)$
(b) Loc min $f(\pi) = -1$
(c) CU on $(\pi/3, 5\pi/3)$; CD on $(0, \pi/3)$, $(5\pi/3, 2\pi)$; IPs $\left(\pi/3, \frac{5}{4}\right)$, $\left(5\pi/3, \frac{5}{4}\right)$
(d) See graph at right.



45. $(3, \infty)$

47. (a) Loc and abs max $f(1) = \sqrt{2}$, no min (b) $\frac{1}{4}\left(3 - \sqrt{17}\right)$

49. (b) CD on (0, 0.85), (1.57, 2.29); CU on (0.85, 1.57), (2.29, π); IP (0.85, 0.74), (1.57, 0), (2.29, −0.74)

51. CU on $(-\infty, -0.6)$, $(0.0, \infty)$; CD on $(-0.6, 0.0)$

53. (a) The rate of increase is initially very small, increases to a maximum at $t \approx 8$ h, then decreases toward 0.
(b) When $t = 8$ (c) CU on (0, 8); CD on (8, 18)
(d) (8, 350)

55. If $D(t)$ is the size of the deficit as a function of time, then at the time of the speech $D'(t) > 0$ and $D''(t) < 0$.

57. $K(3) - K(2)$; CD

59. $f(x) = \frac{1}{9}(2x^3 + 3x^2 - 12x + 7)$

61. (a) $a = 0, b = -1$ (b) $y = -x$ at (0, 0)

EXERCISES 3.4 ■ PAGE 241

1. (a) As x becomes large, $f(x)$ approaches 5.
(b) As x becomes large negative, $f(x)$ approaches 3.

3. (a) −2 (b) 2 (c) ∞ (d) $-\infty$
(e) $x = 1, x = 3, y = -2, y = 2$

5. 0 **7.** $\frac{2}{5}$ **9.** $\frac{3}{2}$ **11.** 0 **13.** −1 **15.** 4

17. −2 **19.** $\frac{\sqrt{3}}{4}$ **21.** $\frac{1}{6}$ **23.** $\frac{1}{2}(a - b)$ **25.** ∞

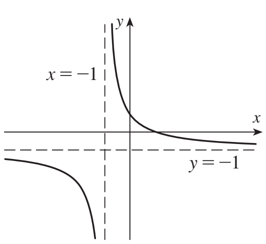
27. $-\infty$ **29.** ∞ **31.** 1 **33.** (a), (b) $-\frac{1}{2}$

35. $y = 4, x = -3$ **37.** $y = 2; x = -2, x = 1$

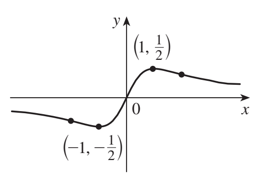
39. $x = 5$ **41.** $y = 3$ **43.** (a) 0 (b) $\pm\infty$

45. $f(x) = \dfrac{2 - x}{x^2(x - 3)}$ **47.** (a) $\frac{5}{4}$ (b) 5

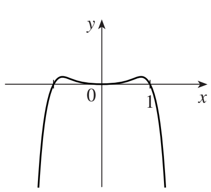
49. $y = -1$



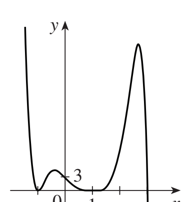
51. $y = 0$



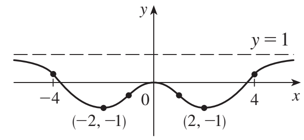
53. $-\infty, -\infty$



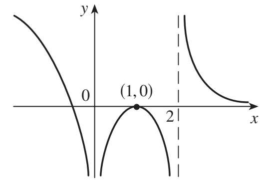
55. $-\infty, \infty$



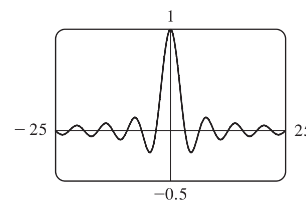
57.



59.



61. (a) 0 (b) An infinite number of times



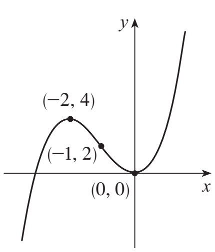
63. 4 **65.** $N \geqslant 15$

67. $N \leqslant -9, N \leqslant -19$ **69.** (a) $x > 100$

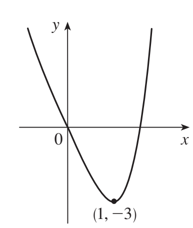
EXERCISES 3.5 ■ PAGE 250

Abbreviation: int, intercept; SA, slant asymptote

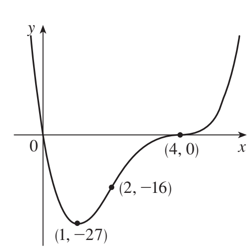
1. A. $\mathbb{R}$ B. y-int 0; x-int −3, 0
C. None D. None
E. Inc on $(-\infty, -2)$, $(0, \infty)$; dec on $(-2, 0)$
F. Loc max $f(-2) = 4$; loc min $f(0) = 0$
G. CU on $(-1, \infty)$; CD on $(-\infty, -1)$; IP $(-1, 2)$
H. See graph at right.



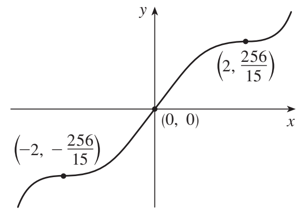
3. A. $\mathbb{R}$ B. y-int 0; x-int 0, $\sqrt[3]{4}$
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -3$
G. CU on $(-\infty, \infty)$
H. See graph at right.



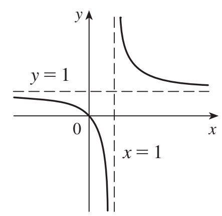
5. A. $\mathbb{R}$ B. y-int 0; x-int 0, 4
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = -27$
G. CU on $(-\infty, 2)$, $(4, \infty)$; CD on (2, 4);
IPs (2, −16), (4, 0)
H. See graph at right.



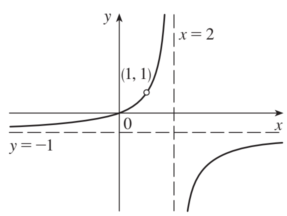
7. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About (0, 0) D. None
E. Inc on $(-\infty, \infty)$
F. None
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$;
IPs $\left(-2, -\frac{256}{15}\right)$, (0, 0), $\left(2, \frac{256}{15}\right)$
H. See graph at right.



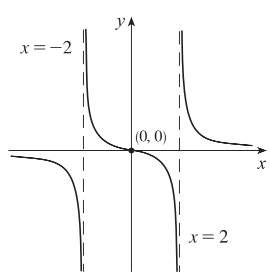
9. A. $\{x \mid x \neq 1\}$
B. y-int 0; x-int 0
C. None D. VA $x = 1$, HA $y = 1$
E. Dec on $(-\infty, 1)$, $(1, \infty)$
F. None
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$
H. See graph at right.



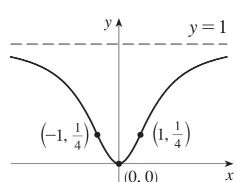
11. A. $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$
B. y-int 0; x-int 0 C. None
D. HA $y = -1$; VA $x = 2$
E. Inc on $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$
F. None
G. CU on $(-\infty, 1)$, $(1, 2)$;
CD on $(2, \infty)$
H. See graph at right.



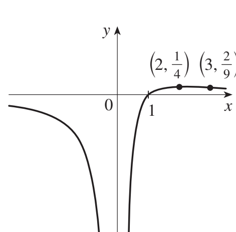
13. A. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
B. y-int 0; x-int 0
C. About (0, 0)
D. VA $x = \pm 2$; HA $y = 0$
E. Dec on $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$
F. No local extrema
G. CU on $(-2, 0)$, $(2, \infty)$;
CD on $(-\infty, -2)$, $(0, 2)$; IP (0, 0)
H. See graph at right.



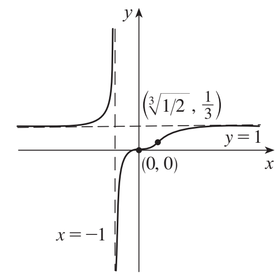
15. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. HA $y = 1$
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $\left(\pm 1, \frac{1}{4}\right)$
H. See graph at right.



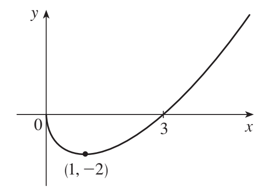
17. A. $(-\infty, 0) \cup (0, \infty)$ B. x-int 1
C. None D. HA $y = 0$; VA $x = 0$
E. Inc on $(0, 2)$;
dec on $(-\infty, 0)$, $(2, \infty)$
F. Loc max $f(2) = \frac{1}{4}$
G. CU on $(3, \infty)$;
CD on $(-\infty, 0)$, $(0, 3)$; IP $\left(3, \frac{2}{9}\right)$
H. See graph at right.



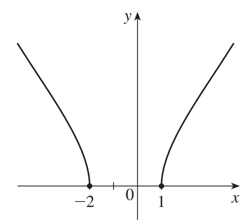
19. A. $(-\infty, -1) \cup (-1, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = -1$; HA $y = 1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$;
F. None
G. CU on $(-\infty, -1)$, $\left(0, \sqrt[3]{\frac{1}{2}}\right)$;
CD on $(-1, 0)$, $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$;
IPs (0, 0), $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$
H. See graph at right.



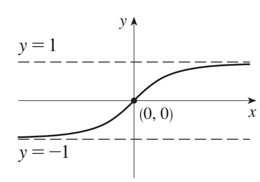
21. A. $[0, \infty)$ B. y-int 0; x-int 0, 3
C. None D. None
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = -2$
G. CU on $(0, \infty)$
H. See graph at right.



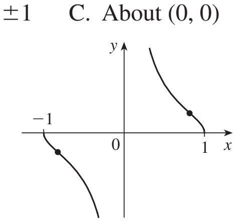
23. A. $(-\infty, -2] \cup [1, \infty)$
B. x-int -2, 1 C. None
D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, -2)$
F. None
G. CD on $(-\infty, -2)$, $(1, \infty)$
H. See graph at right.



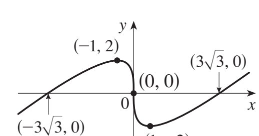
25. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About (0, 0)
D. HA $y = \pm 1$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP (0, 0)
H. See graph at right.



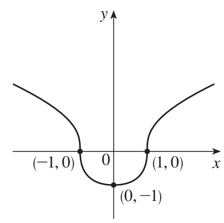
27. A. $[-1, 0) \cup (0, 1]$ B. x-int ± 1 C. About (0, 0)
D. VA $x = 0$
E. Dec on $(-1, 0)$, $(0, 1)$
F. None
G. CU on $\left(-1, -\sqrt{2/3}\right)$,
$\left(0, \sqrt{2/3}\right)$;
CD on $\left(-\sqrt{2/3}, 0\right)$, $\left(\sqrt{2/3}, 1\right)$;
IPs $\left(\pm\sqrt{2/3}, \pm 1/\sqrt{2}\right)$
H. See graph at right.



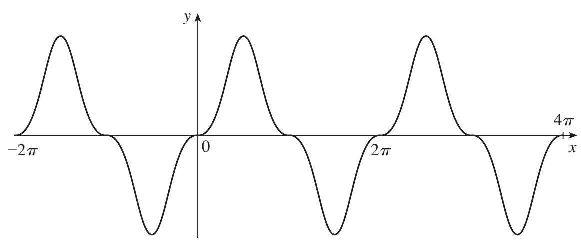
29. A. $\mathbb{R}$ B. y-int 0; x-int $\pm 3\sqrt{3}$, 0 C. About (0, 0)
D. None E. Inc on $(-\infty, -1)$, $(1, \infty)$; dec on $(-1, 1)$
F. Loc max $f(-1) = 2$;
loc min $f(1) = -2$
G. CU on $(0, \infty)$;
CD on $(-\infty, 0)$; IP (0, 0)
H. See graph at right.



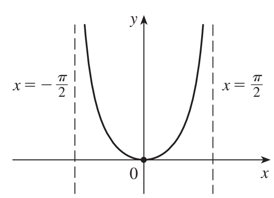
31. A. $\mathbb{R}$ B. y-int -1; x-int ± 1
C. About the y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = -1$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$; IPs $(\pm 1, 0)$
H. See graph at right.



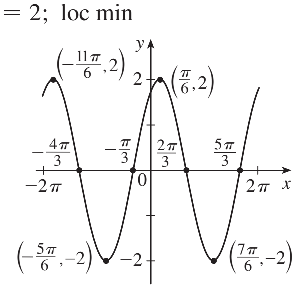
33. A. $\mathbb{R}$ B. y-int 0; x-int $n\pi$ (n an integer)
C. About (0, 0), period 2π D. None
E–G answers for $0 \leqslant x \leqslant \pi$:
E. Inc on $(0, \pi/2)$; dec on $(\pi/2, \pi)$ F. Loc max $f(\pi/2) = 1$
G. Let $\alpha = \sin^{-1}\sqrt{2/3}$; CU on $(0, \alpha)$, $(\pi - \alpha, \pi)$;
CD on $(\alpha, \pi - \alpha)$; IPs at $x = 0, \pi, \alpha, \pi - \alpha$
H.



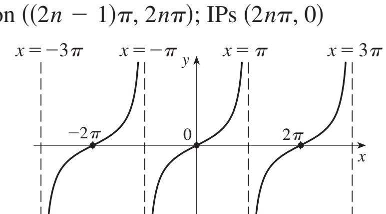
35. A. $(-\pi/2, \pi/2)$ B. y-int 0; x-int 0 C. About y-axis
D. VA $x = \pm\pi/2$
E. Inc on $(0, \pi/2)$;
dec on $(-\pi/2, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-\pi/2, \pi/2)$
H. See graph at right.



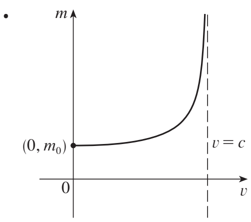
37. A. $[-2\pi, 2\pi]$ B. y-int $\sqrt{3}$; x-int $-4\pi/3, -\pi/3, 2\pi/3, 5\pi/3$ C. Period 2π D. None
E. Inc on $(-2\pi, -11\pi/6)$, $(-5\pi/6, \pi/6)$, $(7\pi/6, 2\pi)$;
dec on $(-11\pi/6, -5\pi/6)$, $(\pi/6, 7\pi/6)$
F. Loc max $f(-11\pi/6) = f(\pi/6) = 2$; loc min
$f(-5\pi/6) = f(7\pi/6) = -2$
G. CU on $(-4\pi/3, -\pi/3)$,
$(2\pi/3, 5\pi/3)$;
CD on $(-2\pi, -4\pi/3)$,
$(-\pi/3, 2\pi/3)$, $(5\pi/3, 2\pi)$;
IPs $(-4\pi/3, 0)$, $(-\pi/3, 0)$,
$(2\pi/3, 0)$, $(5\pi/3, 0)$
H. See graph at right.



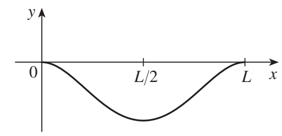
39. A. All reals except $(2n + 1)\pi$ (n an integer)
B. y-int 0; x-int $2n\pi$ C. About the origin, period 2π
D. VA $x = (2n + 1)\pi$ E. Inc on $((2n - 1)\pi, (2n + 1)\pi)$
F. None G. CU on $(2n\pi, (2n + 1)\pi)$;
CD on $((2n - 1)\pi, 2n\pi)$; IPs $(2n\pi, 0)$
H.



41.

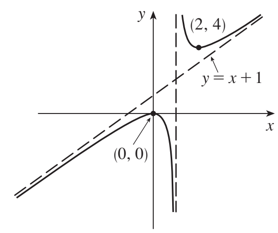


43.

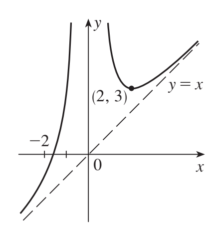


45. $y = x - 1$ **47.** $y = 2x - 3$

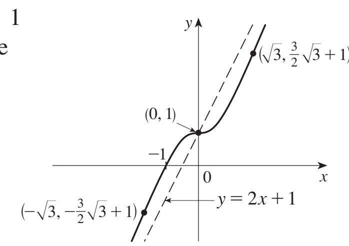
49. A. $(-\infty, 1) \cup (1, \infty)$
B. y-int 0; x-int 0 C. None
D. VA $x = 1$; SA $y = x + 1$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 1)$, $(1, 2)$
F. Loc max $f(0) = 0$;
loc min $f(2) = 4$
G. CU on $(1, \infty)$; CD on $(-\infty, 1)$
H. See graph at right.



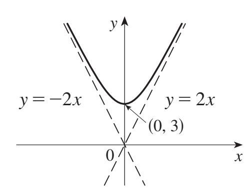
51. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int $-\sqrt[3]{4}$ C. None
D. VA $x = 0$; SA $y = x$
E. Inc on $(-\infty, 0)$, $(2, \infty)$;
dec on $(0, 2)$
F. Loc min $f(2) = 3$
G. CU on $(-\infty, 0)$, $(0, \infty)$
H. See graph at right.



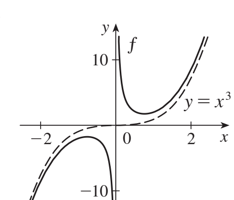
53. A. $\mathbb{R}$ B. y-int 1; x-int -1
C. None D. SA $y = 2x + 1$
E. Inc on $(-\infty, \infty)$ F. None
G. CU on $\left(-\infty, -\sqrt{3}\right)$,
$\left(0, \sqrt{3}\right)$;
CD on $\left(-\sqrt{3}, 0\right)$, $\left(\sqrt{3}, \infty\right)$;
IP $\left(\pm\sqrt{3}, 1 \pm \frac{3}{2}\sqrt{3}\right)$, (0, 1)
H. See graph at right.



55.

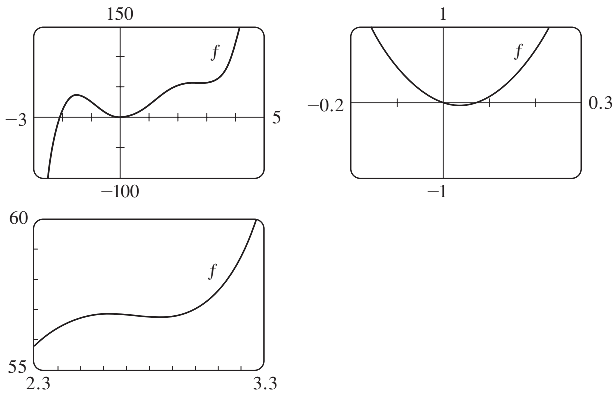


59. VA $x = 0$, asymptotic to $y = x^3$

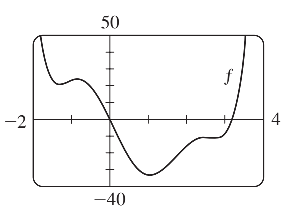


EXERCISES 3.6 ■ PAGE 257

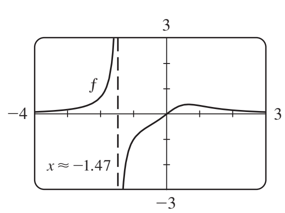
1. Inc on $(-\infty, -1.50)$, $(0.04, 2.62)$, $(2.84, \infty)$; dec on $(-1.50, 0.04)$, $(2.62, 2.84)$; loc max $f(-1.50) \approx 36.47$, $f(2.62) \approx 56.83$; loc min $f(0.04) \approx -0.04$, $f(2.84) \approx 56.73$; CU on $(-0.89, 1.15)$, $(2.74, \infty)$; CD on $(-\infty, -0.89)$, $(1.15, 2.74)$; IPs $(-0.89, 20.90)$, $(1.15, 26.57)$, $(2.74, 56.78)$



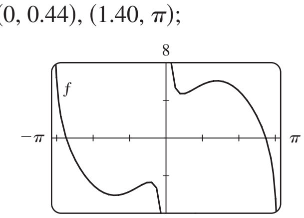
3. Inc on $(-1.31, -0.84)$, $(1.06, 2.50)$, $(2.75, \infty)$; dec on $(-\infty, -1.31)$, $(-0.84, 1.06)$, $(2.50, 2.75)$; loc max $f(-0.84) \approx 23.71$, $f(2.50) \approx -11.02$; loc min $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, $f(2.75) \approx -11.33$; CU on $(-\infty. -1.10)$, $(0.08, 1.72)$, $(2.64, \infty)$; CD on $(-1.10, 0.08)$, $(1.72, 2.64)$; IPs $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, $(2.64, -11.18)$



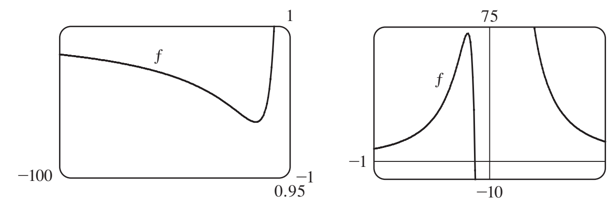
5. Inc on $(-\infty, -1.47)$, $(-1.47, 0.66)$; dec on $(0.66, \infty)$; loc max $f(0.66) \approx 0.38$; CU on $(-\infty, -1.47)$, $(-0.49, 0)$, $(1.10, \infty)$; CD on $(-1.47, -0.49)$, $(0, 1.10)$; IPs $(-0.49, -0.44)$, $(1.10, 0.31)$, $(0, 0)$



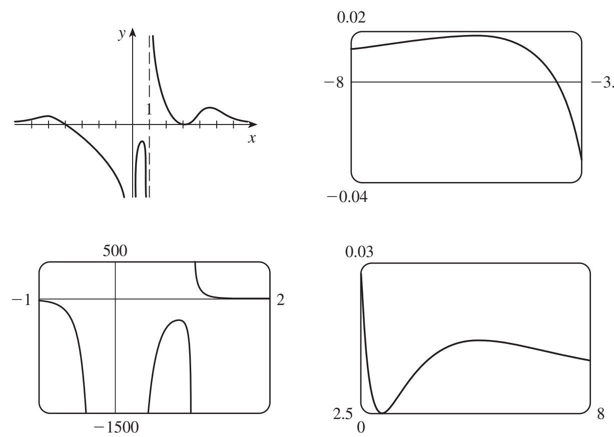
7. Inc on $(-1.40, -0.44)$, $(0.44, 1.40)$; dec on $(-\pi, -1.40)$, $(-0.44, 0)$, $(0, 0.44)$, $(1.40, \pi)$; loc max $f(-0.44) \approx -4.68$, $f(1.40) \approx 6.09$; loc min $f(-1.40) \approx -6.09$, $f(0.44) \approx 4.68$; CU on $(-\pi, -0.77)$, $(0, 0.77)$; CD on $(-0.77, 0)$, $(0.77, \pi)$; IPs $(-0.77, -5.22)$, $(0.77, 5.22)$



9. Inc on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$; dec on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, $(0, \infty)$; CU on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$, $(0, \infty)$; CD on $\left(-\infty, -12 - \sqrt{138}\right)$, $\left(-12 + \sqrt{138}, 0\right)$



11. Loc max $f(-5.6) \approx 0.018$, $f(0.82) \approx -281.5$, $f(5.2) \approx 0.0145$; loc min $f(3) = 0$

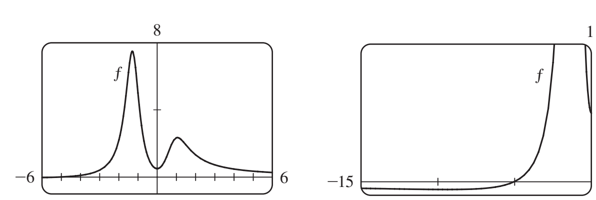


13. $f'(x) = -\dfrac{x(x+1)^2(x^3 + 18x^2 - 44x - 16)}{(x-2)^3(x-4)^5}$

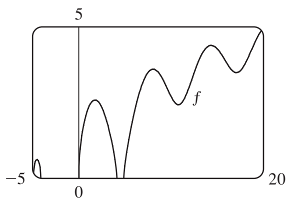
$$f''(x) = 2\frac{(x+1)(x^6 + 36x^5 + 6x^4 - 628x^3 + 684x^2 + 672x + 64)}{(x-2)^4(x-4)^6}$$

CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, $(4, \infty)$; CD on $(-\infty, -35.3)$, $(-5.0, -1)$, $(-0.5, -0.1)$; IPs $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, $(-0.1, 0.0000066)$

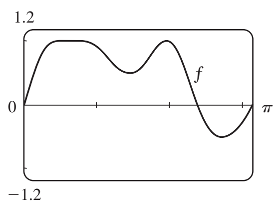
15. Inc on $(-9.41, -1.29)$, $(0, 1.05)$; dec on $(-\infty, -9.41)$, $(-1.29, 0)$, $(1.05, \infty)$; loc max $f(-1.29) \approx 7.49$, $f(1.05) \approx 2.35$; loc min $f(-9.41) \approx -0.056$, $f(0) = 0.5$; CU on $(-13.81, -1.55)$, $(-1.03, 0.60)$, $(1.48, \infty)$; CD on $(-\infty, -13.81)$, $(-1.55, -1.03)$, $(0.60, 1.48)$; IPs $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, $(1.48, 1.93)$

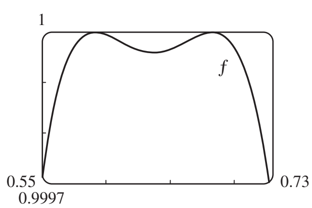


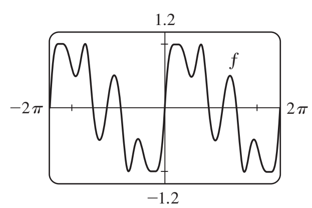
17. Inc on $(-4.91, -4.51)$, $(0, 1.77)$, $(4.91, 8.06)$, $(10.79, 14.34)$, $(17.08, 20)$;
dec on $(-4.51, -4.10)$, $(1.77, 4.10)$, $(8.06, 10.79)$, $(14.34, 17.08)$;
loc max $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, $f(14.34) \approx 4.39$;
loc min $f(10.79) \approx 2.43$, $f(17.08) \approx 3.49$;
CU on $(9.60, 12.25)$, $(15.81, 18.65)$;
CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, $(12.25, 15.81)$, $(18.65, 20)$;
IPs $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, $(18.65, 4.20)$



19. Max $f(0.59) \approx 1$, $f(0.68) \approx 1$, $f(1.96) \approx 1$;
min $f(0.64) \approx 0.99996$, $f(1.46) \approx 0.49$, $f(2.73) \approx -0.51$;
IPs $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, $(2.28, 0.34)$

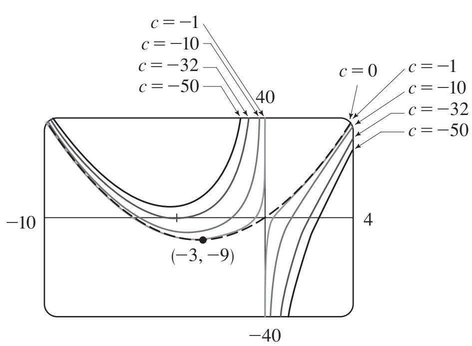




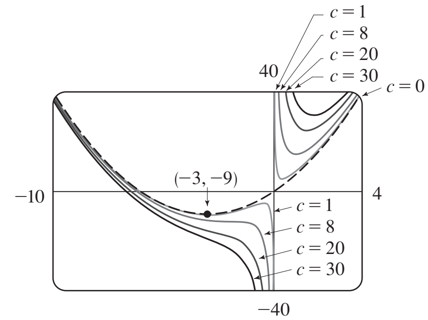


21. For $c < 0$, there is a loc min that moves toward $(-3, -9)$ as c increases. For $0 < c < 8$, there is a loc min that moves toward $(-3, -9)$ and a loc max that moves toward the origin as c decreases. For all $c > 0$, there is a first quadrant loc min that moves toward the origin as c decreases. $c = 0$ is a transitional value that gives the graph of a parabola. For all nonzero c, the y-axis is a VA and there is an IP that moves toward the origin as $|c| \to 0$.

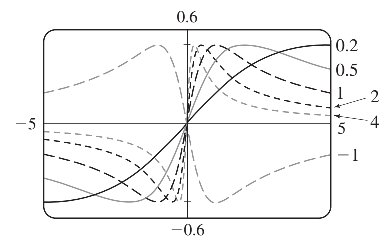
$c \leqslant 0$:



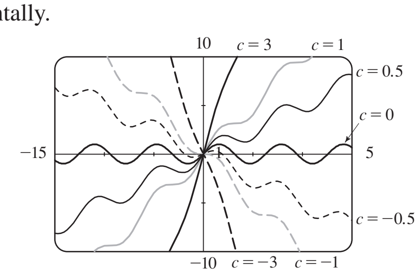
$c \geqslant 0$:



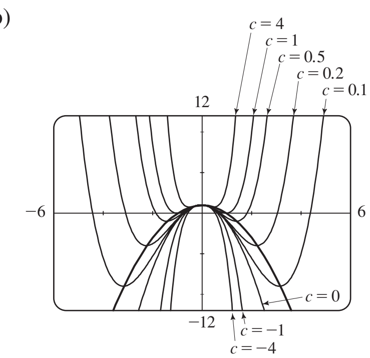
23. For $c > 0$, the maximum and minimum values are always $\pm\frac{1}{2}$, but the extreme points and IPs move closer to the y-axis as c increases. $c = 0$ is a transitional value: when c is replaced by $-c$, the curve is reflected in the x-axis.



25. For $|c| < 1$, the graph has loc max and min values; for $|c| \geqslant 1$ it does not. The function increases for $c \geqslant 1$ and decreases for $c \leqslant -1$. As c changes, the IPs move vertically but not horizontally.



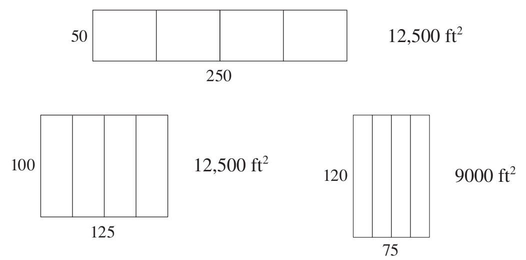
27. (a) Positive (b)



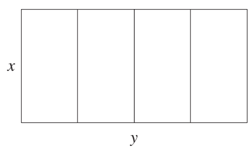
EXERCISES 3.7 ■ PAGE 264

1. (a) 11, 12 (b) 11.5, 11.5 **3.** 10, 10 **5.** $\frac{9}{4}$
7. 25 m by 25 m **9.** $N = 1$

11. (a)



(b)



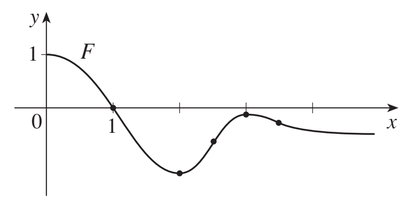
(c) $A = xy$ (d) $5x + 2y = 750$ (e) $A(x) = 375x - \frac{5}{2}x^2$
(f) 14,062.5 ft^2
13. 1000 ft by 1500 ft **15.** 4000 cm^3 **17.** \$191.28
19. $20\sqrt{30}$ ft by $\frac{40}{3}\sqrt{30}$ ft **21.** $\left(-\frac{6}{5}, \frac{3}{5}\right)$
23. $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$ **25.** Square, side $\sqrt{2}\,r$
27. $L/2$, $\sqrt{3}\,L/4$ **29.** Base $\sqrt{3}\,r$, height $3r/2$
31. $4\pi r^3/(3\sqrt{3})$ **33.** $\pi r^2(1 + \sqrt{5})$ **35.** 24 cm by 36 cm
37. (a) Use all of the wire for the square
(b) $40\sqrt{3}/(9 + 4\sqrt{3})$ m for the square
39. 16 in. **41.** $V = 2\pi R^3/(9\sqrt{3})$ **45.** $E^2/(4r)$
47. (a) $\frac{3}{2}s^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta)$ (b) $\cos^{-1}(1/\sqrt{3}) \approx 55°$
(c) $6s\left[h + s/(2\sqrt{2})\right]$
49. Row directly to B **51.** $\approx$4.85 km east of the refinery
53. $10\sqrt[3]{3}/(1 + \sqrt[3]{3})$ ft from the stronger source
55. $(a^{2/3} + b^{2/3})^{3/2}$ **57.** $2\sqrt{6}$
59. (b) (i) \$342,491; \$342/unit; \$390/unit (ii) 400
(iii) \$320/unit
61. (a) $p(x) = 19 - \frac{1}{3000}x$ (b) \$9.50
63. (a) $p(x) = 500 - \frac{1}{8}x$ (b) \$250 (c) \$310
69. 9.35 m **73.** $x = 6$ in. **75.** $\pi/6$ **77.** $\frac{1}{2}(L + W)^2$
79. (a) About 5.1 km from B (b) C is close to B; C is close to D; $W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$
(c) $\approx$1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$

EXERCISES 3.8 ■ PAGE 276

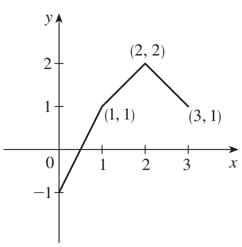
1. (a) $x_2 \approx 7.3$, $x_3 \approx 6.8$ (b) Yes
3. $\frac{9}{2}$ **5.** a, b, c **7.** 1.5215 **9.** -1.25
11. 2.94283096 **13.** (b) 2.630020 **15.** 0.876726
17. $-3.637958, -1.862365, 0.889470$
19. $-0.549700, 2.629658$ **21.** 0.865474
23. $-1.69312029, -0.74466668, 1.26587094$
25. 0.76682579 **27.** (b) 31.622777
33. (a) $-1.293227, -0.441731, 0.507854$ (b) -2.0212
35. (1.519855, 2.306964) **37.** (0.410245, 0.347810)
39. 0.76286%

EXERCISES 3.9 ■ PAGE 282

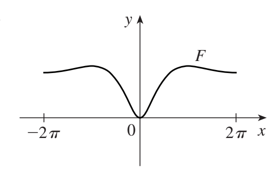
1. $F(x) = 2x^2 + 7x + C$ **3.** $F(x) = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$
5. $F(x) = 4x^3 + 4x^2 + C$ **7.** $F(x) = 5x^{7/5} + 40x^{1/5} + C$
9. $F(x) = \sqrt{2}\,x + C$ **11.** $F(x) = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$
13. $F(x) = \begin{cases} -5/(4x^8) + C_1 & \text{if } x < 0 \\ -5/(4x^8) + C_2 & \text{if } x > 0 \end{cases}$
15. $G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$
17. $H(\theta) = -2\cos\theta - \tan\theta + C_n$ on $(n\pi - \pi/2, n\pi + \pi/2)$, n an integer
19. $F(t) = \frac{16}{3}t^{3/2} - \sec t + C$ **21.** $F(x) = x^5 - \frac{1}{3}x^6 + 4$
23. $f(x) = x^5 - x^4 + x^3 + Cx + D$
25. $f(x) = 2x^2 - \frac{9}{28}x^{7/3} + Cx + D$
27. $f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$
29. $f(x) = x + 2x^{3/2} + 5$ **31.** $f(x) = 4x^{3/2} + 2x^{5/2} + 4$
33. $f(t) = \tan t + \sec t - 2 - \sqrt{2}$
35. $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$
37. $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$
39. $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$
41. $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$
43. 8 **45.** b
47.



49.



51.



53. $s(t) = 1 - \cos t - \sin t$ **55.** $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$
57. $s(t) = -10\sin t - 3\cos t + (6/\pi)t + 3$
59. (a) $s(t) = 450 - 4.9t^2$ (b) $\sqrt{450/4.9} \approx 9.58$ s
(c) $-9.8\sqrt{450/4.9} \approx -93.9$ m/s (d) About 9.09 s
63. 225 ft **65.** \$742.08 **67.** $\frac{130}{11} \approx 11.8$ s
69. $\frac{88}{15} \approx 5.87$ ft/s^2 **71.** 62,500 km/h^2 ≈ 4.82 m/s^2
73. (a) 22.9125 mi (b) 21.675 mi (c) 30 min 33 s
(d) 55.425 mi

CHAPTER 3 REVIEW ■ PAGE 285

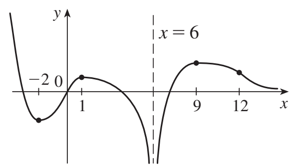
True-False Quiz

1. False **3.** False **5.** True **7.** False **9.** True
11. True **13.** False **15.** True **17.** True
19. True

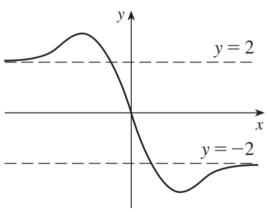
Exercises

1. Abs max $f(2) = f(5) = 18$, abs min $f(0) = -2$, loc max $f(2) = 18$, loc min $f(4) = 14$
3. Abs max $f(2) = \frac{2}{5}$, abs and loc min $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$
5. Abs and loc max $f(\pi/6) = \pi/6 + \sqrt{3}$, abs min $f(-\pi) = -\pi - 2$, loc min $f(5\pi/6) = 5\pi/6 - \sqrt{3}$
7. $\frac{1}{2}$ **9.** $-\frac{2}{3}$ **11.** $\frac{3}{4}$

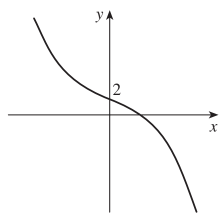
13.



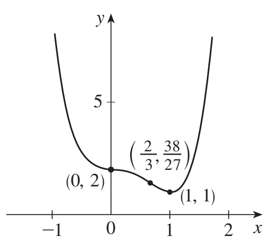
15.



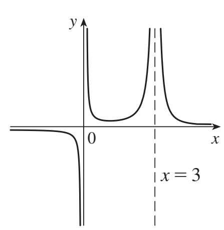
17. A. $\mathbb{R}$ B. y-int 2
C. None D. None
E. Dec on $(-\infty, \infty)$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $(0, 2)$
H. See graph at right.



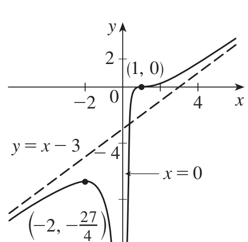
19. A. $\mathbb{R}$ B. y-int 2
C. None D. None
E. Inc on $(1, \infty)$; dec on $(-\infty, 1)$
F. Loc min $f(1) = 1$
G. CU on $(-\infty, 0)$, $\left(\frac{2}{3}, \infty\right)$;
CD on $\left(0, \frac{2}{3}\right)$; IPs $(0, 2)$, $\left(\frac{2}{3}, \frac{38}{27}\right)$
H. See graph at right.



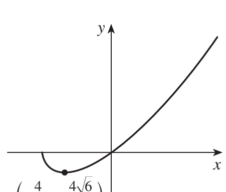
21. A. $\{x \mid x \neq 0, 3\}$
B. None C. None
D. HA $y = 0$; VA $x = 0$, $x = 3$
E. Inc on $(1, 3)$;
dec on $(-\infty, 0)$, $(0, 1)$, $(3, \infty)$
F. Loc min $f(1) = \frac{1}{4}$
G. CU on $(0, 3)$, $(3, \infty)$;
CD on $(-\infty, 0)$
H. See graph at right.



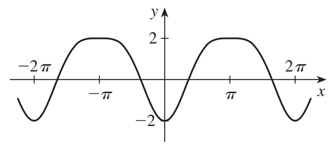
23. A. $(-\infty, 0) \cup (0, \infty)$
B. x-int 1 C. None
D. VA $x = 0$; SA $y = x - 3$
E. Inc on $(-\infty, -2)$, $(0, \infty)$;
dec on $(-2, 0)$
F. Loc max $f(-2) = -\frac{27}{4}$
G. CU on $(1, \infty)$; CD on $(-\infty, 0)$, $(0, 1)$; IP $(1, 0)$
H. See graph at right.



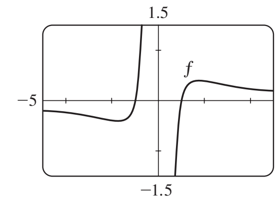
25. A. $[-2, \infty)$
B. y-int 0; x-int -2, 0
C. None D. None
E. Inc on $\left(-\frac{4}{3}, \infty\right)$, dec on $\left(-2, -\frac{4}{3}\right)$
F. Loc min $f\left(-\frac{4}{3}\right) = -\frac{4}{9}\sqrt{6}$
G. CU on $(-2, \infty)$
H. See graph at right.



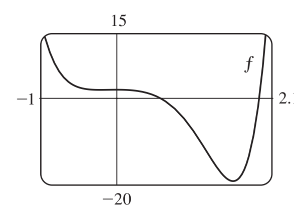
27. A. $\mathbb{R}$ B. y-int -2 C. About y-axis, period 2π
D. None E. Inc on $(2n\pi, (2n + 1)\pi)$, n an integer;
dec on $((2n - 1)\pi, 2n\pi)$
F. Loc max $f((2n + 1)\pi) = 2$; loc min $f(2n\pi) = -2$
G. CU on $(2n\pi - (\pi/3), 2n\pi + (\pi/3))$;
CD on $(2n\pi + (\pi/3), 2n\pi + (5\pi/3))$; IP $\left(2n\pi \pm (\pi/3), -\frac{1}{4}\right)$
H.

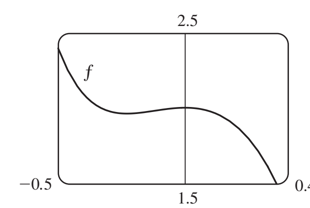


29. Inc on $\left(-\sqrt{3}, 0\right)$, $\left(0, \sqrt{3}\right)$;
dec on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$;
loc max $f\left(\sqrt{3}\right) = \frac{2}{9}\sqrt{3}$,
loc min $f\left(-\sqrt{3}\right) = -\frac{2}{9}\sqrt{3}$;
CU on $\left(-\sqrt{6}, 0\right)$, $\left(\sqrt{6}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{6}\right)$;
IPs $\left(\sqrt{6}, \frac{5}{36}\sqrt{6}\right)$, $\left(-\sqrt{6}, -\frac{5}{36}\sqrt{6}\right)$

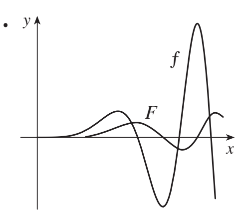


31. Inc on $(-0.23, 0)$, $(1.62, \infty)$; dec on $(-\infty, -0.23)$, $(0, 1.62)$;
loc max $f(0) = 2$; loc min $f(-0.23) \approx 1.96$, $f(1.62) \approx -19.2$; CU on $(-\infty, -0.12)$, $(1.24, \infty)$;
CD on $(-0.12, 1.24)$; IPs $(-0.12, 1.98)$, $(1.24, -12.1)$





37. (a) 0 (b) CU on $\mathbb{R}$ **41.** $3\sqrt{3}\,r^2$
43. $4/\sqrt{3}$ cm from D **45.** $L = C$ **47.** \$11.50
49. 1.297383 **51.** 1.16718557
53. $F(x) = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$
55. $f(t) = t^2 + 3\cos t + 2$
57. $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$
59. $s(t) = t^2 + \cos t + 2$ **61.**



63. No
65. (b) About 8.5 in. by 2 in. (c) $20/\sqrt{3}$ in. by $20\sqrt{2/3}$ in.
67. (a) $20\sqrt{2} \approx 28$ ft
(b) $\dfrac{dI}{dt} = \dfrac{-480k(h - 4)}{[(h - 4)^2 + 1600]^{5/2}}$, where k is the constant of proportionality

PROBLEMS PLUS ■ PAGE 290

5. $(-2, 4)$, $(2, -4)$ **7.** $\frac{4}{3}$ **11.** $(m/2, m^2/4)$
13. $-3.5 < a < -2.5$
15. (a) $x/(x^2 + 1)$ (b) $\frac{1}{2}$

17. (a) $\dfrac{d\theta}{dt} = -\tan\theta\left[\dfrac{1}{c}\dfrac{dc}{dt} + \dfrac{1}{b}\dfrac{db}{dt}\right]$

(b) $\dfrac{da}{dt} = \dfrac{b\dfrac{db}{dt} + c\dfrac{dc}{dt} - \left(b\dfrac{dc}{dt} + c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}$

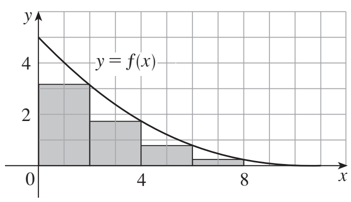
19. (a) $T_1 = D/c_1$, $T_2 = (2h\sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$, $T_3 = \sqrt{4h^2 + D^2}/c_1$
(c) $c_1 \approx 3.85$ km/s, $c_2 \approx 7.66$ km/s, $h \approx 0.42$ km

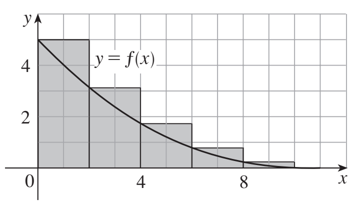
23. $3/(\sqrt[3]{2} - 1) \approx 11\frac{1}{2}$ h

CHAPTER 4

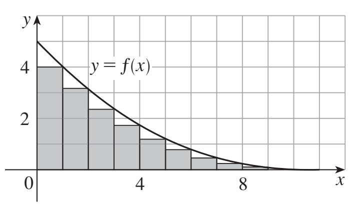
EXERCISES 4.1 ■ PAGE 303

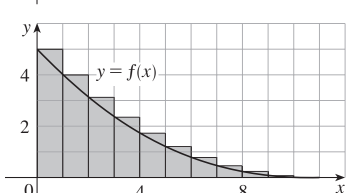
1. (a) $R_5 \approx 12$, $L_5 \approx 22$



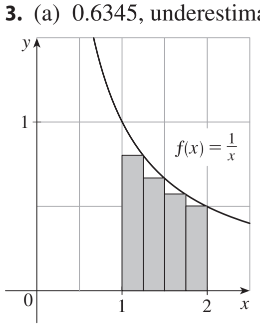


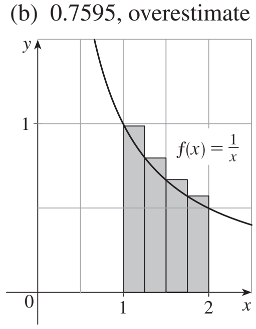
(b) $R_{10} \approx 14.4$, $L_{10} \approx 19.4$



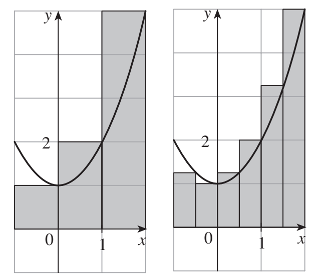


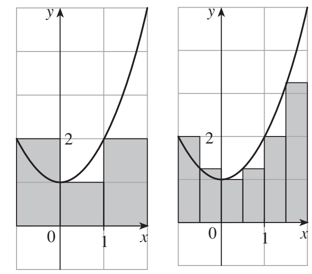
3. (a) 0.6345, underestimate (b) 0.7595, overestimate



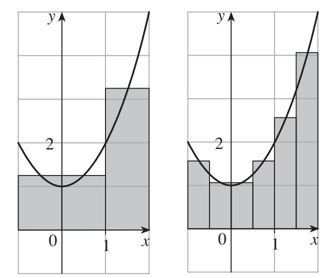


5. (a) 8, 6.875 (b) 5, 5.375



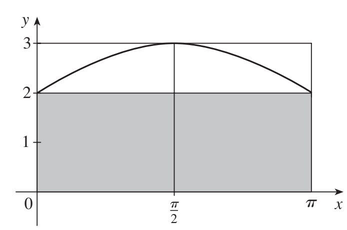


(c) 5.75, 5.9375

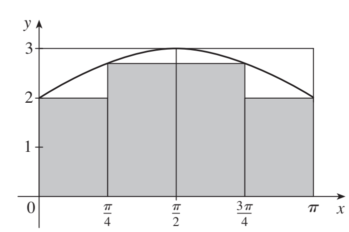


(d) M_6

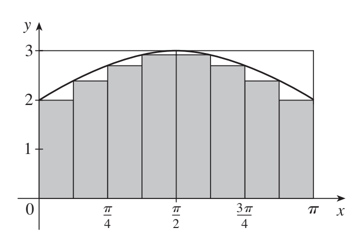
7. $\boldsymbol{n = 2}$**:** upper $= 3\pi \approx 9.42$, lower $= 2\pi \approx 6.28$



$\boldsymbol{n = 4}$**:** upper $= (10 + \sqrt{2})(\pi/4) \approx 8.96$,
lower $= (8 + \sqrt{2})(\pi/4) \approx 7.39$



$\boldsymbol{n = 8}$**:** upper ≈ 8.65, lower ≈ 7.86



9. 0.2533, 0.2170, 0.2101, 0.2050; 0.2

11. (a) Left: 0.8100, 0.7937, 0.7904;
right: 0.7600, 0.7770, 0.7804

13. 34.7 ft, 44.8 ft **15.** 63.2 L, 70 L **17.** 155 ft

19. 7840 **21.** $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+2i/n)}{(1+2i/n)^2+1} \cdot \frac{2}{n}$

23. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin(\pi i/n)} \cdot \frac{\pi}{n}$

25. The region under the graph of $y = \tan x$ from 0 to $\pi/4$

27. (a) $L_n < A < R_n$

29. (a) $\lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5$ (b) $\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

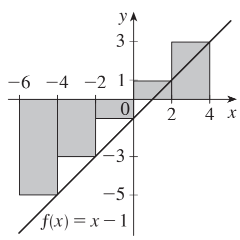
(c) $\frac{32}{3}$

31. $\sin b$, 1

EXERCISES 4.2 ■ PAGE 316

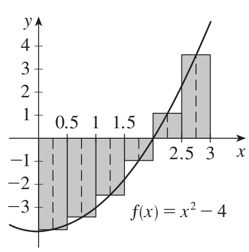
1. -10

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.



3. $-\frac{49}{16}$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis.



5. (a) 6 (b) 4 (c) 2

7. Lower, $L_5 = -64$; upper, $R_5 = 16$

9. 6.1820 **11.** 0.9071 **13.** 0.9029, 0.9018

15.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

17. $\int_0^{\pi} \frac{\sin x}{1+x}\,dx$ **19.** $\int_2^7 (5x^3 - 4x)\,dx$

21. -9 **23.** $\frac{2}{3}$ **25.** $-\frac{3}{4}$

29. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + (1+2i/n)^2} \cdot \frac{2}{n}$

31. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sin \frac{5\pi i}{n}\right) \frac{\pi}{n} = \frac{2}{5}$

33. (a) 4 (b) 10 (c) -3 (d) 2

35. $\frac{3}{2}$ **37.** $3 + \frac{9}{4}\pi$ **39.** $\frac{25}{4}$ **41.** 0 **43.** 3

45. 22.5 **47.** $\int_{-1}^{5} f(x)\,dx$ **49.** 122

51. B < E < A < D < C **53.** 15

59. $0 \leqslant \int_0^1 x^3\,dx \leqslant 1$ **61.** $\frac{\pi}{12} \leqslant \int_{\pi/4}^{\pi/3} \tan x\,dx \leqslant \frac{\pi}{12}\sqrt{3}$

63. $2 \leqslant \int_{-1}^{1} \sqrt{1+x^4}\,dx \leqslant 2\sqrt{2}$ **67.** $\int_1^2 \sqrt{x}\,dx$

73. $\int_0^1 x^4\,dx$ **75.** $\frac{1}{2}$

EXERCISES 4.3 ■ PAGE 327

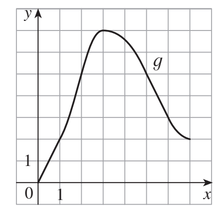
1. One process undoes what the other one does. See the Fundamental Theorem of Calculus, page 326.

3. (a) 0, 2, 5, 7, 3

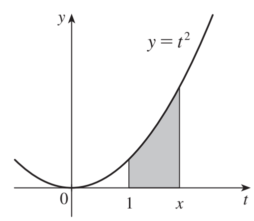
(b) (0, 3)

(c) $x = 3$

(d)



5.



(a), (b) x^2

7. $g'(x) = \sqrt{x + x^3}$ **9.** $g'(s) = (s - s^2)^8$

11. $F'(x) = -\sqrt{1 + \sec x}$ **13.** $h'(x) = -\sin^4(1/x)/x^2$

15. $y' = \frac{3(3x+2)}{1+(3x+2)^3}$

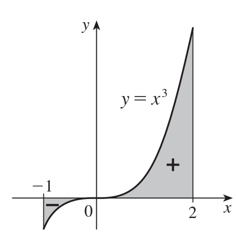
17. $y' = -\frac{1}{2}\tan\sqrt{x}$ **19.** $\frac{26}{3}$ **21.** 2 **23.** $\frac{52}{3}$

25. $1 + \sqrt{3}/2$ **27.** $-\frac{37}{6}$ **29.** $\frac{82}{5}$ **31.** 1 **33.** $\frac{15}{4}$

35. $\frac{17}{2}$ **37.** 0 **39.** $\frac{16}{3}$ **41.** $\frac{32}{3}$

43. $\frac{243}{4}$ **45.** 2

47. 3.75



49. The function $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied.

51. The function $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied.

53. $g'(x) = \frac{-2(4x^2-1)}{4x^2+1} + \frac{3(9x^2-1)}{9x^2+1}$

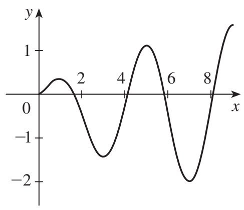
55. $h'(x) = -\frac{1}{2\sqrt{x}}\cos x + 3x^2\cos(x^6)$ **57.** $y = -\frac{1}{\pi}x + 1$

59. $(-4, 0)$ **61.** 29

63. (a) $-2\sqrt{n}$, $\sqrt{4n-2}$, n an integer > 0

(b) $(0, 1)$, $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, and $\left(\sqrt{4n-1}, \sqrt{4n+1}\right)$, n an integer > 0 (c) 0.74

65. (a) Loc max at 1 and 5; loc min at 3 and 7
(b) $x = 9$
(c) $\left(\frac{1}{2}, 2\right)$, (4, 6), (8, 9)
(d) See graph at right.



67. $\frac{7}{10}$ **75.** $f(x) = x^{3/2}$, $a = 9$

77. (b) Average expenditure over $[0, t]$; minimize average expenditure

79. ln 3 **81.** $\pi/3$ **83.** $e^2 - 1$

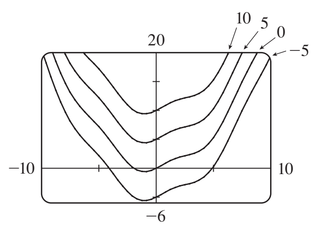
EXERCISES 4.4 ■ PAGE 336

5. $\frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$ **7.** $5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$

9. $\frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$ **11.** $2\sqrt{x} + x + \frac{2}{3}x^{3/2} + C$

13. $\theta + \tan\theta + C$ **15.** $-\cot t + \cos t + C$

17. $\sin x + \frac{1}{4}x^2 + C$



19. $-\frac{10}{3}$ **21.** $\frac{21}{5}$ **23.** -2 **25.** 8 **27.** 36

29. $2\sqrt{5}$ **31.** $\frac{256}{15}$ **33.** $1 + \pi/4$ **35.** $\frac{69}{4}$

37. 1 **39.** $\frac{5}{2}$ **41.** -3.5 **43.** ≈ 1.36 **45.** $\frac{4}{3}$

47. The change in the child's weight (in pounds) between the ages of 5 and 10

49. Number of gallons of oil leaked in the first 2 hours

51. Change in revenue when production is increased from 1000 to 5000 units

53. Newton-meters (or joules) **55.** (a) $-\frac{3}{2}$ m (b) $\frac{41}{6}$ m

57. (a) $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s (b) $416\frac{2}{3}$ m

59. $46\frac{2}{3}$ kg **61.** 1.4 mi

63. 2,954,707,200 ft^3

65. 39.8 ft/s **67.** 4.75×10^5 megawatt-hours

69. $-\cos x + \cosh x + C$ **71.** $\frac{1}{3}x^3 + x + \tan^{-1}x + C$

73. $\pi/6$

EXERCISES 4.5 ■ PAGE 346

1. $\frac{1}{2}\sin 2x + C$ **3.** $\frac{2}{9}(x^3 + 1)^{3/2} + C$

5. $-\dfrac{1}{4(x^4 - 5)} + C$ **7.** $-\frac{1}{3}(1 - x^2)^{3/2} + C$

9. $-\frac{1}{20}(1 - 2x)^{10} + C$ **11.** $-\frac{3}{2}\cos(2\theta/3) + C$

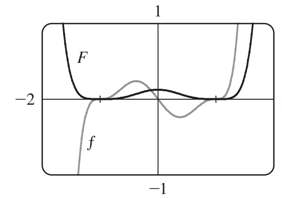
13. $\frac{1}{3}\sec 3t + C$ **15.** $\frac{1}{5}\sin(1 + 5t) + C$ **17.** $\frac{1}{4}\tan^4\theta + C$

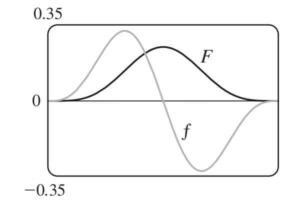
19. $\frac{1}{15}(x^3 + 3x)^5 + C$ **21.** $\frac{2}{3}\sqrt{3ax + bx^3} + C$

23. $\frac{1}{2}(1 + z^3)^{2/3} + C$ **25.** $-\frac{2}{3}(\cot x)^{3/2} + C$

27. $\frac{1}{3}\sec^3 x + C$ **29.** $\frac{1}{40}(2x + 5)^{10} - \frac{5}{36}(2x + 5)^9 + C$

31. $\frac{1}{8}(x^2 - 1)^4 + C$ **33.** $\frac{1}{4}\sin^4 x + C$





35. $2/\pi$ **37.** $\frac{45}{28}$ **39.** $2/\sqrt{3} - 1$

41. 0 **43.** 3 **45.** $\frac{1}{3}(2\sqrt{2} - 1)a^3$ **47.** $\frac{16}{15}$

49. $\frac{1}{2}(\sin 4 - \sin 1)$ **51.** $\frac{1}{6}$ **53.** $\sqrt{3} - \frac{1}{3}$ **55.** 6π

57. $\dfrac{5}{4\pi}\left(1 - \cos\dfrac{2\pi t}{5}\right)$ L **59.** 5

67. $-\frac{1}{3}\ln|5 - 3x| + C$ **69.** $\frac{1}{3}(\ln x)^3 + C$

71. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **73.** $\frac{1}{3}(\arctan x)^3 + C$

75. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$ **77.** $-\ln(1 + \cos^2 x) + C$

79. $\ln|\sin x| + C$ **81.** 2 **83.** $\ln(e + 1)$ **85.** $\pi^2/4$

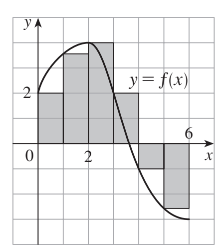
CHAPTER 4 REVIEW ■ PAGE 348

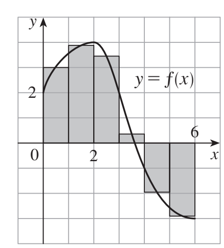
True-False Quiz

1. True **3.** True **5.** False **7.** True **9.** True
11. False **13.** True **15.** False **17.** False

Exercises

1. (a) 8 (b) 5.7





3. $\frac{1}{2} + \pi/4$ **5.** 3 **7.** f is c, f' is b, $\int_0^x f(t)\,dt$ is a

9. 3, 0 **11.** 37 **13.** $\frac{9}{10}$ **15.** -76 **17.** $\frac{21}{4}$

19. Does not exist **21.** $\frac{1}{3}\sin 1$ **23.** 0

25. $[1/(2\pi)]\sin^2\pi t + C$ **27.** $\frac{1}{2}\sqrt{2} - \frac{1}{2}$ **29.** $\frac{23}{3}$

31. $2\sqrt{1 + \sin x} + C$ **33.** $\frac{64}{5}$ **35.** $F'(x) = x^2/(1 + x^3)$

37. $g'(x) = 4x^3\cos(x^8)$ **39.** $y' = \dfrac{2\cos x - \cos\sqrt{x}}{2x}$

41. $4 \leqslant \int_1^3 \sqrt{x^2 + 3}\,dx \leqslant 4\sqrt{3}$ **45.** 0.280981

47. Number of barrels of oil consumed from January 1, 2000, through January 1, 2008

49. 72,400 **51.** 3 **53.** $(1 + x^2)(x\cos x + \sin x)/x^2$

PROBLEMS PLUS ■ PAGE 353

1. $\pi/2$ **3.** $f(x) = \frac{1}{2}x$ **5.** -1 **7.** $[-1, 2]$

9. (a) $\frac{1}{2}(n - 1)n$
(b) $\frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$

15. $y = -\dfrac{2b}{a^2}x^2 + \dfrac{3b}{a}x$ **17.** $2(\sqrt{2} - 1)$

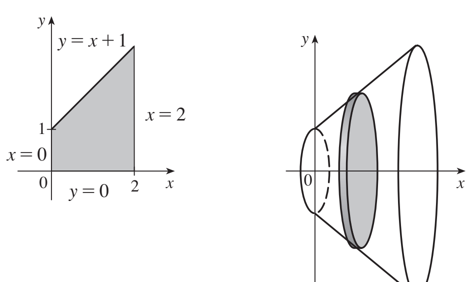
CHAPTER 5

EXERCISES 5.1 ■ PAGE 362

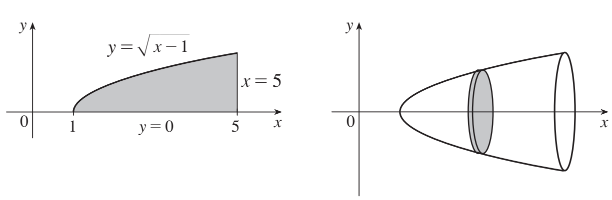
1. $\frac{47}{12}$ **3.** $\frac{4}{3}$ **5.** 19.5 **7.** $\frac{9}{2}$ **9.** $\frac{4}{3}$ **11.** $\frac{8}{3}$
13. 72 **15.** $6\sqrt{3}$ **17.** $\frac{32}{3}$ **19.** $2/\pi + \frac{2}{3}$
21. $2 - \pi/2$ **23.** $\frac{47}{3} - \frac{9}{2}\sqrt[3]{12}$ **25.** $\frac{13}{5}$ **27.** $\frac{3}{4}$
29. (a) 39 (b) 15 **31.** $\frac{4}{3}$ **33.** $\frac{5}{2}$ **35.** $\frac{3}{2}\sqrt{3} - 1$
37. 0, 0.90; 0.04 **39.** −1.11, 1.25, 2.86; 8.38
41. 2.80123 **43.** 0.25142 **45.** $12\sqrt{6} - 9$
47. $117\frac{1}{3}$ ft **49.** 4232 cm^2
51. (a) Twelfth ($t \approx 11.26$) (b) Eighteenth ($t \approx 17.18$)
(c) 706
53. (a) Car A (b) The distance by which A is ahead of B after 1 minute (c) Car A (d) $t \approx 2.2$ min
55. $\frac{24}{5}\sqrt{3}$ **57.** $4^{2/3}$ **59.** ± 6
61. $\ln 2 - \frac{1}{2}$ **63.** $2 - 2\ln 2$

EXERCISES 5.2 ■ PAGE 374

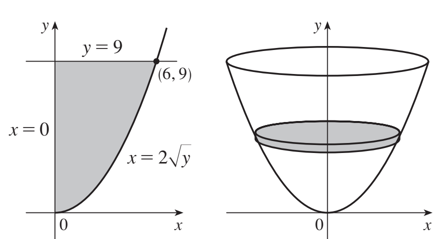
1. $26\pi/3$



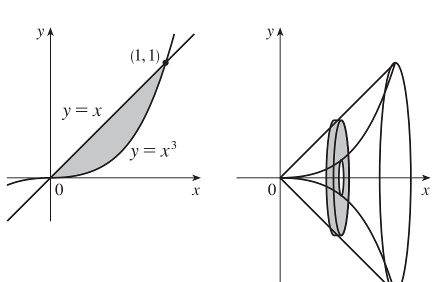
3. 8π



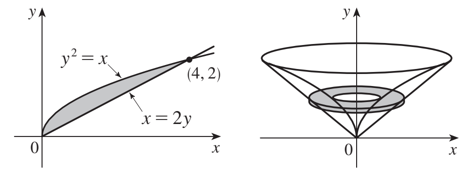
5. 162π



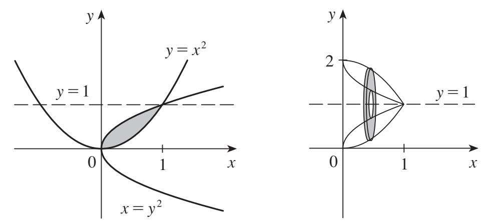
7. $4\pi/21$



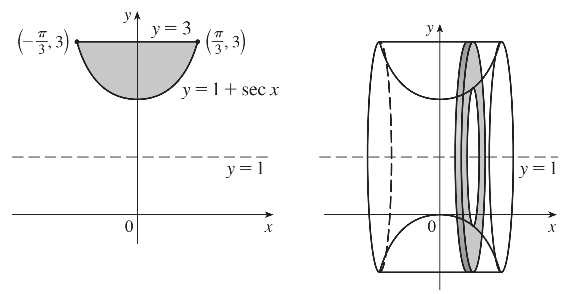
9. $64\pi/15$



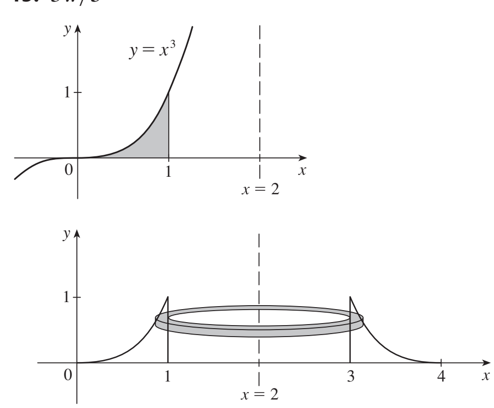
11. $11\pi/30$



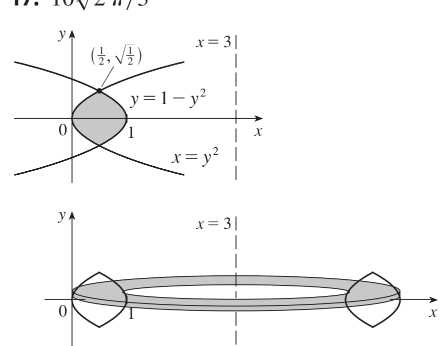
13. $2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$



15. $3\pi/5$



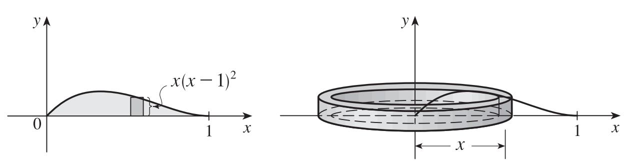
17. $10\sqrt{2}\pi/3$



19. $\pi/3$ **21.** $\pi/3$ **23.** $\pi/3$
25. $13\pi/45$ **27.** $\pi/3$ **29.** $17\pi/45$
31. (a) $\pi\int_0^{\pi/4} \tan^2x\,dx \approx 0.67419$
(b) $\pi\int_0^{\pi/4} (\tan^2x + 2\tan x)\,dx \approx 2.85178$
33. (a) $2\pi\int_0^2 8\sqrt{1 - x^2/4}\,dx \approx 78.95684$
(b) $2\pi\int_0^1 8\sqrt{4 - 4y^2}\,dy \approx 78.95684$
35. $-1, 0.857; 9.756$ **37.** $\frac{11}{8}\pi^2$
39. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \sqrt{\sin x}$ about the x-axis
41. Solid obtained by rotating the region above the x-axis bounded by $x = y^2$ and $x = y^4$ about the y-axis
43. 1110 cm^3 **45.** (a) 196 (b) 838
47. $\frac{1}{3}\pi r^2h$ **49.** $\pi h^2\left(r - \frac{1}{3}h\right)$ **51.** $\frac{2}{3}b^2h$
53. 10 cm^3 **55.** 24 **57.** $\frac{1}{3}$ **59.** $\frac{8}{15}$ **61.** $4\pi/15$
63. (a) $8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy$ (b) $2\pi^2r^2R$
65. (b) πr^2h **67.** $\frac{5}{12}\pi r^3$ **69.** $8\int_0^r \sqrt{R^2 - y^2}\sqrt{r^2 - y^2}\,dy$

EXERCISES 5.3 ■ PAGE 381

1. Circumference $= 2\pi x$, height $= x(x - 1)^2$; $\pi/15$



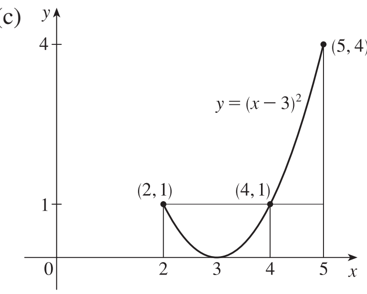
3. $6\pi/7$ **5.** 8π **7.** 8π
9. 4π **11.** 192π **13.** $16\pi/3$
15. $264\pi/5$ **17.** $8\pi/3$ **19.** $13\pi/3$
21. (a) $\int_{2\pi}^{3\pi} 2\pi x\sin x\,dx$ (b) 98.69604
23. (a) $4\pi\int_{-\pi/2}^{\pi/2} (\pi - x)\cos^4x\,dx$ (b) 46.50942
25. (a) $\int_0^{\pi} 2\pi(4 - y)\sqrt{\sin y}\,dy$ (b) 36.57476
27. 3.68
29. Solid obtained (using shells) by rotating the region $0 \leqslant y \leqslant x^4$, $0 \leqslant x \leqslant 3$ about the y-axis
31. Solid obtained (using shells) by rotating the region $0 \leqslant x \leqslant 1/y^2$, $1 \leqslant y \leqslant 4$ about the line $y = -2$
33. $0, 2.175; 14.450$ **35.** $\frac{1}{32}\pi^3$ **37.** 8π
39. $4\sqrt{3}\,\pi$ **41.** $4\pi/3$
43. $117\pi/5$ **45.** $\frac{4}{3}\pi r^3$ **47.** $\frac{1}{3}\pi r^2h$

EXERCISES 5.4 ■ PAGE 386

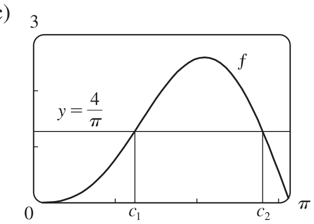
1. (a) 7200 ft-lb (b) 7200 ft-lb
3. 4.5 ft-lb **5.** 180 J **7.** $\frac{15}{4}$ ft-lb
9. (a) $\frac{25}{24} \approx 1.04$ J (b) 10.8 cm **11.** $W_2 = 3W_1$
13. (a) 625 ft-lb (b) $\frac{1875}{4}$ ft-lb **15.** 650,000 ft-lb
17. 3857 J **19.** 62.5 ft-lb **21.** 2450 J
23. $\approx 1.06 \times 10^6$ J **25.** $\approx 1.04 \times 10^5$ ft-lb **27.** 2.0 m
31. (b) $161.\overline{3}$ ft-lb
33. (a) $Gm_1m_2\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$ (b) $\approx 8.50 \times 10^9$ J

EXERCISES 5.5 ■ PAGE 391

1. 7 **3.** $6/\pi$ **5.** 29,524/15 **7.** $2/(5\pi)$
9. (a) 1 (b) 2, 4 (c)



11. (a) $4/\pi$ (b) $\approx 1.24, 2.81$
(c)



15. $\frac{9}{8}$ **17.** $(50 + 28/\pi)°\text{F} \approx 59°\text{F}$ **19.** 6 kg/m
21. $5/(4\pi) \approx 0.4$ L

CHAPTER 5 REVIEW ■ PAGE 393

Exercises

1. $\frac{8}{3}$ **3.** $\frac{7}{12}$ **5.** $\frac{4}{3} + 4/\pi$ **7.** $64\pi/15$ **9.** $1656\pi/5$
11. $\frac{4}{3}\pi(2ah + h^2)^{3/2}$ **13.** $\int_{-\pi/3}^{\pi/3} 2\pi(\pi/2 - x)\left(\cos^2x - \frac{1}{4}\right)dx$
15. (a) $2\pi/15$ (b) $\pi/6$ (c) $8\pi/15$
17. (a) 0.38 (b) 0.87
19. Solid obtained by rotating the region $0 \leqslant y \leqslant \cos x$, $0 \leqslant x \leqslant \pi/2$ about the y-axis
21. Solid obtained by rotating the region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant 2 - \sin x$ about the x-axis
23. 36 **25.** $\frac{125}{3}\sqrt{3}\text{ m}^3$ **27.** 3.2 J
29. (a) $8000\pi/3 \approx 8378$ ft-lb (b) 2.1 ft
31. $4/\pi$ **33.** $f(x)$

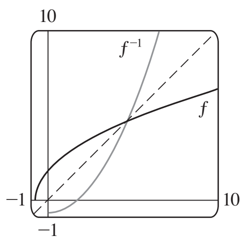
PROBLEMS PLUS ■ PAGE 395

1. (a) $f(t) = 3t^2$ (b) $f(x) = \sqrt{2x/\pi}$ **3.** $\frac{32}{27}$
5. (b) 0.2261 (c) 0.6736 m
(d) (i) $1/(105\pi) \approx 0.003$ in/s (ii) $370\pi/3$ s ≈ 6.5 min
9. $y = \frac{32}{9}x^2$
11. (a) $V = \int_0^h \pi[f(y)]^2\,dy$
(c) $f(y) = \sqrt{kA/(\pi C)}\,y^{1/4}$. Advantage: the markings on the container are equally spaced.
13. $b = 2a$ **15.** $B = 16A$

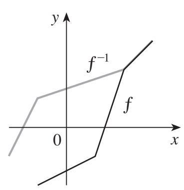
CHAPTER 6

EXERCISES 6.1 ■ PAGE 406

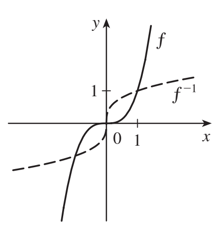
1. (a) See Definition 1.
(b) It must pass the Horizontal Line Test.
3. No **5.** No **7.** Yes **9.** Yes **11.** No **13.** No
15. No **17.** (a) 6 (b) 3 **19.** 4
21. $F = \frac{9}{5}C + 32$; the Fahrenheit temperature as a function of the Celsius temperature; $[-273.15, \infty)$
23. $f^{-1}(x) = \frac{5}{4} - \frac{1}{4}x$ **25.** $f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}, x \geqslant 1$
27. $y = \left(\dfrac{1-x}{1+x}\right)^2, -1 < x \leqslant 1$
29. $f^{-1}(x) = \frac{1}{4}(x^2 - 3), x \geqslant 0$



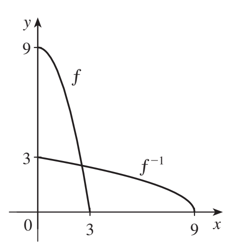
31.



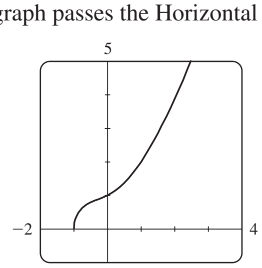
33. (a) $f^{-1}(x) = \sqrt{1-x^2}, 0 \leqslant x \leqslant 1$; f^{-1} and f are the same function. (b) Quarter-circle in the first quadrant
35. (b) $\frac{1}{12}$
(c) $f^{-1}(x) = \sqrt[3]{x}$,
domain $= \mathbb{R} =$ range
(e)



37. (b) $-\frac{1}{2}$
(c) $f^{-1}(x) = \sqrt{9-x}$,
domain $= [0, 9]$, range $= [0, 3]$
(e)



39. $\frac{1}{6}$ **41.** $2/\pi$ **43.** $\frac{3}{2}$ **45.** $1/\sqrt{28}$
47. The graph passes the Horizontal Line Test.

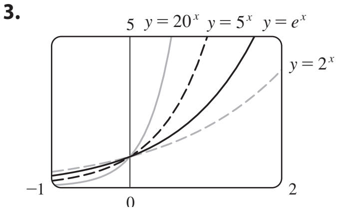


$f^{-1}(x) = -\frac{1}{6}\sqrt[3]{4}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$, where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$; two of the expressions are complex.

49. (a) $g^{-1}(x) = f^{-1}(x) - c$
(b) $h^{-1}(x) = (1/c)f^{-1}(x)$

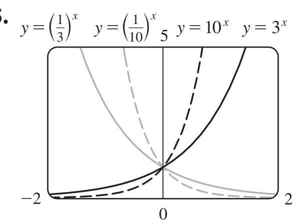
EXERCISES 6.2 ■ PAGE 418

1. (a) $f(x) = b^x, b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$
(d) See Figures 6(c), 6(b), and 6(a), respectively.
3.



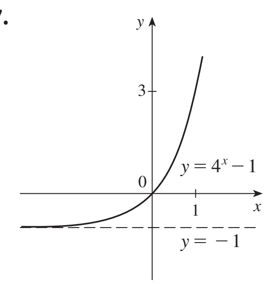
All approach 0 as $x \to -\infty$, all pass through (0, 1), and all are increasing. The larger the base, the faster the rate of increase.

5.

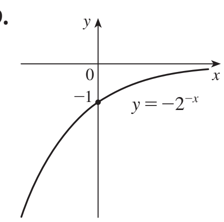


The functions with base greater than 1 are increasing and those with base less than 1 are decreasing. The latter are reflections of the former about the y-axis.

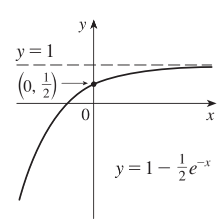
7.



9.



11.

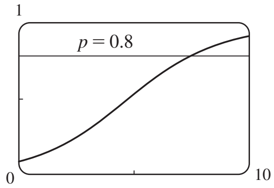


13. (a) $y = e^x - 2$ (b) $y = e^{x-2}$ (c) $y = -e^x$
(d) $y = e^{-x}$ (e) $y = -e^{-x}$
15. (a) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (b) $(-\infty, \infty)$
17. $f(x) = 3 \cdot 2^x$ **21.** At $x \approx 35.8$ **23.** ∞
25. 1 **27.** 0 **29.** 0 **31.** $f'(x) = 0$
33. $f'(x) = e^x(3x^2 + x - 5)$ **35.** $y' = 3ax^2 e^{ax^3}$
37. $y' = (\sec^2\theta)\, e^{\tan\theta}$ **39.** $f'(x) = \dfrac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$
41. $y' = xe^{-3x}(2 - 3x)$ **43.** $f'(t) = e^{at}(b\cos bt + a\sin bt)$
45. $F'(t) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$
47. $g'(u) = ue^{\sqrt{\sec u^2}}\sqrt{\sec u^2}\tan u^2$
49. $y' = \dfrac{4e^{2x}}{(1+e^{2x})^2}\sin\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right)$ **51.** $y = 2x + 1$
53. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **57.** $-4, -2$

59. $f^{(n)}(x) = 2^n e^{2x}$ **61.** (b) -0.567143 **63.** 3.5 days

65. (a) 1 (b) $kae^{-kt}/(1 + ae^{-kt})^2$

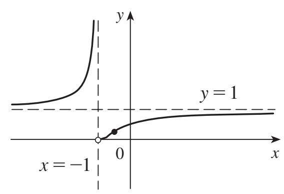
(c) $t \approx 7.4$ h



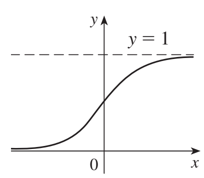
67. -1 **69.** $f(2) = 2/\sqrt{e}$, $f(-1) = -1/\sqrt[8]{e}$

71. (a) Inc on $(2, \infty)$; dec on $(-\infty, 2)$
(b) CU on $(-\infty, 3)$; CD on $(3, \infty)$ (c) $(3, -2e^{-3})$

73. A. $\{x \mid x \neq -1\}$
B. y-int $1/e$ C. None
D. HA $y = 1$; VA $x = -1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$
F. None
G. CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;
CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$
H. See graph at right.



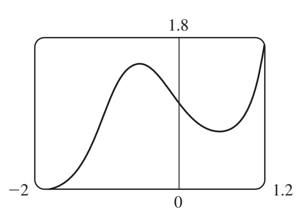
75. A. $\mathbb{R}$ B. y-int $\frac{1}{2}$ C. None
D. HA $y = 0$, $y = 1$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $\left(0, \frac{1}{2}\right)$
H. See graph at right.



77. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when the rate of decrease is greatest

79. 0.177 mg/mL; 21.4 min

81. Loc max $f\left(-1/\sqrt{3}\right) = e^{2\sqrt{3}/9} \approx 1.5$;
loc min $f\left(1/\sqrt{3}\right) = e^{-2\sqrt{3}/9} \approx 0.7$;
IP $(-0.15, 1.15)$, $(-1.09, 0.82)$



83. $\dfrac{1}{e + 1} + e - 1$ **85.** $\dfrac{1}{\pi}(1 - e^{-2\pi})$

87. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **89.** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

91. $\dfrac{1}{1 - e^u} + C$ **93.** $e - \sqrt{e}$

95. 4.644 **97.** $\pi(e^2 - 1)/2$ **101.** ≈ 4512 L

103. $C_0(1 - e^{-30r/V})$; the total amount of urea removed from the blood in the first 30 minutes of dialysis treatment

105. $\frac{1}{2}$

EXERCISES 6.3 ■ PAGE 426

1. (a) It's defined as the inverse of the exponential function with base b, that is, $\log_b x = y \iff b^y = x$.
(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

3. (a) 5 (b) $\frac{1}{3}$

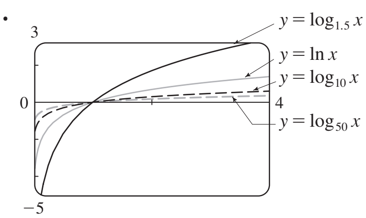
5. (a) 4.5 (b) -4 **7.** (a) 2 (b) $\frac{2}{3}$

9. $\frac{1}{2}\ln a + \frac{1}{2}\ln b$ **11.** $2\ln x - 3\ln y - 4\ln z$

13. $\ln \dfrac{x^2 y^3}{z}$ **15.** $\ln 250$ **17.** $\ln \dfrac{\sqrt{x}}{x + 1}$

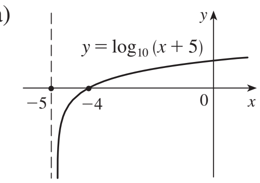
19. (a) 1.430677 (b) 3.680144 (c) 1.651496

21.

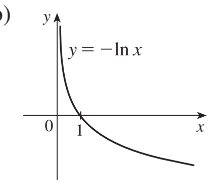


All graphs approach $-\infty$ as $x \to 0^+$, all pass through $(1, 0)$, and all are increasing. The larger the base, the slower the rate of increase.

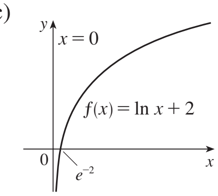
23. (a)



(b)



25. (a) $(0, \infty)$; $(-\infty, \infty)$ (b) e^{-2}
(c)



27. (a) $\frac{1}{4}(7 - \ln 6)$ (b) $\frac{1}{3}(e^2 + 10)$

29. (a) $5 + \log_2 3$ or $5 + (\ln 3)/\ln 2$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$

31. $-\frac{1}{2}\ln(e - 1)$ **33.** e^e **35.** $\ln 3$

37. (a) 3.7704 (b) 0.3285

39. (a) $0 < x < 1$ (b) $x > \ln 5$

41. About 1,084,588 mi **43.** 8.3

45. (a) $f^{-1}(n) = (3/\ln 2)\ln(n/100)$; the time elapsed when there are n bacteria (b) After about 26.9 hours

47. $-\infty$ **49.** 0 **51.** ∞ **53.** $(-2, 2)$

55. (a) $\left(-\infty, \frac{1}{2}\ln 3\right]$ (b) $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$, $\left[0, \sqrt{3}\right)$

57. (a) $(\ln 3, \infty)$ (b) $f^{-1}(x) = \ln(e^x + 3)$; $\mathbb{R}$

59. $y = e^{x/2} + 1$ **61.** $f^{-1}(x) = \sqrt[3]{\ln x}$

63. $y = 2 + \frac{1}{2}\log_3 x$ **65.** $\left(-\frac{1}{2}\ln 3, \infty\right)$

67. (b) $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$

69. f is a constant function.

73. $-1 \leqslant x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \leqslant 3$

EXERCISES 6.4 ■ PAGE 436

1. 1. The differentiation formula is simplest.

3. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **5.** $f'(x) = -\dfrac{1}{x}$

7. $f'(x) = \dfrac{-\sin x}{(1 + \cos x)\ln 10}$ **9.** $g'(x) = \dfrac{1}{x} - 2$

11. $F'(t) = \ln t \left(\ln t \cos t + \dfrac{2 \sin t}{t}\right)$

13. $G'(y) = \dfrac{10}{2y + 1} - \dfrac{y}{y^2 + 1}$ **15.** $f'(u) = \dfrac{1 + \ln 2}{u[1 + \ln(2u)]^2}$

17. $f'(x) = 5x^4 + 5^x \ln 5$ **19.** $T'(z) = 2^z\left(\dfrac{1}{z \ln 2} + \ln z\right)$

21. $y' = \dfrac{-x}{1 + x}$ **23.** $y' = \sec^2[\ln(ax + b)]\dfrac{a}{ax + b}$

25. $G'(x) = -C(\ln 4)\dfrac{4^{C/x}}{x^2}$

27. $y' = (2 + \ln x)/(2\sqrt{x})$; $y'' = -\ln x/(4x\sqrt{x})$

29. $y' = \tan x$; $y'' = \sec^2 x$

31. $f'(x) = \dfrac{2x - 1 - (x - 1)\ln(x - 1)}{(x - 1)[1 - \ln(x - 1)]^2}$;
$(1, 1 + e) \cup (1 + e, \infty)$

33. $f'(x) = \dfrac{2(x - 1)}{x(x - 2)}$; $(-\infty, 0) \cup (2, \infty)$ **35.** 2

37. $y = 3x - 9$ **39.** $\cos x + 1/x$ **41.** 7

43. $y' = (x^2 + 2)^2(x^4 + 4)^4\left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right)$

45. $y' = \sqrt{\dfrac{x - 1}{x^4 + 1}}\left(\dfrac{1}{2x - 2} - \dfrac{2x^3}{x^4 + 1}\right)$

47. $y' = x^x(1 + \ln x)$

49. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$

51. $y' = (\cos x)^x(-x \tan x + \ln \cos x)$

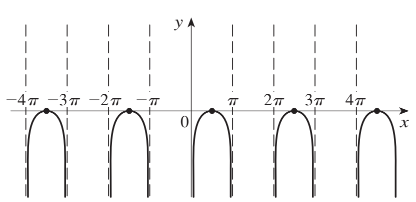
53. $y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right)$

55. $y' = \dfrac{2x}{x^2 + y^2 - 2y}$ **57.** $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n - 1)!}{(x - 1)^n}$

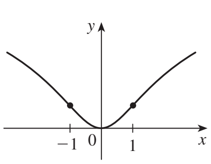
59. 2.958516, 5.290718

61. CU on $(e^{8/3}, \infty)$, CD on $(0, e^{8/3})$, IP $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$

63. A. All x in $(2n\pi, (2n + 1)\pi)$ (n an integer)
B. x-int $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc on $(2n\pi, \pi/2 + 2n\pi)$; dec on $(\pi/2 + 2n\pi, (2n + 1)\pi)$
F. Loc max $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n + 1)\pi)$
H.



65. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$;
IP $(\pm 1, \ln 2)$ H. See graph at right.



67. Inc on (0, 2.7), (4.5, 8.2), (10.9, 14.3);
IP (3.8, 1.7), (5.7, 2.1), (10.0, 2.7), (12.0, 2.9)

69. (a) $Q = ab^t$ where $a \approx 100.01244$ and $b \approx 0.000045146$
(b) -670.63 μA

71. $3 \ln 2$ **73.** $\frac{1}{3}\ln\frac{5}{2}$ **75.** $\frac{1}{2}e^2 + e - \frac{1}{2}$

77. $\frac{1}{3}(\ln x)^3 + C$ **79.** $-\ln(1 + \cos^2 x) + C$

81. $\dfrac{15}{\ln 2}$ **85.** $\pi \ln 2$ **87.** 45,974 J **89.** $\frac{1}{3}$

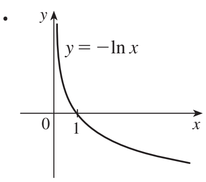
91. $0 < m < 1$; $m - 1 - \ln m$

EXERCISES 6.2* ■ PAGE 445

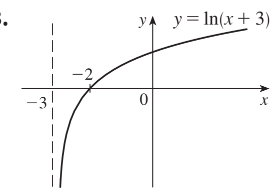
1. $\frac{1}{2}\ln a + \frac{1}{2}\ln b$ **3.** $2\ln x - 3\ln y - 4\ln z$

5. $\ln\dfrac{x^2y^3}{z}$ **7.** $\ln 250$ **9.** $\ln\dfrac{\sqrt{x}}{x + 1}$

11.



13.



15. $-\infty$ **17.** $f'(x) = x^2 + 3x^2 \ln x$

19. $f'(x) = \dfrac{\cos(\ln x)}{x}$ **21.** $f'(x) = -\dfrac{1}{x}$

23. $f'(x) = \dfrac{\sin x}{x} + \cos x \ln(5x)$ **25.** $g'(x) = -\dfrac{2a}{a^2 - x^2}$

27. $G'(y) = \dfrac{10}{2y + 1} - \dfrac{y}{y^2 + 1}$

29. $F'(t) = \ln t\left(\ln t \cos t + \dfrac{2 \sin t}{t}\right)$

31. $f'(u) = \dfrac{1 + \ln 2}{u[1 + \ln(2u)]^2}$ **33.** $y' = \dfrac{10x + 1}{5x^2 + x - 2}$

35. $y' = \sec^2[\ln(ax + b)]\dfrac{a}{ax + b}$

37. $y' = (2 + \ln x)/(2\sqrt{x})$; $y'' = -\ln x/(4x\sqrt{x})$

39. $f'(x) = \dfrac{2x - 1 - (x - 1)\ln(x - 1)}{(x - 1)[1 - \ln(x - 1)]^2}$;
$(1, 1 + e) \cup (1 + e, \infty)$

41. $f'(x) = -\dfrac{1}{2x\sqrt{1 - \ln x}}$; $(0, e]$

43. 2 **45.** $\cos x + 1/x$

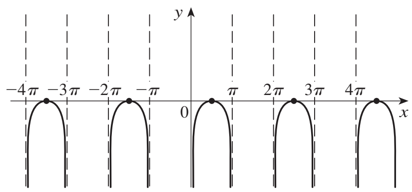
47. $y = 2x - 2$ **49.** $y' = \dfrac{2x}{x^2 + y^2 - 2y}$

51. $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n - 1)!}{(x - 1)^n}$

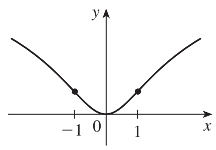
53. 2.958516, 5.290718

55. A. All x in $(2n\pi, (2n + 1)\pi)$ (n an integer)
B. x-int $\pi/2 + 2n\pi$ C. Period 2π D. VA $x = n\pi$
E. Inc on $(2n\pi, \pi/2 + 2n\pi)$; dec on $(\pi/2 + 2n\pi, (2n + 1)\pi)$
F. Loc max $f(\pi/2 + 2n\pi) = 0$ G. CD on $(2n\pi, (2n + 1)\pi)$

H.



57. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. About y-axis D. None
E. Inc on $(0, \infty)$; dec on $(-\infty, 0)$
F. Loc min $f(0) = 0$
G. CU on $(-1, 1)$;
CD on $(-\infty, -1)$, $(1, \infty)$;
IP $(\pm 1, \ln 2)$ H. See graph at right.



59. Inc on (0, 2.7), (4.5, 8.2), (10.9, 14.3);
IP (3.8, 1.7), (5.7, 2.1), (10.0, 2.7), (12.0, 2.9)

61. $y' = (x^2 + 2)^2(x^4 + 4)^4\left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right)$

63. $y' = \sqrt{\dfrac{x - 1}{x^4 + 1}}\left(\dfrac{1}{2x - 2} - \dfrac{2x^3}{x^4 + 1}\right)$

65. $3 \ln 2$ **67.** $\frac{1}{3} \ln \frac{5}{2}$ **69.** $\frac{1}{2}e^2 + e - \frac{1}{2}$

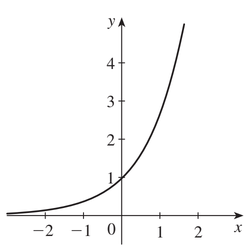
71. $\frac{1}{3}(\ln x)^3 + C$ **73.** $-\ln(1 + \cos^2 x) + C$

77. $\pi \ln 2$ **79.** 45,974 J **81.** $\frac{1}{3}$ **83.** (b) 0.405

87. $0 < m < 1$; $m - 1 - \ln m$

EXERCISES 6.3* ■ PAGE 452

1.



If $f(x) = e^x$, then $f'(0) = 1$.

3. (a) $\frac{1}{2}$ (b) 3

5. (a) $\frac{1}{4}(7 - \ln 6)$ (b) $\frac{1}{3}(e^2 + 10)$

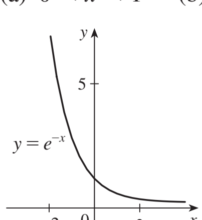
7. (a) $\frac{1}{3}(\ln k - 1)$ (b) $\frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$

9. $-\frac{1}{2}\ln(e - 1)$ **11.** $\ln 3$

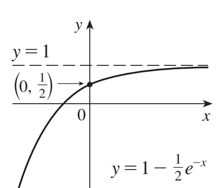
13. (a) 3.7704 (b) 0.3285

15. (a) $0 < x < 1$ (b) $x > \ln 5$

17.



19.



21. (a) $\left(-\infty, \frac{1}{2}\ln 3\right]$ (b) $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$, $\left[0, \sqrt{3}\right)$

23. $y = e^{x/2} + 1$ **25.** $f^{-1}(x) = \sqrt[3]{\ln x}$

27. 1 **29.** 0 **31.** 0

33. $f'(x) = 0$ **35.** $f'(x) = e^x(3x^2 + x - 5)$

37. $y' = 3ax^2 e^{ax^3}$ **39.** $y' = (\sec^2\theta)\, e^{\tan\theta}$

41. $f'(x) = \dfrac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$ **43.** $y' = xe^{-3x}(2 - 3x)$

45. $f'(t) = e^{at}(b \cos bt + a \sin bt)$

47. $F'(t) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

49. $g'(u) = ue^{\sqrt{\sec u^2}}\sqrt{\sec u^2}\tan u^2$

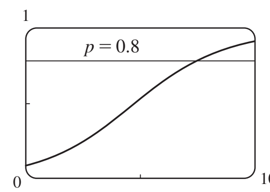
51. $y' = \dfrac{4e^{2x}}{(1 + e^{2x})^2}\sin\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$ **53.** $y = 2x + 1$

55. $y' = \dfrac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$ **59.** $-4, -2$ **61.** $f^{(n)}(x) = 2^n e^{2x}$

63. (b) -0.567143

65. (a) 1 (b) $kae^{-kt}/(1 + ae^{-kt})^2$

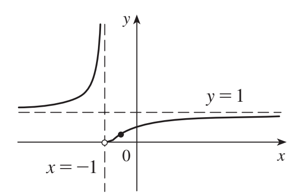
(c) $t \approx 7.4$ h



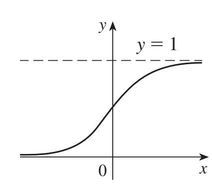
67. -1 **69.** $f(2) = 2/\sqrt{e}$, $f(-1) = -1/\sqrt[8]{e}$

71. (a) Inc on $(2, \infty)$; dec on $(-\infty, 2)$
(b) CU on $(-\infty, 3)$; CD on $(3, \infty)$ (c) $(3, -2e^{-3})$

73. A. $\{x \mid x \neq -1\}$
B. y-int $1/e$ C. None
D. HA $y = 1$; VA $x = -1$
E. Inc on $(-\infty, -1)$, $(-1, \infty)$
F. None
G. CU on $(-\infty, -1)$, $\left(-1, -\frac{1}{2}\right)$;
CD on $\left(-\frac{1}{2}, \infty\right)$; IP $\left(-\frac{1}{2}, 1/e^2\right)$
H. See graph at right.



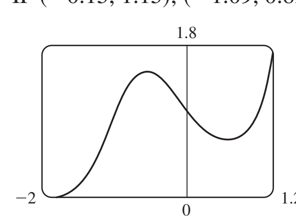
75. A. $\mathbb{R}$ B. y-int $\frac{1}{2}$ C. None
D. HA $y = 0$, $y = 1$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(-\infty, 0)$;
CD on $(0, \infty)$; IP $\left(0, \frac{1}{2}\right)$
H. See graph at right.



77. 28.57 min, when the rate of increase of drug level in the bloodstream is greatest; 85.71 min, when the rate of decrease is greatest

79. 0.177 mg/mL; 21.4 min

81. Loc max $f\left(-1/\sqrt{3}\right) = e^{2\sqrt{3}/9} \approx 1.5$;
loc min $f\left(1/\sqrt{3}\right) = e^{-2\sqrt{3}/9} \approx 0.7$;
IP $(-0.15, 1.15)$, $(-1.09, 0.82)$



83. $\dfrac{1}{e + 1} + e - 1$ **85.** $\dfrac{1}{\pi}(1 - e^{-2\pi})$

87. $\frac{2}{3}(1 + e^x)^{3/2} + C$ **89.** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

91. $\dfrac{1}{1 - e^u} + C$ **93.** $e - \sqrt{e}$ **95.** 4.644

97. $\pi(e^2 - 1)/2$ **101.** ≈ 4512 L

103. $C_0(1 - e^{-30r/V})$; the total amount of urea removed from the blood in the first 30 minutes of dialysis treatment

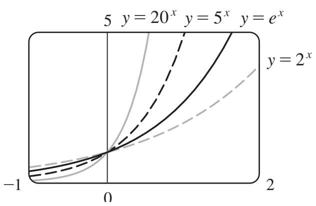
EXERCISES 6.4* ■ PAGE 463

1. (a) $b^x = e^{x \ln b}$ (b) $(-\infty, \infty)$ (c) $(0, \infty)$
(d) See Figures 1, 3, and 2.

3. $e^{-\pi \ln 4}$ **5.** $e^{x^2 \ln 10}$

7. (a) 5 (b) $\frac{1}{3}$ **9.** (a) 2 (b) $\frac{2}{3}$

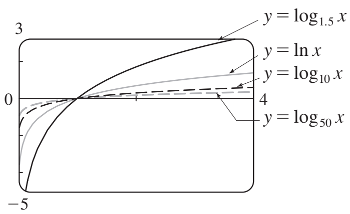
11.



All approach 0 as $x \to -\infty$, all pass through (0, 1), and all are increasing. The larger the base, the faster the rate of increase.

13. (a) 1.430677 (b) 3.680144 (c) 1.651496

15.



All graphs approach $-\infty$ as $x \to 0^+$, all pass through (1, 0), and all are increasing. The larger the base, the slower the rate of increase.

17. $f(x) = 3 \cdot 2^x$ **19.** (b) About 1,084,588 mi

21. ∞ **23.** 0 **25.** $f'(x) = 5x^4 + 5^x \ln 5$

27. $G'(x) = -C(\ln 4)\dfrac{4^{C/x}}{x^2}$ **29.** $L'(v) = 2v \ln 4 \sec^2(4^{v^2}) \cdot 4^{v^2}$

31. $f'(x) = \dfrac{3}{(3x - 1)\ln 2}$ **33.** $y' = \dfrac{x \cot x}{\ln 4} + \log_4 \sin x$

35. $y' = x^x(1 + \ln x)$

37. $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)$

39. $y' = (\cos x)^x(-x \tan x + \ln \cos x)$

41. $y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x \tan x} - \dfrac{\ln \tan x}{x^2}\right)$

43. $y = (10 \ln 10)x + 10(1 - \ln 10)$ **45.** $\dfrac{15}{\ln 2}$

47. $(\ln x)^2/(2 \ln 10) + C$ $\left[\text{or } \frac{1}{2}(\ln 10)(\log_{10} x)^2 + C\right]$

49. $3^{\sin\theta}/\ln 3 + C$ **51.** $16/(5 \ln 5) - 1/(2 \ln 2)$

53. 0.600967 **55.** $g^{-1}(x) = \sqrt[3]{4^x - 2}$ **57.** 8.3

59. $10^8/\ln 10$ dB/(watt/m²) **61.** 0.177 mg/mL; 21.4 min

63. 3.5 days

65. (a) $Q = ab^t$ where $a \approx 100.01244$ and $b \approx 0.000045146$
(b) -670.63 μA

EXERCISES 6.5 ■ PAGE 471

1. About 235

3. (a) $100(4.2)^t$ (b) ≈ 7409 (c) $\approx 10{,}632$ bacteria/h
(d) $(\ln 100)/(\ln 4.2) \approx 3.2$ h

5. (a) 1508 million, 1871 million (b) 2161 million
(c) 3972 million; wars in the first half of century, increased life expectancy in second half

7. (a) $Ce^{-0.0005t}$ (b) $-2000 \ln 0.9 \approx 211$ s

9. (a) $100 \times 2^{-t/30}$ mg (b) ≈ 9.92 mg (c) ≈ 199.3 years

11. ≈ 2500 years **13.** Yes; 12.5 billion years

15. (a) ≈ 137°F (b) ≈ 116 min

17. (a) $13.\overline{3}$°C (b) ≈ 67.74 min

19. (a) ≈ 64.5 kPa (b) ≈ 39.9 kPa

21. (a) (i) \$3828.84 (ii) \$3840.25 (iii) \$3850.08
(iv) \$3851.61 (v) \$3852.01 (vi) \$3852.08
(b) $dA/dt = 0.05A$, $A(0) = 3000$

EXERCISES 6.6 ■ PAGE 481

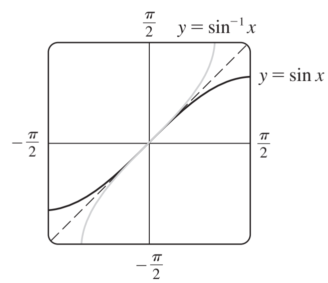
1. (a) $\pi/6$ (b) π **3.** (a) $\pi/4$ (b) $\pi/6$

5. (a) 10 (b) $-\pi/4$

7. $2/\sqrt{5}$ **9.** $\frac{119}{169}$

13. $x/\sqrt{1 + x^2}$

15.



The second graph is the reflection of the first graph about the line $y = x$

23. $y' = \dfrac{2 \tan^{-1} x}{1 + x^2}$ **25.** $y' = \dfrac{1}{\sqrt{-x^2 - x}}$

27. $y' = \sin^{-1} x$ **29.** $F'(x) = \dfrac{3}{\sqrt{x^6 - 1}} + \sec^{-1}(x^3)$

31. $y' = -\dfrac{\sin\theta}{1 + \cos^2\theta}$ **33.** $h'(t) = 0$

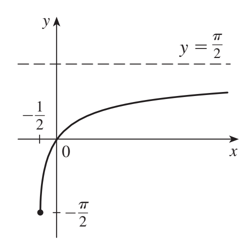
35. $y' = \dfrac{\sqrt{a^2 - b^2}}{a + b\cos x}$

37. $g'(x) = \dfrac{2}{\sqrt{1 - (3 - 2x)^2}}$; [1, 2], (1, 2) **39.** $\pi/6$

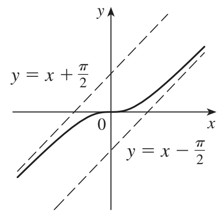
41. $1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$ **43.** $-\pi/2$ **45.** $\pi/2$

47. At a distance $5 - 2\sqrt{5}$ from A **49.** $\frac{1}{4}$ rad/s

51. A. $\left[-\frac{1}{2}, \infty\right)$
B. y-int 0; x-int 0
C. None
D. HA $y = \pi/2$
E. Inc on $\left(-\frac{1}{2}, \infty\right)$
F. None
G. CD on $\left(-\frac{1}{2}, \infty\right)$
H. See graph at right.



53. A. $\mathbb{R}$
B. y-int 0; x-int 0
C. About (0, 0)
D. SA $y = x \pm \pi/2$
E. Inc on $\mathbb{R}$ F. None
G. CU on $(0, \infty)$; CD on $(-\infty, 0)$;
IP (0, 0)
H. See graph at right.



55. Max at $x = 0$, min at $x \approx \pm 0.87$, IP at $x \approx \pm 0.52$
57. $F(x) = 2x + 3\tan^{-1}x + C$
59. $4\pi/3$ **61.** $\pi^2/72$
63. $\tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$ **65.** $\ln|\sin^{-1}x| + C$
67. $\frac{1}{3}\sin^{-1}(t^3) + C$ **69.** $2\tan^{-1}\sqrt{x} + C$ **73.** $\pi/2 - 1$

EXERCISES 6.7 ■ PAGE 489

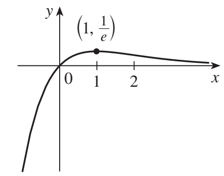
1. (a) 0 (b) 1 **3.** (a) $\frac{13}{5}$ (b) $\frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$
5. (a) 1 (b) 0
21. $\operatorname{sech} x = \frac{3}{5}$, $\sinh x = \frac{4}{3}$, $\operatorname{csch} x = \frac{3}{4}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$
23. (a) 1 (b) -1 (c) ∞ (d) $-\infty$ (e) 0 (f) 1
(g) ∞ (h) $-\infty$ (i) 0 (j) $\frac{1}{2}$
31. $f'(x) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$ **33.** $h'(x) = 2x\cosh(x^2)$
35. $G'(t) = \dfrac{t^2 + 1}{2t^2}$
37. $y' = 3e^{\cosh 3x}\sinh 3x$
39. $g'(t) = \coth\sqrt{t^2 + 1} - \dfrac{t^2}{\sqrt{t^2 + 1}}\operatorname{csch}^2\sqrt{t^2 + 1}$
41. $y' = \dfrac{1}{2\sqrt{x}(x - 1)}$
43. $y' = \sinh^{-1}(x/3)$ **45.** $y' = -\csc x$
51. (a) ≈ 0.3572 (b) $\approx 70.34°$
53. (a) ≈ 164.50 m (b) 120 m; ≈ 164.13 m
55. (b) $y = 2\sinh 3x - 4\cosh 3x$
57. $\left(\ln\left(1 + \sqrt{2}\right), \sqrt{2}\right)$
59. $\frac{1}{3}\cosh^3 x + C$ **61.** $2\cosh\sqrt{x} + C$ **63.** $-\operatorname{csch} x + C$
65. $\ln\left(\dfrac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right)$ **67.** $\tanh^{-1}e^x + C$
69. (a) 0, 0.48 (b) 0.04

EXERCISES 6.8 ■ PAGE 499

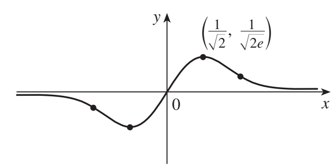
1. (a) Indeterminate (b) 0 (c) 0
(d) ∞, $-\infty$, or does not exist (e) Indeterminate
3. (a) $-\infty$ (b) Indeterminate (c) ∞
5. $\frac{9}{4}$ **7.** 1 **9.** 6 **11.** $-\frac{1}{3}$
13. $-\infty$ **15.** 2 **17.** $\frac{1}{4}$ **19.** 0 **21.** $-\infty$
23. $\frac{8}{5}$ **25.** 3 **27.** $\frac{1}{2}$ **29.** 1 **31.** 1
33. $1/\ln 3$ **35.** 0 **37.** 0 **39.** a/b
41. $\frac{1}{24}$ **43.** π **45.** $\frac{5}{3}$ **47.** 0 **49.** $-2/\pi$
51. $\frac{1}{2}$ **53.** $\frac{1}{2}$ **55.** ∞ **57.** 1 **59.** e^{-2}
61. $1/e$ **63.** 1 **65.** e^4 **67.** e^3 **69.** e^2

71. $\frac{1}{4}$ **75.** 1

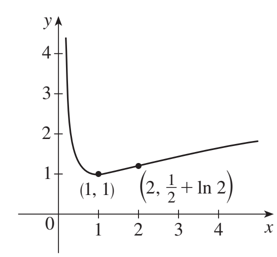
77. A. $\mathbb{R}$ B. y-int 0; x-int 0
C. None D. HA $y = 0$
E. Inc on $(-\infty, 1)$, dec on $(1, \infty)$
F. Loc max $f(1) = 1/e$
G. CU on $(2, \infty)$; CD on $(-\infty, 2)$
IP $(2, 2/e^2)$
H. See graph at right.



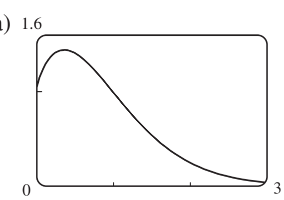
79. A. $\mathbb{R}$ B. y-int 0; x-int 0 C. About (0, 0) D. HA $y = 0$
E. Inc on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$; dec on $\left(-\infty, -1/\sqrt{2}\right)$, $\left(1/\sqrt{2}, \infty\right)$
F. Loc min $f\left(-1/\sqrt{2}\right) = -1/\sqrt{2e}$; loc max $f\left(1/\sqrt{2}\right) = 1/\sqrt{2e}$
G. CU on $\left(-\sqrt{3/2}, 0\right)$, $\left(\sqrt{3/2}, \infty\right)$;
CD on $\left(-\infty, -\sqrt{3/2}\right)$, $\left(0, \sqrt{3/2}\right)$; IP $\left(\pm\sqrt{3/2}, \pm\sqrt{3/2}e^{-3/2}\right)$, (0, 0)
H.



81. A. $(0, \infty)$ B. None
C. None D. VA $x = 0$
E. Inc on $(1, \infty)$; dec on $(0, 1)$
F. Loc min $f(1) = 1$
G. CU on $(0, 2)$; CD on $(2, \infty)$;
IP $\left(2, \frac{1}{2} + \ln 2\right)$
H. See graph at right.

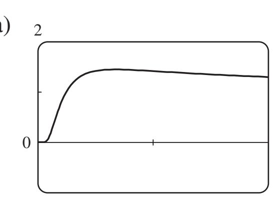


83. (a)



(b) $\lim_{x \to 0^+} x^{-x} = 1$
(c) Max value $f(1/e) = e^{1/e} \approx 1.44$ (d) 1.0

85. (a)



(b) $\lim_{x \to 0^+} x^{1/x} = 0$, $\lim_{x \to \infty} x^{1/x} = 1$
(c) Loc max $f(e) = e^{1/e}$ (d) IPs at $x \approx 0.58, 4.37$
87. f has an absolute minimum for $c > 0$. As c increases, the minimum points get farther away from the origin.
91. (a) M; the population should approach its maximum size as time increases (b) P_0e^{kt}; exponential
93. $\pi/6$ **95.** $\frac{16}{9}a$ **97.** $\frac{1}{2}$ **99.** 56 **103.** (a) 0

CHAPTER 6 REVIEW ■ PAGE 504

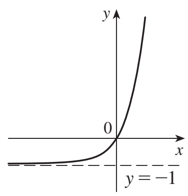
True-False Quiz

1. True **3.** False **5.** True **7.** True **9.** False
11. False **13.** False **15.** True **17.** True **19.** False

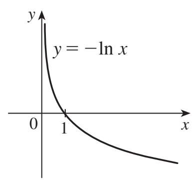
Exercises

1. No **3.** (a) 7 (b) $\frac{1}{8}$

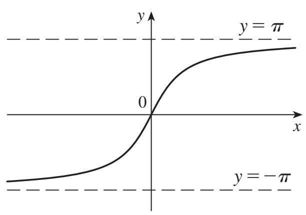
5.



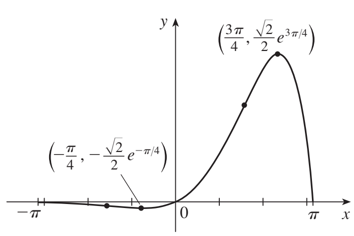
7.



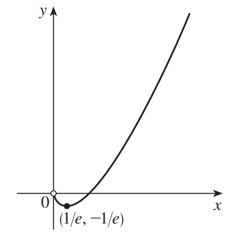
9.



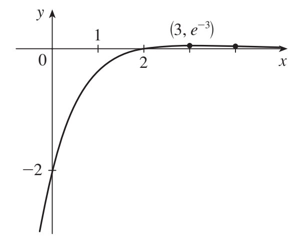
11. (a) 9 (b) 2 **13.** $e^{1/3}$
15. ln ln 17 **17.** $\sqrt{1+e}$
19. tan 1 **21.** $f'(t) = t + 2t \ln t$
23. $h'(\theta) = 2\sec^2(2\theta)e^{\tan 2\theta}$ **25.** $y' = 5 \sec 5x$
27. $y' = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$ **29.** $y' = 2\tan x$
31. $y' = -\dfrac{e^{1/x}(1+2x)}{x^4}$ **33.** $y' = 3^{x \ln x}(\ln 3)(1 + \ln x)$
35. $H'(v) = \dfrac{v}{1+v^2} + \tan^{-1}v$
37. $y' = 2x^2\cosh(x^2) + \sinh(x^2)$
39. $y' = \cot x - \sin x \cos x$ **41.** $y' = -(1/x)[1 + 1/(\ln x)^2]$
43. $y' = 3\tanh 3x$ **45.** $y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$
47. $y' = \dfrac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)e^{\sqrt{\tan 3x}}\sec^2(3x)}{2\sqrt{\tan 3x}}$ **49.** $e^{g(x)}g'(x)$
51. $g'(x)/g(x)$ **53.** $2^x(\ln 2)^n$ **57.** $y = -x + 2$
59. $(-3, 0)$ **61.** (a) $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$ (b) $y = ex$
63. 0 **65.** 0 **67.** $-\infty$ **69.** -1
71. 1 **73.** 4 **75.** 0 **77.** $\frac{1}{2}$
79. A. $[-\pi, \pi]$ B. y-int 0; x-int $-\pi, 0, \pi$
C. None D. None
E. Inc on $(-\pi/4, 3\pi/4)$; dec on $(-\pi, -\pi/4)$, $(3\pi/4, \pi)$
F. Loc max $f(3\pi/4) = \frac{1}{2}\sqrt{2}\,e^{3\pi/4}$,
loc min $f(-\pi/4) = -\frac{1}{2}\sqrt{2}\,e^{-\pi/4}$
G. CU on $(-\pi/2, \pi/2)$; CD on $(-\pi, -\pi/2)$, $(\pi/2, \pi)$;
IPs $(-\pi/2, -e^{-\pi/2})$, $(\pi/2, e^{\pi/2})$
H.



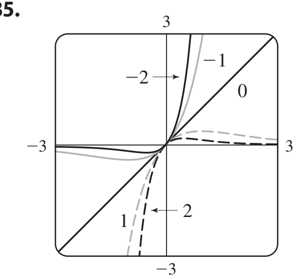
81. A. $(0, \infty)$ B. x-int 1
C. None D. None
E. Inc on $(1/e, \infty)$; dec on $(0, 1/e)$
F. Loc min $f(1/e) = -1/e$
G. CU on $(0, \infty)$
H. See graph at right.



83. A. $\mathbb{R}$
B. y-int -2; x-int 2
C. None D. HA $y = 0$
E. Inc on $(-\infty, 3)$; dec on $(3, \infty)$
F. Loc max $f(3) = e^{-3}$
G. CU on $(4, \infty)$;
CD on $(-\infty, 4)$; IP $(4, 2e^{-4})$
H. See graph at right.



85.



For $c > 0$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.
For $c < 0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.
As $|c|$ increases, the max and min points and the IPs get closer to the origin.
87. $v(t) = -Ae^{-ct}[c\cos(\omega t + \delta) + \omega\sin(\omega t + \delta)]$,
$a(t) = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)]$
89. (a) $200(3.24)^t$ (b) $\approx 22{,}040$
(c) $\approx 25{,}910$ bacteria/h (d) $(\ln 50)/(\ln 3.24) \approx 3.33$ h
91. ≈ 4.32 days **93.** $\frac{1}{4}(1 - e^{-2})$
95. $\arctan e - \pi/4$
97. $2e^{\sqrt{x}} + C$ **99.** $\frac{1}{2}\ln|x^2 + 2x| + C$
101. $-\frac{1}{2}[\ln(\cos x)]^2 + C$ **103.** $2^{\tan\theta}/\ln 2 + C$
105. $-(1/x) - 2\ln|x| + x + C$
109. $e^{\sqrt{x}}/(2x)$
111. $\frac{1}{3}\ln 4$ **113.** $\pi^2/4$ **115.** $\frac{2}{3}$ **117.** $2/e$
121. $e^{2x}(1 + 2x)/(1 - e^{-x})$

PROBLEMS PLUS ■ PAGE 509

3. Abs max $f(-5) = e^{45}$, no abs min **9.** $1/\sqrt{2}$
11. $a = \frac{1}{2}$ **13.** e^{-2} **17.** $2\sqrt{e}$ **19.** $a \leqslant e^{1/e}$

CHAPTER 7

EXERCISES 7.1 ■ PAGE 516

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ **3.** $\frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$
5. $-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C$

7. $(x^2 + 2x)\sin x + (2x + 2)\cos x - 2\sin x + C$

9. $x\cos^{-1}x - \sqrt{1 - x^2} + C$ **11.** $\frac{1}{5}t^5 \ln t - \frac{1}{25}t^5 + C$

13. $-t\cot t + \ln|\sin t| + C$

15. $x(\ln x)^2 - 2x\ln x + 2x + C$

17. $\frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$

19. $z^3e^z - 3z^2e^z + 6ze^z - 6e^z + C$

21. $\dfrac{e^{2x}}{4(2x + 1)} + C$ **23.** $\dfrac{\pi - 2}{2\pi^2}$

25. $2\cosh 2 - \sinh 2$ **27.** $\frac{4}{5} - \frac{1}{5}\ln 5$ **29.** $-\pi/4$

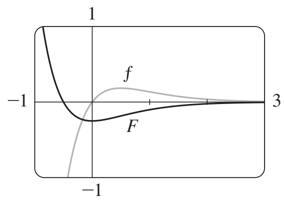
31. $2e^{-1} - 6e^{-5}$ **33.** $\frac{1}{2}\ln 2 - \frac{1}{2}$

35. $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$

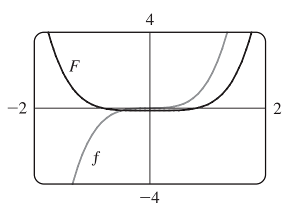
37. $2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$ **39.** $-\frac{1}{2} - \pi/4$

41. $\frac{1}{2}(x^2 - 1)\ln(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$

43. $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$



45. $\frac{1}{3}x^2(1 + x^2)^{3/2} - \frac{2}{15}(1 + x^2)^{5/2} + C$



47. (b) $-\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$

49. (b) $\frac{2}{3}, \frac{8}{15}$

55. $x[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6] + C$

57. $\frac{16}{3}\ln 2 - \frac{29}{9}$ **59.** $-1.75119, 1.17210; 3.99926$

61. $4 - 8/\pi$ **63.** $2\pi e$

65. (a) $2\pi(2\ln 2 - \frac{3}{4})$ (b) $2\pi[(\ln 2)^2 - 2\ln 2 + 1]$

67. $xS(x) + \frac{1}{\pi}\cos(\frac{1}{2}\pi x^2) + C$

69. $2 - e^{-t}(t^2 + 2t + 2)$ m **71.** 2

EXERCISES 7.2 ■ PAGE 524

1. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$ **3.** $\frac{1}{120}$

5. $-\frac{1}{14}\cos^7(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{6}\cos^3(2t) + C$

7. $\pi/4$ **9.** $3\pi/8$ **11.** $\pi/16$

13. $\frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

15. $\ln|\sin x| - \frac{1}{2}\sin^2 x + C$ **17.** $\frac{1}{2}\sin^4 x + C$

19. $\frac{1}{4}t^2 - \frac{1}{4}t\sin 2t - \frac{1}{8}\cos 2t + C$ **21.** $\frac{1}{3}\sec^3 x + C$

23. $\tan x - x + C$ **25.** $\frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

27. $\frac{1}{3}\sec^3 x - \sec x + C$ **29.** $\frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$

31. $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

33. $x\sec x - \ln|\sec x + \tan x| + C$ **35.** $\sqrt{3} - \frac{1}{3}\pi$

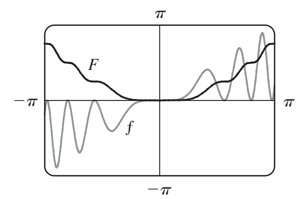
37. $\frac{22}{105}\sqrt{2} - \frac{8}{105}$ **39.** $\ln|\csc x - \cot x| + C$

41. $-\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$ **43.** $\frac{1}{15}$

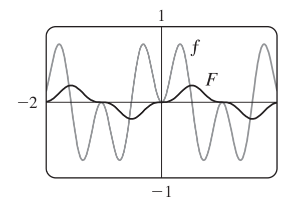
45. $\frac{1}{2}\sqrt{2}$ **47.** $\frac{1}{2}\sin 2x + C$

49. $x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$

51. $\frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$



53. $\frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$



55. 0 **57.** $\frac{1}{2}\pi - \frac{4}{3}$ **59.** 0 **61.** $\pi^2/4$

63. $\pi(2\sqrt{2} - \frac{5}{2})$ **65.** $s = (1 - \cos^3\omega t)/(3\omega)$

EXERCISES 7.3 ■ PAGE 531

1. $-\dfrac{\sqrt{4 - x^2}}{4x} + C$ **3.** $\sqrt{x^2 - 4} - 2\sec^{-1}\left(\dfrac{x}{2}\right) + C$

5. $\dfrac{1}{3}\dfrac{(x^2 - 1)^{3/2}}{x^3} + C$ **7.** $\dfrac{1}{\sqrt{2}\,a^2}$

9. $\frac{2}{3}\sqrt{3} - \frac{3}{4}\sqrt{2}$ **11.** $\frac{1}{12}$

13. $\frac{1}{6}\sec^{-1}(x/3) - \sqrt{x^2 - 9}/(2x^2) + C$

15. $\frac{1}{16}\pi a^4$ **17.** $\sqrt{x^2 - 7} + C$

19. $\ln\left|(\sqrt{1 + x^2} - 1)/x\right| + \sqrt{1 + x^2} + C$ **21.** $\frac{9}{500}\pi$

23. $\ln\left|\sqrt{x^2 + 2x + 5} + x + 1\right| + C$

25. $4\sin^{-1}\left(\dfrac{x - 1}{2}\right) + \dfrac{1}{4}(x - 1)^3\sqrt{3 + 2x - x^2}$
$- \frac{2}{3}(3 + 2x - x^2)^{3/2} + C$

27. $\frac{1}{2}(x + 1)\sqrt{x^2 + 2x} - \frac{1}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x}\right| + C$

29. $\frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1 - x^4} + C$

33. $\frac{1}{6}(\sqrt{48} - \sec^{-1} 7)$ **37.** $\frac{3}{8}\pi^2 + \frac{3}{4}\pi$

41. $2\pi^2 Rr^2$ **43.** $r\sqrt{R^2 - r^2} + \pi r^2/2 - R^2 \arcsin(r/R)$

EXERCISES 7.4 ■ PAGE 541

1. (a) $\dfrac{A}{1 + 2x} + \dfrac{B}{3 - x}$ (b) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{1 + x}$

3. (a) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{1 + x^2}$ (b) $1 + \dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{C}{x - 2}$

5. (a) $x^4 + 4x^2 + 16 + \dfrac{A}{x + 2} + \dfrac{B}{x - 2}$

(b) $\dfrac{Ax + B}{x^2 - x + 1} + \dfrac{Cx + D}{x^2 + 2} + \dfrac{Ex + F}{(x^2 + 2)^2}$

7. $\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x - 1| + C$

9. $\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$ **11.** $2\ln\frac{3}{2}$

13. $a\ln|x-b| + C$ **15.** $\frac{5}{2} - \ln 2 - \ln 3$ $\left(\text{or } \frac{5}{2} - \ln 6\right)$

17. $\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3$ $\left(\text{or } \frac{9}{5}\ln\frac{8}{3}\right)$

19. $\frac{1}{2} - 5\ln 2 + 3\ln 3$ $\left(\text{or } \frac{1}{2} + \ln\frac{27}{32}\right)$

21. $\dfrac{1}{4}\left[\ln|t+1| - \dfrac{1}{t+1} - \ln|t-1| - \dfrac{1}{t-1}\right] + C$

23. $\ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C$

25. $-2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C$

27. $\frac{1}{2}\ln(x^2+1) + \tan^{-1}x - \frac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

29. $\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right) + C$

31. $\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}} + C$

33. $\frac{1}{4}\ln\frac{8}{3}$

35. $2\ln|x| + \frac{3}{2}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \dfrac{x}{2(x^2+1)} + C$

37. $\frac{7}{8}\sqrt{2}\tan^{-1}\left(\dfrac{x-2}{\sqrt{2}}\right) + \dfrac{3x-8}{4(x^2-4x+6)} + C$

39. $2\tan^{-1}\sqrt{x-1} + C$

41. $-2\ln\sqrt{x} - \dfrac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C$

43. $\frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$

45. $2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$

47. $\ln\left[\dfrac{(e^x+2)^2}{(e^x+1)}\right] + C$

49. $\ln|\tan t + 1| - \ln|\tan t + 2| + C$

51. $x - \ln(e^x+1) + C$

53. $\left(x - \frac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{7}}\right) + C$

55. $-\frac{1}{2}\ln 3 \approx -0.55$

57. $\frac{1}{2}\ln\left|\dfrac{x-2}{x}\right| + C$ **61.** $\frac{1}{5}\ln\left|\dfrac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$

63. $4\ln\frac{2}{3} + 2$ **65.** $-1 + \frac{11}{3}\ln 2$

67. $t = \ln\dfrac{10{,}000}{P} + 11\ln\dfrac{P-9000}{1000}$

69. (a) $\dfrac{24{,}110}{4879}\dfrac{1}{5x+2} - \dfrac{668}{323}\dfrac{1}{2x+1} - \dfrac{9438}{80{,}155}\dfrac{1}{3x-7} + \dfrac{1}{260{,}015}\dfrac{22{,}098x + 48{,}935}{x^2+x+5}$

(b) $\dfrac{4822}{4879}\ln|5x+2| - \dfrac{334}{323}\ln|2x+1| - \dfrac{3146}{80{,}155}\ln|3x-7| + \dfrac{11{,}049}{260{,}015}\ln(x^2+x+5) + \dfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\dfrac{2x+1}{\sqrt{19}} + C$

The CAS omits the absolute value signs and the constant of integration.

75. $\dfrac{1}{a^n(x-a)} - \dfrac{1}{a^n x} - \dfrac{1}{a^{n-1}x^2} - \cdots - \dfrac{1}{ax^n}$

EXERCISES 7.5 ■ PAGE 547

1. $-\ln(1 - \sin x) + C$ **3.** $\frac{32}{3}\ln 2 - \frac{28}{9}$

5. $\dfrac{1}{2\sqrt{2}}\tan^{-1}\left(\dfrac{t^2}{\sqrt{2}}\right) + C$ **7.** $e^{\pi/4} - e^{-\pi/4}$

9. $\frac{4}{5}\ln 2 + \frac{1}{5}\ln 3$ $\left(\text{or } \frac{1}{5}\ln 48\right)$ **11.** $\frac{1}{2}\sec^{-1}x + \dfrac{\sqrt{x^2-1}}{2x^2} + C$

13. $-\frac{1}{5}\cos^5 t + \frac{2}{7}\cos^7 t - \frac{1}{9}\cos^9 t + C$

15. $x\sec x - \ln|\sec x + \tan x| + C$

17. $\frac{1}{4}\pi^2$ **19.** $e^{e^x} + C$ **21.** $(x+1)\arctan\sqrt{x} - \sqrt{x} + C$

23. $\frac{4097}{45}$ **25.** $4 - \ln 4$ **27.** $x - \ln(1 + e^x) + C$

29. $x\ln\left(x + \sqrt{x^2-1}\right) - \sqrt{x^2-1} + C$

31. $\sin^{-1}x - \sqrt{1-x^2} + C$

33. $2\sin^{-1}\left(\dfrac{x+1}{2}\right) + \dfrac{x+1}{2}\sqrt{3-2x-x^2} + C$

35. 0 **37.** $\frac{1}{4}$ **39.** $\ln|\sec\theta - 1| - \ln|\sec\theta| + C$

41. $\theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$ **43.** $\frac{2}{3}\tan^{-1}(x^{3/2}) + C$

45. $-\frac{1}{3}(x^3+1)e^{-x^3} + C$

47. $\ln|x-1| - 3(x-1)^{-1} - \frac{3}{2}(x-1)^{-2} - \frac{1}{3}(x-1)^{-3} + C$

49. $\ln\left|\dfrac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$ **51.** $-\ln\left|\dfrac{\sqrt{4x^2+1}+1}{2x}\right| + C$

53. $\dfrac{1}{m}x^2\cosh mx - \dfrac{2}{m^2}x\sinh mx + \dfrac{2}{m^3}\cosh mx + C$

55. $2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

57. $\frac{3}{7}(x+c)^{7/3} - \frac{3}{4}c(x+c)^{4/3} + C$

59. $\dfrac{1}{32}\ln\left|\dfrac{x-2}{x+2}\right| - \dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

61. $\csc\theta - \cot\theta + C$ or $\tan(\theta/2) + C$

63. $2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$

65. $-\tan^{-1}(\cos^2 x) + C$ **67.** $\frac{2}{3}[(x+1)^{3/2} - x^{3/2}] + C$

69. $\sqrt{2} - 2/\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)$

71. $e^x - \ln(1+e^x) + C$

73. $-\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + C$ **75.** $\ln|\ln x - 1| + C$

77. $2(x-2)\sqrt{1+e^x} + 2\ln\dfrac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$

79. $\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$

81. $2\sqrt{1+\sin x} + C$ **83.** $xe^{x^2} + C$

EXERCISES 7.6 ■ PAGE 552

1. $-\frac{5}{21}$ **3.** $\sqrt{13} - \frac{3}{4}\ln\left(4+\sqrt{13}\right) - \frac{1}{2} + \frac{3}{4}\ln 3$

5. $\dfrac{\pi}{8}\arctan\dfrac{\pi}{4} - \frac{1}{4}\ln\left(1 + \frac{1}{16}\pi^2\right)$ **7.** $\frac{1}{6}\ln\left|\dfrac{\sin x - 3}{\sin x + 3}\right| + C$

9. $-\dfrac{\sqrt{9x^2+4}}{x} + 3\ln\left(3x + \sqrt{9x^2+4}\right) + C$

11. $5\pi/16$ **13.** $2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C$

15. $-\ln|\sinh(1/y)| + C$

17. $\dfrac{2y-1}{8}\sqrt{6+4y-4y^2} + \dfrac{7}{8}\sin^{-1}\left(\dfrac{2y-1}{\sqrt{7}}\right) - \frac{1}{12}(6+4y-4y^2)^{3/2} + C$

19. $\frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$

21. $\dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{e^x + \sqrt{3}}{e^x - \sqrt{3}}\right| + C$

23. $\frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$

25. $\frac{1}{2}(\ln x)\sqrt{4 + (\ln x)^2} + 2\ln\left[\ln x + \sqrt{4 + (\ln x)^2}\right] + C$

27. $-\frac{1}{2}x^{-2}\cos^{-1}(x^{-2}) + \frac{1}{2}\sqrt{1 - x^{-4}} + C$

29. $\sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C$

31. $\frac{1}{5}\ln|x^5 + \sqrt{x^{10} - 2}| + C$ **33.** $\frac{3}{8}\pi^2$

37. $\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x + C$

39. $\frac{1}{4}x(x^2 + 2)\sqrt{x^2 + 4} - 2\ln\left(\sqrt{x^2 + 4} + x\right) + C$

41. $\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{8}\sin x\cos x + C$

43. $-\ln|\cos x| - \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$

45. (a) $-\ln\left|\dfrac{1 + \sqrt{1 - x^2}}{x}\right| + C$;
both have domain $(-1, 0) \cup (0, 1)$

EXERCISES 7.7 ■ PAGE 564

1. (a) $L_2 = 6$, $R_2 = 12$, $M_2 \approx 9.6$
(b) L_2 is an underestimate, R_2 and M_2 are overestimates.
(c) $T_2 = 9 < I$ (d) $L_n < T_n < I < M_n < R_n$

3. (a) $T_4 \approx 0.895759$ (underestimate)
(b) $M_4 \approx 0.908907$ (overestimate); $T_4 < I < M_4$

5. (a) $M_{10} \approx 0.806598$, $E_M \approx -0.001879$
(b) $S_{10} \approx 0.804779$, $E_S \approx -0.000060$

7. (a) 1.506361 (b) 1.518362 (c) 1.511519

9. (a) 2.660833 (b) 2.664377 (c) 2.663244

11. (a) −7.276910 (b) −4.818251 (c) −5.605350

13. (a) −2.364034 (b) −2.310690 (c) −2.346520

15. (a) 0.243747 (b) 0.243748 (c) 0.243751

17. (a) 8.814278 (b) 8.799212 (c) 8.804229

19. (a) $T_8 \approx 0.902333$, $M_8 \approx 0.905620$
(b) $|E_T| \leqslant 0.0078$, $|E_M| \leqslant 0.0039$
(c) $n = 71$ for T_n, $n = 50$ for M_n

21. (a) $T_{10} \approx 1.983524$, $E_T \approx 0.016476$;
$M_{10} \approx 2.008248$, $E_M \approx -0.008248$;
$S_{10} \approx 2.000110$, $E_S \approx -0.000110$
(b) $|E_T| \leqslant 0.025839$, $|E_M| \leqslant 0.012919$, $|E_S| \leqslant 0.000170$
(c) $n = 509$ for T_n, $n = 360$ for M_n, $n = 22$ for S_n

23. (a) 2.8 (b) 7.954926518 (c) 0.2894
(d) 7.954926521 (e) The actual error is much smaller.
(f) 10.9 (g) 7.953789422 (h) 0.0593
(i) The actual error is smaller. (j) $n \geqslant 50$

25.

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations are the same as after Example 1.

27.

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations are the same as after Example 1.

29. (a) 19 (b) 18.6 (c) $18.\overline{6}$

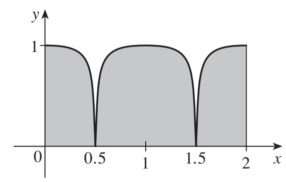
31. (a) 14.4 (b) $\frac{1}{2}$

33. 70.8°F **35.** $37.7\overline{3}$ ft/s **37.** 10,177 megawatt-hours

39. (a) 190 (b) 828

41. 28 **43.** 59.4

45.



EXERCISES 7.8 ■ PAGE 574

Abbreviations: C, convergent; D, divergent

1. (a), (d) Infinite discontinuity (b), (c) Infinite interval

3. $\frac{1}{2} - 1/(2t^2)$; 0.495, 0.49995, 0.4999995; 0.5

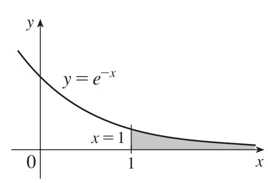
5. 2 **7.** D **9.** $\frac{1}{5}e^{-10}$ **11.** D **13.** 0 **15.** D

17. ln 2 **19.** $-\frac{1}{4}$ **21.** D **23.** $-\pi/8$ **25.** 2

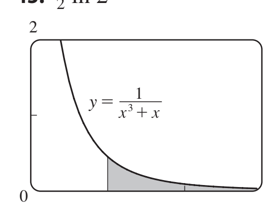
27. D **29.** $\frac{32}{3}$ **31.** D **33.** $\frac{9}{2}$ **35.** D **37.** $-\frac{1}{4}$

39. $-2/e$

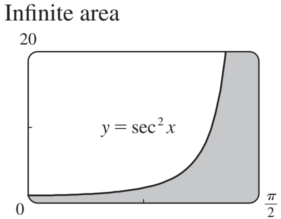
41. $1/e$



43. $\frac{1}{2}\ln 2$



45. Infinite area

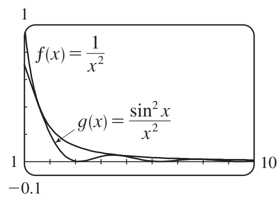


47. (a)

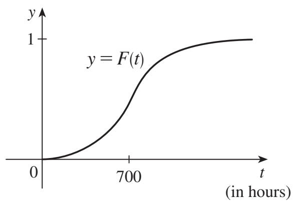
t	$\int_1^t [(\sin^2 x)/x^2]\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1,000	0.672957
10,000	0.673407

It appears that the integral is convergent.

(c)



49. C **51.** D **53.** D **55.** π **57.** $p < 1,\ 1/(1-p)$
59. $p > -1,\ -1/(p+1)^2$ **63.** π **65.** $\sqrt{2GM/R}$
67. (a)



(b) The rate at which the fraction $F(t)$ increases as t increases
(c) 1; all bulbs burn out eventually

69. $\gamma = \dfrac{cN}{\lambda(k+\lambda)}$ **71.** 1000
73. (a) $F(s) = 1/s,\ s > 0$ (b) $F(s) = 1/(s-1),\ s > 1$
(c) $F(s) = 1/s^2,\ s > 0$
79. $C = 1$; $\ln 2$ **81.** No

CHAPTER 7 REVIEW ■ PAGE 577

True-False Quiz

1. False **3.** False **5.** False **7.** False
9. (a) True (b) False **11.** False **13.** False

Exercises

1. $\frac{7}{2} + \ln 2$ **3.** $e^{\sin x} + C$ **5.** $\ln|2t+1| - \ln|t+1| + C$
7. $\frac{2}{15}$ **9.** $-\cos(\ln t) + C$ **11.** $\sqrt{3} - \frac{1}{3}\pi$
13. $3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$
15. $-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x+2| + C$
17. $x\sinh x - \cosh x + C$
19. $\frac{1}{18}\ln(9x^2 + 6x + 5) + \frac{1}{9}\tan^{-1}\left[\frac{1}{2}(3x+1)\right] + C$
21. $\ln\left|x - 2 + \sqrt{x^2 - 4x}\right| + C$
23. $\ln\left|\dfrac{\sqrt{x^2+1} - 1}{x}\right| + C$
25. $\frac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}(x/\sqrt{2}) + C$
27. $\frac{2}{5}$ **29.** 0 **31.** $6 - \frac{3}{2}\pi$
33. $\dfrac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$
35. $4\sqrt{1+\sqrt{x}} + C$ **37.** $\frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$
39. $\frac{1}{8}e - \frac{1}{4}$ **41.** $\frac{1}{36}$ **43.** D
45. $4\ln 4 - 8$ **47.** $-\frac{4}{3}$ **49.** $\pi/4$
51. $(x+1)\ln(x^2+2x+2) + 2\arctan(x+1) - 2x + C$
53. 0
55. $\frac{1}{4}(2x-1)\sqrt{4x^2-4x-3} - \ln\left|2x - 1 + \sqrt{4x^2-4x-3}\right| + C$
57. $\frac{1}{2}\sin x\sqrt{4+\sin^2 x} + 2\ln\left(\sin x + \sqrt{4+\sin^2 x}\right) + C$
61. No
63. (a) 1.925444 (b) 1.920915 (c) 1.922470
65. (a) $0.01348,\ n \geqslant 368$ (b) $0.00674,\ n \geqslant 260$
67. 8.6 mi
69. (a) 3.8 (b) 1.7867, 0.000646 (c) $n \geqslant 30$
71. (a) D (b) C
73. 2 **75.** $\frac{3}{16}\pi^2$

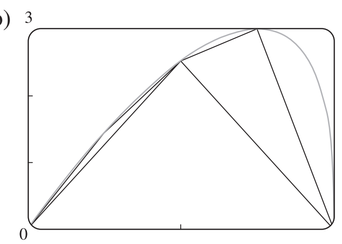
PROBLEMS PLUS ■ PAGE 581

1. About 1.85 inches from the center **3.** 0
7. $f(\pi) = -\pi/2$ **11.** $(b^b a^{-a})^{1/(b-a)}e^{-1}$ **13.** $\frac{1}{8}\pi - \frac{1}{12}$
15. $2 - \sin^{-1}(2/\sqrt{5})$

CHAPTER 8

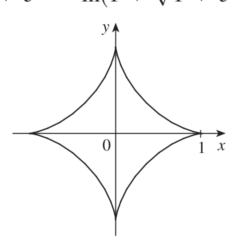
EXERCISES 8.1 ■ PAGE 588

1. $4\sqrt{5}$ **3.** 3.8202 **5.** 3.4467 **7.** 3.6095
9. $\frac{2}{243}(82\sqrt{82} - 1)$ **11.** $\frac{59}{24}$ **13.** $\frac{32}{3}$
15. $\ln(\sqrt{2} + 1)$ **17.** $\frac{3}{4} + \frac{1}{2}\ln 2$ **19.** $\ln 3 - \frac{1}{2}$
21. $\sqrt{2} + \ln(1 + \sqrt{2})$ **23.** 10.0556
25. 15.498085; 15.374568 **27.** 7.094570; 7.118819
29. (a), (b)



$L_1 = 4$,
$L_2 \approx 6.43$,
$L_4 \approx 7.50$

(c) $\int_0^4 \sqrt{1 + [4(3-x)/(3(4-x)^{2/3})]^2}\,dx$ (d) 7.7988
31. $\sqrt{1+e^4} - \ln(1 + \sqrt{1+e^4}) + 2 - \sqrt{2} + \ln(1 + \sqrt{2})$
33. 6



35. $s(x) = \frac{2}{27}\left[(1+9x)^{3/2} - 10\sqrt{10}\right]$ **37.** $2\sqrt{2}\left(\sqrt{1+x} - 1\right)$
41. 209.1 m **43.** 29.36 in. **45.** 12.4

EXERCISES 8.2 ■ PAGE 595

1. (a) (i) $\int_0^{\pi/3} 2\pi\tan x\sqrt{1+\sec^4 x}\,dx$
(ii) $\int_0^{\pi/3} 2\pi x\sqrt{1+\sec^4 x}\,dx$ (b) (i) 10.5017 (ii) 7.9353
3. (a) (i) $\int_{-1}^1 2\pi e^{-x^2}\sqrt{1+4x^2e^{-2x^2}}\,dx$
(ii) $\int_0^1 2\pi x\sqrt{1+4x^2e^{-2x^2}}\,dx$ (b) (i) 11.0753 (ii) 3.9603
5. (a) (i) $\int_0^1 2\pi y\sqrt{1+(1+3y^2)^2}\,dy$
(ii) $\int_0^1 2\pi(y+y^3)\sqrt{1+(1+3y^2)^2}\,dy$
(b) (i) 8.5302 (ii) 13.5134
7. $\frac{1}{27}\pi(145\sqrt{145} - 1)$ **9.** $\frac{1}{6}\pi(27\sqrt{27} - 5\sqrt{5})$
11. $\pi\sqrt{5} + 4\pi\ln\left(\dfrac{1+\sqrt{5}}{2}\right)$ **13.** $\frac{21}{2}\pi$ **15.** $\frac{3712}{15}\pi$

17. πa^2 **19.** 1,230,507 **21.** 24.145807

23. $\frac{1}{4}\pi\left[4\ln(\sqrt{17}+4) - 4\ln(\sqrt{2}+1) - \sqrt{17} + 4\sqrt{2}\right]$

25. $\frac{1}{6}\pi\left[\ln(\sqrt{10}+3) + 3\sqrt{10}\right]$

29. (a) $\frac{1}{3}\pi a^2$ (b) $\frac{56}{45}\pi\sqrt{3}\,a^2$

31. (a) $2\pi\left[b^2 + \dfrac{a^2 b \sin^{-1}(\sqrt{a^2-b^2}/a)}{\sqrt{a^2-b^2}}\right]$

(b) $2\pi a^2 + \dfrac{2\pi a b^2}{\sqrt{a^2-b^2}} \ln \dfrac{a+\sqrt{a^2-b^2}}{b}$

33. $\int_a^b 2\pi[c - f(x)]\sqrt{1+[f'(x)]^2}\,dx$ **35.** $4\pi^2 r^2$

37. Both equal $\pi \int_a^b (e^{x/2} + e^{-x/2})^2\,dx$.

EXERCISES 8.3 ■ PAGE 605

1. (a) 187.5 lb/ft^2 (b) 1875 lb (c) 562.5 lb
3. 7000 lb **5.** 2.36×10^7 N **7.** 9.8×10^3 N
9. 889 lb **11.** $\frac{2}{3}\delta a h^2$ **13.** 5.27×10^5 N
15. (a) 314 N (b) 353 N
17. (a) 5.63×10^3 lb (b) 5.06×10^4 lb
(c) 4.88×10^4 lb (d) 3.03×10^5 lb
19. 4148 lb **21.** 330; 22
23. 10; 14; (1.4, 1) **25.** $\left(\frac{2}{3}, \frac{2}{3}\right)$

27. $\left(\dfrac{1}{e-1}, \dfrac{e+1}{4}\right)$ **29.** $\left(\frac{9}{20}, \frac{9}{20}\right)$

31. $\left(\dfrac{\pi\sqrt{2}-4}{4(\sqrt{2}-1)}, \dfrac{1}{4(\sqrt{2}-1)}\right)$ **33.** $\left(\frac{8}{5}, -\frac{1}{2}\right)$

35. $\left(\dfrac{28}{3(\pi+2)}, \dfrac{10}{3(\pi+2)}\right)$ **37.** $\left(-\frac{1}{5}, -\frac{12}{35}\right)$

41. $\left(0, \frac{1}{12}\right)$ **45.** $\frac{1}{3}\pi r^2 h$ **47.** $\left(\dfrac{8}{\pi}, \dfrac{8}{\pi}\right)$

49. $4\pi^2 r R$

EXERCISES 8.4 ■ PAGE 612

1. \$21,104 **3.** \$140,000; \$60,000 **5.** \$407.25
7. \$166,666.67 **9.** (a) 3800 (b) \$324,900
11. 3727; \$37,753 **13.** $\frac{2}{3}(16\sqrt{2} - 8) \approx$ \$9.75 million

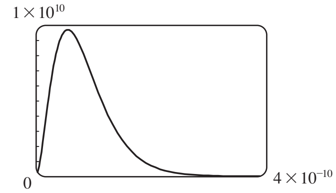
15. \$65,230.48 **17.** $\dfrac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}$

19. 1.19×10^{-4} cm^3/s **21.** 6.59 L/min **23.** 5.77 L/min

EXERCISES 8.5 ■ PAGE 619

1. (a) The probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles
(b) The probability that a randomly chosen tire will have a lifetime of at least 25,000 miles
3. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) $\frac{17}{81}$
5. (a) $1/\pi$ (b) $\frac{1}{2}$
7. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) 5
11. (a) $\approx$46.5% (b) $\approx$15.3% (c) About 4.8 s
13. $\approx$59.4% (b) 40 min **15.** $\approx$44%

17. (a) 0.0668 (b) $\approx$5.21% **19.** $\approx$0.9545
21. (b) 0; a_0
(c)



(d) $1 - 41e^{-8} \approx 0.986$ (e) $\frac{3}{2}a_0$

CHAPTER 8 REVIEW ■ PAGE 621

Exercises

1. $\frac{1}{54}(109\sqrt{109} - 1)$ **3.** $\frac{53}{6}$
5. (a) 3.5121 (b) 22.1391 (c) 29.8522
7. 3.8202 **9.** $\frac{124}{5}$ **11.** $\approx$458 lb **13.** $\left(\frac{8}{5}, 1\right)$
15. $\left(\frac{4}{3}, \frac{4}{3}\right)$ **17.** $2\pi^2$ **19.** \$7166.67
21. (a) $f(x) \geqslant 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$
(b) $\approx$0.3455 (c) 5; yes
23. (a) $1 - e^{-3/8} \approx 0.31$ (b) $e^{-5/4} \approx 0.29$
(c) $8 \ln 2 \approx 5.55$ min

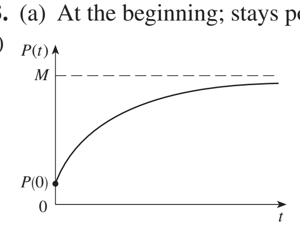
PROBLEMS PLUS ■ PAGE 623

1. $\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$
3. (a) $2\pi r(r \pm d)$ (b) $\approx 3.36 \times 10^6$ mi^2
(d) $\approx 7.84 \times 10^7$ mi^2
5. (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$
(b) $(P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$
7. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$ **9.** 0.14 m
11. $2/\pi$; $1/\pi$ **13.** $(0, -1)$

CHAPTER 9

EXERCISES 9.1 ■ PAGE 630

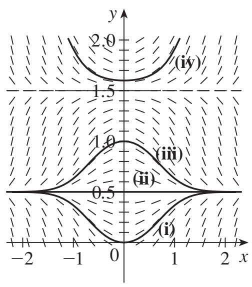
3. (a) $\frac{1}{2}, -1$ **5.** (d)
7. (a) It must be either 0 or decreasing
(c) $y = 0$ (d) $y = 1/(x + 2)$
9. (a) $0 < P < 4200$ (b) $P > 4200$
(c) $P = 0$, $P = 4200$
13. (a) III (b) I (c) IV (d) II
15. (a) At the beginning; stays positive, but decreases
(c)



17. It approaches 0 as c approaches c_s.

EXERCISES 9.2 ■ PAGE 637

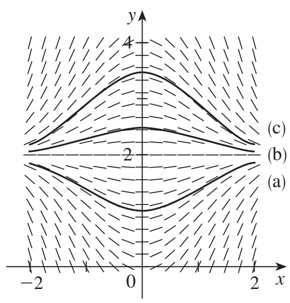
1. (a)



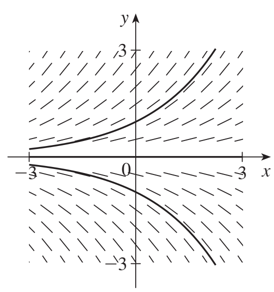
(b) $y = 0.5$, $y = 1.5$

3. III **5.** IV

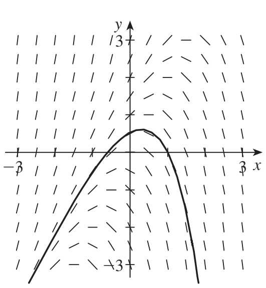
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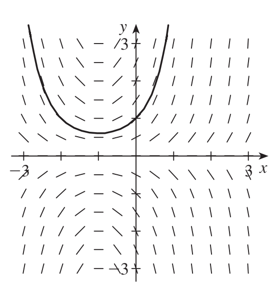
9.



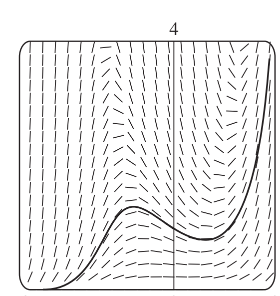
11.



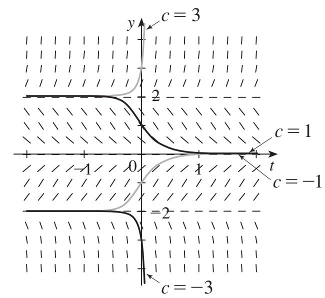
13.



15.



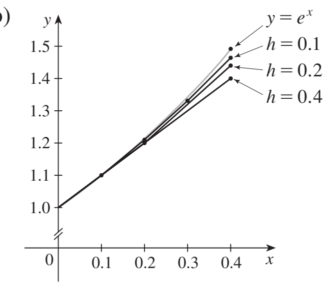
17.



$-2 \leqslant c \leqslant 2$; $-2, 0, 2$

19. (a) (i) 1.4 (ii) 1.44 (iii) 1.4641

(b)



Underestimates

(c) (i) 0.0918 (ii) 0.0518 (iii) 0.0277

It appears that the error is also halved (approximately).

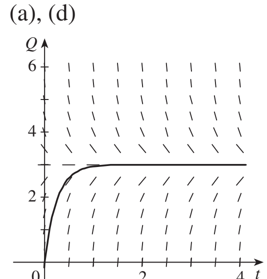
21. $-1, -3, -6.5, -12.25$ **23.** 1.7616

25. (a) (i) 3 (ii) 2.3928 (iii) 2.3701 (iv) 2.3681

(c) (i) -0.6321 (ii) -0.0249 (iii) -0.0022 (iv) -0.0002

It appears that the error is also divided by 10 (approximately).

27. (a), (d)



(b) 3

(c) Yes, $Q = 3$

(e) 2.77 C

EXERCISES 9.3 ■ PAGE 645

1. $y = -1/(x^3 + C)$, $y = 0$

3. $y = \pm\sqrt{x^2 + 2\ln|x| + C}$

5. $e^y - y = 2x + \sin x + C$

7. $\theta \sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C$ **9.** $p = Ke^{t^3/3-t} - 1$

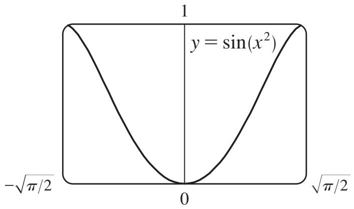
11. $y = -\ln\left(1 - \frac{1}{2}x^2\right)$ **13.** $u = -\sqrt{t^2 + \tan t + 25}$

15. $\frac{1}{2}y^2 + \frac{1}{3}(3 + y^2)^{3/2} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{41}{12}$

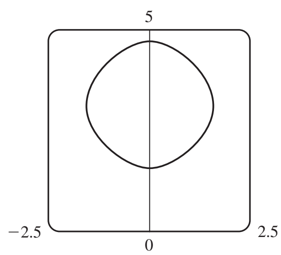
17. $y = \dfrac{4a}{\sqrt{3}}\sin x - a$

19. $y = \sqrt{x^2 + 4}$ **21.** $y = Ke^x - x - 1$

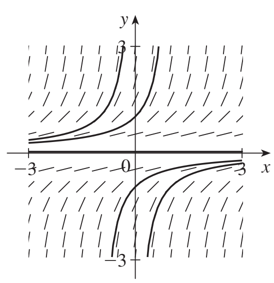
23. (a) $\sin^{-1}y = x^2 + C$
(b) $y = \sin(x^2)$, $-\sqrt{\pi/2} \leqslant x \leqslant \sqrt{\pi/2}$ (c) No



25. $\cos y = \cos x - 1$

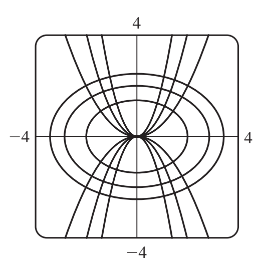


27. (a)

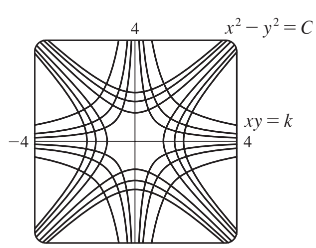


(b) $y = \dfrac{1}{K - x}$

29. $y = Cx^2$



31. $x^2 - y^2 = C$



33. $y = 1 + e^{2 - x^2/2}$ **35.** $y = \left(\frac{1}{2}x^2 + 2\right)^2$

37. $Q(t) = 3 - 3e^{-4t}$; 3 **39.** $P(t) = M - Me^{-kt}$; M

41. (a) $x = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$

(b) $t = \dfrac{2}{k\sqrt{a - b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a - b}} - \tan^{-1}\sqrt{\dfrac{b - x}{a - b}}\right)$

43. (a) $C(t) = (C_0 - r/k)e^{-kt} + r/k$ (b) r/k; the concentration approaches r/k regardless of the value of C_0

45. (a) $15e^{-t/100}$ kg (b) $15e^{-0.2} \approx 12.3$ kg

47. About 4.9% **49.** g/k

51. (a) $L_1 = KL_2^k$ (b) $B = KV^{0.0794}$

53. (a) $dA/dt = k\sqrt{A}\,(M - A)$ (b) $A(t) = M\left(\dfrac{Ce^{\sqrt{M}\,kt} - 1}{Ce^{\sqrt{M}\,kt} + 1}\right)^2$,

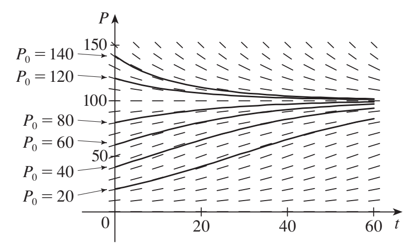
where $C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$ and $A_0 = A(0)$

EXERCISES 9.4 ■ PAGE 657

1. (a) 1200; 0.04 (b) $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$ (c) 87

3. (a) 100; 0.05 (b) Where P is close to 0 or 100; on the line $P = 50$; $0 < P_0 < 100$; $P_0 > 100$

(c)



Solutions approach 100; some increase and some decrease, some have an inflection point but others don't; solutions with $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$

(d) $P = 0$, $P = 100$; other solutions move away from $P = 0$ and toward $P = 100$

5. (a) 3.23×10^7 kg (b) $\approx$1.55 years **7.** 9000

9. (a) $\dfrac{dP}{dt} = \dfrac{1}{305}P\left(1 - \dfrac{P}{20}\right)$

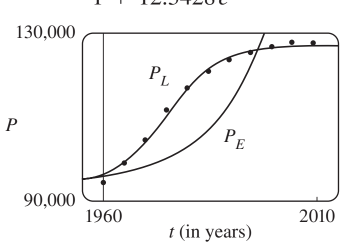
(b) 6.24 billion (c) 7.57 billion; 13.87 billion

11. (a) $dy/dt = ky(1 - y)$ (b) $y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$

(c) 3:36 PM

15. $P_E(t) = 1909.7761\,(1.0796)^t + 94{,}000$;

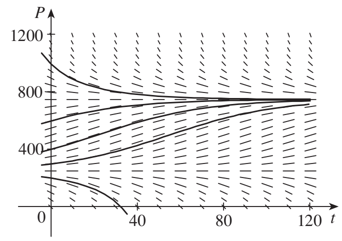
$P_L(t) = \dfrac{33{,}086.4394}{1 + 12.3428\,e^{-0.1657t}} + 94{,}000$



17. (a) $P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$ (b) $m < kP_0$

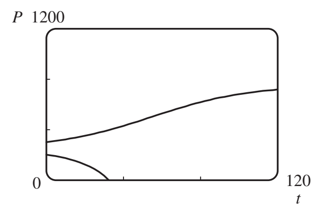
(c) $m = kP_0$, $m > kP_0$ (d) Declining

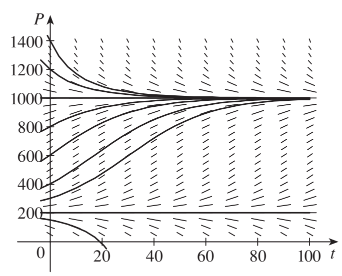
19. (a) Fish are caught at a rate of 15 per week.
(b) See part (d). (c) $P = 250$, $P = 750$

(d) 

$0 < P_0 < 250$: $P \to 0$;
$P_0 = 250$: $P \to 250$;
$P_0 > 250$: $P \to 750$

(e) $P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$
where $k = \frac{1}{11}, -\frac{1}{9}$



21. (b) 

$0 < P_0 < 200$: $P \to 0$
;
$P_0 = 200$: $P \to 200$;
$P_0 > 200$: $P \to 1000$

(c) $P(t) = \dfrac{m(M - P_0) + M(P_0 - m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0 - m)e^{(M-m)(k/M)t}}$

23. (a) $P(t) = P_0 e^{(k/r)[\sin(rt - \phi) + \sin\phi]}$ (b) Does not exist

EXERCISES 9.5 ■ PAGE 665

1. No **3.** Yes **5.** $y = 1 + Ce^{-x}$

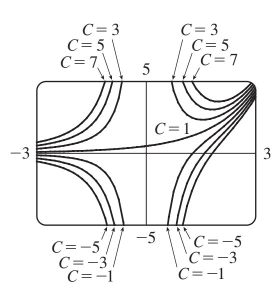
7. $y = x - 1 + Ce^{-x}$ **9.** $y = \frac{2}{3}\sqrt{x} + C/x$

11. $y = x^2(\ln x + C)$ **13.** $y = \frac{1}{3}t^{-3}(1 + t^2)^{3/2} + Ct^{-3}$

15. $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$ **17.** $u = -t^2 + t^3$

19. $y = -x\cos x - x$

21. $y = \dfrac{(x - 1)e^x + C}{x^2}$

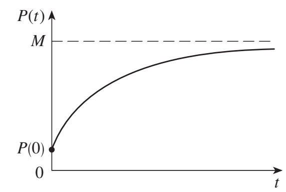


25. $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$

27. (a) $I(t) = 4 - 4e^{-5t}$ (b) $4 - 4e^{-1/2} \approx 1.57$ A

29. $Q(t) = 3(1 - e^{-4t})$, $I(t) = 12e^{-4t}$

31. $P(t) = M + Ce^{-kt}$



33. $y = \frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}$; 0.2275 kg/L

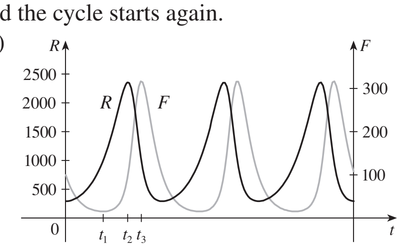
35. (b) mg/c (c) $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

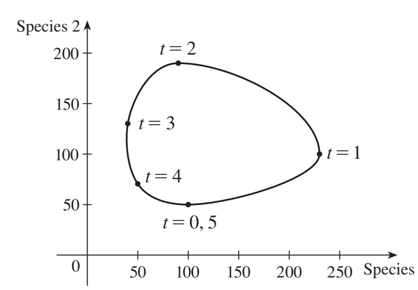
37. (b) $P(t) = \dfrac{M}{1 + MCe^{-kt}}$

EXERCISES 9.6 ■ PAGE 671

1. (a) x = predators, y = prey; growth is restricted only by predators, which feed only on prey.
(b) x = prey, y = predators; growth is restricted by carrying capacity and by predators, which feed only on prey.

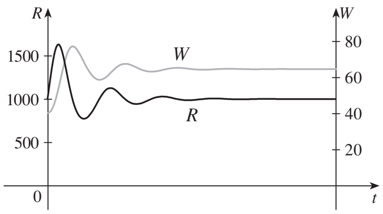
3. (a) Competition
(b) (i) $x = 0$, $y = 0$: zero populations
(ii) $x = 0$, $y = 400$: In the absence of an x-population, the y-population stabilizes at 400.
(iii) $x = 125$, $y = 0$: In the absence of a y-population, the x-population stabilizes at 125.
(iv) $x = 50$, $y = 300$: Both populations are stable.

5. (a) The rabbit population starts at about 300, increases to 2400, then decreases back to 300. The fox population starts at 100, decreases to about 20, increases to about 315, decreases to 100, and the cycle starts again.
(b) 

7. 

11. (a) Population stabilizes at 5000.
(b) (i) $W = 0$, $R = 0$: Zero populations
(ii) $W = 0$, $R = 5000$: In the absence of wolves, the rabbit population is always 5000.
(iii) $W = 64$, $R = 1000$: Both populations are stable.
(c) The populations stabilize at 1000 rabbits and 64 wolves.

(d)



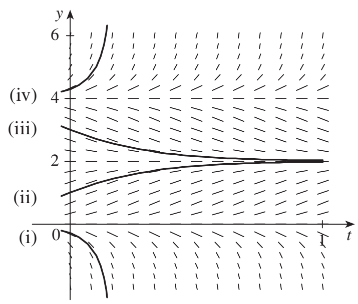
CHAPTER 9 REVIEW ■ PAGE 674

True-False Quiz

1. True **3.** False **5.** True **7.** True

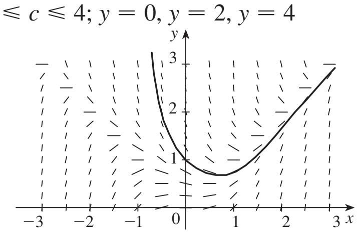
Exercises

1. (a)



(b) $0 \leqslant c \leqslant 4$; $y = 0$, $y = 2$, $y = 4$

3. (a)



$y(0.3) \approx 0.8$

(b) 0.75676

(c) $y = x$ and $y = -x$; there is a loc max or loc min

5. $y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$

7. $y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

9. $r(t) = 5e^{t-t^2}$ **11.** $y = \frac{1}{2}x(\ln x)^2 + 2x$

13. $x = C - \frac{1}{2}y^2$

15. (a) $P(t) = \dfrac{2000}{1 + 19e^{-0.1t}}$; ≈ 560 (b) $t = -10 \ln \frac{2}{57} \approx 33.5$

17. (a) $L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$ (b) $L(t) = 53 - 43e^{-0.2t}$

19. 15 days **21.** $k \ln h + h = (-R/V)t + C$

23. (a) Stabilizes at 200,000

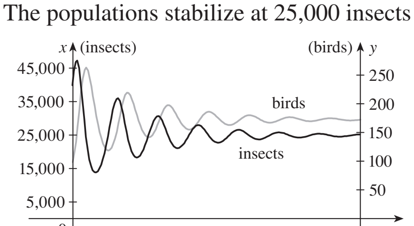
(b) (i) $x = 0$, $y = 0$: Zero populations

(ii) $x = 200{,}000$, $y = 0$: In the absence of birds, the insect population is always 200,000.

(iii) $x = 25{,}000$, $y = 175$: Both populations are stable.

(c) The populations stabilize at 25,000 insects and 175 birds.

(d)



PROBLEMS PLUS ■ PAGE 677

1. $f(x) = \pm 10e^x$ **5.** $y = x^{1/n}$ **7.** 20°C

9. (b) $f(x) = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$ (c) No

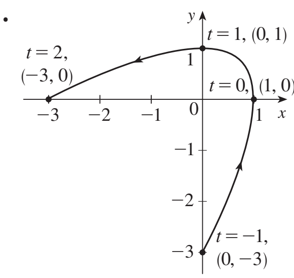
11. (a) 9.8 h (b) $31{,}900\pi$ ft^2; 2000π ft^2/h (c) 5.1 h

13. $x^2 + (y - 6)^2 = 25$ **15.** $y = K/x$, $K \neq 0$

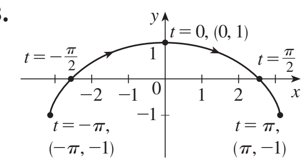
CHAPTER 10

EXERCISES 10.1 ■ PAGE 685

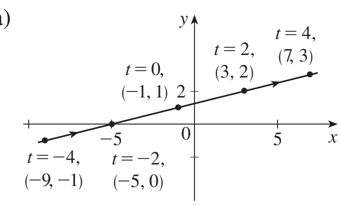
1.



3.

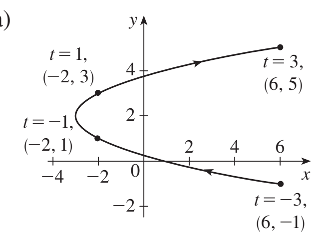


5. (a)



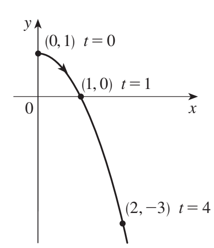
(b) $y = \frac{1}{4}x + \frac{5}{4}$

7. (a)



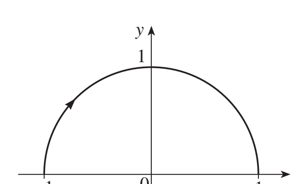
(b) $x = y^2 - 4y + 1$, $-1 \leqslant y \leqslant 5$

9. (a)

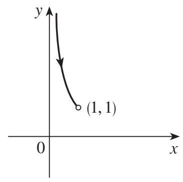


(b) $y = 1 - x^2, x \geqslant 0$

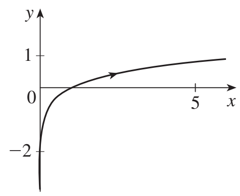
11. (a) $x^2 + y^2 = 1, y \geqslant 0$ (b)



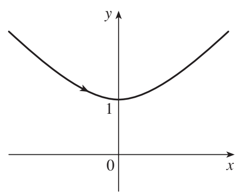
13. (a) $y = 1/x, y > 1$ (b)



15. (a) $x = e^{2y}$ (b)

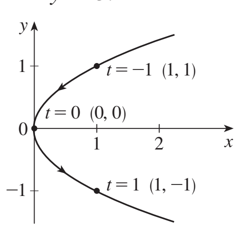


17. (a) $y^2 - x^2 = 1, y \geqslant 1$ (b)

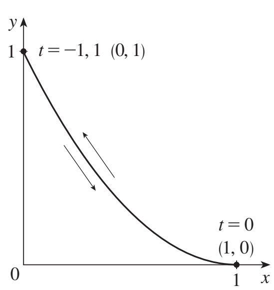


19. Moves counterclockwise along the circle
$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$ from (3, 3) to (7, 3)
21. Moves 3 times clockwise around the ellipse $(x^2/25) + (y^2/4) = 1$, starting and ending at $(0, -2)$
23. It is contained in the rectangle described by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 3$.

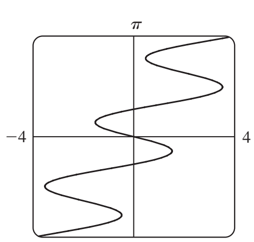
25.



27.

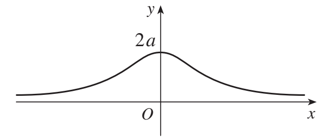


29.

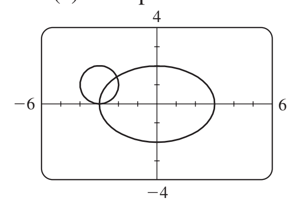


31. (b) $x = -2 + 5t, y = 7 - 8t, 0 \leqslant t \leqslant 1$
33. (a) $x = 2\cos t, y = 1 - 2\sin t, 0 \leqslant t \leqslant 2\pi$
(b) $x = 2\cos t, y = 1 + 2\sin t, 0 \leqslant t \leqslant 6\pi$
(c) $x = 2\cos t, y = 1 + 2\sin t, \pi/2 \leqslant t \leqslant 3\pi/2$
37. The curve $y = x^{2/3}$ is generated in (a). In (b), only the portion with $x \geqslant 0$ is generated, and in (c) we get only the portion with $x > 0$.

41. $x = a\cos\theta, y = b\sin\theta; (x^2/a^2) + (y^2/b^2) = 1$, ellipse
43.

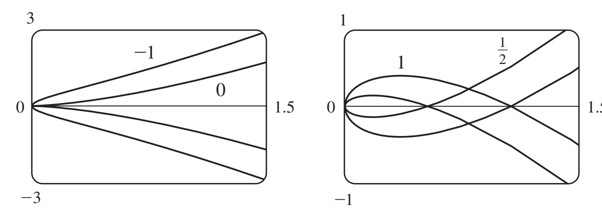


45. (a) Two points of intersection



(b) One collision point at $(-3, 0)$ when $t = 3\pi/2$
(c) There are still two intersection points, but no collision point.

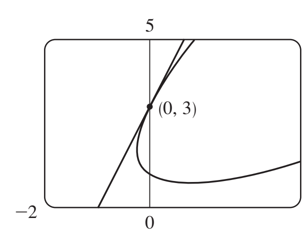
47. For $c = 0$, there is a cusp; for $c > 0$, there is a loop whose size increases as c increases.



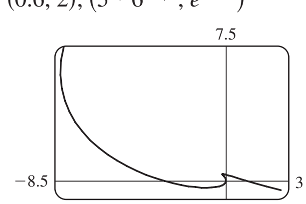
49. The curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6 (more precisely, when $a > \sqrt{2}$). The loops increase in size as a increases.
51. As n increases, the number of oscillations increases; a and b determine the width and height.

EXERCISES 10.2 ■ PAGE 695

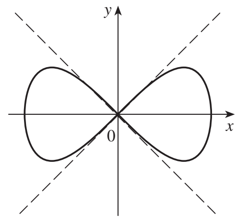
1. $\frac{1}{2}(1 + t)^{3/2}$ **3.** $y = -x$ **5.** $y = \pi x + \pi^2$
7. $y = 2x + 1$
9. $y = 3x + 3$



11. $\frac{2t+1}{2t}, -\frac{1}{4t^3}, t < 0$ **13.** $e^{-2t}(1 - t), e^{-3t}(2t - 3),\ t > \frac{3}{2}$
15. $\frac{t+1}{t-1}, \frac{-2t}{(t-1)^3}, 0 < t < 1$
17. Horizontal at $(0, -3)$, vertical at $(\pm 2, -2)$
19. Horizontal at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, no vertical
21. $(0.6, 2)$; $\left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right)$
23.



25. $y = x, y = -x$



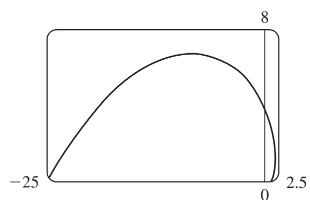
27. (a) $d \sin\theta/(r - d\cos\theta)$ **29.** $(4, 0)$ **31.** πab

33. $\frac{24}{5}$ **35.** $2\pi r^2 + \pi d^2$ **37.** $\int_0^2 \sqrt{2 + 2e^{-2t}}\, dt \approx 3.1416$

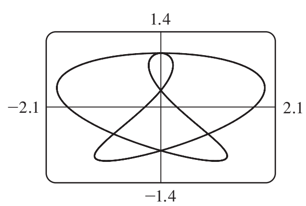
39. $\int_0^{4\pi} \sqrt{5 - 4\cos t}\, dt \approx 26.7298$ **41.** $4\sqrt{2} - 2$

43. $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1 + \sqrt{2})$

45. $\sqrt{2}\,(e^{\pi} - 1)$

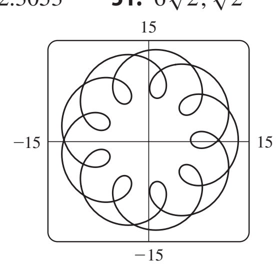


47. 16.7102



49. 612.3053 **51.** $6\sqrt{2}, \sqrt{2}$

55. (a) $t \in [0, 4\pi]$



(b) 294

57. $\int_0^{\pi/2} 2\pi t \cos t \sqrt{t^2 + 1}\, dt \approx 4.7394$

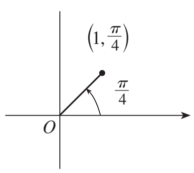
59. $\int_0^1 2\pi e^{-t}\sqrt{1 + 2e^t + e^{2t} + e^{-2t}}\, dt \approx 10.6705$

61. $\frac{2}{1215}\pi(247\sqrt{3} + 64)$ **63.** $\frac{6}{5}\pi a^2$

65. $\frac{24}{5}\pi(949\sqrt{26} + 1)$ **71.** $\frac{1}{4}$

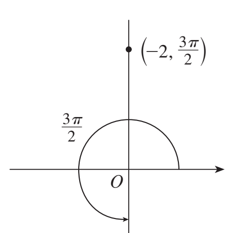
EXERCISES 10.3 ■ PAGE 706

1. (a)



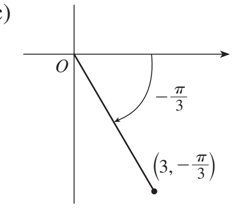
$(1, 9\pi/4), (-1, 5\pi/4)$

(b)



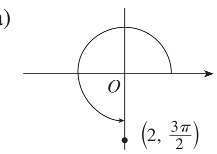
$(2, \pi/2), (-2, 7\pi/2)$

(c)



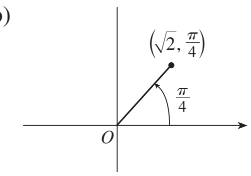
$(3, 5\pi/3), (-3, 2\pi/3)$

3. (a)



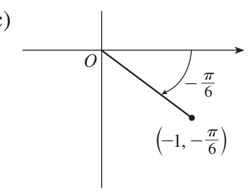
$(0, -2)$

(b)



$(1, 1)$

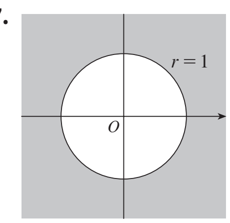
(c)



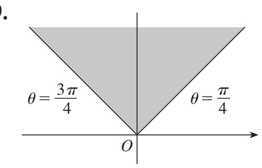
$(-\sqrt{3}/2, 1/2)$

5. (a) (i) $(4\sqrt{2}, 3\pi/4)$ (ii) $(-4\sqrt{2}, 7\pi/4)$
(b) (i) $(6, \pi/3)$ (ii) $(-6, 4\pi/3)$

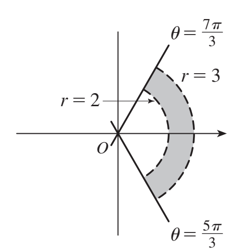
7.



9.



11.



13. $2\sqrt{7}$ **15.** Circle, center O, radius $\sqrt{5}$

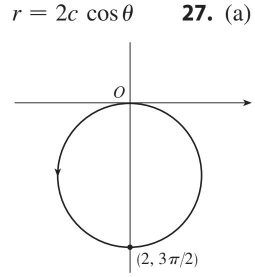
17. Circle, center $(5/2, 0)$, radius $5/2$

19. Hyperbola, center O, foci on x-axis

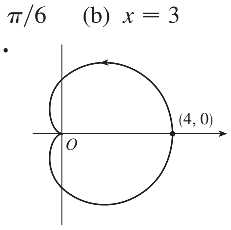
21. $r = 2\csc\theta$ **23.** $r = 1/(\sin\theta - 3\cos\theta)$

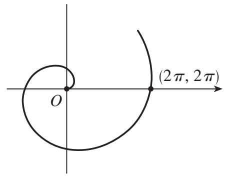
25. $r = 2c\cos\theta$ **27.** (a) $\theta = \pi/6$ (b) $x = 3$

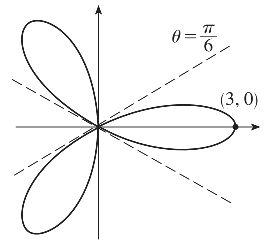
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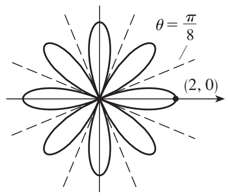


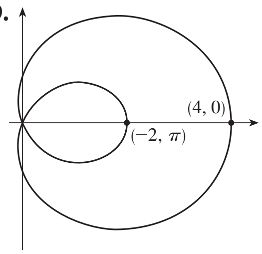
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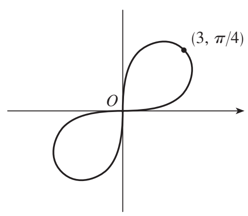


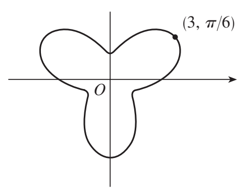
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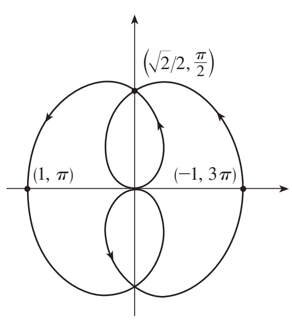
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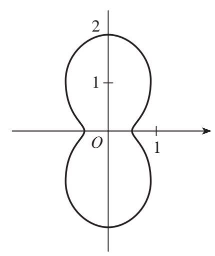
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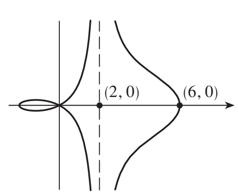
39.


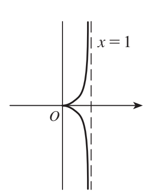
41.


43.


45.


47.


49.


51.


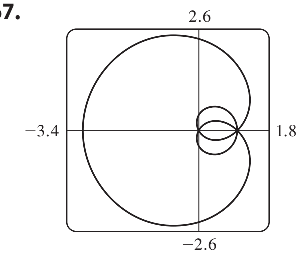
53. (a) For $c < -1$, the inner loop begins at $\theta = \sin^{-1}(-1/c)$ and ends at $\theta = \pi - \sin^{-1}(-1/c)$; for $c > 1$, it begins at $\theta = \pi + \sin^{-1}(1/c)$ and ends at $\theta = 2\pi - \sin^{-1}(1/c)$.

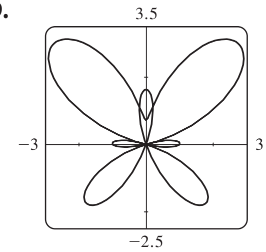
55. $1/\sqrt{3}$ **57.** $-\pi$ **59.** 1

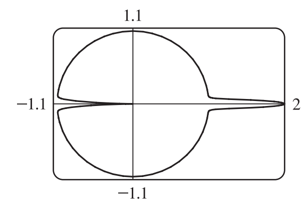
61. Horizontal at $(3/\sqrt{2}, \pi/4)$, $(-3/\sqrt{2}, 3\pi/4)$; vertical at $(3, 0)$, $(0, \pi/2)$

63. Horizontal at $(\frac{3}{2}, \pi/3)$, $(0, \pi)$ [the pole], and $(\frac{3}{2}, 5\pi/3)$; vertical at $(2, 0)$, $(\frac{1}{2}, 2\pi/3)$, $(\frac{1}{2}, 4\pi/3)$

65. Center $(b/2, a/2)$, radius $\sqrt{a^2 + b^2}/2$

67.


69.


71.


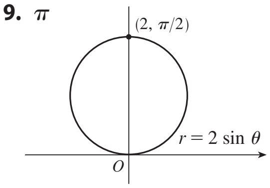
73. By counterclockwise rotation through angle $\pi/6$, $\pi/3$, or α about the origin

75. For $c = 0$, the curve is a circle. As c increases, the left side gets flatter, then has a dimple for $0.5 < c < 1$, a cusp for $c = 1$, and a loop for $c > 1$.

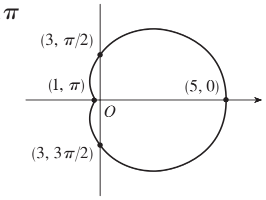
EXERCISES 10.4 ■ PAGE 712

1. $e^{-\pi/4} - e^{-\pi/2}$ **3.** $\pi/2$ **5.** $\frac{1}{2}$ **7.** $\frac{41}{4}\pi$

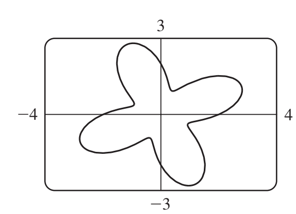
9. π



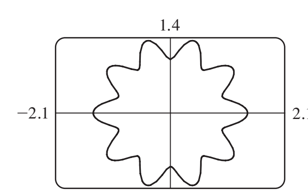
11. 11π



13. $\frac{9}{2}\pi$



15. $\frac{3}{2}\pi$



17. $\frac{4}{3}\pi$ **19.** $\frac{1}{16}\pi$ **21.** $\pi - \frac{3}{2}\sqrt{3}$ **23.** $\frac{4}{3}\pi + 2\sqrt{3}$

25. $4\sqrt{3} - \frac{4}{3}\pi$ **27.** π **29.** $\frac{9}{8}\pi - \frac{9}{4}$ **31.** $\frac{1}{2}\pi - 1$

33. $-\sqrt{3} + 2 + \frac{1}{3}\pi$ **35.** $\frac{1}{4}(\pi + 3\sqrt{3})$

37. $(\frac{1}{2}, \pi/6)$, $(\frac{1}{2}, 5\pi/6)$, and the pole

39. $(1, \theta)$ where $\theta = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$ and $(-1, \theta)$ where $\theta = 7\pi/12, 11\pi/12, 19\pi/12, 23\pi/12$

41. $(\frac{1}{2}\sqrt{3}, \pi/3)$, $(\frac{1}{2}\sqrt{3}, 2\pi/3)$, and the pole

43. Intersection at $\theta \approx 0.89, 2.25$; area ≈ 3.46

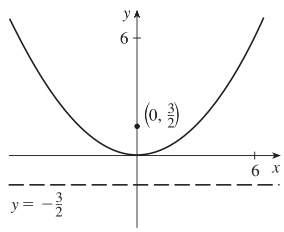
45. 2π **47.** $\frac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$ **49.** $\frac{16}{3}$

51. 2.4221 **53.** 8.0091

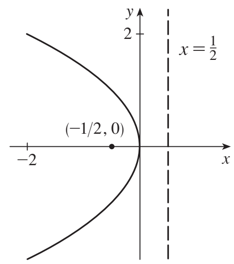
55. (b) $2\pi(2 - \sqrt{2})$

EXERCISES 10.5 ■ PAGE 720

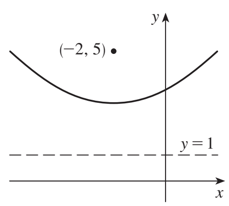
1. $(0, 0), (0, \frac{3}{2}), y = -\frac{3}{2}$



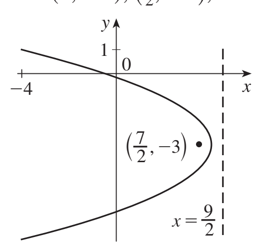
3. $(0, 0), (-\frac{1}{2}, 0), x = \frac{1}{2}$



5. $(-2, 3)\ (-2, 5), y = 1$

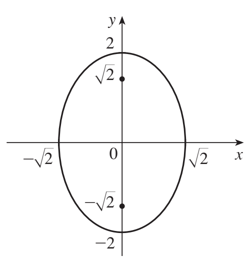


7. $(4, -3), (\frac{7}{2}, -3), x = \frac{9}{2}$

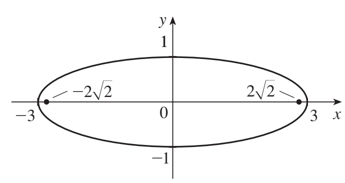


9. $x = -y^2$, focus $(\frac{1}{4}, 0)$, directrix $x = \frac{1}{4}$

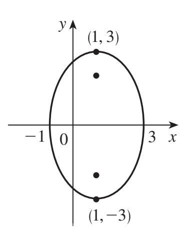
11. $(0, \pm 2), (0, \pm\sqrt{2})$



13. $(\pm 3, 0), (\pm 2\sqrt{2}, 0)$

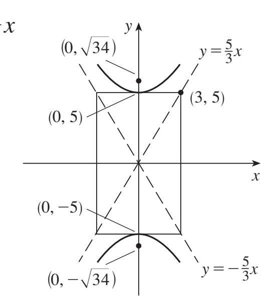


15. $(1, \pm 3), (1, \pm\sqrt{5})$

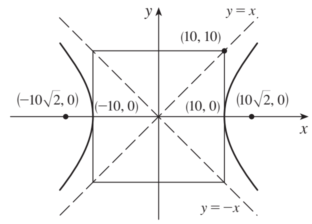


17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, foci $(0, \pm\sqrt{5})$

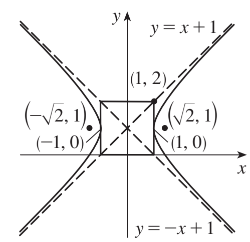
19. $(0, \pm 5); (0, \pm\sqrt{34}); y = \pm\frac{5}{3}x$



21. $(\pm 10, 0), (\pm 10\sqrt{2}, 0), y = \pm x$



23. $(\pm 1, 1), (\pm\sqrt{2}, 1), y - 1 = \pm x$



25. Hyperbola, $(\pm 1, 0), (\pm\sqrt{5}, 0)$

27. Ellipse, $(\pm\sqrt{2}, 1), (\pm 1, 1)$

29. Parabola, $(1, -2), (1, -\frac{11}{6})$

31. $y^2 = 4x$ **33.** $y^2 = -12(x + 1)$

35. $(y + 1)^2 = -\frac{1}{2}(x - 3)$

37. $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$ **39.** $\dfrac{x^2}{12} + \dfrac{(y - 4)^2}{16} = 1$

41. $\dfrac{(x + 1)^2}{12} + \dfrac{(y - 4)^2}{16} = 1$ **43.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

45. $\dfrac{(y - 1)^2}{25} - \dfrac{(x + 3)^2}{39} = 1$ **47.** $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$

49. $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$

51. (a) $\dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$ (b) $\approx$248 mi

55. (a) Ellipse (b) Hyperbola (c) No curve

59. 15.9

61. $\dfrac{b^2c}{a} + ab \ln\left(\dfrac{a}{b + c}\right)$ where $c^2 = a^2 + b^2$

63. $(0, 4/\pi)$

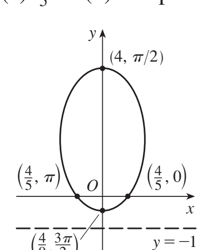
EXERCISES 10.6 ■ PAGE 728

1. $r = \dfrac{4}{2 + \cos\theta}$ **3.** $r = \dfrac{6}{2 + 3\sin\theta}$

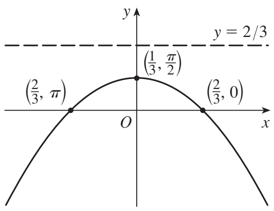
5. $r = \dfrac{10}{3 - 2\cos\theta}$ **7.** $r = \dfrac{6}{1 + \sin\theta}$

9. (a) $\frac{4}{5}$ (b) Ellipse (c) $y = -1$

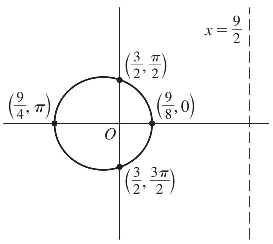
(d)



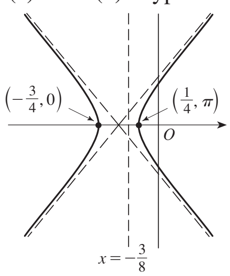
11. (a) 1 (b) Parabola (c) $y = \frac{2}{3}$
(d)



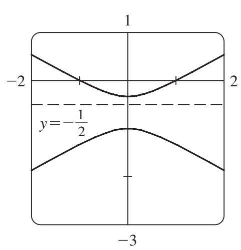
13. (a) $\frac{1}{3}$ (b) Ellipse (c) $x = \frac{9}{2}$
(d)



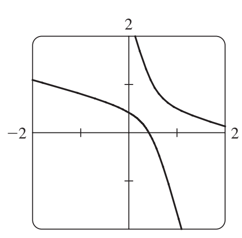
15. (a) 2 (b) Hyperbola (c) $x = -\frac{3}{8}$
(d)



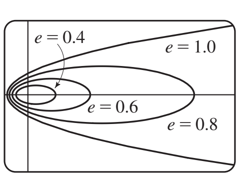
17. (a) $2, y = -\frac{1}{2}$



(b) $r = \dfrac{1}{1 - 2\sin(\theta - 3\pi/4)}$



19. The ellipse is nearly circular when e is close to 0 and becomes more elongated as $e \to 1^-$. At $e = 1$, the curve becomes a parabola.



25. $r = \dfrac{2.26 \times 10^8}{1 + 0.093\cos\theta}$ **27.** $r = \dfrac{1.07}{1 + 0.97\cos\theta}$; 35.64 AU

29. 7.0×10^7 km **31.** 3.6×10^8 km

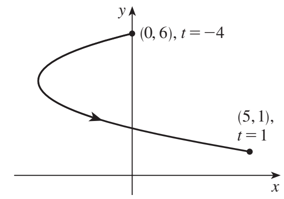
CHAPTER 10 REVIEW ■ PAGE 729

True-False Quiz

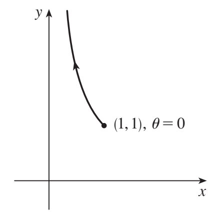
1. False **3.** False **5.** True **7.** False **9.** True

Exercises

1. $x = y^2 - 8y + 12$

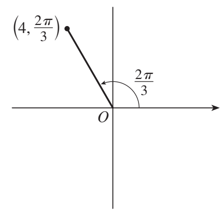


3. $y = 1/x$



5. $x = t, y = \sqrt{t}$; $x = t^4, y = t^2$; $x = \tan^2 t, y = \tan t, 0 \leqslant t < \pi/2$

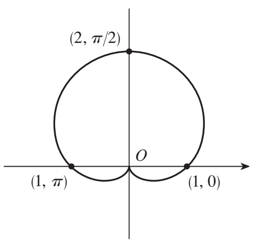
7. (a)



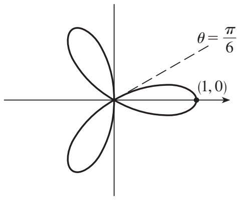
$(-2, 2\sqrt{3})$

(b) $(3\sqrt{2}, 3\pi/4)$, $(-3\sqrt{2}, 7\pi/4)$

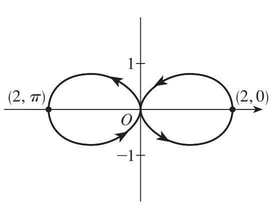
9.



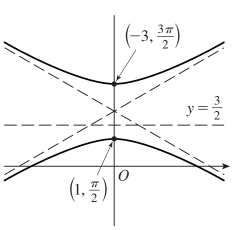
11.



13.

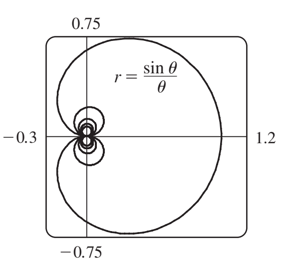


15.



17. $r = \dfrac{2}{\cos\theta + \sin\theta}$

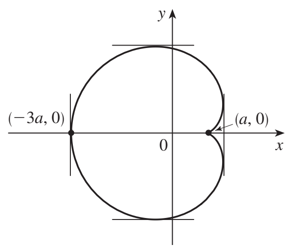
19.



21. 2 **23.** −1

25. $\dfrac{1 + \sin t}{1 + \cos t}, \dfrac{1 + \cos t + \sin t}{(1 + \cos t)^3}$ **27.** $\left(\frac{11}{8}, \frac{3}{4}\right)$

29. Vertical tangent at $\left(\frac{3}{2}a, \pm\frac{1}{2}\sqrt{3}\,a\right), (-3a, 0)$; horizontal tangent at $(a, 0), \left(-\frac{1}{2}a, \pm\frac{3}{2}\sqrt{3}\,a\right)$



31. 18 **33.** $(2, \pm\pi/3)$ **35.** $\frac{1}{2}(\pi - 1)$

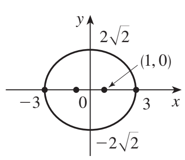
37. $2(5\sqrt{5} - 1)$

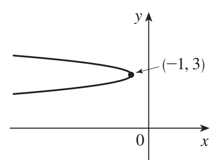
39. $\dfrac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right)$

41. $471{,}295\pi/1024$

43. All curves have the vertical asymptote $x = 1$. For $c < -1$, the curve bulges to the right. At $c = -1$, the curve is the line $x = 1$. For $-1 < c < 0$, it bulges to the left. At $c = 0$ there is a cusp at $(0, 0)$. For $c > 0$, there is a loop.

45. $(\pm 1, 0), (\pm 3, 0)$ **47.** $\left(-\frac{25}{24}, 3\right), (-1, 3)$





49. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ **51.** $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$

53. $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$ **55.** $r = \dfrac{4}{3 + \cos\theta}$

57. $x = a(\cot\theta + \sin\theta\cos\theta),\ y = a(1 + \sin^2\theta)$

PROBLEMS PLUS ■ PAGE 732

1. $\ln(\pi/2)$ **3.** $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$

CHAPTER 11

EXERCISES 11.1 ■ PAGE 744

Abbreviations: C, convergent; D, divergent

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large.
(c) The terms a_n become large as n becomes large.

3. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$ **5.** $\frac{1}{5}, -\frac{1}{25}, \frac{1}{125}, -\frac{1}{625}, \frac{1}{3125}$ **7.** $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}$

9. 1, 2, 7, 32, 157 **11.** $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}$ **13.** $a_n = 1/(2n)$

15. $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$ **17.** $a_n = (-1)^{n+1}\dfrac{n^2}{n + 1}$

19. 0.4286, 0.4615, 0.4737, 0.4800, 0.4839, 0.4865, 0.4884, 0.4898, 0.4909, 0.4918; yes; $\frac{1}{2}$

21. 0.5000, 1.2500, 0.8750, 1.0625, 0.9688, 1.0156, 0.9922, 1.0039, 0.9980, 1.0010; yes; 1

23. 5 **25.** D **27.** 0 **29.** 1 **31.** 2

33. D **35.** 0 **37.** 0 **39.** D **41.** 0 **43.** 0

45. 1 **47.** e^2 **49.** ln 2 **51.** $\pi/2$ **53.** D **55.** D

57. D **59.** $\pi/4$ **61.** D **63.** 0

65. (a) 1060, 1123.60, 1191.02, 1262.48, 1338.23 (b) D

67. (b) 5734 **69.** $-1 < r < 1$

71. Convergent by the Monotonic Sequence Theorem; $5 \leqslant L < 8$

73. Decreasing; yes **75.** Not monotonic; no

77. Increasing; yes

79. 2 **81.** $\frac{1}{2}(3 + \sqrt{5})$ **83.** (b) $\frac{1}{2}(1 + \sqrt{5})$

85. (a) 0 (b) 9, 11

EXERCISES 11.2 ■ PAGE 755

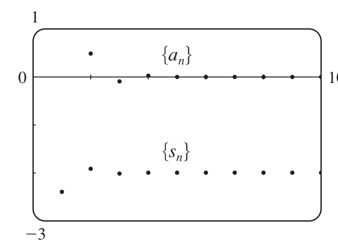
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.
(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

3. 2

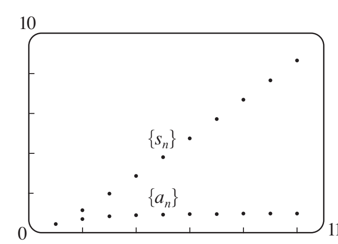
5. 0.5, 0.55, 0.5611, 0.5648, 0.5663, 0.5671, 0.5675, 0.5677; C

7. 1, 1.7937, 2.4871, 3.1170, 3.7018, 4.2521, 4.7749, 5.2749; D

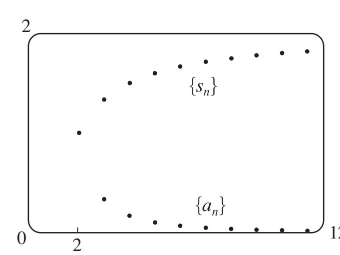
9. −2.40000, −1.92000, −2.01600, −1.99680, −2.00064, −1.99987, −2.00003, −1.99999, −2.00000, −2.00000; convergent, sum = −2



11. 0.44721, 1.15432, 1.98637, 2.88080, 3.80927, 4.75796, 5.71948, 6.68962, 7.66581, 8.64639; divergent



13. 1.00000, 1.33333, 1.50000, 1.60000, 1.66667, 1.71429, 1.75000, 1.77778, 1.80000, 1.81818; convergent, sum = 2



15. (a) Yes (b) No **17.** D **19.** $\frac{25}{3}$ **21.** $\frac{400}{9}$

23. $\frac{1}{7}$ **25.** D **27.** D **29.** D **31.** 9 **33.** D

35. $\dfrac{\sin 100}{1 - \sin 100}$

37. D **39.** D **41.** $e/(e - 1)$ **43.** $\frac{3}{2}$ **45.** $\frac{11}{6}$

47. $e - 1$

49. (b) 1 (c) 2 (d) All rational numbers with a terminating decimal representation, except 0

51. $\frac{8}{9}$ **53.** $\frac{838}{333}$ **55.** 45,679/37,000

57. $-\dfrac{1}{5} < x < \dfrac{1}{5};\ \dfrac{-5x}{1 + 5x}$

59. $-1 < x < 5$; $\dfrac{3}{5 - x}$

61. $x > 2$ or $x < -2$; $\dfrac{x}{x - 2}$ **63.** $x < 0$; $\dfrac{1}{1 - e^x}$

65. 1 **67.** $a_1 = 0$, $a_n = \dfrac{2}{n(n + 1)}$ for $n > 1$, sum $= 1$

69. (a) 120 mg; 124 mg
(b) $Q_{n+1} = 100 + 0.20Q_n$ (c) 125 mg

71. (a) 157.875 mg; $\frac{3000}{19}(1 - 0.05^n)$ (b) 157.895 mg

73. (a) $S_n = \dfrac{D(1 - c^n)}{1 - c}$ (b) 5 **75.** $\frac{1}{2}(\sqrt{3} - 1)$

79. $\dfrac{1}{n(n + 1)}$ **81.** The series is divergent.

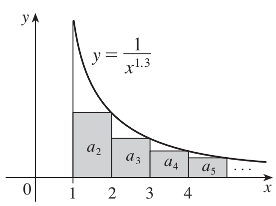
87. $\{s_n\}$ is bounded and increasing.

89. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$

91. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $\dfrac{(n + 1)! - 1}{(n + 1)!}$ (c) 1

EXERCISES 11.3 ■ PAGE 765

1. C



3. C **5.** D **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** D **21.** D **23.** C **25.** C
27. f is neither positive nor decreasing.
29. $p > 1$ **31.** $p < -1$ **33.** $(1, \infty)$
35. (a) $\frac{9}{10}\pi^4$ (b) $\frac{1}{90}\pi^4 - \frac{17}{16}$
37. (a) 1.54977, error $\leqslant 0.1$ (b) 1.64522, error $\leqslant 0.005$
(c) 1.64522 compared to 1.64493 (d) $n > 1000$
39. 0.00145 **45.** $b < 1/e$

EXERCISES 11.4 ■ PAGE 771

1. (a) Nothing (b) C **3.** C **5.** D **7.** C **9.** D
11. C **13.** C **15.** D **17.** D **19.** C **21.** D
23. C **25.** D **27.** C **29.** C **31.** D
33. 0.1993, error $< 2.5 \times 10^{-5}$
35. 0.0739, error $< 6.4 \times 10^{-8}$
45. Yes

EXERCISES 11.5 ■ PAGE 776

1. (a) A series whose terms are alternately positive and negative (b) $0 < b_{n+1} \leqslant b_n$ and $\lim_{n\to\infty} b_n = 0$, where $b_n = |a_n|$ (c) $|R_n| \leqslant b_{n+1}$
3. D **5.** C **7.** D **9.** C **11.** C **13.** D
15. C **17.** C **19.** D **21.** -0.5507 **23.** 5
25. 5 **27.** -0.4597 **29.** -0.1050
31. An underestimate
33. p is not a negative integer. **35.** $\{b_n\}$ is not decreasing.

EXERCISES 11.6 ■ PAGE 782

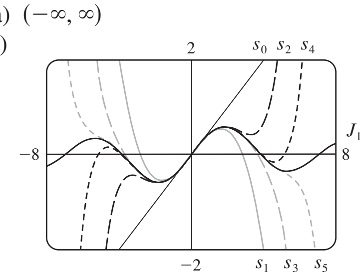
Abbreviations: AC, absolutely convergent; CC, conditionally convergent
1. (a) D (b) C (c) May converge or diverge
3. CC **5.** AC **7.** AC **9.** D **11.** AC
13. AC **15.** D **17.** AC **19.** AC **21.** AC
23. D **25.** AC **27.** AC **29.** D **31.** CC
33. AC **35.** D **37.** AC **39.** D **41.** AC
43. (a) and (d)
47. (a) $\frac{661}{960} \approx 0.68854$, error < 0.00521
(b) $n \geqslant 11$, 0.693109
53. (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$; $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

EXERCISES 11.7 ■ PAGE 786

1. D **3.** CC **5.** D **7.** D **9.** C **11.** C
13. C **15.** C **17.** C **19.** C **21.** D **23.** D
25. C **27.** C **29.** C **31.** D
33. C **35.** D **37.** C

EXERCISES 11.8 ■ PAGE 791

1. A series of the form $\sum_{n=0}^{\infty} c_n(x - a)^n$, where x is a variable and a and the c_n's are constants
3. 1, $(-1, 1)$ **5.** 1, $[-1, 1)$
7. ∞, $(-\infty, \infty)$ **9.** 4, $[-4, 4]$
11. $\frac{1}{4}$, $\left(-\frac{1}{4}, \frac{1}{4}\right]$ **13.** 2, $[-2, 2)$
15. 1, $[1, 3]$ **17.** 2, $[-4, 0)$
19. ∞, $(-\infty, \infty)$ **21.** b, $(a - b, a + b)$ **23.** 0, $\left\{\frac{1}{2}\right\}$
25. $\frac{1}{5}$, $\left[\frac{3}{5}, 1\right]$ **27.** ∞, $(-\infty, \infty)$
29. (a) Yes (b) No
31. k^k **33.** No
35. (a) $(-\infty, \infty)$
(b), (c)



37. $(-1, 1)$, $f(x) = (1 + 2x)/(1 - x^2)$ **41.** 2

EXERCISES 11.9 ■ PAGE 797

1. 10 **3.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$, $(-1, 1)$ **5.** $2\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n$, $(-3, 3)$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{4n+4}}$, $(-2, 2)$ **9.** $-\dfrac{1}{2} - \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$, $(-2, 2)$

11. $\displaystyle\sum_{n=0}^{\infty} \left(-1 - \frac{1}{3^{n+1}}\right) x^n$, $(-1, 1)$

13. (a) $\sum_{n=0}^{\infty} (-1)^n (n+1)x^n,\ R = 1$

(b) $\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)x^n,\ R = 1$

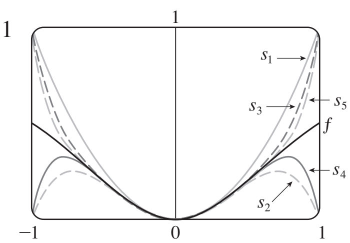
(c) $\frac{1}{2}\sum_{n=2}^{\infty} (-1)^n n(n-1)x^n,\ R = 1$

15. $\ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n},\ R = 5$

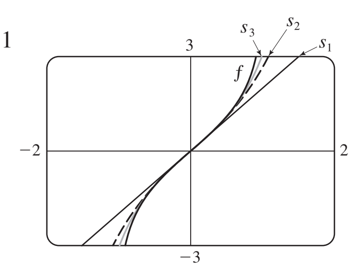
17. $\sum_{n=0}^{\infty} (-1)^n 4^n (n+1)x^{n+1},\ R = \frac{1}{4}$

19. $\sum_{n=0}^{\infty} (2n+1)x^n,\ R = 1$

21. $\sum_{n=0}^{\infty} (-1)^n x^{2n+2},\ R = 1$



23. $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1},\ R = 1$



25. $C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2},\ R = 1$

27. $C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+3}}{n(n+3)},\ R = 1$

29. 0.044522 **31.** 0.000395
33. 0.19740
35. (b) 0.920 **39.** $[-1, 1], [-1, 1), (-1, 1)$

EXERCISES 11.10 ■ PAGE 811

1. $b_8 = f^{(8)}(5)/8!$ **3.** $\sum_{n=0}^{\infty} (n+1)x^n,\ R = 1$

5. $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

7. $2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$

9. $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$

11. $\sum_{n=0}^{\infty} (n+1)x^n,\ R = 1$ **13.** $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},\ R = \infty$

15. $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n,\ R = \infty$ **17.** $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!},\ R = \infty$

19. $50 + 105(x-2) + 92(x-2)^2 + 42(x-2)^3 + 10(x-2)^4 + (x-2)^5,\ R = \infty$

21. $\ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n}(x-2)^n,\ R = 2$

23. $\sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n,\ R = \infty$

25. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1},\ R = \infty$

31. $1 - \frac{1}{4}x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{4^n \cdot n!} x^n,\ R = 1$

33. $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2^{n+4}} x^n,\ R = 2$

35. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{4n+2},\ R = 1$

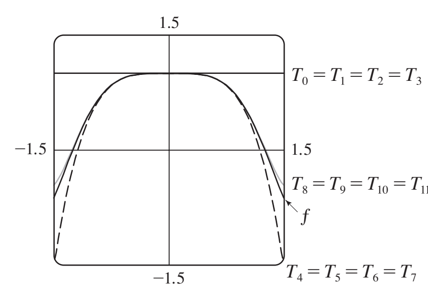
37. $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n+1},\ R = \infty$

39. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!} x^{4n+1},\ R = \infty$

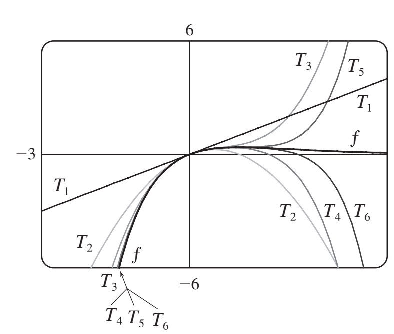
41. $\frac{1}{2}x + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, 2^{3n+1}} x^{2n+1},\ R = 2$

43. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} x^{2n},\ R = \infty$

45. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{4n},\ R = \infty$



47. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} x^n,\ R = \infty$



49. 0.99619

51. (a) $1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}$

(b) $x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n n!} x^{2n+1}$

53. $C + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^{3n+1}}{3n+1}, R = 1$

55. $C + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n(2n)!} x^{2n}, R = \infty$

57. 0.0059 **59.** 0.40102 **61.** $\frac{1}{2}$ **63.** $\frac{1}{120}$ **65.** $\frac{3}{5}$

67. $1 - \frac{3}{2}x^2 + \frac{25}{24}x^4$ **69.** $1 + \frac{1}{6}x^2 + \frac{7}{360}x^4$

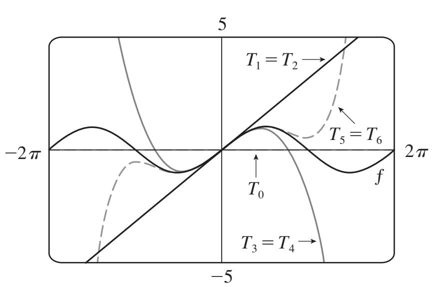
71. $x - \frac{2}{3}x^4 + \frac{23}{45}x^6$

73. e^{-x^4} **75.** $\ln \frac{8}{5}$

77. $1/\sqrt{2}$ **79.** $e^3 - 1$

EXERCISES 11.11 ■ PAGE 820

1. (a) $T_0(x) = 0, T_1(x) = T_2(x) = x, T_3(x) = T_4(x) = x - \frac{1}{6}x^3$, $T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

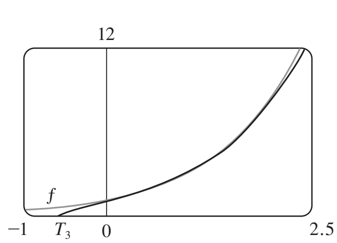


(b)

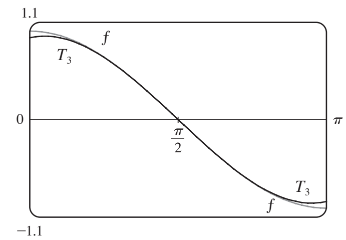
x	f	T_0	$T_1 = T_2$	$T_3 = T_4$	T_5
$\pi/4$	0.7071	0	0.7854	0.7047	0.7071
$\pi/2$	1	0	1.5708	0.9248	1.0045
π	0	0	3.1416	−2.0261	0.5240

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

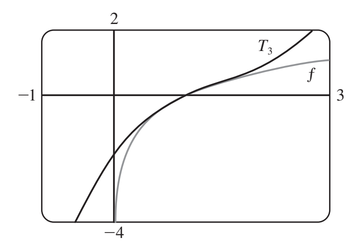
3. $e + e(x-1) + \frac{1}{2}e(x-1)^2 + \frac{1}{6}e(x-1)^3$



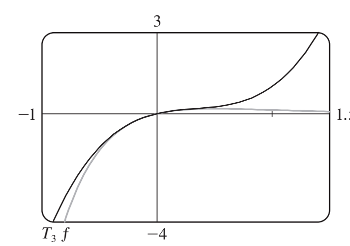
5. $-\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3$



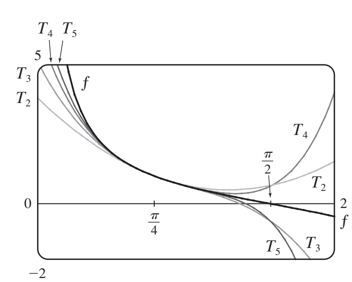
7. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$



9. $x - 2x^2 + 2x^3$



11. $T_5(x) = 1 - 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 - \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5$



13. (a) $1 - (x-1) + (x-1)^2$ (b) 0.006 482 7

15. (a) $1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$ (b) 0.000 097

17. (a) $1 + \frac{1}{2}x^2$ (b) 0.0015

19. (a) $1 + x^2$ (b) 0.000 06 **21.** (a) $x^2 - \frac{1}{6}x^4$ (b) 0.042

23. 0.17365 **25.** Four **27.** $-1.037 < x < 1.037$
29. $-0.86 < x < 0.86$ **31.** 21 m, no
37. (c) They differ by about 8×10^{-9} km.

CHAPTER 11 REVIEW ■ PAGE 824

True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True
19. True **21.** True

Exercises

1. $\frac{1}{2}$ **3.** D **5.** 0 **7.** e^{12} **9.** 2 **11.** C
13. C **15.** D **17.** C **19.** C **21.** C **23.** CC
25. AC **27.** $\frac{1}{11}$ **29.** $\pi/4$ **31.** e^{-e} **35.** 0.9721
37. 0.189 762 24, error $< 6.4 \times 10^{-7}$
41. 4, $[-6, 2)$ **43.** 0.5, $[2.5, 3.5)$

45. $\displaystyle \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n}+\frac{\sqrt{3}}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\right]$

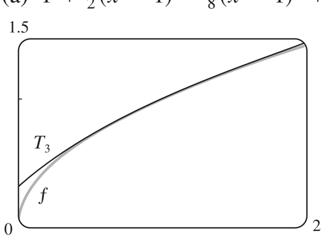
47. $\displaystyle \sum_{n=0}^{\infty}(-1)^n x^{n+2}$, $R = 1$ **49.** $\displaystyle \ln 4 - \sum_{n=1}^{\infty}\frac{x^n}{n4^n}$, $R = 4$

51. $\displaystyle \sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+4}}{(2n+1)!}$, $R = \infty$

53. $\displaystyle \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-3)}{n!\,2^{6n+1}}x^n$, $R = 16$

55. $\displaystyle C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$

57. (a) $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
(b)



(c) 0.000 006

59. $-\frac{1}{6}$

PROBLEMS PLUS ■ PAGE 827

1. $15!/5! = 10{,}897{,}286{,}400$
3. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, k an integer
5. (a) $s_n = 3\cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $\frac{2}{5}\sqrt{3}$

9. $\dfrac{3\pi}{4}$ **11.** $(-1, 1)$, $\dfrac{x^3+4x^2+x}{(1-x)^4}$ **13.** $\ln\frac{1}{2}$

17. (a) $\frac{250}{101}\pi(e^{-(n-1)\pi/5} - e^{-n\pi/5})$ (b) $\frac{250}{101}\pi$

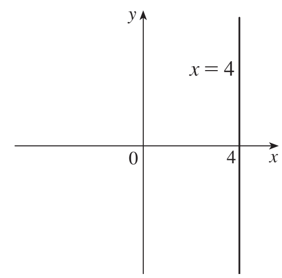
19. $\dfrac{\pi}{2\sqrt{3}} - 1$

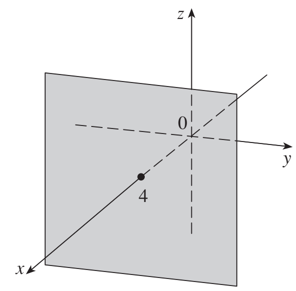
21. $-\left(\dfrac{\pi}{2} - \pi k\right)^2$, where k is a positive integer

CHAPTER 12

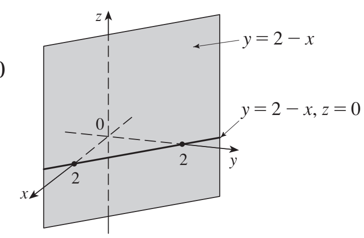
EXERCISES 12.1 ■ PAGE 836

1. $(4, 0, -3)$ **3.** C; A
5. A line parallel to the y-axis and 4 units to the right of it; a vertical plane parallel to the yz-plane and 4 units in front of it.

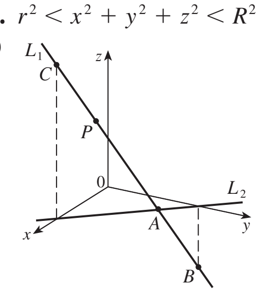




7. A vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$



9. (a) $|PQ| = 6$, $|QR| = 2\sqrt{10}$, $|RP| = 6$; isosceles triangle
11. (a) No (b) Yes
13. $(x+3)^2 + (y-2)^2 + (z-5)^2 = 16$;
$(y-2)^2 + (z-5)^2 = 7$, $x = 0$ (a circle)
15. $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$
17. $(1, 2, -4)$, 6 **19.** $(2, 0, -6)$, $9/\sqrt{2}$
21. (b) $\frac{5}{2}$, $\frac{1}{2}\sqrt{94}$, $\frac{1}{2}\sqrt{85}$
23. (a) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 36$
(b) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 4$
(c) $(x-2)^2 + (y+3)^2 + (z-6)^2 = 9$
25. A plane parallel to the yz-plane and 5 units in front of it
27. A half-space consisting of all points to the left of the plane $y = 8$
29. All points on or between the horizontal planes $z = 0$ and $z = 6$
31. All points on a circle with radius 2 with center on the z-axis that is contained in the plane $z = -1$
33. All point on a sphere with radius 2 and center $(0, 0, 0)$
35. All points on or between spheres with radii 1 and $\sqrt{5}$ and centers $(0, 0, 0)$
37. All points on or inside a circular cylinder of radius 3 with axis the y-axis
39. $0 < x < 5$ **41.** $r^2 < x^2 + y^2 + z^2 < R^2$
43. (a) $(2, 1, 4)$ (b)



45. $14x - 6y - 10z = 9$, a plane perpendicular to AB

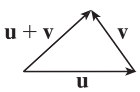
47. $2\sqrt{3} - 3$

EXERCISES 12.2 ■ PAGE 845

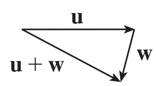
1. (a) Scalar (b) Vector (c) Vector (d) Scalar

3. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, $\overrightarrow{EA} = \overrightarrow{CE}$

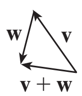
5. (a)



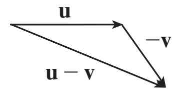
(b)



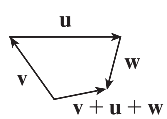
(c)



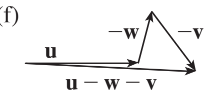
(d)



(e)

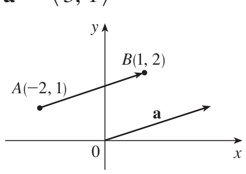


(f)

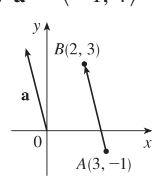


7. $\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{d} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$

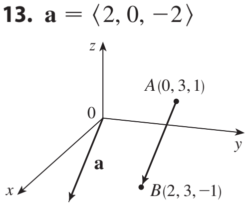
9. $\mathbf{a} = \langle 3, 1 \rangle$



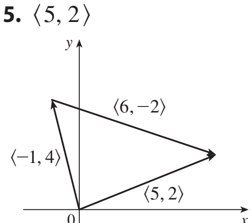
11. $\mathbf{a} = \langle -1, 4 \rangle$



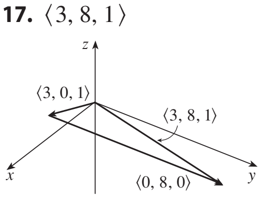
13. $\mathbf{a} = \langle 2, 0, -2 \rangle$



15. $\langle 5, 2 \rangle$



17. $\langle 3, 8, 1 \rangle$



19. $\langle 6, 3 \rangle$, $\langle 6, 14 \rangle$, 5, 13

21. $6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, $20\mathbf{i} - 12\mathbf{j}$, $\sqrt{29}$, 7

23. $\left\langle \dfrac{3}{\sqrt{10}}, -\dfrac{1}{\sqrt{10}} \right\rangle$ **25.** $\frac{8}{9}\mathbf{i} - \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k}$ **27.** 60°

29. $\langle 2, 2\sqrt{3} \rangle$ **31.** ≈ 45.96 ft/s, ≈ 38.57 ft/s

33. $100\sqrt{7} \approx 264.6$ N, $\approx 139.1°$

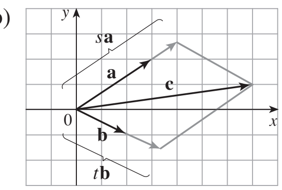
35. $\sqrt{493} \approx 22.2$ mi/h, N8°W

37. $\approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$, $\approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$; ≈ 275.97 N, ≈ 225.11 N

39. (a) At an angle of 43.4° from the bank, toward upstream (b) 20.2 min

41. $\pm(\mathbf{i} + 4\,\mathbf{j})/\sqrt{17}$ **43.** $\mathbf{0}$

45. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

47. A sphere with radius 1, centered at (x_0, y_0, z_0)

EXERCISES 12.3 ■ PAGE 852

1. (b), (c), (d) are meaningful **3.** -3.6 **5.** 19 **7.** 1

9. $14\sqrt{3}$ **11.** $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$, $\mathbf{u} \cdot \mathbf{w} = -\frac{1}{2}$

15. $\cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) \approx 63°$ **17.** $\cos^{-1}\left(-\frac{5}{6}\right) \approx 146°$

19. $\cos^{-1}\left(\dfrac{7}{\sqrt{130}}\right) \approx 52°$ **21.** 48°, 75°, 57°

23. (a) Orthogonal (b) Neither (c) Parallel (d) Orthogonal

25. Yes **27.** $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ $\left[\text{or } (-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}\right]$

29. 45° **31.** 0° at (0, 0), $\approx 8.1°$ at (1, 1)

33. $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$; 48°, 71°, 48°

35. $1/\sqrt{14}$, $-2/\sqrt{14}$, $-3/\sqrt{14}$; 74°, 122°, 143°

37. $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$; 55°, 55°, 55° **39.** 4, $\left\langle -\frac{20}{13}, \frac{48}{13} \right\rangle$

41. $\frac{1}{9}$, $\left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle$ **43.** $-7/\sqrt{19}$, $-\frac{21}{19}\mathbf{i} + \frac{21}{19}\mathbf{j} - \frac{7}{19}\mathbf{k}$

47. $\left\langle 0, 0, -2\sqrt{10} \right\rangle$ or any vector of the form $\left\langle s, t, 3s - 2\sqrt{10} \right\rangle$, $s, t \in \mathbb{R}$

49. 144 J **51.** $2400\cos(40°) \approx 1839$ ft-lb

53. $\frac{13}{5}$ **55.** $\cos^{-1}\left(1/\sqrt{3}\right) \approx 55°$

EXERCISES 12.4 ■ PAGE 861

1. $15\,\mathbf{i} - 10\,\mathbf{j} - 3\,\mathbf{k}$ **3.** $14\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

5. $-\frac{3}{2}\mathbf{i} + \frac{7}{4}\mathbf{j} + \frac{2}{3}\mathbf{k}$ **7.** $(1 - t)\,\mathbf{i} + (t^3 - t^2)\,\mathbf{k}$

9. $\mathbf{0}$ **11.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

13. (a) Scalar (b) Meaningless (c) Vector (d) Meaningless (e) Meaningless (f) Scalar

15. $96\sqrt{3}$; into the page **17.** $\langle -7, 10, 8 \rangle$, $\langle 7, -10, -8 \rangle$

19. $\left\langle -\dfrac{1}{3\sqrt{3}}, -\dfrac{1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}} \right\rangle$, $\left\langle \dfrac{1}{3\sqrt{3}}, \dfrac{1}{3\sqrt{3}}, -\dfrac{5}{3\sqrt{3}} \right\rangle$

27. 20 **29.** (a) $\langle 0, 18, -9 \rangle$ (b) $\frac{9}{2}\sqrt{5}$

31. (a) $\langle 13, -14, 5 \rangle$ (b) $\frac{1}{2}\sqrt{390}$

33. 9 **35.** 16 **39.** $10.8 \sin 80° \approx 10.6$ N·m

41. ≈ 417 N **43.** 60°

45. (b) $\sqrt{97/3}$ **53.** (a) No (b) No (c) Yes

EXERCISES 12.5 ■ PAGE 871

1. (a) True (b) False (c) True (d) False (e) False (f) True (g) False (h) True (i) True (j) False (k) True

3. $\mathbf{r} = (2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}) + t(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$; $x = 2 + 3t$, $y = 2.4 + 2t$, $z = 3.5 - t$

5. $\mathbf{r} = (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k})$; $x = 1 + t$, $y = 3t$, $z = 6 + t$

7. $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$; $(x - 2)/2 = 2y - 2 = (z + 3)/(-4)$

9. $x = -8 + 11t$, $y = 1 - 3t$, $z = 4$; $\dfrac{x + 8}{11} = \dfrac{y - 1}{-3}$, $z = 4$

11. $x = -6 + 2t$, $y = 2 + 3t$, $z = 3 + t$; $(x + 6)/2 = (y - 2)/3 = z - 3$

13. Yes

15. (a) $(x - 1)/(-1) = (y + 5)/2 = (z - 6)/(-3)$ (b) $(-1, -1, 0)$, $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$, $(0, -3, 3)$

17. $\mathbf{r}(t) = (6\mathbf{i} - \mathbf{j} + 9\mathbf{k}) + t(\mathbf{i} + 7\mathbf{j} - 9\mathbf{k})$, $0 \leqslant t \leqslant 1$

19. Skew **21.** $(4, -1, -5)$ **23.** $x - 2y + 5z = 0$

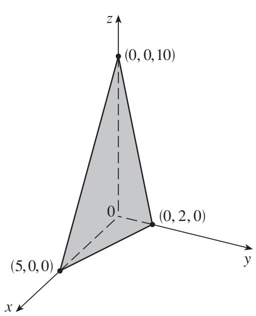
25. $x + 4y + z = 4$ **27.** $5x - y - z = 7$

29. $6x + 6y + 6z = 11$ **31.** $x + y + z = 2$

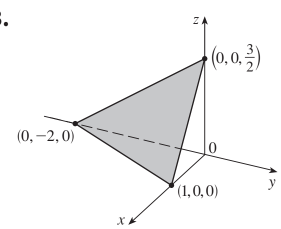
33. $5x - 3y - 8z = -9$ **35.** $8x + y - 2z = 31$

37. $x - 2y - z = -3$ **39.** $3x - 8y - z = -38$

41.



43.



45. $(-2, 6, 3)$ **47.** $\left(\frac{2}{5}, 4, 0\right)$ **49.** $1, 0, -1$

51. Perpendicular **53.** Neither, $\cos^{-1}\left(-\dfrac{1}{\sqrt{6}}\right) \approx 114.1°$

55. Parallel

57. (a) $x = 1$, $y = -t$, $z = t$ (b) $\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8°$

59. $x = 1$, $y - 2 = -z$

61. $x + 2y + z = 5$

63. $(x/a) + (y/b) + (z/c) = 1$

65. $x = 3t$, $y = 1 - t$, $z = 2 - 2t$

67. P_2 and P_3 are parallel, P_1 and P_4 are identical

69. $\sqrt{61/14}$ **71.** $\frac{18}{7}$ **73.** $5/\left(2\sqrt{14}\right)$

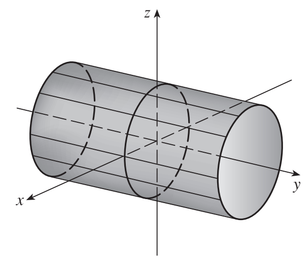
77. $1/\sqrt{6}$ **79.** $13/\sqrt{69}$

81. (a) $x = 325 + 440t$, $y = 810 - 135t$, $z = 561 + 38t$, $0 \leqslant t \leqslant 1$ (b) No

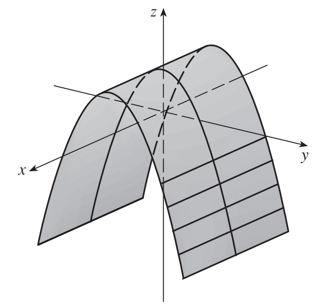
EXERCISES 12.6 ■ PAGE 879

1. (a) Parabola
(b) Parabolic cylinder with rulings parallel to the z-axis
(c) Parabolic cylinder with rulings parallel to the x-axis

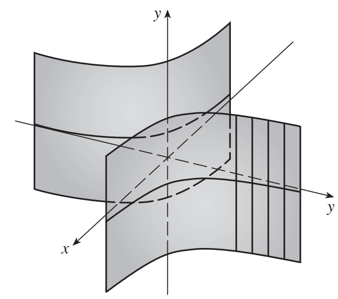
3. Circular cylinder



5. Parabolic cylinder

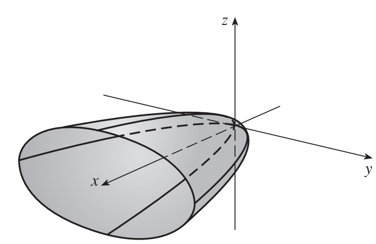


7. Hyperbolic cylinder

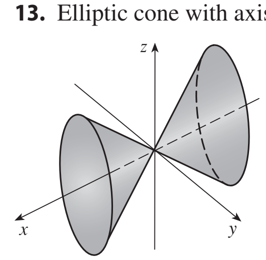


9. (a) $x = k$, $y^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$; $y = k$, $x^2 - z^2 = 1 - k^2$, hyperbola $(k \neq \pm 1)$; $z = k$, $x^2 + y^2 = 1 + k^2$, circle
(b) The hyperboloid is rotated so that it has axis the y-axis
(c) The hyperboloid is shifted one unit in the negative y-direction

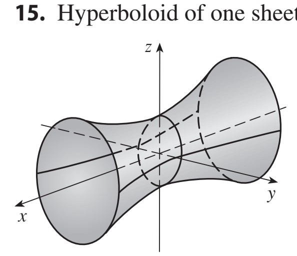
11. Elliptic paraboloid with axis the x-axis



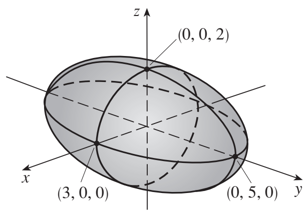
13. Elliptic cone with axis the x-axis



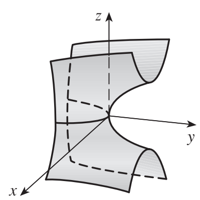
15. Hyperboloid of one sheet with axis the x-axis



17. Ellipsoid

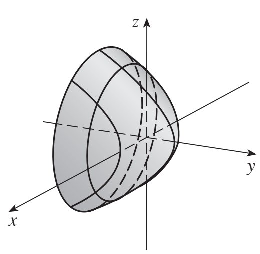


19. Hyperbolic paraboloid



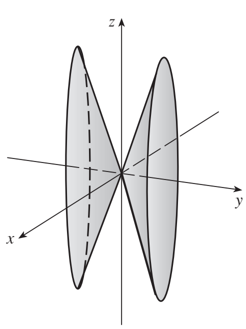
21. VII **23.** II **25.** VI **27.** VIII

29. Circular paraboloid



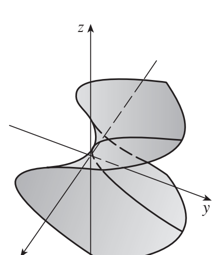
31. $y^2 = x^2 + \dfrac{z^2}{9}$

Elliptic cone with axis the y-axis

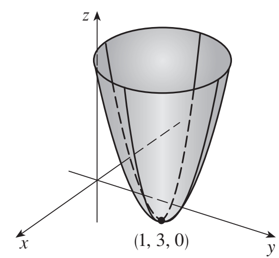


33. $y = z^2 - \dfrac{x^2}{2}$

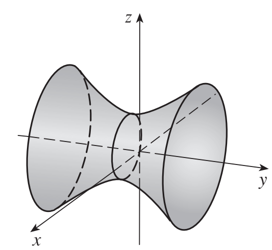
Hyperbolic paraboloid



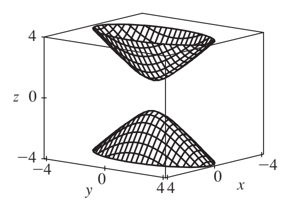
35. $z = (x-1)^2 + (y-3)^2$
Circular paraboloid with vertex $(1, 3, 0)$ and axis the vertical line $x = 1$, $y = 3$



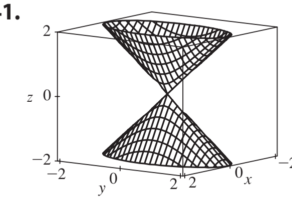
37. $\dfrac{(x-2)^2}{5} - \dfrac{y^2}{5} + \dfrac{(z-1)^2}{5} = 1$
Hyperboloid of one sheet with center $(2, 0, 1)$ and axis the horizontal line $x = 2$, $z = 1$



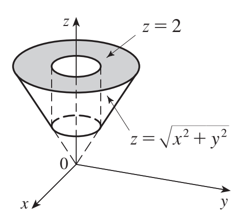
39.



41.



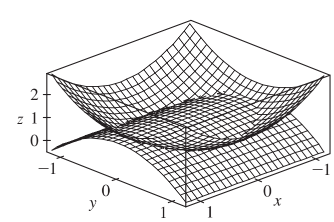
43.



45. $x = y^2 + z^2$ **47.** $-4x = y^2 + z^2$, paraboloid

49. (a) $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} + \dfrac{z^2}{(6356.523)^2} = 1$
(b) Circle (c) Ellipse

53.



CHAPTER 12 REVIEW ■ PAGE 882

True-False Quiz

1. False **3.** False **5.** True **7.** True **9.** True
11. True **13.** True **15.** False **17.** False
19. False **21.** True

Exercises

1. (a) $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$
(b) $(y-2)^2 + (z-1)^2 = 68,\ x = 0$
(c) Center $(4, -1, -3)$, radius 5

3. $\mathbf{u} \cdot \mathbf{v} = 3\sqrt{2}$; $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{2}$; out of the page

5. $-2, -4$ **7.** (a) 2 (b) -2 (c) -2 (d) 0

9. $\cos^{-1}\left(\frac{1}{3}\right) \approx 71°$ **11.** (a) $\langle 4, -3, 4 \rangle$ (b) $\sqrt{41}/2$

13. ≈ 166 N, ≈ 114 N

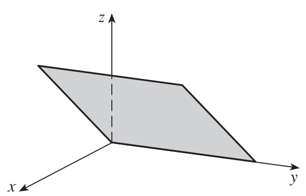
15. $x = 4 - 3t,\ y = -1 + 2t,\ z = 2 + 3t$

17. $x = -2 + 2t,\ y = 2 - t,\ z = 4 + 5t$

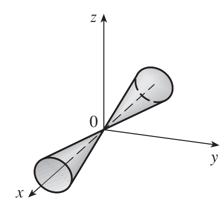
19. $-4x + 3y + z = -14$ **21.** $(1, 4, 4)$ **23.** Skew

25. $x + y + z = 4$ **27.** $22/\sqrt{26}$

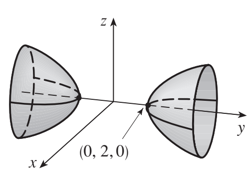
29. Plane



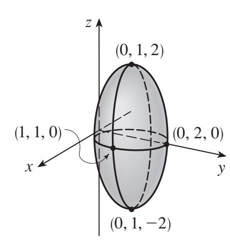
31. Cone



33. Hyperboloid of two sheets



35. Ellipsoid



37. $4x^2 + y^2 + z^2 = 16$

PROBLEMS PLUS ■ PAGE 884

1. $\left(\sqrt{3} - \frac{3}{2}\right)$ m

3. (a) $(x+1)/(-2c) = (y-c)/(c^2-1) = (z-c)/(c^2+1)$
(b) $x^2 + y^2 = t^2 + 1,\ z = t$ (c) $4\pi/3$

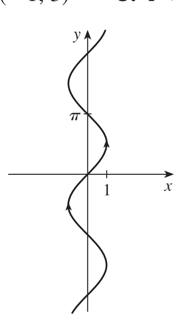
5. 20

CHAPTER 13

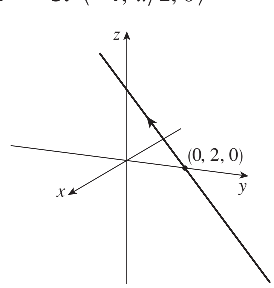
EXERCISES 13.1 ■ PAGE 893

1. $(-1, 3)$ **3.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$ **5.** $\langle -1, \pi/2, 0 \rangle$

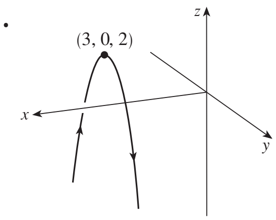
7.



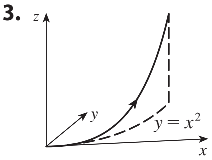
9.



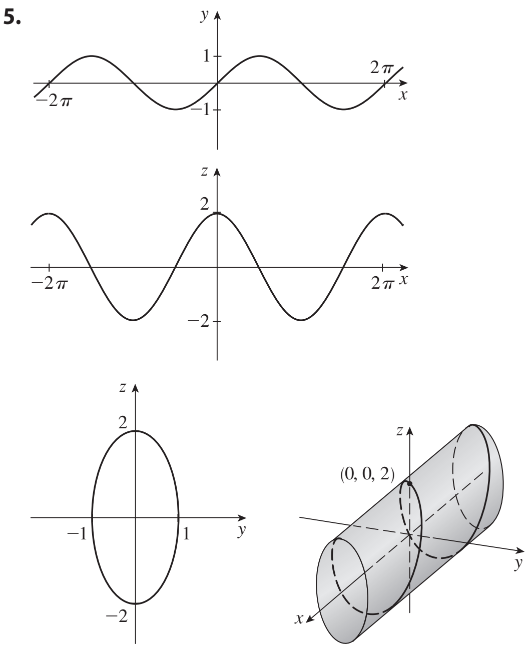
11.



13.



15.

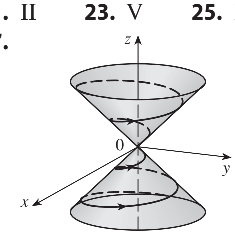


17. $\mathbf{r}(t) = \langle 2 + 4t, 2t, -2t \rangle,\ 0 \leqslant t \leqslant 1$;
$x = 2 + 4t,\ y = 2t,\ z = -2t,\ 0 \leqslant t \leqslant 1$

19. $\mathbf{r}(t) = \left\langle \frac{1}{2}t, -1 + \frac{4}{3}t, 1 - \frac{3}{4}t \right\rangle,\ 0 \leqslant t \leqslant 1$;
$x = \frac{1}{2}t,\ y = -1 + \frac{4}{3}t,\ z = 1 - \frac{3}{4}t,\ 0 \leqslant t \leqslant 1$

21. II **23.** V **25.** IV

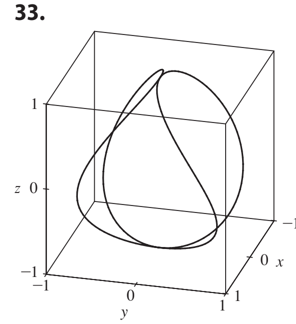
27.



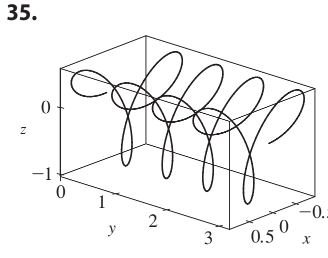
29. $y = e^{x/2},\ z = e^x,\ z = y^2$

31. $(0, 0, 0), (1, 0, 1)$

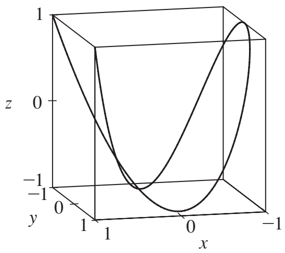
33.



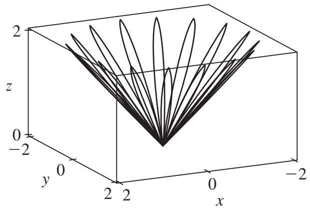
35.



37.



39.

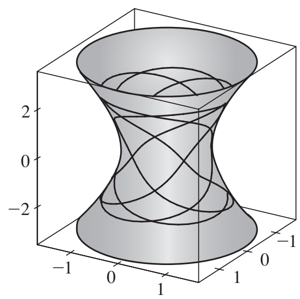


43. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$

45. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k},\ 0 \leqslant t \leqslant 2\pi$

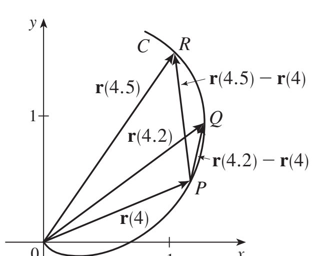
47. $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \leqslant t \leqslant 2\pi$ **49.** Yes

51. (a)

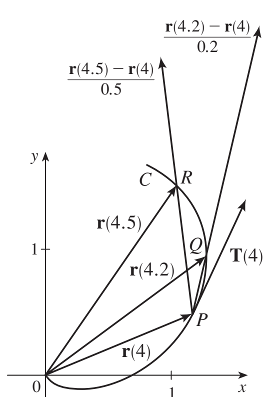


EXERCISES 13.2 ■ PAGE 900

1. (a)

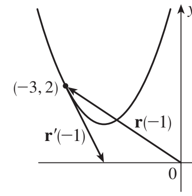


(b), (d)



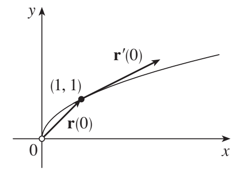
(c) $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$; $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$

3. (a), (c)



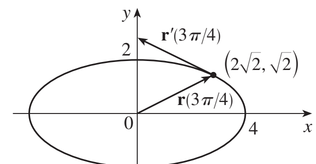
(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$

5. (a), (c)



(b) $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$

7. (a), (c)



(b) $\mathbf{r}'(t) = 4\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t-2}}, 0, -\dfrac{2}{t^3} \right\rangle$

11. $\mathbf{r}'(t) = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$

13. $\mathbf{r}'(t) = (t\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j} + (\cos^2 t - \sin^2 t)\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{b} + 2t\,\mathbf{c}$ **17.** $\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$ **19.** $\frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$

21. $\langle 1, 2t, 3t^2 \rangle, \langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle, \langle 0, 2, 6t \rangle, \langle 6t^2, -6t, 2 \rangle$

23. $x = 2 + 2t,\ y = 4 + 2t,\ z = 1 + t$

25. $x = 1 - t,\ y = t,\ z = 1 - t$

27. $\mathbf{r}(t) = (3 - 4t)\,\mathbf{i} + (4 + 3t)\,\mathbf{j} + (2 - 6t)\,\mathbf{k}$

29. $x = t,\ y = 1 - t,\ z = 2t$

31. $x = -\pi - t,\ y = \pi + t,\ z = -\pi t$

33. $66°$ **35.** $2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}$

37. $(\ln 2)\,\mathbf{i} + (\pi/4)\,\mathbf{j} + \frac{1}{2}\ln 2\,\mathbf{k}$

39. $\tan t\,\mathbf{i} + \frac{1}{8}(t^2 + 1)^4\,\mathbf{j} + \left(\frac{1}{3}t^3 \ln t - \frac{1}{9}t^3\right)\mathbf{k} + \mathbf{C}$

41. $t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$

47. $2t\cos t + 2\sin t - 2\cos t\sin t$ **49.** 35

EXERCISES 13.3 ■ PAGE 908

1. $10\sqrt{10}$ **3.** $e - e^{-1}$ **5.** $\frac{1}{27}(13^{3/2} - 8)$

7. 18.6833 **9.** 10.3311 **11.** 42

13. (a) $s(t) = \sqrt{26}\,(t - 1)$;

$\mathbf{r}(t(s)) = \left(4 - \dfrac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\dfrac{4s}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\dfrac{3s}{\sqrt{26}} + 3\right)\mathbf{k}$

(b) $\left(4 - \dfrac{4}{\sqrt{26}}, \dfrac{16}{\sqrt{26}} + 1, \dfrac{12}{\sqrt{26}} + 3\right)$

15. $(3\sin 1, 4, 3\cos 1)$

17. (a) $\langle 1/\sqrt{10}, (-3/\sqrt{10})\sin t, (3/\sqrt{10})\cos t \rangle$, $\langle 0, -\cos t, -\sin t \rangle$ (b) $\frac{3}{10}$

19. (a) $\dfrac{1}{e^{2t}+1}\langle \sqrt{2}e^t, e^{2t}, -1 \rangle$, $\dfrac{1}{e^{2t}+1}\langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t \rangle$

(b) $\sqrt{2}e^{2t}/(e^{2t} + 1)^2$

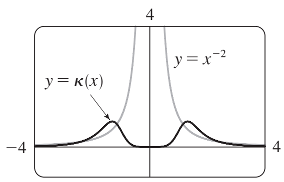
21. $6t^2/(9t^4 + 4t^2)^{3/2}$ **23.** $\dfrac{\sqrt{6}}{2(3t^2 + 1)^2}$

25. $\frac{1}{7}\sqrt{\frac{19}{14}}$ **27.** $12x^2/(1 + 16x^6)^{3/2}$

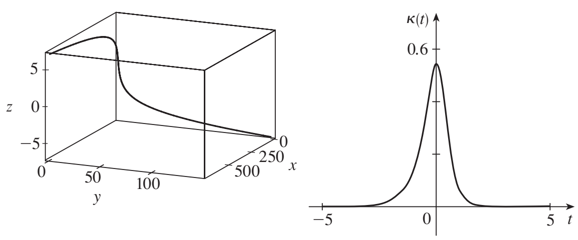
29. $e^x|x + 2|/[1 + (xe^x + e^x)^2]^{3/2}$

31. $\left(-\frac{1}{2}\ln 2, 1/\sqrt{2}\right)$; approaches 0 **33.** (a) P (b) 1.3, 0.7

35.

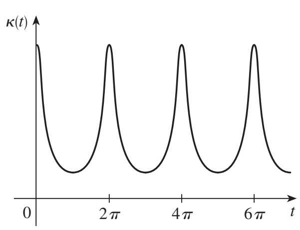


37.



39. a is $y = f(x)$, b is $y = \kappa(x)$

41. $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$



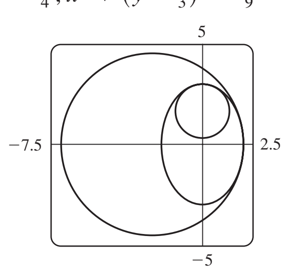
integer multiples of 2π

43. $6t^2/(4t^2 + 9t^4)^{3/2}$

45. $1/\left(\sqrt{2}e^t\right)$ **47.** $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle, \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle, \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

49. $x - 2z = -4\pi,\ 2x + z = 2\pi$

51. $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4},\ x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$



53. $(-1, -3, 1)$

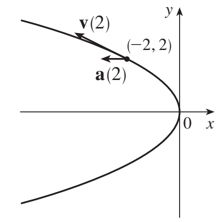
55. $2x + y + 4z = 7,\ 6x - 8y - z = -3$

65. $2/(t^4 + 4t^2 + 1)$ **67.** 2.07×10^{10} Å $\approx$ 2 m

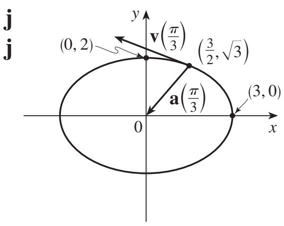
EXERCISES 13.4 ■ PAGE 918

1. (a) $1.8\mathbf{i} - 3.8\mathbf{j} - 0.7\mathbf{k},\ 2.0\mathbf{i} - 2.4\mathbf{j} - 0.6\mathbf{k},$
$2.8\mathbf{i} + 1.8\mathbf{j} - 0.3\mathbf{k},\ 2.8\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$
(b) $2.4\mathbf{i} - 0.8\mathbf{j} - 0.5\mathbf{k}$, 2.58

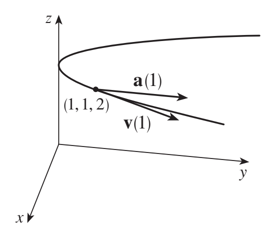
3. $\mathbf{v}(t) = \langle -t, 1 \rangle$
$\mathbf{a}(t) = \langle -1, 0 \rangle$
$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



5. $\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{5\sin^2 t + 4}$



7. $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$
$\mathbf{a}(t) = 2\,\mathbf{j}$
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$



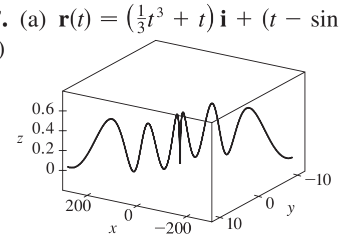
9. $\langle 2t + 1, 2t - 1, 3t^2 \rangle,\ \langle 2, 2, 6t \rangle,\ \sqrt{9t^4 + 8t^2 + 2}$

11. $\sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k},\ e^t\,\mathbf{j} + e^{-t}\,\mathbf{k},\ e^t + e^{-t}$

13. $e^t[(\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j} + (t + 1)\mathbf{k}],$
$e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t + 2)\mathbf{k}],\ e^t\sqrt{t^2 + 2t + 3}$

15. $\mathbf{v}(t) = (2t + 3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k},$
$\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} + (1 - t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$
(b)



19. $t = 4$

21. $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k},\ |\mathbf{v}(t)| = \sqrt{25t^2 + 2}$

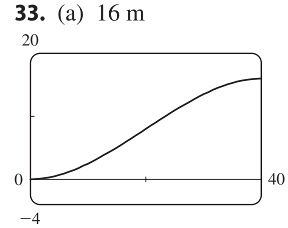
23. (a) $\approx$3535 m (b) $\approx$1531 m (c) 200 m/s

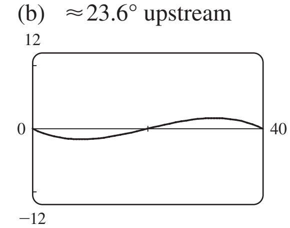
25. $\approx$30 m/s **27.** $\approx$ 544 ft/s

29. $13.0° < \theta < 36.0°,\ 55.4° < \theta < 85.5°$

31. $(250, -50, 0)$; $10\sqrt{93} \approx 96.4$ ft/s

33. (a) 16 m (b) $\approx$23.6° upstream





35. The path is contained in a circle that lies in a plane perpendicular to **c** with center on a line through the origin in the direction of **c**.

37. $\dfrac{4 + 18t^2}{\sqrt{4 + 9t^2}}, \dfrac{6t}{\sqrt{4 + 9t^2}}$ **39.** 0, 1 **41.** $\dfrac{7}{\sqrt{30}}, \sqrt{\dfrac{131}{30}}$

43. 4.5 cm/s^2, 9.0 cm/s^2 **45.** $t = 1$

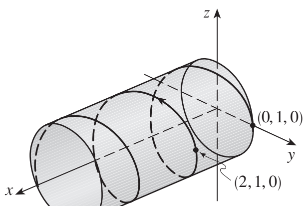
CHAPTER 13 REVIEW ■ PAGE 921

True-False Quiz

1. True **3.** False **5.** False **7.** False
9. True **11.** False **13.** True

Exercises

1. (a)



(b) $\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\, \mathbf{j} + \pi \cos \pi t\, \mathbf{k}$,
$\mathbf{r}''(t) = -\pi^2 \cos \pi t\, \mathbf{j} - \pi^2 \sin \pi t\, \mathbf{k}$

3. $\mathbf{r}(t) = 4 \cos t\, \mathbf{i} + 4 \sin t\, \mathbf{j} + (5 - 4 \cos t)\mathbf{k}$, $0 \leqslant t \leqslant 2\pi$

5. $\frac{1}{3}\mathbf{i} - (2/\pi^2)\mathbf{j} + (2/\pi)\mathbf{k}$ **7.** 86.631 **9.** 90°

11. (a) $\dfrac{1}{\sqrt{13}}\langle 3 \sin t, -3 \cos t, 2\rangle$ (b) $\langle \cos t, \sin t, 0\rangle$

(c) $\dfrac{1}{\sqrt{13}}\langle -2 \sin t, 2 \cos t, 3\rangle$

(d) $\dfrac{3}{13 \sin t \cos t}$ or $\dfrac{3}{13} \sec t \csc t$

13. $12/17^{3/2}$ **15.** $x - 2y + 2\pi = 0$

17. $\mathbf{v}(t) = (1 + \ln t)\mathbf{i} + \mathbf{j} - e^{-t}\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{2 + 2 \ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = (1/t)\mathbf{i} + e^{-t}\mathbf{k}$

19. $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$

21. $\approx 37.3°$, ≈ 157.4 m

23. (c) $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$

PROBLEMS PLUS ■ PAGE 924

1. (a) $\mathbf{v} = \omega R(-\sin \omega t\, \mathbf{i} + \cos \omega t\, \mathbf{j})$ (c) $\mathbf{a} = -\omega^2\mathbf{r}$

3. (a) 90°, $v_0^2/(2g)$

5. (a) ≈ 0.94 ft to the right of the table's edge, ≈ 15 ft/s
(b) $\approx 7.6°$ (c) ≈ 2.13 ft to the right of the table's edge

7. 56°

9. $(a_2b_3 - a_3b_2)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2) + (a_1b_2 - a_2b_1)(z - c_3) = 0$

CHAPTER 14

EXERCISES 14.1 ■ PAGE 939

1. (a) -27; a temperature of -15°C with wind blowing at 40 km/h feels equivalent to about -27°C without wind.
(b) When the temperature is -20°C, what wind speed gives a wind chill of -30°C? 20 km/h
(c) With a wind speed of 20 km/h, what temperature gives a wind chill of -49°C? -35°C
(d) A function of wind speed that gives wind-chill values when the temperature is -5°C
(e) A function of temperature that gives wind-chill values when the wind speed is 50 km/h

3. ≈ 94.2; the manufacturer's yearly production is valued at \$94.2 million when 120,000 labor hours are spent and \$20 million in capital is invested.

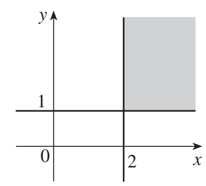
5. (a) ≈ 20.5; the surface area of a person 70 inches tall who weighs 160 pounds is approximately 20.5 square feet.

7. (a) 25; a 40-knot wind blowing in the open sea for 15 h will create waves about 25 ft high.
(b) $f(30, t)$ is a function of t giving the wave heights produced by 30-knot winds blowing for t hours.
(c) $f(v, 30)$ is a function of v giving the wave heights produced by winds of speed v blowing for 30 hours.

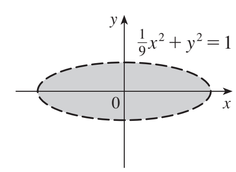
9. (a) 1 (b) $\mathbb{R}^2$ (c) $[-1, 1]$

11. (a) 3
(b) $\{(x, y, z) \mid x^2 + y^2 + z^2 < 4, x \geqslant 0, y \geqslant 0, z \geqslant 0\}$, interior of a sphere of radius 2, center the origin, in the first octant

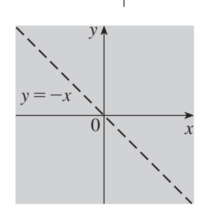
13. $\{(x, y) \mid x \geqslant 2, y \geqslant 1\}$



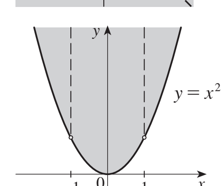
15. $\left\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\right\}$



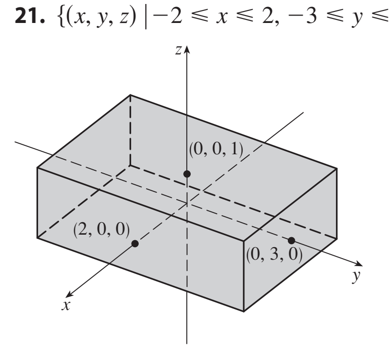
17. $\{(x, y) \mid y \neq -x\}$



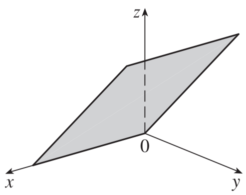
19. $\{(x, y) \mid y \geqslant x^2, x \neq \pm 1\}$



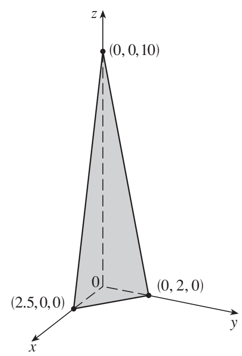
21. $\{(x, y, z) \mid -2 \leqslant x \leqslant 2, -3 \leqslant y \leqslant 3, -1 \leqslant z \leqslant 1\}$



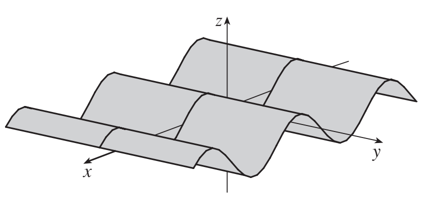
23. $z = y$, plane through the x-axis



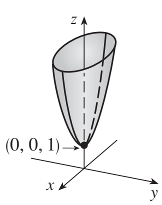
25. $4x + 5y + z = 10$, plane



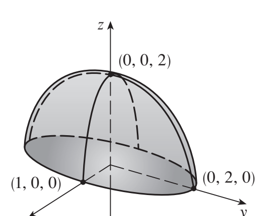
27. $z = \sin x$, cylinder



29. $z = x^2 + 4y^2 + 1$, elliptic paraboloid

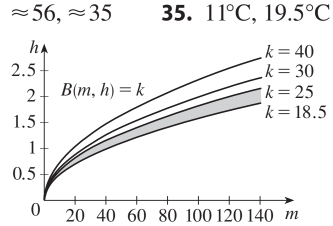


31. $z = \sqrt{4 - 4x^2 - y^2}$, top half of ellipsoid

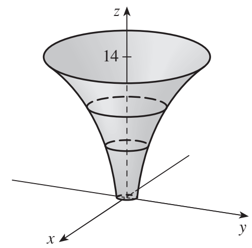


33. $\approx 56, \approx 35$ **35.** 11°C, 19.5°C **37.** Steep; nearly flat

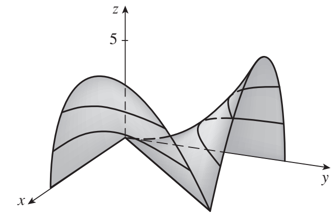
39. No



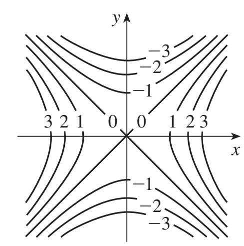
41.



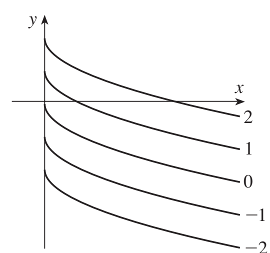
43.



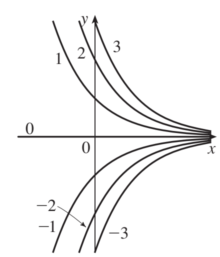
45. $x^2 - y^2 = k$



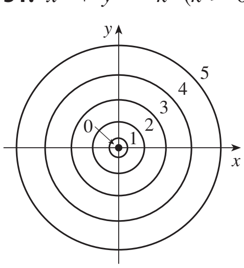
47. $y = -\sqrt{x} + k$



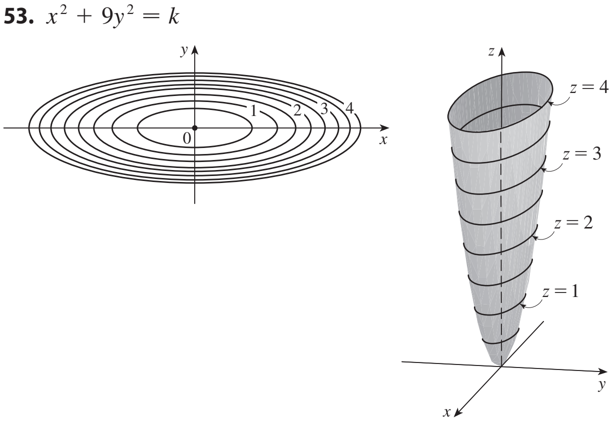
49. $y = ke^{-x}$



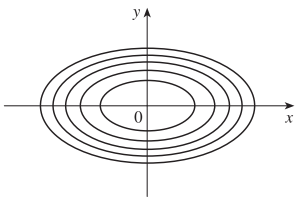
51. $x^2 + y^2 = k^3$ $(k \geqslant 0)$



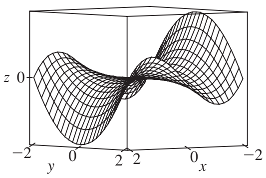
53. $x^2 + 9y^2 = k$



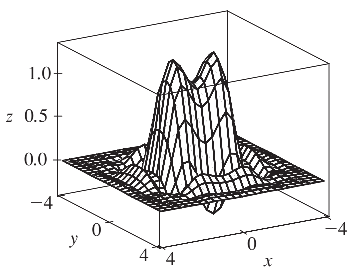
55.



57.



59.



61. (a) C (b) II **63.** (a) F (b) I

65. (a) B (b) VI **67.** Family of parallel planes

69. Family of circular cylinders with axis the x-axis ($k > 0$)

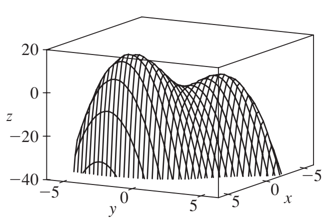
71. (a) Shift the graph of f upward 2 units

(b) Stretch the graph of f vertically by a factor of 2

(c) Reflect the graph of f about the xy-plane

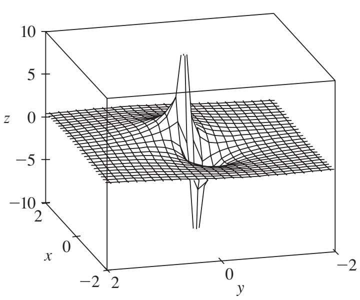
(d) Reflect the graph of f about the xy-plane and then shift it upward 2 units

73.



f appears to have a maximum value of about 15. There are two local maximum points but no local minimum point.

75.



The function values approach 0 as x, y become large; as (x, y) approaches the origin, f approaches $\pm\infty$ or 0, depending on the direction of approach.

77. If $c = 0$, the graph is a cylindrical surface. For $c > 0$, the level curves are ellipses. The graph curves upward as we leave the origin, and the steepness increases as c increases. For $c < 0$, the level curves are hyperbolas. The graph curves upward in the y-direction and downward, approaching the xy-plane, in the x-direction giving a saddle-shaped appearance near $(0, 0, 1)$.

79. $c = -2, 0, 2$ **81.** (b) $y = 0.75x + 0.01$

EXERCISES 14.2 ■ PAGE 950

1. Nothing; if f is continuous, $f(3, 1) = 6$ **3.** $-\frac{5}{2}$

5. 56 **7.** $\pi/2$ **9.** Does not exist **11.** Does not exist

13. 0 **15.** Does not exist **17.** 2 **19.** $\sqrt{3}$

21. Does not exist

23. The graph shows that the function approaches different numbers along different lines.

25. $h(x, y) = (2x + 3y - 6)^2 + \sqrt{2x + 3y - 6}$; $\{(x, y) \mid 2x + 3y \geqslant 6\}$

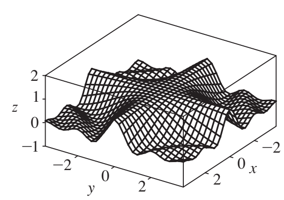
27. Along the line $y = x$ **29.** $\mathbb{R}^2$

31. $\{(x, y) \mid x^2 + y^2 \neq 1\}$ **33.** $\{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

35. $\{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$

37. $\{(x, y) \mid (x, y) \neq (0, 0)\}$ **39.** 0 **41.** -1

43.



f is continuous on $\mathbb{R}^2$

EXERCISES 14.3 ■ PAGE 963

1. (a) The rate of change of temperature as longitude varies, with latitude and time fixed; the rate of change as only latitude varies; the rate of change as only time varies

(b) Positive, negative, positive

3. (a) $f_T(-15, 30) \approx 1.3$; for a temperature of $-15°C$ and wind speed of 30 km/h, the wind-chill index rises by 1.3°C for each degree the temperature increases. $f_v(-15, 30) \approx -0.15$; for a temperature of $-15°C$ and wind speed of 30 km/h, the wind-chill index decreases by 0.15°C for each km/h the wind speed increases.

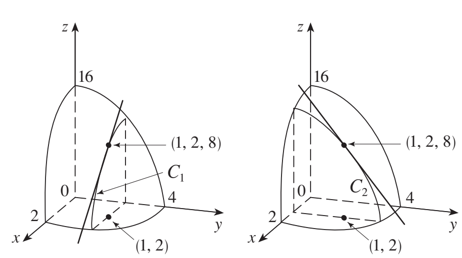
(b) Positive, negative (c) 0

5. (a) Positive (b) Negative

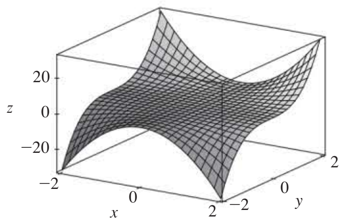
7. (a) Positive (b) Negative

9. $c = f$, $b = f_x$, $a = f_y$

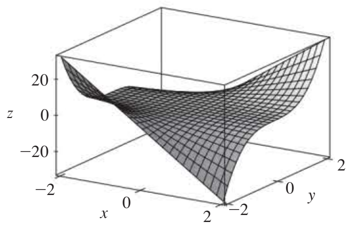
11. $f_x(1, 2) = -8 =$ slope of C_1, $f_y(1, 2) = -4 =$ slope of C_2



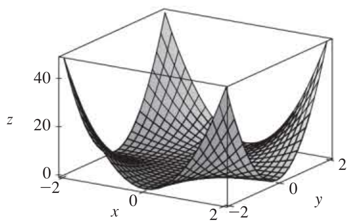
13.



$f(x, y) = x^2y^3$



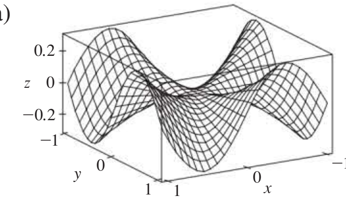
$f_x(x, y) = 2xy^3$



$f_y(x, y) = 3x^2y^2$

15. $f_x(x, y) = 4x^3 + 5y^3$, $f_y(x, y) = 15xy^2$
17. $f_x(x, t) = -t^2e^{-x}$, $f_t(x, t) = 2te^{-x}$
19. $\dfrac{\partial z}{\partial x} = \dfrac{1}{x + t^2}$, $\dfrac{\partial z}{\partial t} = \dfrac{2t}{x + t^2}$
21. $f_x(x, y) = 1/y$, $f_y(x, y) = -x/y^2$
23. $f_x(x, y) = \dfrac{(ad - bc)y}{(cx + dy)^2}$, $f_y(x, y) = \dfrac{(bc - ad)x}{(cx + dy)^2}$
25. $g_u(u, v) = 10uv(u^2v - v^3)^4$, $g_v(u, v) = 5(u^2 - 3v^2)(u^2v - v^3)^4$
27. $R_p(p, q) = \dfrac{q^2}{1 + p^2q^4}$, $R_q(p, q) = \dfrac{2pq}{1 + p^2q^4}$
29. $F_x(x, y) = \cos(e^x)$, $F_y(x, y) = -\cos(e^y)$
31. $f_x = 3x^2yz^2$, $f_y = x^3z^2 + 2z$, $f_z = 2x^3yz + 2y$
33. $\partial w/\partial x = 1/(x + 2y + 3z)$, $\partial w/\partial y = 2/(x + 2y + 3z)$, $\partial w/\partial z = 3/(x + 2y + 3z)$
35. $\partial p/\partial t = 2t^3/\sqrt{t^4 + u^2 \cos v}$,
$\partial p/\partial u = u \cos v/\sqrt{t^4 + u^2 \cos v}$,
$\partial p/\partial v = -u^2 \sin v/\left(2\sqrt{t^4 + u^2 \cos v}\right)$
37. $h_x = 2xy\cos(z/t)$, $h_y = x^2\cos(z/t)$,
$h_z = (-x^2y/t)\sin(z/t)$, $h_t = (x^2yz/t^2)\sin(z/t)$
39. $\partial u/\partial x_i = x_i/\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
41. 1 **43.** $\frac{1}{6}$ **45.** $f_x(x, y) = y^2 - 3x^2y$, $f_y(x, y) = 2xy - x^3$
47. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{3z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2y}{3z}$
49. $\dfrac{\partial z}{\partial x} = \dfrac{yz}{e^z - xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz}{e^z - xy}$

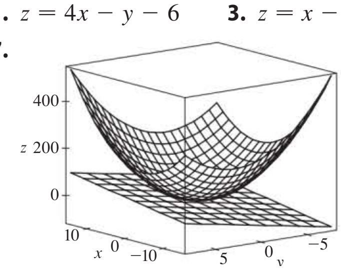
51. (a) $f'(x), g'(y)$ (b) $f'(x + y), f'(x + y)$
53. $f_{xx} = 12x^2y - 12xy^2$, $f_{xy} = 4x^3 - 12x^2y = f_{yx}$, $f_{yy} = -4x^3$
55. $z_{xx} = \dfrac{8y}{(2x + 3y)^3}$, $z_{xy} = \dfrac{6y - 4x}{(2x + 3y)^3} = z_{yx}$,
$z_{yy} = -\dfrac{12x}{(2x + 3y)^3}$
57. $v_{ss} = 2\cos(s^2 - t^2) - 4s^2\sin(s^2 - t^2)$,
$v_{st} = 4st\sin(s^2 - t^2) = v_{ts}$,
$v_{tt} = -2\cos(s^2 - t^2) - 4t^2\sin(s^2 - t^2)$
63. $24xy^2 - 6y$, $24x^2y - 6x$ **65.** $(2x^2y^2z^5 + 6xyz^3 + 2z)e^{xyz^2}$
67. $\frac{3}{4}v(u + v^2)^{-5/2}$ **69.** $4/(y + 2z)^3$, 0 **71.** $6yz^2$
73. ≈ 12.2, ≈ 16.8, ≈ 23.25 **83.** R^2/R_1^2
87. $\dfrac{\partial T}{\partial P} = \dfrac{V - nb}{nR}$, $\dfrac{\partial P}{\partial V} = \dfrac{2n^2a}{V^3} - \dfrac{nRT}{(V - nb)^2}$
91. (a) ≈ 0.0545; for a person 70 inches tall who weighs 160 pounds, an increase in weight causes the surface area to increase at a rate of about 0.0545 square feet per pound.
(b) ≈ 0.213; for a person 70 inches tall who weighs 160 pounds, an increase in height (with no change in weight) causes the surface area to increase at a rate of about 0.213 square feet per inch of height.
93. $\partial P/\partial v = 3Av^2 - \dfrac{B(mg/x)^2}{v^2}$ is the rate of change of the power needed during flapping mode with respect to the bird's velocity when the mass and fraction of flapping time remain constant; $\partial P/\partial x = -\dfrac{2Bm^2g^2}{x^3v}$ is the rate at which the power changes when only the fraction of time spent in flapping mode varies; $\partial P/\partial m = \dfrac{2Bmg^2}{x^2v}$ is the rate of change of the power when only the mass varies.
97. No **99.** $x = 1 + t$, $y = 2$, $z = 2 - 2t$ **103.** -2
105. (a)



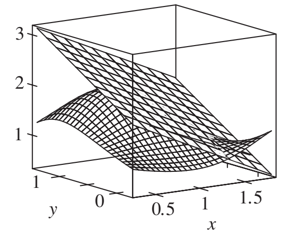
(b) $f_x(x, y) = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$, $f_y(x, y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$
(c) 0, 0 (e) No, since f_{xy} and f_{yx} are not continuous.

EXERCISES 14.4 ■ PAGE 974

1. $z = 4x - y - 6$ **3.** $z = x - y + 1$ **5.** $x + y + z = 0$
7.



9.



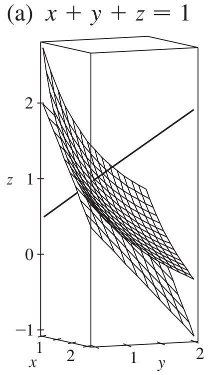
11. $6x + 4y - 23$ **13.** $2x + y - 1$
15. $2x + 2y + \pi - 4$ **19.** 6.3
21. $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$; 6.9914 **23.** $4T + H - 329$; 129°F
25. $dz = -2e^{-2x}\cos 2\pi t\, dx - 2\pi e^{-2x}\sin 2\pi t\, dt$
27. $dm = 5p^4q^3\, dp + 3p^5q^2\, dq$
29. $dR = \beta^2\cos\gamma\, d\alpha + 2\alpha\beta\cos\gamma\, d\beta - \alpha\beta^2\sin\gamma\, d\gamma$
31. $\Delta z = 0.9225$, $dz = 0.9$ **33.** 5.4 cm^2 **35.** 16 cm^3
37. $\approx -0.0165mg$; decrease **39.** $\frac{1}{17} \approx 0.059\ \Omega$
41. (a) $0.8264m - 34.56h + 38.02$ (b) 18.801
43. $\varepsilon_1 = \Delta x$, $\varepsilon_2 = \Delta y$

EXERCISES 14.5 ■ PAGE 983

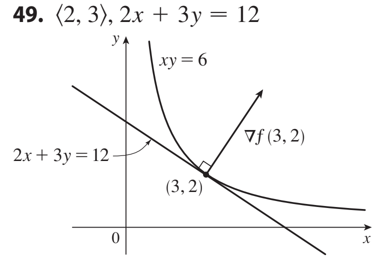
1. $2t(y^3 - 2xy + 3xy^2 - x^2)$
3. $\dfrac{1}{2\sqrt{t}}\cos x\cos y + \dfrac{1}{t^2}\sin x\sin y$
5. $e^{y/z}[2t - (x/z) - (2xy/z^2)]$
7. $\partial z/\partial s = 5(x-y)^4(2st - t^2)$, $\partial z/\partial t = 5(x-y)^4(s^2 - 2st)$
9. $\dfrac{\partial z}{\partial s} = \dfrac{3\sin t - 2t\sin s}{3x + 2y}$, $\dfrac{\partial z}{\partial t} = \dfrac{3s\cos t + 2\cos s}{3x + 2y}$
11. $\dfrac{\partial z}{\partial s} = e^r\left(t\cos\theta - \dfrac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$,
$\dfrac{\partial z}{\partial t} = e^r\left(s\cos\theta - \dfrac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$
13. 42 **15.** 7, 2
17. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}$,
$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$
19. $\dfrac{\partial T}{\partial x} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial x} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial x} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial x}$,
$\dfrac{\partial T}{\partial y} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial y} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial y} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial y}$,
$\dfrac{\partial T}{\partial z} = \dfrac{\partial T}{\partial p}\dfrac{\partial p}{\partial z} + \dfrac{\partial T}{\partial q}\dfrac{\partial q}{\partial z} + \dfrac{\partial T}{\partial r}\dfrac{\partial r}{\partial z}$
21. 1582, 3164, -700 **23.** 2π, -2π
25. $\frac{5}{144}$, $-\frac{5}{96}$, $\frac{5}{144}$ **27.** $\dfrac{2x + y\sin x}{\cos x - 2y}$
29. $\dfrac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$
31. $-\dfrac{x}{3z}$, $-\dfrac{2y}{3z}$ **33.** $\dfrac{yz}{e^z - xy}$, $\dfrac{xz}{e^z - xy}$
35. 2°C/s **37.** ≈ -0.33 m/s per minute
39. (a) 6 m^3/s (b) 10 m^2/s (c) 0 m/s
41. ≈ -0.27 L/s **43.** $-1/(12\sqrt{3})$ rad/s
45. (a) $\partial z/\partial r = (\partial z/\partial x)\cos\theta + (\partial z/\partial y)\sin\theta$,
$\partial z/\partial\theta = -(\partial z/\partial x)\, r\sin\theta + (\partial z/\partial y)\, r\cos\theta$
51. $4rs\,\partial^2 z/\partial x^2 + (4r^2 + 4s^2)\partial^2 z/\partial x\,\partial y + 4rs\,\partial^2 z/\partial y^2 + 2\,\partial z/\partial y$

EXERCISES 14.6 ■ PAGE 996

1. ≈ -0.08 mb/km **3.** ≈ 0.778 **5.** $\sqrt{2}/2$
7. (a) $\nabla f(x, y) = (1/y)\mathbf{i} - (x/y^2)\mathbf{j}$ (b) $\mathbf{i} - 2\mathbf{j}$ (c) -1
9. (a) $\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\rangle$
(b) $\langle -3, 2, 2\rangle$ (c) $\frac{2}{5}$
11. $\dfrac{4 - 3\sqrt{3}}{10}$ **13.** $7/(2\sqrt{5})$ **15.** 1 **17.** $\frac{23}{42}$
19. $\frac{2}{5}$ **21.** $\sqrt{65}$, $\langle 1, 8\rangle$ **23.** 1, $\langle 0, 1\rangle$
25. $\frac{3}{4}$, $\langle 1, -2, -2\rangle$ **27.** (b) $\langle -12, 92\rangle$
29. All points on the line $y = x + 1$ **31.** (a) $-40/(3\sqrt{3})$
33. (a) $32/\sqrt{3}$ (b) $\langle 38, 6, 12\rangle$ (c) $2\sqrt{406}$
35. $\frac{327}{13}$ **39.** $\frac{774}{25}$
41. (a) $x + y + z = 11$ (b) $x - 3 = y - 3 = z - 5$
43. (a) $x + 2y + 6z = 12$ (b) $x - 2 = \dfrac{y - 2}{2} = \dfrac{z - 1}{6}$
45. (a) $x + y + z = 1$ (b) $x = y = z - 1$
47.



49. $\langle 2, 3\rangle$, $2x + 3y = 12$



55. No **59.** $\left(-\frac{5}{4}, -\frac{5}{4}, \frac{25}{8}\right)$
63. $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$
65. $(-1, 0, 1)$; $\approx 7.8°$
69. If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $af_x + bf_y$ and $cf_x + df_y$ are known, so we solve linear equations for f_x and f_y.

EXERCISES 14.7 ■ PAGE 1007

1. (a) f has a local minimum at (1, 1).
(b) f has a saddle point at (1, 1).
3. Local minimum at (1, 1), saddle point at (0, 0)
5. Minimum $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$
7. Saddle points at $(1, 1)$, $(-1, -1)$
9. Minima $f\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right) = f\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{4}$,
saddle point at (0, 0)
11. Maximum $f(-1, 0) = 2$, minimum $f(1, 0) = -2$,
saddle points at $(0, \pm 1)$
13. Maximum $f(0, -1) = 2$, minima $f(\pm 1, 1) = -3$,
saddle points at $(0, 1)$, $(\pm 1, -1)$
15. None
17. Minima $f(x, y) = 1$ at all points (x, y) on x- and y-axes
19. Minima $f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1$,
saddle points at $(\pi/2, 0)$, $(3\pi/2, 0)$

23. Minima $f(1, \pm 1) = f(-1, \pm 1) = 3$

25. Maximum $f(\pi/3, \pi/3) = 3\sqrt{3}/2$,
minimum $f(5\pi/3, 5\pi/3) = -3\sqrt{3}/2$, saddle point at (π, π)

27. Minima $f(0, -0.794) \approx -1.191$, $f(\pm 1.592, 1.267) \approx -1.310$,
saddle points $(\pm 0.720, 0.259)$,
lowest points $(\pm 1.592, 1.267, -1.310)$

29. Maximum $f(0.170, -1.215) \approx 3.197$,
minima $f(-1.301, 0.549) \approx -3.145$, $f(1.131, 0.549) \approx -0.701$,
saddle points $(-1.301, -1.215)$, $(0.170, 0.549)$, $(1.131, -1.215)$,
no highest or lowest point

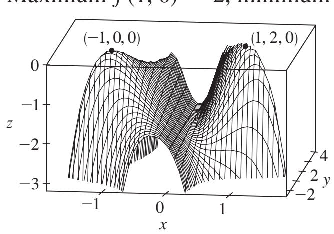
31. Maximum $f(0, \pm 2) = 4$, minimum $f(1, 0) = -1$

33. Maximum $f(\pm 1, 1) = 7$, minimum $f(0, 0) = 4$

35. Maximum $f(0, 3) = f(2, 3) = 7$, minimum $f(1, 1) = -2$

37. Maximum $f(1, 0) = 2$, minimum $f(-1, 0) = -2$

39.



41. $2/\sqrt{3}$ **43.** $(2, 1, \sqrt{5}), (2, 1, -\sqrt{5})$ **45.** $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$

47. $8r^3/(3\sqrt{3})$ **49.** $\frac{4}{3}$ **51.** Cube, edge length $c/12$

53. Square base of side 40 cm, height 20 cm **55.** $L^3/(3\sqrt{3})$

57. (a) $H = -p_1 \ln p_1 - p_2 \ln p_2 - (1 - p_1 - p_2) \ln(1 - p_1 - p_2)$
(b) $\{(p_1, p_2) \mid 0 < p_1 < 1, p_2 < 1 - p_1\}$
(c) $\ln 3$; $p_1 = p_2 = p_3 = \frac{1}{3}$

EXERCISES 14.8 ■ PAGE 1017

1. $\approx 59, 30$

3. Maximum $f(\pm 1, 0) = 1$, minimum $f(0, \pm 1) = -1$

5. Maximum $f(1, 2) = f(-1, -2) = 2$,
minimum $f(1, -2) = f(-1, 2) = -2$

7. Maximum $f(2, 2, 1) = 9$, minimum $f(-2, -2, -1) = -9$

9. Maximum $f(1, \pm\sqrt{2}, 1) = f(-1, \pm\sqrt{2}, -1) = 2$,
minimum $f(1, \pm\sqrt{2}, -1) = f(-1, \pm\sqrt{2}, 1) = -2$

11. Maximum $\sqrt{3}$, minimum 1

13. Maximum $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 2$,
minimum $f(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = -2$

15. Minimum $f(1, 1) = f(-1, -1) = 2$

17. Maximum $f(0, 1, \sqrt{2}) = 1 + \sqrt{2}$,
minimum $f(0, 1, -\sqrt{2}) = 1 - \sqrt{2}$

19. Maximum $\frac{3}{2}$, minimum $\frac{1}{2}$

21. Maximum $f(3/\sqrt{2}, -3/\sqrt{2}) = 9 + 12\sqrt{2}$,
minimum $f(-2, 2) = -8$

23. Maximum $f(\pm 1/\sqrt{2}, \mp 1/(2\sqrt{2})) = e^{1/4}$,
minimum $f(\pm 1/\sqrt{2}, \pm 1/(2\sqrt{2})) = e^{-1/4}$

31–43. See Exercises 41–55 in Section 14.7.

45. Nearest $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, farthest $(-1, -1, 2)$

47. Maximum ≈ 9.7938, minimum ≈ -5.3506

49. (a) c/n (b) When $x_1 = x_2 = \cdots = x_n$

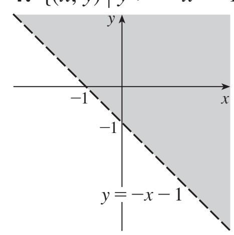
CHAPTER 14 REVIEW ■ PAGE 1022

True-False Quiz

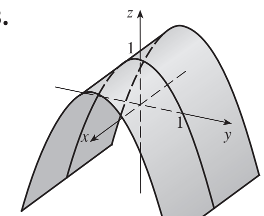
1. True **3.** False **5.** False **7.** True **9.** False
11. True

Exercises

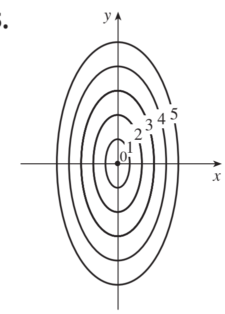
1. $\{(x, y) \mid y > -x - 1\}$



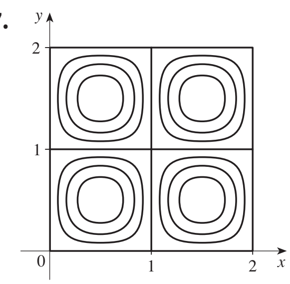
3.



5.



7.



9. $\frac{2}{3}$

11. (a) $\approx 3.5°\text{C/m}$, $-3.0°\text{C/m}$
(b) $\approx 0.35°\text{C/m}$ by Equation 14.6.9 (Definition 14.6.2 gives $\approx 1.1°\text{C/m}$.)
(c) -0.25

13. $f_x = 32xy(5y^3 + 2x^2y)^7$, $f_y = (16x^2 + 120y^2)(5y^3 + 2x^2y)^7$

15. $F_\alpha = \dfrac{2\alpha^3}{\alpha^2 + \beta^2} + 2\alpha \ln(\alpha^2 + \beta^2)$, $F_\beta = \dfrac{2\alpha^2\beta}{\alpha^2 + \beta^2}$

17. $S_u = \arctan(v\sqrt{w})$, $S_v = \dfrac{u\sqrt{w}}{1 + v^2w}$, $S_w = \dfrac{uv}{2\sqrt{w}(1 + v^2w)}$

19. $f_{xx} = 24x$, $f_{xy} = -2y = f_{yx}$, $f_{yy} = -2x$

21. $f_{xx} = k(k-1)x^{k-2}y^lz^m$, $f_{xy} = klx^{k-1}y^{l-1}z^m = f_{yx}$,
$f_{xz} = kmx^{k-1}y^lz^{m-1} = f_{zx}$, $f_{yy} = l(l-1)x^ky^{l-2}z^m$,
$f_{yz} = lmx^ky^{l-1}z^{m-1} = f_{zy}$, $f_{zz} = m(m-1)x^ky^lz^{m-2}$

25. (a) $z = 8x + 4y + 1$ (b) $\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$

27. (a) $2x - 2y - 3z = 3$ (b) $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$

29. (a) $x + 2y + 5z = 0$
(b) $x = 2 + t$, $y = -1 + 2t$, $z = 5t$

31. $(2, \frac{1}{2}, -1), (-2, -\frac{1}{2}, 1)$

33. $60x + \frac{24}{5}y + \frac{32}{5}z - 120$; 38.656

35. $2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

37. $-47, 108$

43. $\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \rangle$ **45.** $-\frac{4}{5}$

47. $\sqrt{145}/2$, $\langle 4, \frac{9}{2} \rangle$ **49.** $\approx \frac{5}{8}$ knots/mi

51. Minimum $f(-4, 1) = -11$

53. Maximum $f(1, 1) = 1$; saddle points $(0, 0)$, $(0, 3)$, $(3, 0)$

55. Maximum $f(1, 2) = 4$, minimum $f(2, 4) = -64$

57. Maximum $f(-1, 0) = 2$, minima $f(1, \pm 1) = -3$,
saddle points $(-1, \pm 1)$, $(1, 0)$

59. Maximum $f(\pm\sqrt{2/3}, 1/\sqrt{3}) = 2/(3\sqrt{3})$, minimum $f(\pm\sqrt{2/3}, -1/\sqrt{3}) = -2/(3\sqrt{3})$
61. Maximum 1, minimum -1
63. $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4})$, $(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$
65. $P(2-\sqrt{3})$, $P(3-\sqrt{3})/6$, $P(2\sqrt{3}-3)/3$

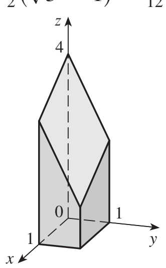
PROBLEMS PLUS ■ PAGE 1025

1. $L^2W^2, \frac{1}{4}L^2W^2$ **3.** (a) $x = w/3$, base $= w/3$ (b) Yes
7. $\sqrt{3/2}, 3/\sqrt{2}$

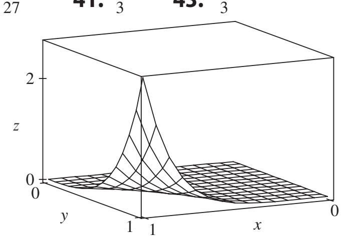
CHAPTER 15

EXERCISES 15.1 ■ PAGE 1039

1. (a) 288 (b) 144 **3.** (a) 0.990 (b) 1.151
5. $U < V < L$ **7.** (a) ≈ 248 (b) ≈ 15.5
9. $24\sqrt{2}$ **11.** 3 **13.** $2 + 8y^2, 3x + 27x^2$
15. 222 **17.** $\frac{5}{2} - e^{-1}$ **19.** 18
21. $\frac{15}{2}\ln 2 + \frac{3}{2}\ln 4$ or $\frac{21}{2}\ln 2$ **23.** 6
25. $\frac{31}{30}$ **27.** 2 **29.** $9\ln 2$
31. $\frac{1}{2}(\sqrt{3}-1) - \frac{1}{12}\pi$ **33.** $\frac{1}{2}e^{-6} + \frac{5}{2}$
35.



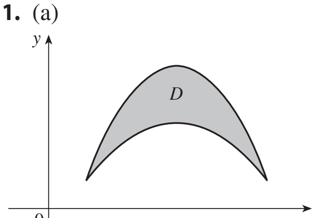
37. 51 **39.** $\frac{166}{27}$ **41.** $\frac{8}{3}$ **43.** $\frac{64}{3}$
45. $21e - 57$



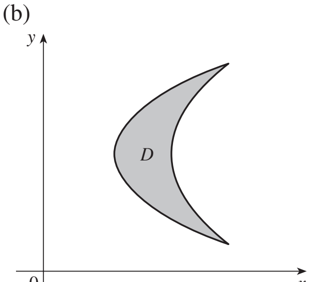
47. $\frac{5}{6}$ **49.** 0
51. Fubini's Theorem does not apply. The integrand has an infinite discontinuity at the origin.

EXERCISES 15.2 ■ PAGE 1048

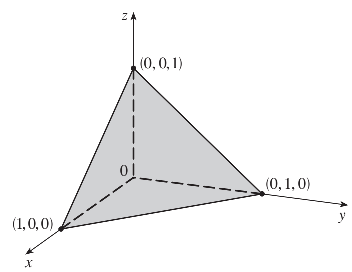
1. $\frac{868}{3}$ **3.** $\frac{1}{6}(e-1)$ **5.** $\frac{1}{3}\sin 1$
7. $\frac{1}{4}\ln 17$ **9.** $\frac{1}{2}(1 - e^{-9})$
11. (a)



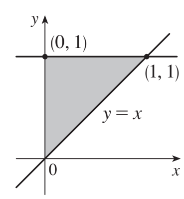
(b)



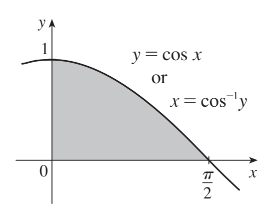
13. Type I: $D = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x\}$, type II: $D = \{(x, y) \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1\}$; $\frac{1}{3}$
15. $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \frac{9}{4}$
17. $\frac{1}{2}(1 - \cos 1)$ **19.** $\frac{11}{3}$ **21.** 0 **23.** $\frac{3}{4}$
25. $\frac{31}{8}$ **27.** $\frac{16}{3}$ **29.** $\frac{128}{15}$ **31.** $\frac{1}{3}$
33. 0, 1.213; 0.713 **35.** $\frac{64}{3}$
37. $\dfrac{10}{3\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{3}$
39.



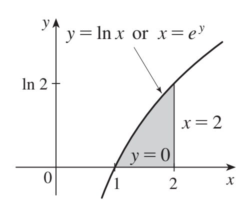
41. 13,984,735,616/14,549,535 **43.** $\pi/2$
45. $\int_0^1 \int_x^1 f(x, y)\,dy\,dx$



47. $\int_0^1 \int_0^{\cos^{-1} y} f(x, y)\,dx\,dy$



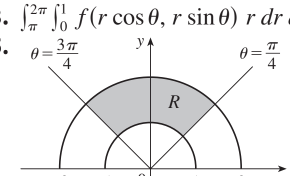
49. $\int_0^{\ln 2} \int_{e^y}^2 f(x, y)\,dx\,dy$



51. $\frac{1}{6}(e^9 - 1)$ **53.** $\frac{2}{9}(2\sqrt{2} - 1)$
55. $\frac{1}{3}(2\sqrt{2} - 1)$ **57.** 1
59. $\dfrac{\sqrt{3}}{2}\pi \leqslant \iint_S \sqrt{4 - x^2y^2}\,dA \leqslant \pi$
61. $\frac{3}{4}$ **65.** 9π **67.** $a^2b + \frac{3}{2}ab^2$ **69.** $\pi a^2 b$

EXERCISES 15.3 ■ PAGE 1054

1. $\int_0^{2\pi} \int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
3. $\int_{\pi}^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
5.



$3\pi/4$

7. $\frac{1250}{3}$ **9.** $(\pi/4)(\cos 1 - \cos 9)$
11. $(\pi/2)(1 - e^{-4})$ **13.** $\frac{3}{64}\pi^2$ **15.** $\pi/12$
17. $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$ **19.** $\frac{625}{2}\pi$ **21.** 4π **23.** $\frac{4}{3}\pi a^3$
25. $(\pi/3)(2 - \sqrt{2})$ **27.** $(8\pi/3)(64 - 24\sqrt{3})$
29. $(\pi/4)(1 - e^{-4})$ **31.** $\frac{1}{120}$ **33.** 4.5951
35. 1800π ft^3 **37.** $2/(a + b)$ **39.** $\frac{15}{16}$
41. (a) $\sqrt{\pi}/4$ (b) $\sqrt{\pi}/2$

EXERCISES 15.4 ■ PAGE 1064

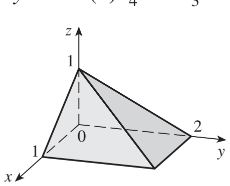
1. 285 C **3.** $42k$, $(2, \frac{85}{28})$ **5.** 6, $(\frac{3}{4}, \frac{3}{2})$ **7.** $\frac{8}{15}k$, $(0, \frac{4}{7})$
9. $\frac{1}{8}(1 - 3e^{-2})$, $\left(\dfrac{e^2 - 5}{e^2 - 3}, \dfrac{8(e^3 - 4)}{27(e^3 - 3e)}\right)$
11. $(\frac{3}{8}, 3\pi/16)$ **13.** $(0, 45/(14\pi))$
15. $(2a/5, 2a/5)$ if vertex is $(0, 0)$ and sides are along positive axes
17. $409.2k$, $182k$, $591.2k$
19. $7ka^6/180$, $7ka^6/180$, $7ka^6/90$ if vertex is $(0, 0)$ and sides are along positive axes
21. $\rho bh^3/3$, $\rho b^3h/3$; $b/\sqrt{3}$, $h/\sqrt{3}$
23. $\rho a^4\pi/16$, $\rho a^4\pi/16$; $a/2$, $a/2$
25. $m = 3\pi/64$, $(\bar{x}, \bar{y}) = \left(\dfrac{16384\sqrt{2}}{10395\pi}, 0\right)$,
$I_x = \dfrac{5\pi}{384} - \dfrac{4}{105}$, $I_y = \dfrac{5\pi}{384} + \dfrac{4}{105}$, $I_0 = \dfrac{5\pi}{192}$
27. (a) $\frac{1}{2}$ (b) 0.375 (c) $\frac{5}{48} \approx 0.1042$
29. (b) (i) $e^{-0.2} \approx 0.8187$
(ii) $1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$ (c) 2, 5
31. (a) ≈ 0.500 (b) ≈ 0.632
33. (a) $\iint_D k\left[1 - \frac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$, where D is the disk with radius 10 mi centered at the center of the city
(b) $200\pi k/3 \approx 209k$, $200(\pi/2 - \frac{8}{9})k \approx 136k$, on the edge

EXERCISES 15.5 ■ PAGE 1068

1. $12\sqrt{35}$ **3.** $3\sqrt{14}$ **5.** $(\pi/6)(13\sqrt{13} - 1)$
7. $(\pi/6)(17\sqrt{17} - 5\sqrt{5})$ **9.** $(2\pi/3)(2\sqrt{2} - 1)$
11. $a^2(\pi - 2)$ **13.** 3.6258
15. (a) ≈ 1.83 (b) ≈ 1.8616
17. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$
19. 3.3213 **23.** $(\pi/6)(101\sqrt{101} - 1)$

EXERCISES 15.6 ■ PAGE 1077

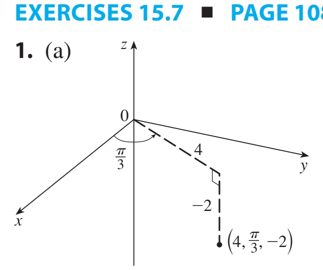
1. $\frac{27}{4}$ **3.** $\frac{16}{15}$ **5.** $\frac{5}{3}$ **7.** $\frac{2}{3}$ **9.** $\frac{27}{2}$ **11.** $9\pi/8$
13. $\frac{65}{28}$ **15.** $\frac{8}{15}$ **17.** $16\pi/3$ **19.** $\frac{16}{3}$ **21.** $\frac{8}{15}$
23. (a) $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx$ (b) $\frac{1}{4}\pi - \frac{1}{3}$
25. ≈ 0.985 **27.**



29. $\int_{-2}^{2} \int_0^{4-x^2} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\,dz\,dy\,dx$
$= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x, y, z)\,dz\,dx\,dy$
$= \int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\,dx\,dy\,dz$
$= \int_0^4 \int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x, y, z)\,dx\,dz\,dy$
$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_0^{4-x^2-4z^2} f(x, y, z)\,dy\,dz\,dx$
$= \int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x, y, z)\,dy\,dx\,dz$
31. $\int_{-2}^{2} \int_{x^2}^{4} \int_0^{2-y/2} f(x, y, z)\,dz\,dy\,dx$
$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{2-y/2} f(x, y, z)\,dz\,dx\,dy$
$= \int_0^2 \int_0^{4-2z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dy\,dz$
$= \int_0^4 \int_0^{2-y/2} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\,dx\,dz\,dy$
$= \int_{-2}^{2} \int_0^{2-x^2/2} \int_{x^2}^{4-2z} f(x, y, z)\,dy\,dz\,dx$
$= \int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} f(x, y, z)\,dy\,dx\,dz$
33. $\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(x, y, z)\,dz\,dy\,dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z)\,dz\,dx\,dy$
$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z)\,dx\,dy\,dz = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\,dx\,dz\,dy$
$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\,dy\,dz\,dx = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x, y, z)\,dy\,dx\,dz$
35. $\int_0^1 \int_y^1 \int_0^y f(x, y, z)\,dz\,dx\,dy = \int_0^1 \int_0^x \int_0^y f(x, y, z)\,dz\,dy\,dx$
$= \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\,dx\,dy\,dz = \int_0^1 \int_0^y \int_y^1 f(x, y, z)\,dx\,dz\,dy$
$= \int_0^1 \int_0^x \int_z^x f(x, y, z)\,dy\,dz\,dx = \int_0^1 \int_z^1 \int_z^x f(x, y, z)\,dy\,dx\,dz$
37. 64π **39.** $\frac{3}{2}\pi$, $(0, 0, \frac{1}{3})$
41. a^5, $(7a/12, 7a/12, 7a/12)$
43. $I_x = I_y = I_z = \frac{2}{3}kL^5$ **45.** $\frac{1}{2}\pi kha^4$
47. (a) $m = \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} \sqrt{x^2 + y^2}\,dz\,dy\,dx$
(b) $(\bar{x}, \bar{y}, \bar{z})$, where
$\bar{x} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} x\sqrt{x^2 + y^2}\,dz\,dy\,dx$,
$\bar{y} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} y\sqrt{x^2 + y^2}\,dz\,dy\,dx$,
and $\bar{z} = (1/m) \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} z\sqrt{x^2 + y^2}\,dz\,dy\,dx$
(c) $\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} (x^2 + y^2)^{3/2}\,dz\,dy\,dx$
49. (a) $\frac{3}{32}\pi + \frac{11}{24}$
(b) $\left(\dfrac{28}{9\pi + 44}, \dfrac{30\pi + 128}{45\pi + 220}, \dfrac{45\pi + 208}{135\pi + 660}\right)$
(c) $\frac{1}{240}(68 + 15\pi)$
51. (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{1}{5760}$ **53.** $L^3/8$
55. (a) The region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$
(b) $4\sqrt{6}\pi/45$

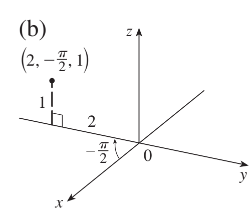
EXERCISES 15.7 ■ PAGE 1083

1. (a)



$(2, 2\sqrt{3}, -2)$

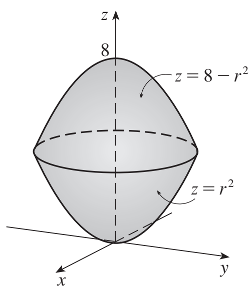
(b)



$(0, -2, 1)$

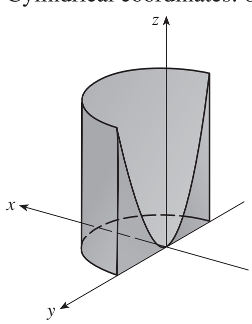
3. (a) $(\sqrt{2}, 3\pi/4, 1)$ (b) $(4, 2\pi/3, 3)$
5. Circular cylinder with radius 2 and axis the z-axis
7. Sphere, radius 2, centered at the origin
9. (a) $z^2 = 1 + r\cos\theta - r^2$ (b) $z = r^2\cos 2\theta$

11.



13. Cylindrical coordinates: $6 \leqslant r \leqslant 7$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant 20$

15. 4π



17. 384π **19.** $\frac{8}{3}\pi + \frac{128}{15}$ **21.** $2\pi/5$ **23.** $\frac{4}{3}\pi(\sqrt{2} - 1)$

25. (a) $\frac{512}{3}\pi$ (b) $\left(0, 0, \frac{23}{2}\right)$

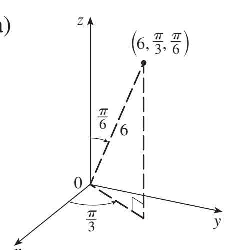
27. $\pi K a^2/8$, $(0, 0, 2a/3)$ **29.** 0

31. (a) $\iiint_C h(P)g(P)\, dV$, where C is the cone

(b) $\approx 3.1 \times 10^{19}$ ft-lb

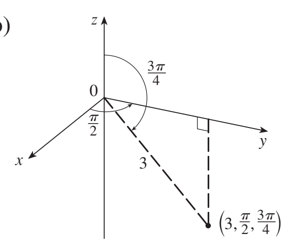
EXERCISES 15.8 ■ PAGE 1089

1. (a)



$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$

(b)



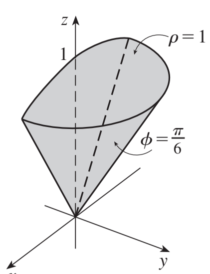
$\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

3. (a) $(2, 3\pi/2, \pi/2)$ (b) $(2, 3\pi/4, 3\pi/4)$

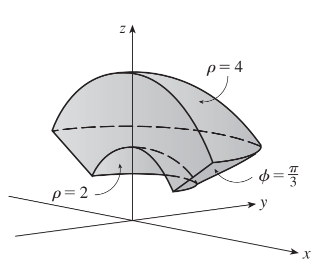
5. Half-cone **7.** Horizontal plane

9. (a) $\rho = 3$ (b) $\rho^2(\sin^2\phi \cos 2\theta - \cos^2\phi) = 1$

11.

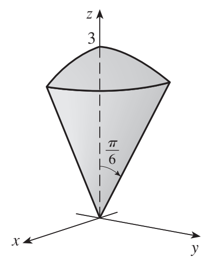


13.



15. $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \rho \leqslant \cos\phi$

17. $(9\pi/4)(2 - \sqrt{3})$



19. $\int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$

21. $312{,}500\pi/7$ **23.** $1688\pi/15$ **25.** $\pi/8$

27. $(\sqrt{3} - 1)\pi a^3/3$ **29.** (a) 10π (b) $(0, 0, 2.1)$

31. (a) $\left(0, 0, \frac{7}{12}\right)$ (b) $11K\pi/960$

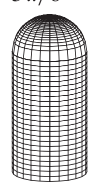
33. (a) $\left(0, 0, \frac{3}{8}a\right)$ (b) $4K\pi a^5/15$ (K is the density)

35. $\frac{1}{3}\pi(2 - \sqrt{2})$, $\left(0, 0, 3/\left[8(2 - \sqrt{2})\right]\right)$

37. (a) $\pi K a^4 h/2$ (K is the density) (b) $\pi K a^2 h(3a^2 + 4h^2)/12$

39. $5\pi/6$ **41.** $(4\sqrt{2} - 5)/15$ **43.** $4096\pi/21$

45.



47. $136\pi/99$

EXERCISES 15.9 ■ PAGE 1100

1. -6 **3.** s **5.** $2uvw$

7. The parallelogram with vertices $(0, 0)$, $(6, 3)$, $(12, 1)$, $(6, -2)$

9. The region bounded by the line $y = 1$, the y-axis, and $y = \sqrt{x}$

11. $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$ is one possible transformation, where $S = \{(u, v) \mid -1 \leqslant u \leqslant 1, 1 \leqslant v \leqslant 3\}$

13. $x = u\cos v$, $y = u\sin v$ is one possible transformation, where $S = \left\{(u, v) \mid 1 \leqslant u \leqslant \sqrt{2}, 0 \leqslant v \leqslant \pi/2\right\}$

15. -3 **17.** 6π **19.** $2\ln 3$

21. (a) $\frac{4}{3}\pi abc$ (b) $1.083 \times 10^{12}\ \text{km}^3$ (c) $\frac{4}{15}\pi(a^2 + b^2)abck$

23. $\frac{8}{5}\ln 8$ **25.** $\frac{3}{2}\sin 1$ **27.** $e - e^{-1}$

CHAPTER 15 REVIEW ■ PAGE 1101

True-False Quiz

1. True **3.** True **5.** True **7.** True **9.** False

Exercises

1. ≈ 64.0 **3.** $4e^2 - 4e + 3$ **5.** $\frac{1}{2}\sin 1$ **7.** $\frac{2}{3}$

9. $\int_0^{\pi} \int_2^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

11. $(\sqrt{3}, 3, 2)$, $(4, \pi/3, \pi/3)$

13. $(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$, $(4, \pi/4, 4\sqrt{3})$

15. (a) $r^2 + z^2 = 4$, $\rho = 2$ (b) $r = 2$, $\rho\sin\phi = 2$

17. The region inside the loop of the four-leaved rose $r = \sin 2\theta$ in the first quadrant

19. $\frac{1}{2}\sin 1$ **21.** $\frac{1}{2}e^6 - \frac{7}{2}$ **23.** $\frac{1}{4}\ln 2$ **25.** 8

27. $81\pi/5$ **29.** $\frac{81}{2}$ **31.** $\pi/96$ **33.** $\frac{64}{15}$

35. 176 **37.** $\frac{2}{3}$ **39.** $2ma^3/9$

41. (a) $\frac{1}{4}$ (b) $\left(\frac{1}{3}, \frac{8}{15}\right)$

(c) $I_x = \frac{1}{12}$, $I_y = \frac{1}{24}$; $\bar{\bar{y}} = 1/\sqrt{3}$, $\bar{\bar{x}} = 1/\sqrt{6}$

43. (a) $(0, 0, h/4)$ (b) $\pi a^5 h/15$

45. $\ln(\sqrt{2} + \sqrt{3}) + \sqrt{2}/3$ **47.** $\frac{486}{5}$ **49.** 0.0512

51. (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{45}$

53. $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$ **55.** $-\ln 2$ **57.** 0

PROBLEMS PLUS ■ PAGE 1105

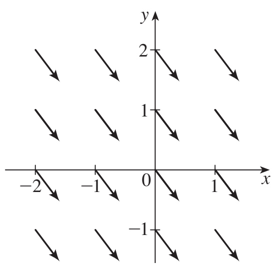
1. 30 **3.** $\frac{1}{2}\sin 1$ **7.** (b) 0.90

13. $abc\pi\left(\dfrac{2}{3} - \dfrac{8}{9\sqrt{3}}\right)$

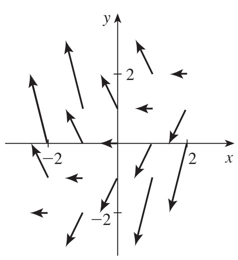
CHAPTER 16

EXERCISES 16.1 ■ PAGE 1113

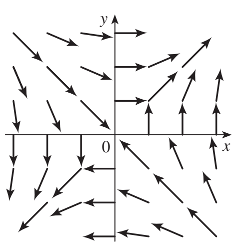
1.



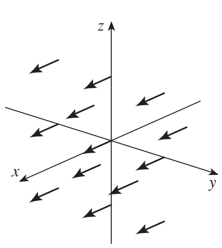
3.



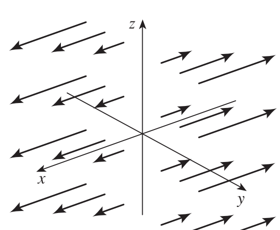
5.



7.

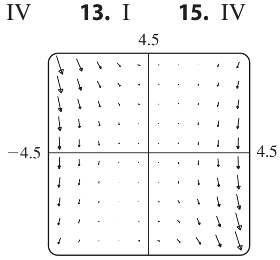


9.



11. IV **13.** I **15.** IV **17.** III

19.

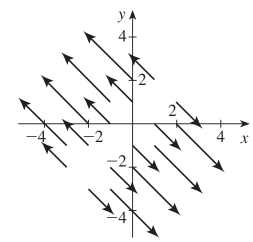


The line $y = 2x$

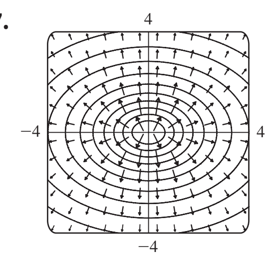
21. $\nabla f(x, y) = y^2 \cos(xy)\,\mathbf{i} + [xy\cos(xy) + \sin(xy)]\,\mathbf{j}$

23. $\nabla f(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$

25. $\nabla f(x, y) = (x - y)\,\mathbf{i} + (y - x)\,\mathbf{j}$

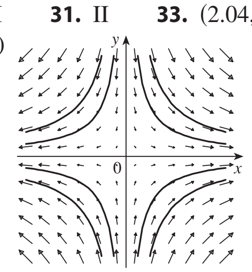


27.



29. III **31.** II **33.** (2.04, 1.03)

35. (a)



$y = C/x$

(b) $y = 1/x,\ x > 0$

EXERCISES 16.2 ■ PAGE 1124

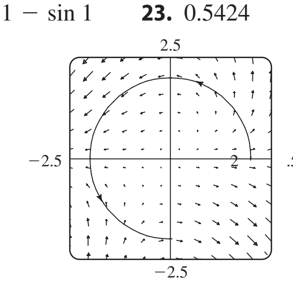
1. $\frac{4}{3}(10^{3/2} - 1)$ **3.** 1638.4 **5.** $\frac{1}{3}\pi^6 + 2\pi$ **7.** $\frac{5}{2}$

9. $\sqrt{2}/3$ **11.** $\frac{1}{12}\sqrt{14}(e^6 - 1)$ **13.** $\frac{2}{5}(e - 1)$ **15.** $\frac{35}{3}$

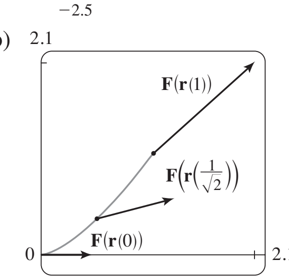
17. (a) Positive (b) Negative **19.** $\frac{1}{20}$

21. $\frac{6}{5} - \cos 1 - \sin 1$ **23.** 0.5424 **25.** 94.8231

27. $3\pi + \frac{2}{3}$



29. (a) $\frac{11}{8} - 1/e$ (b)



31. $\frac{172{,}704}{5{,}632{,}705}\sqrt{2}(1 - e^{-14\pi})$ **33.** $2\pi k$, $(4/\pi, 0)$

35. (a) $\bar{x} = (1/m)\int_C x\rho(x, y, z)\, ds$,
$\bar{y} = (1/m)\int_C y\rho(x, y, z)\, ds$,
$\bar{z} = (1/m)\int_C z\rho(x, y, z)\, ds$, where $m = \int_C \rho(x, y, z)\, ds$
(b) $(0, 0, 3\pi)$

37. $I_x = k\left(\frac{1}{2}\pi - \frac{4}{3}\right)$, $I_y = k\left(\frac{1}{2}\pi - \frac{2}{3}\right)$ **39.** $2\pi^2$ **41.** $\frac{7}{3}$

43. (a) $2ma\,\mathbf{i} + 6mbt\,\mathbf{j}$, $0 \leqslant t \leqslant 1$ (b) $2ma^2 + \frac{9}{2}mb^2$

45. $\approx 1.67 \times 10^4$ ft-lb **47.** (b) Yes **51.** ≈ 22 J

EXERCISES 16.3 ■ PAGE 1134

1. 40 **3.** Not conservative

5. $f(x, y) = ye^{xy} + K$ **7.** $f(x, y) = ye^x + x \sin y + K$

9. $f(x, y) = y^2 \sin x + x \cos y + K$

11. (b) 16 **13.** (a) $f(x, y) = \frac{1}{3}x^3y^3$ (b) -9

15. (a) $f(x, y, z) = xyz + z^2$ (b) 77

17. (a) $f(x, y, z) = ye^{xz}$ (b) 4 **19.** $4/e$

21. It doesn't matter which curve is chosen.

23. $\frac{31}{4}$ **25.** No **27.** Conservative

31. (a) Yes (b) Yes (c) Yes

33. (a) No (b) Yes (c) Yes

EXERCISES 16.4 ■ PAGE 1141

1. 120 **3.** $\frac{2}{3}$ **5.** $4(e^3 - 1)$ **7.** $\frac{1}{3}$

9. -24π **11.** $-\frac{16}{3}$ **13.** 4π

15. $\frac{1}{15}\pi^4 - \frac{4144}{1125}\pi^2 + \frac{7{,}578{,}368}{253{,}125} \approx 0.0779$

17. $-\frac{1}{12}$ **19.** 3π **21.** (c) $\frac{9}{2}$

23. $(4a/3\pi, 4a/3\pi)$ if the region is the portion of the disk $x^2 + y^2 = a^2$ in the first quadrant

27. 0

EXERCISES 16.5 ■ PAGE 1149

1. (a) $\mathbf{0}$ (b) $y^2z^2 + x^2z^2 + x^2y^2$

3. (a) $ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$ (b) $y(e^z + e^x)$

5. (a) $-\dfrac{\sqrt{z}}{(1 + y)^2}\,\mathbf{i} - \dfrac{\sqrt{x}}{(1 + z)^2}\,\mathbf{j} - \dfrac{\sqrt{y}}{(1 + x)^2}\,\mathbf{k}$

(b) $\dfrac{1}{2\sqrt{x}\,(1 + z)} + \dfrac{1}{2\sqrt{y}\,(1 + x)} + \dfrac{1}{2\sqrt{z}\,(1 + y)}$

7. (a) $\langle -e^y \cos z, -e^z \cos x, -e^x \cos y \rangle$
(b) $e^x \sin y + e^y \sin z + e^z \sin x$

9. (a) Negative (b) curl $\mathbf{F} = \mathbf{0}$

11. (a) Zero (b) curl $\mathbf{F}$ points in the negative z-direction.

13. $f(x, y, z) = xy^2z^3 + K$ **15.** Not conservative

17. $f(x, y, z) = xe^{yz} + K$ **19.** No

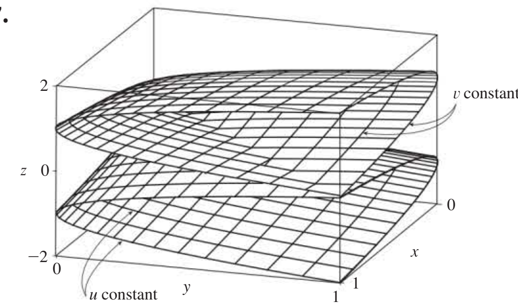
EXERCISES 16.6 ■ PAGE 1160

1. P: yes; Q: no

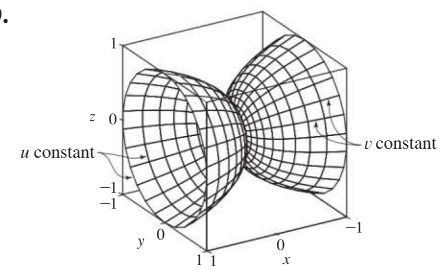
3. Plane through $(0, 3, 1)$ containing vectors $\langle 1, 0, 4\rangle$, $\langle 1, -1, 5\rangle$

5. Circular cone with axis the z-axis

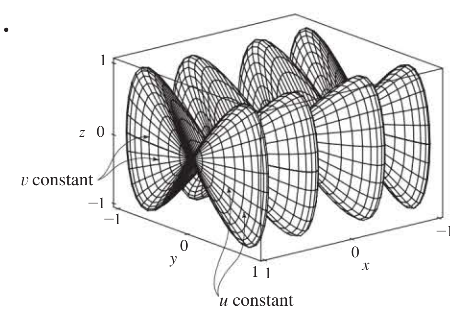
7.



9.



11.



13. IV **15.** I **17.** III

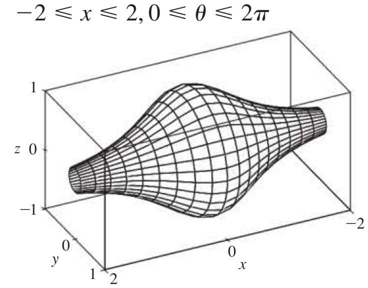
19. $x = u$, $y = v - u$, $z = -v$

21. $y = y$, $z = z$, $x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$

23. $x = 2 \sin \phi \cos \theta$, $y = 2 \sin \phi \sin \theta$,
$z = 2 \cos \phi$, $0 \leqslant \phi \leqslant \pi/4$, $0 \leqslant \theta \leqslant 2\pi$
$\left[\text{or } x = x, y = y, z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 \leqslant 2\right]$

25. $x = 6 \sin \phi \cos \theta$, $y = 6 \sin \phi \sin \theta$, $z = 6 \cos \phi$,
$\pi/6 \leqslant \phi \leqslant \pi/2$, $0 \leqslant \theta \leqslant 2\pi$

29. $x = x$, $y = \dfrac{1}{1 + x^2} \cos \theta$, $y = \dfrac{1}{1 + x^2} \sin \theta$,
$-2 \leqslant x \leqslant 2$, $0 \leqslant \theta \leqslant 2\pi$

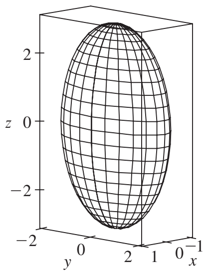


31. (a) Direction reverses (b) Number of coils doubles

33. $3x - y + 3z = 3$ **35.** $\dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y + z = \dfrac{\pi}{3}$

37. $-x + 2z = 1$ **39.** $3\sqrt{14}$ **41.** $\sqrt{14}\,\pi$

43. $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1)$ **45.** $(2\pi/3)(2\sqrt{2} - 1)$
47. $(\pi/6)(65^{3/2} - 1)$ **49.** 4 **51.** $\pi R^2 \leqslant A(S) \leqslant \sqrt{3}\,\pi R^2$
53. 3.5618 **55.** (a) ≈ 24.2055 (b) 24.2476
57. $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left[(11\sqrt{5} + 3\sqrt{70})/(3\sqrt{5} + \sqrt{70})\right]$
59. (b)



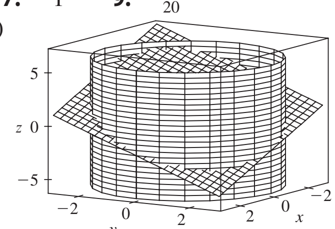
(c) $\int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\;du\,dv$
61. 4π **63.** $2a^2(\pi - 2)$

EXERCISES 16.7 ■ PAGE 1172

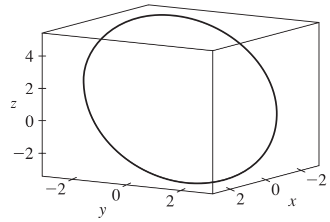
1. ≈ -6.93 **3.** 900π **5.** $11\sqrt{14}$ **7.** $\frac{2}{3}(2\sqrt{2} - 1)$
9. $171\sqrt{14}$ **11.** $\sqrt{21}/3$ **13.** $(\pi/120)(25\sqrt{5} + 1)$
15. $\frac{7}{4}\sqrt{21} - \frac{17}{12}\sqrt{17}$ **17.** 16π **19.** 0 **21.** 4
23. $\frac{713}{180}$ **25.** $\frac{8}{3}\pi$ **27.** 0 **29.** 48 **31.** $2\pi + \frac{8}{3}$
33. 4.5822 **35.** 3.4895
37. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [P(\partial h/\partial x) - Q + R(\partial h/\partial z)]\,dA$, where D = projection of S onto xz-plane
39. $(0, 0, a/2)$
41. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$ (b) $4329\sqrt{2}\,\pi/5$
43. 0 kg/s **45.** $\frac{8}{3}\pi a^3\varepsilon_0$ **47.** 1248π

EXERCISES 16.8 ■ PAGE 1179

3. 16π **5.** 0 **7.** -1 **9.** $-\frac{17}{20}$
11. (a) $81\pi/2$ (b)



(c) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3(\cos t + \sin t)$, $0 \leqslant t \leqslant 2\pi$



13. -32π **15.** $-\pi$ **17.** 3

EXERCISES 16.9 ■ PAGE 1185

1. $\frac{9}{2}$ **3.** $256\pi/3$ **5.** $\frac{9}{2}$ **7.** $9\pi/2$ **9.** 0
11. π **13.** 2π **15.** $341\sqrt{2}/60 + \frac{81}{20}\arcsin(\sqrt{3}/3)$
17. $13\pi/20$ **19.** Negative at P_1, positive at P_2
21. div $\mathbf{F} > 0$ in quadrants I, II; div $\mathbf{F} < 0$ in quadrants III, IV

CHAPTER 16 REVIEW ■ PAGE 1188

True-False Quiz

1. False **3.** True **5.** False **7.** False
9. True **11.** True **13.** False

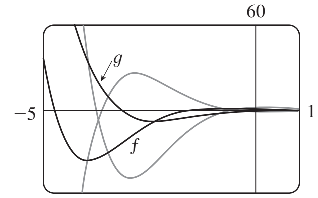
Exercises

1. (a) Negative (b) Positive **3.** $6\sqrt{10}$ **5.** $\frac{4}{15}$ **7.** $\frac{110}{3}$
9. $\frac{11}{12} - 4/e$ **11.** $f(x, y) = e^y + xe^{xy} + K$ **13.** 0
15. 0 **17.** -8π **25.** $\frac{1}{6}(27 - 5\sqrt{5})$
27. $(\pi/60)(391\sqrt{17} + 1)$ **29.** $-64\pi/3$ **31.** 0
33. $-\frac{1}{2}$ **35.** 4π **37.** -4 **39.** 21

CHAPTER 17

EXERCISES 17.1 ■ PAGE 1200

1. $y = c_1e^{3x} + c_2e^{-2x}$ **3.** $y = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)$
5. $y = c_1e^{-x/2} + c_2xe^{-x/2}$ **7.** $y = c_1 + c_2e^{4x/3}$
9. $y = e^{2x}(c_1\cos 3x + c_2\sin 3x)$
11. $y = c_1e^{(\sqrt{3}-1)t/2} + c_2e^{-(\sqrt{3}+1)t/2}$
13. $V = e^{-2t/3}\left[c_1\cos\left(\frac{\sqrt{5}}{3}t\right) + c_2\sin\left(\frac{\sqrt{5}}{3}t\right)\right]$
15. $f(x) = e^{-x}\cos x$, $g(x) = e^{-x}\sin x$. All solution curves approach 0 as $x \to \infty$ and oscillate with amplitudes that become arbitrarily large as $x \to -\infty$.

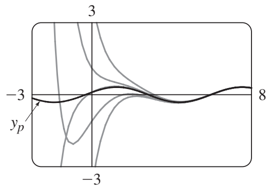


17. $y = \cos(\sqrt{3}x) + \sqrt{3}\sin(\sqrt{3}x)$ **19.** $y = e^{-2x/3} + \frac{2}{3}xe^{-2x/3}$
21. $y = e^{3x}(2\cos x - 3\sin x)$
23. $y = \frac{1}{7}e^{4x-4} - \frac{1}{7}e^{3-3x}$ **25.** $y = -3\cos 4x + 2\sin 4x$
27. $y = 2e^{-2x} - 2xe^{-2x}$ **29.** $y = \dfrac{e-2}{e-1} + \dfrac{e^x}{e-1}$
31. No solution
33. (b) $\lambda = n^2\pi^2/L^2$, n a positive integer; $y = C\sin(n\pi x/L)$
35. (a) $b - a \neq n\pi$, n any integer
(b) $b - a = n\pi$ and $\dfrac{c}{d} \neq e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b = 0$, then $\dfrac{c}{d} \neq e^{a-b}\dfrac{\sin a}{\sin b}$
(c) $b - a = n\pi$ and $\dfrac{c}{d} = e^{a-b}\dfrac{\cos a}{\cos b}$ unless $\cos b = 0$, then $\dfrac{c}{d} = e^{a-b}\dfrac{\sin a}{\sin b}$

EXERCISES 17.2 ■ PAGE 1207

1. $y = c_1e^{2x} + c_2e^{-4x} + \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}$
3. $y = c_1\cos(\frac{1}{3}x) + c_2\sin(\frac{1}{3}x) + \frac{1}{37}e^{2x}$
5. $y = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$
7. $y = e^x(\frac{9}{10}\cos 2x - \frac{1}{20}\sin 2x) + \frac{1}{10}\cos x + \frac{1}{5}\sin x$
9. $y = e^x(\frac{1}{2}x^2 - x + 2)$

11.



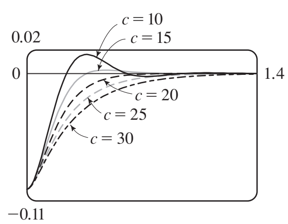
The solutions are all asymptotic to $y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x$ as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.

13. $y_p = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$
15. $y_p = Axe^x + B\cos x + C\sin x$
17. $y_p = xe^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$
19. $y = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right) - \frac{1}{3}\cos x$
21. $y = c_1e^x + c_2xe^x + e^{2x}$
23. $y = c_1\sin x + c_2\cos x + \sin x\ln(\sec x + \tan x) - 1$
25. $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$
27. $y = e^x\left[c_1 + c_2x - \frac{1}{2}\ln(1 + x^2) + x\tan^{-1}x\right]$

EXERCISES 17.3 ■ PAGE 1215

1. $x = 0.35\cos\left(2\sqrt{5}\,t\right)$ **3.** $x = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$ **5.** $\frac{49}{12}$ kg

7.



13. $Q(t) = (-e^{-10t}/250)(6\cos 20t + 3\sin 20t) + \frac{3}{125}$, $I(t) = \frac{3}{5}e^{-10t}\sin 20t$
15. $Q(t) = e^{-10t}\left[\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right] - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$

EXERCISES 17.4 ■ PAGE 1220

1. $c_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = c_0e^x$ **3.** $c_0\sum_{n=0}^{\infty}\frac{x^{3n}}{3^n n!} = c_0e^{x^3/3}$
5. $c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n} + c_1\sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}$
7. $c_0 + c_1\sum_{n=1}^{\infty}\frac{x^n}{n} = c_0 - c_1\ln(1 - x)$ for $|x| < 1$
9. $\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!} = e^{x^2/2}$
11. $x + \sum_{n=1}^{\infty}\frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}x^{3n+1}$

CHAPTER 17 REVIEW ■ PAGE 1221

True-False Quiz

1. True **3.** True

Exercises

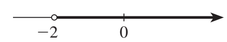
1. $y = c_1e^{x/2} + c_2e^{-x/2}$
3. $y = c_1\cos\left(\sqrt{3}x\right) + c_2\sin\left(\sqrt{3}x\right)$
5. $y = e^{2x}(c_1\cos x + c_2\sin x + 1)$
7. $y = c_1e^x + c_2xe^x - \frac{1}{2}\cos x - \frac{1}{2}(x + 1)\sin x$
9. $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$
11. $y = 5 - 2e^{-6(x-1)}$ **13.** $y = (e^{4x} - e^x)/3$
15. No solution **17.** $\sum_{n=0}^{\infty}\frac{(-2)^n n!}{(2n+1)!}x^{2n+1}$
19. $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$
21. (c) $2\pi/k \approx 85$ min (d) $\approx$17,600 mi/h

APPENDIXES

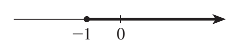
EXERCISES A ■ PAGE A9

1. 18 **3.** π **5.** $5 - \sqrt{5}$ **7.** $2 - x$
9. $|x + 1| = \begin{cases} x + 1 & \text{for } x \geqslant -1 \\ -x - 1 & \text{for } x < -1 \end{cases}$ **11.** $x^2 + 1$

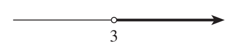
13. $(-2, \infty)$



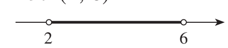
15. $[-1, \infty)$



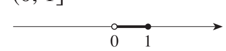
17. $(3, \infty)$



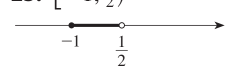
19. $(2, 6)$



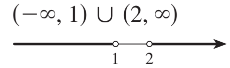
21. $(0, 1]$



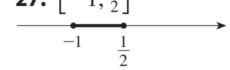
23. $\left[-1, \frac{1}{2}\right)$



25. $(-\infty, 1) \cup (2, \infty)$



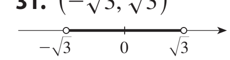
27. $\left[-1, \frac{1}{2}\right]$



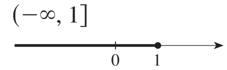
29. $(-\infty, \infty)$



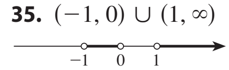
31. $\left(-\sqrt{3}, \sqrt{3}\right)$



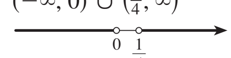
33. $(-\infty, 1]$



35. $(-1, 0) \cup (1, \infty)$



37. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$

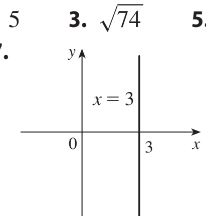


39. $10 \leqslant C \leqslant 35$ **41.** (a) $T = 20 - 10h$, $0 \leqslant h \leqslant 12$
(b) $-30°\text{C} \leqslant T \leqslant 20°\text{C}$ **43.** $\pm\frac{3}{2}$ **45.** $2, -\frac{4}{3}$
47. $(-3, 3)$ **49.** $(3, 5)$ **51.** $(-\infty, -7] \cup [-3, \infty)$
53. $[1.3, 1.7]$ **55.** $[-4, -1] \cup [1, 4]$
57. $x \geqslant (a + b)c/(ab)$ **59.** $x > (c - b)/a$

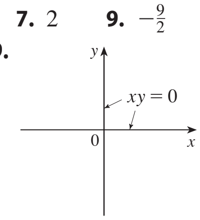
EXERCISES B ■ PAGE A15

1. 5 **3.** $\sqrt{74}$ **5.** $2\sqrt{37}$ **7.** 2 **9.** $-\frac{9}{2}$

17.



19.

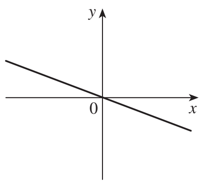


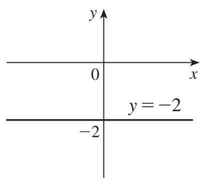
21. $y = 6x - 15$ **23.** $2x - 3y + 19 = 0$

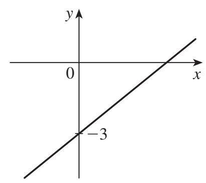
25. $5x + y = 11$ **27.** $y = 3x - 2$ **29.** $y = 3x - 3$

31. $y = 5$ **33.** $x + 2y + 11 = 0$ **35.** $5x - 2y + 1 = 0$

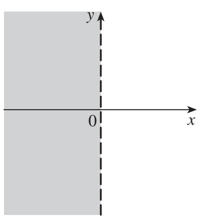
37. $m = -\frac{1}{3}$, $b = 0$ **39.** $m = 0$, $b = -2$ **41.** $m = \frac{3}{4}$, $b = -3$



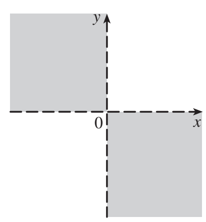




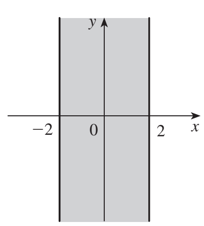
43.



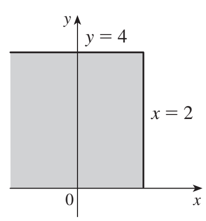
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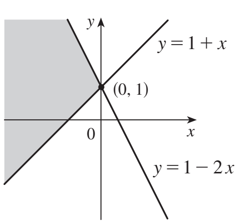
47.



49.



51.



53. $(0, -4)$ **55.** (a) $(4, 9)$ (b) $(3.5, -3)$ **57.** $(1, -2)$

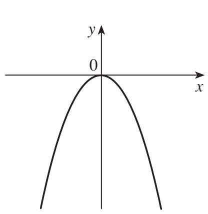
59. $y = x - 3$ **61.** (b) $4x - 3y - 24 = 0$

EXERCISES C ■ PAGE A23

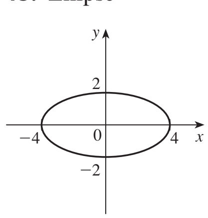
1. $(x - 3)^2 + (y + 1)^2 = 25$ **3.** $x^2 + y^2 = 65$

5. $(2, -5)$, 4 **7.** $\left(-\frac{1}{2}, 0\right)$, $\frac{1}{2}$ **9.** $\left(\frac{1}{4}, -\frac{1}{4}\right)$, $\sqrt{10}/4$

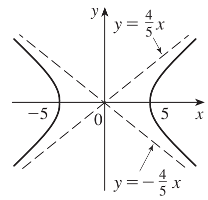
11. Parabola



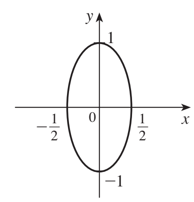
13. Ellipse



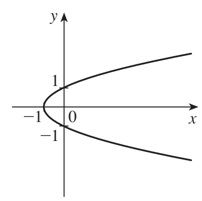
15. Hyperbola



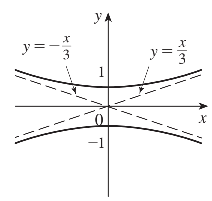
17. Ellipse



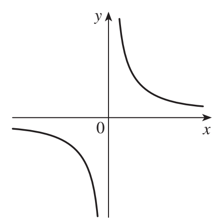
19. Parabola



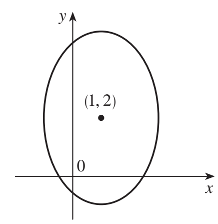
21. Hyperbola



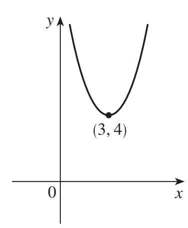
23. Hyperbola



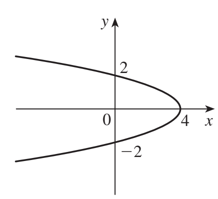
25. Ellipse



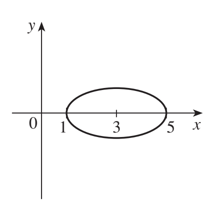
27. Parabola



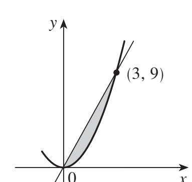
29. Parabola



31. Ellipse

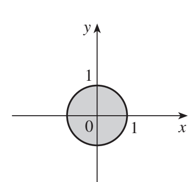


33.

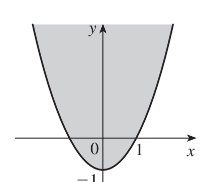


35. $y = x^2 - 2x$

37.



39.

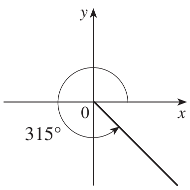


EXERCISES D ■ PAGE A32

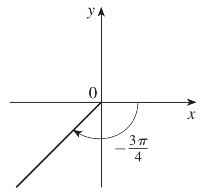
1. $7\pi/6$ **3.** $\pi/20$ **5.** 5π **7.** $720°$ **9.** $75°$

11. $-67.5°$ **13.** 3π cm **15.** $\frac{2}{3}$ rad $= (120/\pi)°$

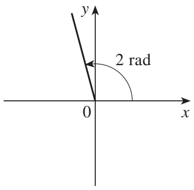
17.



19.



21.



23. $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$

25. $\sin(9\pi/2) = 1$, $\cos(9\pi/2) = 0$, $\csc(9\pi/2) = 1$, $\cot(9\pi/2) = 0$, $\tan(9\pi/2)$ and $\sec(9\pi/2)$ undefined

27. $\sin(5\pi/6) = \frac{1}{2}$, $\cos(5\pi/6) = -\sqrt{3}/2$, $\tan(5\pi/6) = -1/\sqrt{3}$, $\csc(5\pi/6) = 2$, $\sec(5\pi/6) = -2/\sqrt{3}$, $\cot(5\pi/6) = -\sqrt{3}$

29. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

31. $\sin\phi = \sqrt{5}/3$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\sqrt{5}/2$, $\csc\phi = 3/\sqrt{5}$, $\cot\phi = -2/\sqrt{5}$

33. $\sin\beta = -1/\sqrt{10}$, $\cos\beta = -3/\sqrt{10}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, $\sec\beta = -\sqrt{10}/3$

35. 5.73576 cm **37.** 24.62147 cm **59.** $\frac{1}{15}(4 + 6\sqrt{2})$

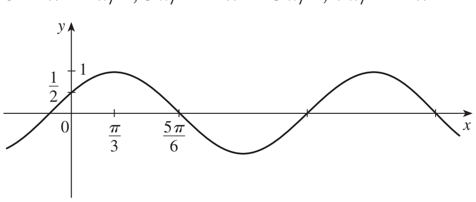
61. $\frac{1}{15}(3 + 8\sqrt{2})$ **63.** $\frac{24}{25}$ **65.** $\pi/3, 5\pi/3$

67. $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ **69.** $\pi/6, \pi/2, 5\pi/6, 3\pi/2$

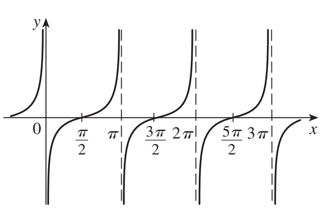
71. $0, \pi, 2\pi$ **73.** $0 \leqslant x \leqslant \pi/6$ and $5\pi/6 \leqslant x \leqslant 2\pi$

75. $0 \leqslant x < \pi/4$, $3\pi/4 < x < 5\pi/4$, $7\pi/4 < x \leqslant 2\pi$

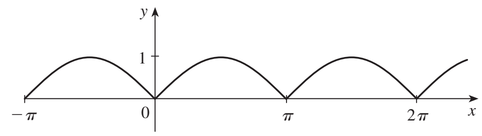
77.



79.



81.



89. 14.34457 cm^2

EXERCISES E ■ PAGE A38

1. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$ **3.** $3^4 + 3^5 + 3^6$

5. $-1 + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$ **7.** $1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

9. $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1}$ **11.** $\sum_{i=1}^{10} i$

13. $\sum_{i=1}^{19} \frac{i}{i+1}$ **15.** $\sum_{i=1}^{n} 2i$ **17.** $\sum_{i=0}^{5} 2^i$ **19.** $\sum_{i=1}^{n} x^i$

21. 80 **23.** 3276 **25.** 0 **27.** 61 **29.** $n(n+1)$

31. $n(n^2 + 6n + 17)/3$ **33.** $n(n^2 + 6n + 11)/3$

35. $n(n^3 + 2n^2 - n - 10)/4$

41. (a) n^4 (b) $5^{100} - 1$ (c) $\frac{97}{300}$ (d) $a_n - a_0$

43. $\frac{1}{3}$ **45.** 14 **49.** $2^{n+1} + n^2 + n - 2$

EXERCISES G ■ PAGE A55

1. $8 - 4i$ **3.** $13 + 18i$ **5.** $12 - 7i$ **7.** $\frac{11}{13} + \frac{10}{13}i$

9. $\frac{1}{2} - \frac{1}{2}i$ **11.** $-i$ **13.** $5i$ **15.** $12 + 5i$, 13

17. $4i$, 4 **19.** $\pm\frac{3}{2}i$ **21.** $-1 \pm 2i$

23. $-\frac{1}{2} \pm (\sqrt{7}/2)i$ **25.** $3\sqrt{2}\,[\cos(3\pi/4) + i\sin(3\pi/4)]$

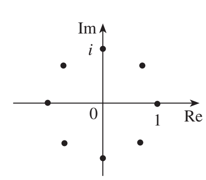
27. $5\{\cos[\tan^{-1}(\frac{4}{3})] + i\sin[\tan^{-1}(\frac{4}{3})]\}$

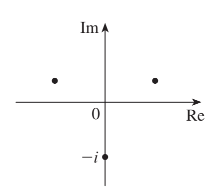
29. $4[\cos(\pi/2) + i\sin(\pi/2)]$, $\cos(-\pi/6) + i\sin(-\pi/6)$, $\frac{1}{2}[\cos(-\pi/6) + i\sin(-\pi/6)]$

31. $4\sqrt{2}\,[\cos(7\pi/12) + i\sin(7\pi/12)]$, $(2\sqrt{2})[\cos(13\pi/12) + i\sin(13\pi/12)]$, $\frac{1}{4}[\cos(\pi/6) + i\sin(\pi/6)]$

33. -1024 **35.** $-512\sqrt{3} + 512i$

37. $\pm1, \pm i, (1/\sqrt{2})(\pm1 \pm i)$ **39.** $\pm(\sqrt{3}/2) + \frac{1}{2}i, -i$





41. i **43.** $\frac{1}{2} + (\sqrt{3}/2)i$ **45.** $-e^2$

47. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

Index

RP denotes Reference Page numbers.
Numbers followed by asterisks denote pages in Sections 6.2*, 6.3*, and 6.4*

Cut here and keep for reference

REFERENCE page 1

ALGEBRA

Arithmetic Operations

$a(b + c) = ab + ac$ $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

$\dfrac{a + c}{b} = \dfrac{a}{b} + \dfrac{c}{b}$ $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

Exponents and Radicals

$x^m x^n = x^{m+n}$ $\dfrac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$ $x^{-n} = \dfrac{1}{x^n}$

$(xy)^n = x^n y^n$ $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

$x^{1/n} = \sqrt[n]{x}$ $x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$ $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

Factoring Special Polynomials

$x^2 - y^2 = (x + y)(x - y)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\dbinom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

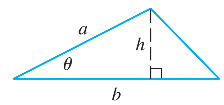
Geometric Formulas

Formulas for area A, circumference C, and volume V:

Triangle

$A = \frac{1}{2}bh$

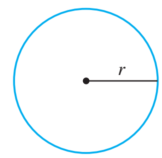
$= \frac{1}{2}ab\sin\theta$



Circle

$A = \pi r^2$

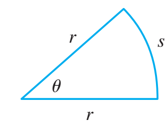
$C = 2\pi r$



Sector of Circle

$A = \frac{1}{2}r^2\theta$

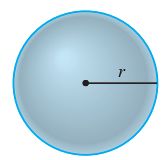
$s = r\theta$ (θ in radians)



Sphere

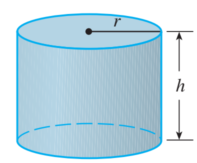
$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$



Cylinder

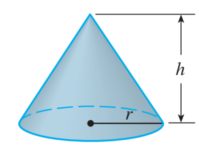
$V = \pi r^2 h$



Cone

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r\sqrt{r^2 + h^2}$



Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

TRIGONOMETRY

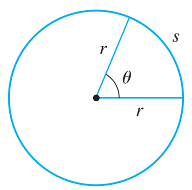
Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad$ 1 rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

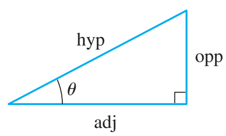


Right Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\qquad$ $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\qquad$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\qquad$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

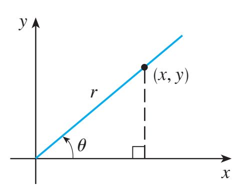


Trigonometric Functions

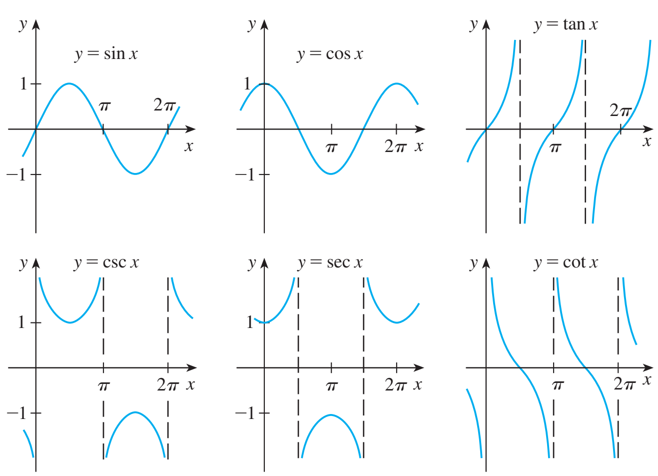
$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

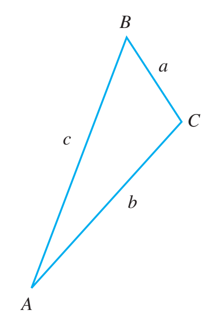
$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$ $\qquad$ $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$

Double-Angle Formulas

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

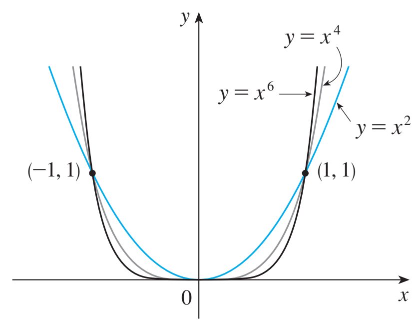
Half-Angle Formulas

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\qquad$ $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

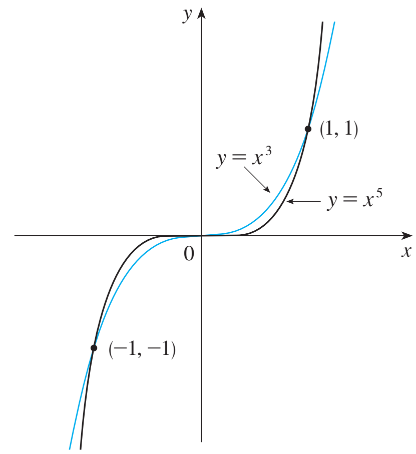
SPECIAL FUNCTIONS

Power Functions $f(x) = x^a$

(i) $f(x) = x^n$, n a positive integer

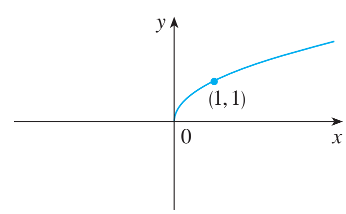


n even

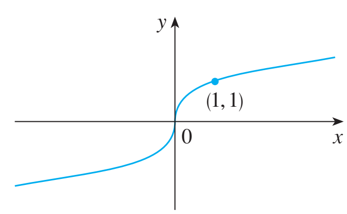


n odd

(ii) $f(x) = x^{1/n} = \sqrt[n]{x}$, n a positive integer

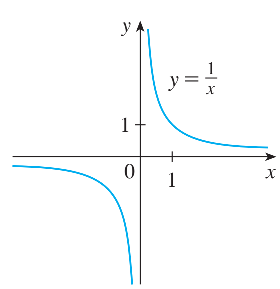


$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

(iii) $f(x) = x^{-1} = \dfrac{1}{x}$

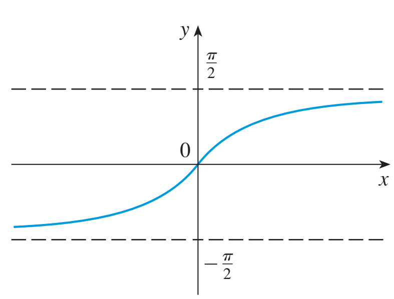


Inverse Trigonometric Functions

$$\arcsin x = \sin^{-1}x = y \quad \Longleftrightarrow \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$$\arccos x = \cos^{-1}x = y \quad \Longleftrightarrow \quad \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

$$\arctan x = \tan^{-1}x = y \quad \Longleftrightarrow \quad \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$y = \tan^{-1}x = \arctan x$

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

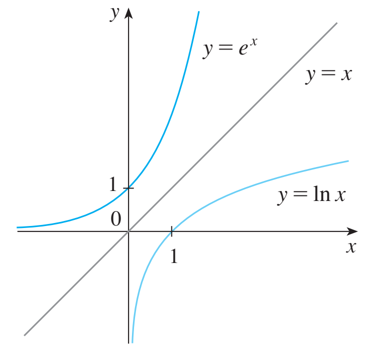
SPECIAL FUNCTIONS

Exponential and Logarithmic Functions

$\log_b x = y \iff b^y = x$

$\ln x = \log_e x, \quad \text{where} \quad \ln e = 1$

$\ln x = y \iff e^y = x$



Cancellation Equations

$\log_b(b^x) = x \qquad b^{\log_b x} = x$

$\ln(e^x) = x \qquad e^{\ln x} = x$

Laws of Logarithms

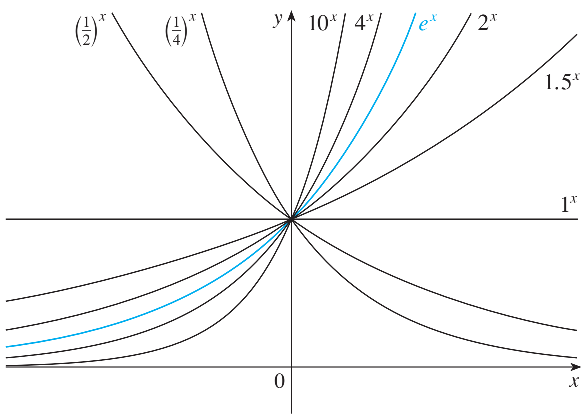
1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

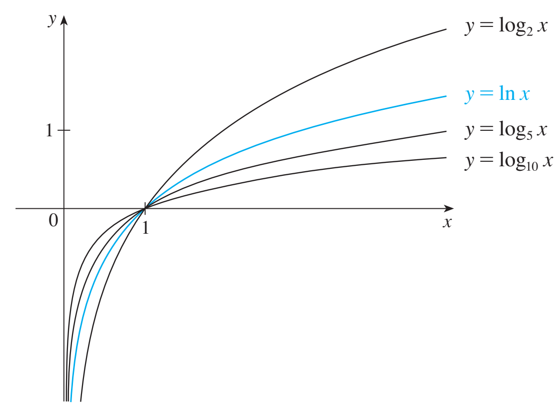
3. $\log_b(x^r) = r \log_b x$

$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$



Exponential functions



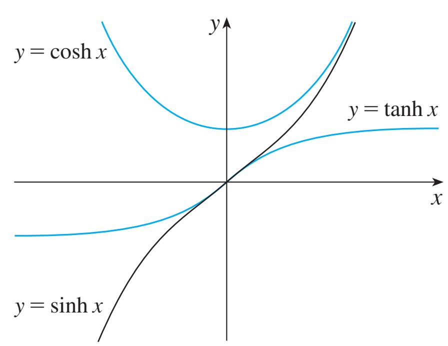
Logarithmic functions

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x} \qquad \coth x = \dfrac{\cosh x}{\sinh x}$



Inverse Hyperbolic Functions

$y = \sinh^{-1}x \iff \sinh y = x$

$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$

$y = \tanh^{-1}x \iff \tanh y = x$

$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$

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DIFFERENTIATION RULES

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}[cf(x)] = cf'(x)$

3. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

4. $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

5. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

7. $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

8. $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\frac{d}{dx}(e^x) = e^x$

10. $\frac{d}{dx}(b^x) = b^x \ln b$

11. $\frac{d}{dx} \ln|x| = \frac{1}{x}$

12. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Trigonometric Functions

13. $\frac{d}{dx}(\sin x) = \cos x$

14. $\frac{d}{dx}(\cos x) = -\sin x$

15. $\frac{d}{dx}(\tan x) = \sec^2 x$

16. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

17. $\frac{d}{dx}(\sec x) = \sec x \tan x$

18. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

19. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$

20. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}$

21. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$

22. $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$

23. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$

24. $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\cosh x) = \sinh x$

27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

28. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

29. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}}$

32. $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}}$

33. $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2}$

34. $\frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$

35. $\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$

36. $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$

TABLE OF INTEGRALS

Basic Forms

1. $\int u\,dv = uv - \int v\,du$

2. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \dfrac{du}{u} = \ln|u| + C$

4. $\int e^u\,du = e^u + C$

5. $\int b^u\,du = \dfrac{b^u}{\ln b} + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\dfrac{u}{a} + C, \quad a > 0$

17. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$

18. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} + C$

19. $\int \dfrac{du}{a^2 - u^2} = \dfrac{1}{2a}\ln\left|\dfrac{u+a}{u-a}\right| + C$

20. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right| + C$

Forms Involving $\sqrt{a^2 + u^2}$, $a > 0$

21. $\int \sqrt{a^2 + u^2}\,du = \dfrac{u}{2}\sqrt{a^2 + u^2} + \dfrac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

22. $\int u^2\sqrt{a^2 + u^2}\,du = \dfrac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \dfrac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

23. $\int \dfrac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\dfrac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

24. $\int \dfrac{\sqrt{a^2 + u^2}}{u^2}\,du = -\dfrac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

25. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

26. $\int \dfrac{u^2\,du}{\sqrt{a^2 + u^2}} = \dfrac{u}{2}\sqrt{a^2 + u^2} - \dfrac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

27. $\int \dfrac{du}{u\sqrt{a^2 + u^2}} = -\dfrac{1}{a}\ln\left|\dfrac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

28. $\int \dfrac{du}{u^2\sqrt{a^2 + u^2}} = -\dfrac{\sqrt{a^2 + u^2}}{a^2 u} + C$

29. $\int \dfrac{du}{(a^2 + u^2)^{3/2}} = \dfrac{u}{a^2\sqrt{a^2 + u^2}} + C$

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TABLE OF INTEGRALS

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

30. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31. $\displaystyle\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34. $\displaystyle\int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36. $\displaystyle\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37. $\displaystyle\int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38. $\displaystyle\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

39. $\displaystyle\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40. $\displaystyle\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

42. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44. $\displaystyle\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

45. $\displaystyle\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

46. $\displaystyle\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

(continued)

TABLE OF INTEGRALS

Forms Involving $a + bu$

47. $\displaystyle\int \frac{u\,du}{a + bu} = \frac{1}{b^2}\left(a + bu - a \ln|a + bu|\right) + C$

48. $\displaystyle\int \frac{u^2\,du}{a + bu} = \frac{1}{2b^3}\left[(a + bu)^2 - 4a(a + bu) + 2a^2 \ln|a + bu|\right] + C$

49. $\displaystyle\int \frac{du}{u(a + bu)} = \frac{1}{a} \ln\left|\frac{u}{a + bu}\right| + C$

50. $\displaystyle\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln\left|\frac{a + bu}{u}\right| + C$

51. $\displaystyle\int \frac{u\,du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2} \ln|a + bu| + C$

52. $\displaystyle\int \frac{du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2} \ln\left|\frac{a + bu}{u}\right| + C$

53. $\displaystyle\int \frac{u^2\,du}{(a + bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C$

54. $\displaystyle\int u\sqrt{a + bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a + bu)^{3/2} + C$

55. $\displaystyle\int \frac{u\,du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + C$

56. $\displaystyle\int \frac{u^2\,du}{\sqrt{a + bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a + bu} + C$

57. $\displaystyle\int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}} \ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C, \quad \text{if } a > 0$

$$= \frac{2}{\sqrt{-a}} \tan^{-1}\sqrt{\frac{a + bu}{-a}} + C, \qquad \text{if } a < 0$$

58. $\displaystyle\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}}$

59. $\displaystyle\int \frac{\sqrt{a + bu}}{u^2}\,du = -\frac{\sqrt{a + bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a + bu}}$

60. $\displaystyle\int u^n\sqrt{a + bu}\,du = \frac{2}{b(2n + 3)}\left[u^n(a + bu)^{3/2} - na\int u^{n-1}\sqrt{a + bu}\,du\right]$

61. $\displaystyle\int \frac{u^n\,du}{\sqrt{a + bu}} = \frac{2u^n\sqrt{a + bu}}{b(2n + 1)} - \frac{2na}{b(2n + 1)}\int \frac{u^{n-1}\,du}{\sqrt{a + bu}}$

62. $\displaystyle\int \frac{du}{u^n\sqrt{a + bu}} = -\frac{\sqrt{a + bu}}{a(n - 1)u^{n-1}} - \frac{b(2n - 3)}{2a(n - 1)}\int \frac{du}{u^{n-1}\sqrt{a + bu}}$

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TABLE OF INTEGRALS

Trigonometric Forms

63. $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

64. $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

65. $\int \tan^2 u\, du = \tan u - u + C$

66. $\int \cot^2 u\, du = -\cot u - u + C$

67. $\int \sin^3 u\, du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

68. $\int \cos^3 u\, du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

69. $\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

70. $\int \cot^3 u\, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

71. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

72. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

73. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1}u \cos u + \frac{n-1}{n}\int \sin^{n-2}u\, du$

74. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1}u \sin u + \frac{n-1}{n}\int \cos^{n-2}u\, du$

75. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\, du$

76. $\int \cot^n u\, du = \frac{-1}{n-1}\cot^{n-1}u - \int \cot^{n-2}u\, du$

77. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2}u + \frac{n-2}{n-1}\int \sec^{n-2}u\, du$

78. $\int \csc^n u\, du = \frac{-1}{n-1}\cot u \csc^{n-2}u + \frac{n-2}{n-1}\int \csc^{n-2}u\, du$

79. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\int u \sin u\, du = \sin u - u\cos u + C$

83. $\int u \cos u\, du = \cos u + u\sin u + C$

84. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1}\cos u\, du$

85. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du$

86. $\int \sin^n u \cos^m u\, du = -\frac{\sin^{n-1}u \cos^{m+1}u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}u \cos^m u\, du$

$$= \frac{\sin^{n+1}u \cos^{m-1}u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2}u\, du$$

Inverse Trigonometric Forms

87. $\int \sin^{-1}u\, du = u\sin^{-1}u + \sqrt{1-u^2} + C$

88. $\int \cos^{-1}u\, du = u\cos^{-1}u - \sqrt{1-u^2} + C$

89. $\int \tan^{-1}u\, du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + C$

90. $\int u\sin^{-1}u\, du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\int u\cos^{-1}u\, du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\int u\tan^{-1}u\, du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

93. $\int u^n \sin^{-1}u\, du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

94. $\int u^n \cos^{-1}u\, du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int \frac{u^{n+1}\, du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

95. $\int u^n \tan^{-1}u\, du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int \frac{u^{n+1}\, du}{1+u^2}\right], \quad n \neq -1$

(continued)

TABLE OF INTEGRALS

Exponential and Logarithmic Forms

96. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

97. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

99. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

100. $\int \ln u\,du = u\ln u - u + C$

101. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n + 1)^2}[(n + 1)\ln u - 1] + C$

102. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

103. $\int \sinh u\,du = \cosh u + C$

104. $\int \cosh u\,du = \sinh u + C$

105. $\int \tanh u\,du = \ln\cosh u + C$

106. $\int \coth u\,du = \ln|\sinh u| + C$

107. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

108. $\int \operatorname{csch} u\,du = \ln\left|\tanh \tfrac{1}{2}u\right| + C$

109. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

110. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

111. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

113. $\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

114. $\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

115. $\int \frac{\sqrt{2au - u^2}}{u}\,du = \sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$

116. $\int \frac{\sqrt{2au - u^2}}{u^2}\,du = -\frac{2\sqrt{2au - u^2}}{u} - \cos^{-1}\left(\frac{a - u}{a}\right) + C$

117. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

118. $\int \frac{u\,du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a\cos^{-1}\left(\frac{a - u}{a}\right) + C$

119. $\int \frac{u^2\,du}{\sqrt{2au - u^2}} = -\frac{(u + 3a)}{2}\sqrt{2au - u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

120. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

Cut here and keep for reference

CHAPTER 1 CONCEPT CHECK ANSWERS

1. (a) What is a function? What are its domain and range?

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The domain is the set D and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) What is the graph of a function?

The graph of a function f consists of all points (x, y) such that $y = f(x)$ and x is in the domain of f.

(c) How can you tell whether a given curve is the graph of a function?

Use the Vertical Line Test: a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

A function can be represented verbally, numerically, visually, or algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between the area A and side length s of a square: $A = s^2$ (an explicit formula)

3. (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.

A function f is even if it satisfies $f(-x) = f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the y-axis, then f is even. Examples are $f(x) = x^2$, $f(x) = \cos x$, $f(x) = |x|$.

(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.

A function f is odd if it satisfies $f(-x) = -f(x)$ for every number x in its domain. If the graph of a function is symmetric with respect to the origin, then f is odd. Examples are $f(x) = x^3$, $f(x) = \sin x$, $f(x) = 1/x$.

4. What is an increasing function?

A function f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. What is a mathematical model?

A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon. (See the discussion on pages 23–24.)

6. Give an example of each type of function.

(a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^n$

(c) Exponential function: $f(x) = 2^x$, $f(x) = b^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

(e) Polynomial of degree 5: $f(x) = x^5 + 2x^4 - 3x^2 + 7$

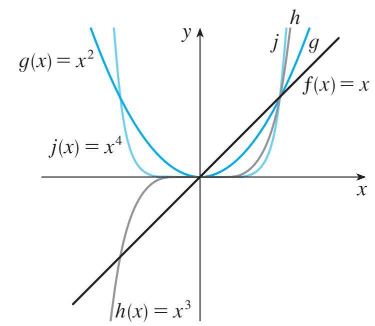
(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$

where $P(x)$ and $Q(x)$ are polynomials

7. Sketch by hand, on the same axes, the graphs of the following functions.

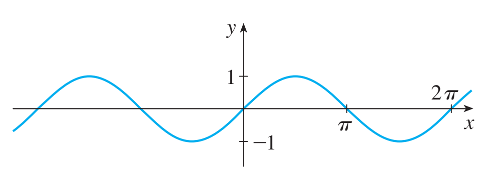
(a) $f(x) = x$ **(b)** $g(x) = x^2$

(c) $h(x) = x^3$ **(d)** $j(x) = x^4$

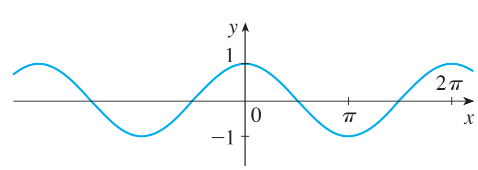


8. Draw, by hand, a rough sketch of the graph of each function.

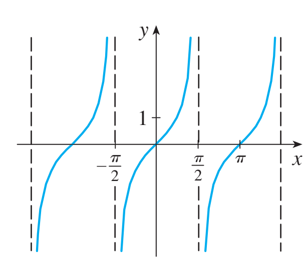
(a) $y = \sin x$



(b) $y = \cos x$

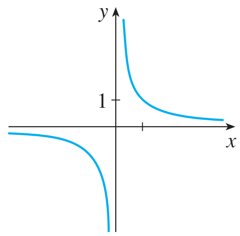


(c) $y = \tan x$

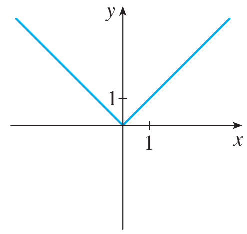


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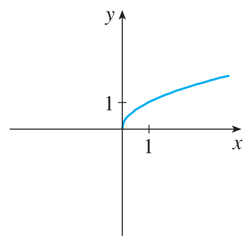
(d) $y = 1/x$



(e) $y = |x|$



(f) $y = \sqrt{x}$



9. Suppose that f has domain A and g has domain B.

(a) What is the domain of $f + g$?

The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) What is the domain of fg?

The domain of fg is also $A \cap B$.

(c) What is the domain of f/g?

The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. How is the composite function $f \circ g$ defined? What is its domain?

The composition of f and g is defined by $(f \circ g)(x) = f(g(x))$. The domain is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.

(a) Shift 2 units upward: $y = f(x) + 2$

(b) Shift 2 units downward: $y = f(x) - 2$

(c) Shift 2 units to the right: $y = f(x - 2)$

(d) Shift 2 units to the left: $y = f(x + 2)$

(e) Reflect about the x-axis: $y = -f(x)$

(f) Reflect about the y-axis: $y = f(-x)$

(g) Stretch vertically by a factor of 2: $y = 2f(x)$

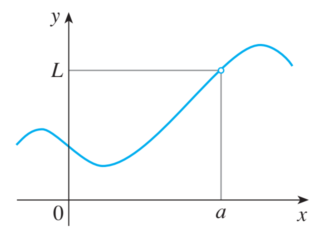
(h) Shrink vertically by a factor of 2: $y = \frac{1}{2}f(x)$

(i) Stretch horizontally by a factor of 2: $y = f\left(\frac{1}{2}x\right)$

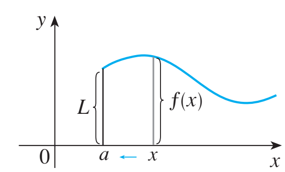
(j) Shrink horizontally by a factor of 2 $y = f(2x)$

12. Explain what each of the following means and illustrate with a sketch.

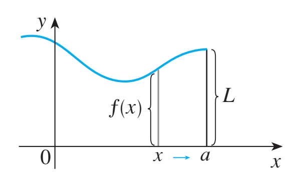
(a) $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a (but $x \neq a$).



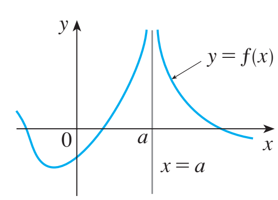
(b) $\lim_{x \to a^+} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values greater than a.



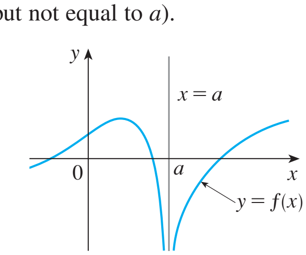
(c) $\lim_{x \to a^-} f(x) = L$ means that the values of $f(x)$ approach L as the values of x approach a through values less than a.



(d) $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a (but not equal to a).



(e) $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a (but not equal to a).

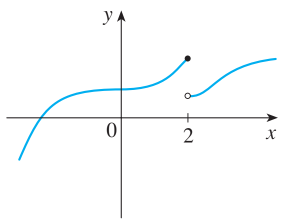


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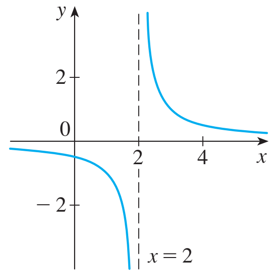
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13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

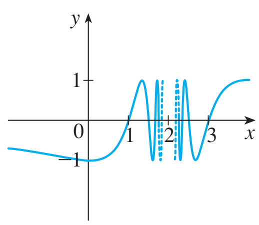
In general, the limit of a function fails to exist when the function values do not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left and right limits are not equal.



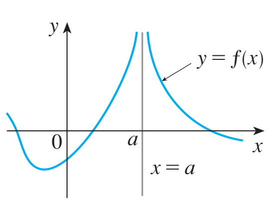
There is an infinite discontinuity.

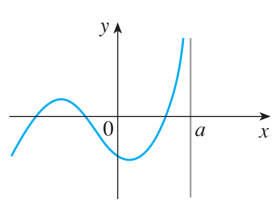


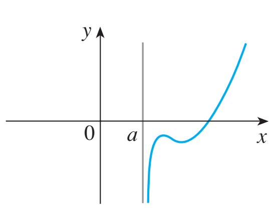
The function values oscillate between 1 and -1 infinitely often.

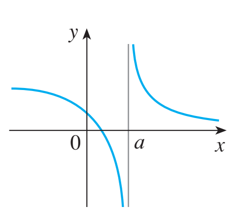
14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

It means that the limit of $f(x)$ as x approaches a from one or both sides is positive or negative infinity.









15. State the following Limit Laws.

(a) Sum Law

The limit of a sum is the sum of the limits:
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(b) Difference Law

The limit of a difference is the difference of the limits:
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

(c) Constant Multiple Law

The limit of a constant times a function is the constant times the limit of the function: $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

(d) Product Law

The limit of a product is the product of the limits:
$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

(e) Quotient Law

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not 0:
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

(f) Power Law

The limit of a power is the power of the limit:
$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \quad \text{(for } n \text{ a positive integer)}$$

(g) Root Law

The limit of a root is the root of the limit:
$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \quad \text{(for } n \text{ a positive integer)}$$

16. What does the Squeeze Theorem say?

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$. In other words, if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

17. (a) What does it mean for f to be continuous at a?

A function f is continuous at a number a if the value of the function at $x = a$ is the same as the limit when x approaches a; that is, $\lim_{x\to a} f(x) = f(a)$.

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

A function f is continuous on the interval $(-\infty, \infty)$ if it is continuous at every real number a.

The graph of such a function has no hole or break in it.

(continued)

18. (a) Give examples of functions that are continuous on $[-1, 1]$.

$f(x) = x^3 - x$, $g(x) = \sqrt{x + 2}$, $y = \sin x$, $y = \tan x$, $y = 1/(x - 3)$, and $h(x) = |x|$ are all continuous on $[-1, 1]$.

(b) Give an example of a function that is not continuous on $[0, 1]$.

$$f(x) = \frac{1}{x - \frac{1}{2}} \qquad \left[f(x) \text{ is not defined at } x = \tfrac{1}{2} \right]$$

19. What does the Intermediate Value Theorem say?

If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$ $[f(a) \neq f(b)]$, Then there exists a number c in (a, b) such that $f(c) = N$. In other words, a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$.

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CHAPTER 2 CONCEPT CHECK ANSWERS

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

The slope of the tangent line is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

2. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

The instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

It is equal to the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.

(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

(b) The instantaneous rate of change of y with respect to x at $x = x_1$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

4. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

or, equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ (with respect to x) when $x = a$ and also represents the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

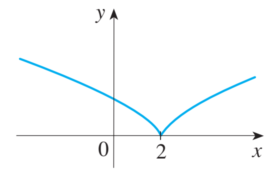
5. (a) What does it mean for f to be differentiable at a?

f is differentiable at a if the derivative $f'(a)$ exists.

(b) What is the relation between the differentiability and continuity of a function?

If f is differentiable at a, then f is continuous at a.

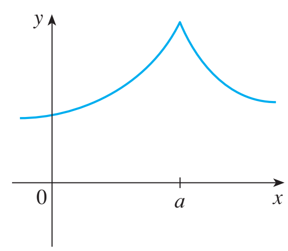
(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.



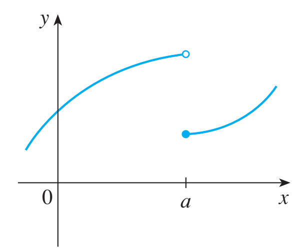
The graph of f changes direction abruptly at $x = 2$, so f has no tangent line there.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

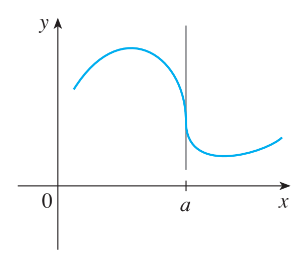
A function is not differentiable at any value where the graph has a "corner," where the graph has a discontinuity, or where it has a vertical tangent line.



A corner



A discontinuity



A vertical tangent

7. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

The second derivative f'' is the derivative of f', and the third derivative f''' is the derivative of f''.

If f is the postition function of an object, then f' is the velocity function of the object, f'' is the acceleration function, and f''' is the jerk function (the rate of change of acceleration).

(continued)

8. State each differentiation rule both in symbols and in words.

(a) The Power Rule

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

To find the derivative of a variable raised to a constant power, we multiply the expression by the exponent and then subtract one from the exponent.

(b) The Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

9. State the derivative of each function.

(a) $y = x^n$: $\quad y' = nx^{n-1}$

(b) $y = \sin x$: $\quad y' = \cos x$

(c) $y = \cos x$: $\quad y' = -\sin x$

(d) $y = \tan x$: $\quad y' = \sec^2 x$

(e) $y = \csc x$: $\quad y' = -\csc x \cot x$

(f) $y = \sec x$: $\quad y' = \sec x \tan x$

(g) $y = \cot x$: $\quad y' = -\csc^2 x$

10. Explain how implicit differentiation works.

Implicit differentiation consists of differentiating both sides of an equation with respect to x, treating y as a function of x. Then we solve the resulting equation for y'.

11. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

In physics, interpretations of the derivative include velocity, linear density, electrical current, power (the rate of change of work), and the rate of radioactive decay. Chemists can use derivatives to measure reaction rates and the compressibility of a substance under pressure. In biology the derivative measures rates of population growth and blood flow. In economics, the derivative measures marginal cost (the rate of change of cost as more items are produced) and marginal profit. Other examples include the rate of heat flow in geology, the rate of performance improvement in psychology, and the rate at which a rumor spreads in sociology.

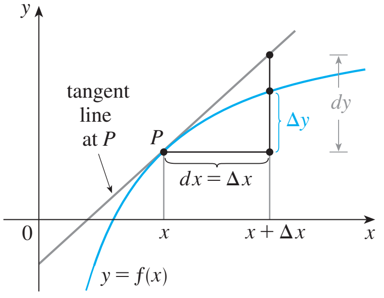
12. (a) Write an expression for the linearization of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

(b) If $y = f(x)$, write an expression for the differential dy.

$$dy = f'(x)\,dx$$

(c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.

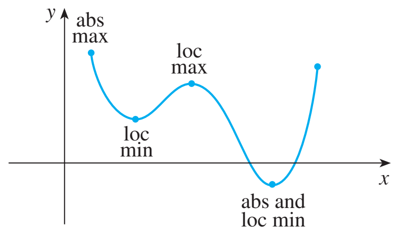


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CHAPTER 3 CONCEPT CHECK ANSWERS

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

The function value $f(c)$ is the absolute maximum value of f if $f(c)$ is the largest function value on the entire domain of f, whereas $f(c)$ is a local maximum value if it is the largest function value when x is near c.



2. What does the Extreme Value Theorem say?

If f is a continuous function on a closed interval $[a, b]$, then it always attains an absolute maximum and an absolute minimum value on that interval.

3. (a) State Fermat's Theorem.

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

(b) Define a critical number of f.

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

4. Explain how the Closed Interval Method works.

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$, we follow these three steps:

- Find the critical numbers of f in the interval (a, b) and compute the values of f at these numbers.
- Find the values of f at the endpoints of the interval.
- The largest of the values from the previous two steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

5. (a) State Rolle's Theorem.

Let f be a function that satisfies the following three hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).
- $f(a) = f(b)$

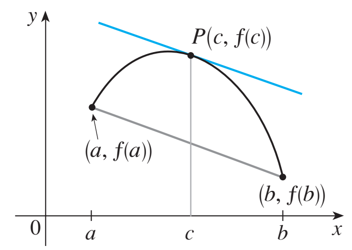
Then there is a number c in (a, b) such that $f'(c) = 0$.

(b) State the Mean Value Theorem and give a geometric interpretation.

If f is continuous on the interval $[a, b]$ and differentiable on (a, b), then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically, the theorem says that there is a point $P(c, f(c))$, where $a < c < b$, on the graph of f where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.



6. (a) State the Increasing/Decreasing Test.

If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

(b) What does it mean to say that f is concave upward on an interval I?

f is concave upward on an interval if the graph of f lies above all of its tangents on that interval.

(c) State the Concavity Test.

If $f''(x) > 0$ on an interval, then the graph of f is concave upward on that interval.

If $f''(x) < 0$ on an interval, then the graph of f is concave downward on that interval.

(d) What are inflection points? How do you find them?

Inflection points on the graph of a continuous function f are points where the curve changes from concave upward to concave downward or from concave downward to concave upward. They can be found by determining the values at which the second derivative changes sign.

7. (a) State the First Derivative Test.

Suppose that c is a critical number of a continuous function f.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

(b) State the Second Derivative Test.

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(continued)

(c) What are the relative advantages and disadvantages of these tests?

The Second Derivative Test is sometimes easier to use, but it is inconclusive when $f''(c) = 0$ and fails if $f''(c)$ does not exist. In either case the First Derivative Test must be used.

8. Explain the meaning of each of the following statements.

(a) $\lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

(b) $\lim_{x \to -\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

(c) $\lim_{x \to \infty} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large.

(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

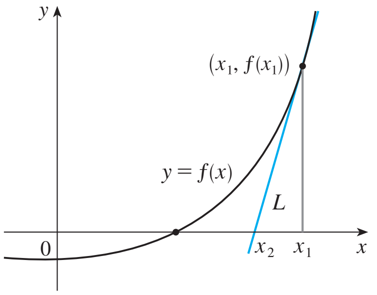
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

9. If you have a graphing calculator or computer, why do you need calculus to graph a function?

Calculus reveals all the important aspects of a graph, such as local extreme values and inflection points, that can be missed when relying solely on technology. In many cases we can find exact locations of these key points rather than approximations. Using derivatives to identify the behavior of the graph also helps us choose an appropriate viewing window and alerts us to where we may wish to zoom in on a graph.

10. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.

We find the tangent line L to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$. Then x_2 is the x-intercept of L.



(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0. It also fails when $f'(x_i)$ is undefined.

11. (a) What is an antiderivative of a function f?

A function F is an antiderivative of f if $F'(x) = f(x)$.

(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

They are identical or they differ by a constant.

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CHAPTER 4 CONCEPT CHECK ANSWERS

1. (a) Write an expression for a Riemann sum of a function f on an interval $[a, b]$. Explain the meaning of the notation that you use.

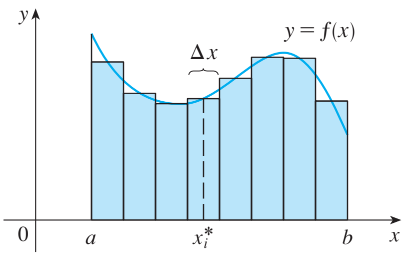
If f is defined for $a \leqslant x \leqslant b$ and we divide the interval $[a, b]$ into n subintervals of equal width Δx, then a Riemann sum of f is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is a point in the ith subinterval.

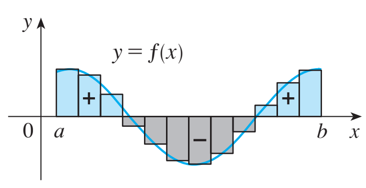
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f is positive, then a Riemann sum can be interpreted as the sum of areas of approximating rectangles, as shown in the figure.



(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

If f takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles minus the areas of the gray rectangles).



2. (a) Write the definition of the definite integral of a continuous function from a to b.

If f is a continuous function on the interval $[a, b]$, then we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

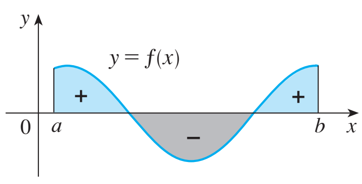
where x_i^* is any sample point in the ith subinterval $[x_{i-1}, x_i]$.

(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?

If f is positive, then $\int_a^b f(x)\,dx$ can be interpreted as the area under the graph of $y = f(x)$ and above the x-axis for $a \leqslant x \leqslant b$.

(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

In this case $\int_a^b f(x)\,dx$ can be interpreted as a "net area," that is, the area of the region above the x-axis and below the graph of f (labeled "+" in the figure) minus the area of the region below the x-axis and above the graph of f (labeled "–").



3. State the Midpoint Rule.

If f is a continuous function on the interval $[a, b]$ and we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$, then

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

where $\bar{x}_i = $ midpoint of $[x_{i-1}, x_i] = \frac{1}{2}(x_{i-1} + x_i)$.

4. State both parts of the Fundamental Theorem of Calculus.

Suppose f is continuous on $[a, b]$.

Part 1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Part 2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

5. (a) State the Net Change Theorem.

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

$\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

(continued)

6. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.

(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?

$\int_{60}^{120} v(t)\,dt$ represents the net change in position (the displacement) of the particle from $t = 60$ s to $t = 120$ s, in other words, in the second minute.

(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?

$\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle in the second minute.

(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

$\int_{60}^{120} a(t)\,dt$ represents the change in velocity of the particle in the second minute.

7. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.

The indefinite integral $\int f(x)\,dx$ is another name for an antiderivative of f, so $\int f(x)\,dx = F(x)$ means that $F'(x) = f(x)$.

(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

The connection is given by Part 2 of the Fundamental Theorem:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

if f is continuous on $[a, b]$.

8. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

Part 1 of the Fundamental Theorem of Calculus can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f.

Since $F'(x) = f(x)$, Part 2 of the theorem (or, equivalently, the Net Change Theorem) states that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function, but in the form $F(b) - F(a)$.

Also, the indefinite integral $\int f(x)\,dx$ represents an antiderivative of f, so

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

9. State the Substitution Rule. In practice, how do you use it?

If $u = g(x)$ is a differentiable function and f is continuous on the range of g, then

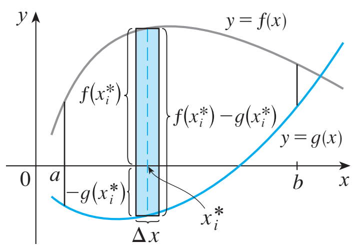
$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

In practice, we make the substitutions $u = g(x)$ and $du = g'(x)\,dx$ in the integrand in order to make the integral simpler to evaluate.

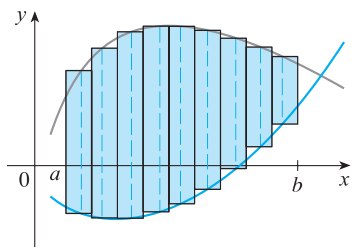
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CHAPTER 5 CONCEPT CHECK ANSWERS

1. **(a)** Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.



A Riemann sum that approximates the area between these curves is $\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$. A sketch of the corresponding approximating rectangles:

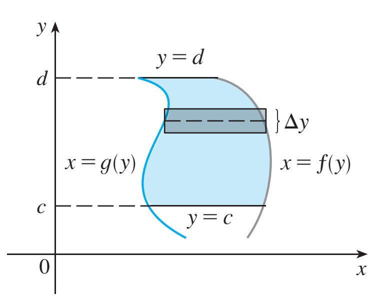


An expression for the exact area is

$$\lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx$$

(b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top: $A = \int_c^d [f(y) - g(y)]\,dy$



2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

It represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. **(a)** Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.

We slice S into n "slabs" of equal width Δx. The volume of the ith slab is approximately $A(x_i^*)\,\Delta x$, where x_i^* is a sample point in the ith slab and $A(x_i^*)$ is the cross-sectional area of S at x_i^*. Then the volume of S is approximately $\sum_{i=1}^{n} A(x_i^*)\,\Delta x$ and the exact volume is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

(b) If S is a solid of revolution, how do you find the cross-sectional areas?

If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi(r_{\text{out}}^2) - \pi(r_{\text{in}}^2)$.

4. **(a)** What is the volume of a cylindrical shell?

$$V = 2\pi rh\,\Delta r = (\text{circumference})(\text{height})(\text{thickness})$$

(b) Explain how to use cylindrical shells to find the volume of a solid of revolution.

We approximate the region to be revolved by rectangles, oriented so that revolution forms cylindrical shells rather than disks or washers. For a typical shell, find the circumference and height in terms of x or y and calculate

$$V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$$

(c) Why might you want to use the shell method instead of slicing?

Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

$\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

6. **(a)** What is the average value of a function f on an interval $[a, b]$?

$$f_{\text{ave}} = \frac{1}{b - a}\int_a^b f(x)\,dx$$

(b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

If f is continuous on $[a, b]$, then there is a number c in $[a, b]$ at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. This means that for positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.

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CHAPTER 6 CONCEPT CHECK ANSWERS

1. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?

A function f is one-to-one if it never takes on the same value twice; that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. When looking at a graph, use the Horizontal Line Test: a function is one-to-one if and only if no horizontal line intersects its graph more than once.

(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

If f is a one-to-one function with domain A and range B, then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

(c) If f is a one-to-one function and $f'(f^{-1}(a)) \neq 0$, write a formula for $(f^{-1})'(a)$.

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

2. (a) What are the domain and range of the natural exponential function $f(x) = e^x$?

Domain: $\mathbb{R}$ Range: $(0, \infty)$

(b) What are the domain and range of the natural logarithmic function $g(x) = \ln x$?

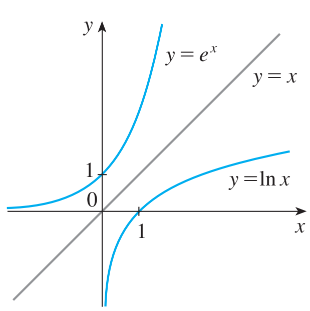
Domain: $(0, \infty)$ Range: $\mathbb{R}$

(c) How are the functions $f(x) = e^x$ and $g(x) = \ln x$ related?

They are inverses of each other.

(d) How are the graphs of these functions related? Sketch these graphs by hand, using the same axes.

The graphs are reflections of one another about the line $y = x$.



(e) If b is a positive number, $b \neq 1$, write an equation that expresses $\log_b x$ in terms of $\ln x$.

$$\log_b x = \frac{\ln x}{\ln b}$$

3. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?

The inverse sine function $f(x) = \sin^{-1}x$ is defined as

$$\sin^{-1}x = y \iff \sin y = x$$

and $$-\pi/2 \leqslant y \leqslant \pi/2$$

Its domain is $-1 \leqslant x \leqslant 1$ and its range is $-\pi/2 \leqslant y \leqslant \pi/2$.

(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?

The inverse cosine function $f(x) = \cos^{-1}x$ is defined as

$$\cos^{-1}x = y \iff \cos y = x$$

and $$0 \leqslant y \leqslant \pi$$

Its domain is $-1 \leqslant x \leqslant 1$ and its range is $0 \leqslant y \leqslant \pi$.

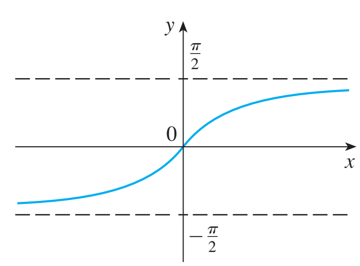
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range? Sketch its graph.

The inverse tangent function $f(x) = \tan^{-1}x$ is defined as

$$\tan^{-1}x = y \iff \tan y = x$$

and $$-\pi/2 < y < \pi/2$$

Its domain is $\mathbb{R}$ and its range is $-\pi/2 < y < \pi/2$.



4. Write the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, and $\tanh x$.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

5. State the derivative of each function.

(a) $y = e^x$: $y' = e^x$

(b) $y = b^x$: $y' = b^x \ln b$

(c) $y = \ln x$: $y' = 1/x$

(d) $y = \log_b x$: $y' = 1/(x \ln b)$

(e) $y = \sin^{-1}x$: $y' = 1/\sqrt{1 - x^2}$

(f) $y = \cos^{-1}x$: $y' = -1/\sqrt{1 - x^2}$

(continued)

(g) $y = \tan^{-1}x$: $y' = 1/(1 + x^2)$

(h) $y = \sinh x$: $y' = \cosh x$

(i) $y = \cosh x$: $y' = \sinh x$

(j) $y = \tanh x$: $y' = \text{sech}^2 x$

(k) $y = \sinh^{-1}x$: $y' = 1/\sqrt{1 + x^2}$

(l) $y = \cosh^{-1}x$: $y' = 1/\sqrt{x^2 - 1}$

(m) $y = \tanh^{-1}x$: $y' = 1/(1 - x^2)$

6. (a) How is the number e defined?

e is the number such that $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

(b) Express e as a limit.

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

(c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = b^x$?

The differentiation formula for $y = b^x$ $[dy/dx = b^x \ln b]$ is simplest when $b = e$ because $\ln e = 1$.

(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_b x$?

The differentiation formula for $y = \log_b x$ $[dy/dx = 1/(x \ln b)]$ is simplest when $b = e$ because $\ln e = 1$.

7. (a) Write a differential equation that expresses the law of natural growth.

If $y(t)$ is the value of a quantity y at time t, then

$$\frac{dy}{dt} = ky \qquad \text{where } k > 0 \text{ is a constant}$$

(b) Under what circumstances is this an appropriate model for population growth?

The equation in part (a) is an appropriate model when there is enough room and nutrition to support growth.

(c) What are the solutions of this equation?

If $y(0) = y_0$, then the solutions are $y(t) = y_0 e^{kt}$.

8. (a) What does l'Hospital's Rule say?

L'Hospital's Rule says that *if* the limit of a quotient of functions is an indeterminate form of type $0/0$ or ∞/∞, then the limit is equal to the limit of the quotient of their derivatives:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(b) How can you use l'Hospital's Rule if you have a product $f(x)\,g(x)$, where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?

Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$ so that the limit becomes an indeterminate form of type $0/0$ or ∞/∞.

(c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$, where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?

Convert the difference into a quotient by using a common denominator, rationalizing, factoring, or by some other method.

(d) How can you use l'Hospital's Rule if you have a power $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

By taking the natural logarithm of both sides of $y = [f(x)]^{g(x)}$, we can convert the power to a product by writing $\ln y = g(x) \ln f(x)$. Alternatively, we can write the function as an exponential: $[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$.

9. State whether each of the following limit forms is indeterminate. Where possible, state the limit.

(a) $\dfrac{0}{0}$: indeterminate. L'Hospital's Rule can be applied to this form. Note that every derivative is a limit of this form.

(b) $\dfrac{\infty}{\infty}$: indeterminate. L'Hospital's Rule can be applied to this form.

(c) $\dfrac{0}{\infty}$: not indeterminate. A limit of this form has value 0.

(d) $\dfrac{\infty}{0}$: not indeterminate. A limit of this form could equal ∞, $-\infty$, or may not exist (but it cannot equal a finite value).

(e) $\infty + \infty$: not indeterminate. A limit of this form is equal to ∞.

(f) $\infty - \infty$: indeterminate

(g) $\infty \cdot \infty$: not indeterminate. A limit of this form is equal to ∞.

(h) $\infty \cdot 0$: indeterminate

(i) 0^0: indeterminate

(j) 0^∞: not indeterminate. A limit of this form has value 0.

(k) ∞^0: indeterminate

(l) 1^∞: indeterminate

CHAPTER 7 CONCEPT CHECK ANSWERS

1. State the rule for integration by parts. In practice, how do you use it?

To integrate $\int f(x)g'(x)\,dx$, let $u = f(x)$ and $v = g(x)$. Then $\int u\,dv = uv - \int v\,du$.

In practice, try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) at long as $dv = g'(x)\,dx$ can be readily integrated to give v.

2. How do you evaluate $\int \sin^m x \cos^n x\,dx$ if m is odd? What if n is odd? What if m and n are both even?

If m is odd, use $\sin^2 x = 1 - \cos^2 x$ to write all sine factors except one in terms of cosine. Then substitute $u = \cos x$.

If n is odd, use $\cos^2 x = 1 - \sin^2 x$ to write all cosine factors except one in terms of sine. Then substitute $u = \sin x$.

If m and n are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

If $\sqrt{a^2 - x^2}$ occurs, try $x = a \sin\theta$; if $\sqrt{a^2 + x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2 - a^2}$ occurs, try $x = a\sec\theta$.

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

For distinct linear factors,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

If the linear factor $a_1x + b_1$ is repeated r times, then we must include all the terms

$$\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \cdots + \frac{B_r}{(a_1x + b_1)^r}$$

If $Q(x)$ has an irreducible quadratic factor (not repeated), then we include a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

If the irreducible quadratic factor is repeated r times, then we include all the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

Let $a \leqslant x \leqslant b$, $I = \int_a^b f(x)\,dx$, and $\Delta x = (b - a)/n$.

Midpoint Rule:

$$I \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$.

Trapezoidal Rule:

$$I \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $x_i = a + i\,\Delta x$.

Simpson's Rule:

$$I \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even.

We would expect the best estimate to be given by Simpson's Rule.

Suppose $|f''(x)| \leqslant K$ and $|f^{(4)}(x)| \leqslant L$ for $a \leqslant x \leqslant b$. The errors in the Midpoint, Trapezoidal, and Simpson's Rules are given by, respectively,

$$|E_M| \leqslant \frac{K(b-a)^3}{24n^2} \qquad |E_T| \leqslant \frac{K(b-a)^3}{12n^2}$$

$$|E_S| \leqslant \frac{L(b-a)^5}{180n^4}$$

6. Define the following improper integrals.

(a) $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

(b) $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$

(c) $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$, where a is any real number (assuming that both integrals are convergent)

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.

(a) f has an infinite discontinuity at a.

If f is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

(b) f has an infinite discontinuity at b.

If f is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(c) f has an infinite discontinuity at c, where $a < c < b$.

If both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

8. State the Comparison Theorem for improper integrals.

Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

CHAPTER 8 CONCEPT CHECK ANSWERS

1. (a) How is the length of a curve defined?

We can approximate a curve C by a polygon with vertices P_i along C. The length L of C is defined to be the limit of the lengths of these inscribed polygons:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

(b) Write an expression for the length of a smooth curve given by $y = f(x)$, $a \leqslant x \leqslant b$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

(c) What if x is given as a function of y?

If $x = g(y)$, $c \leqslant y \leqslant d$, then $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$.

2. (a) Write an expression for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis.

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

(b) What if x is given as a function of y?

If $x = g(y)$, $c \leqslant y \leqslant d$, then $S = \int_c^d 2\pi y\sqrt{1 + [g'(y)]^2}\, dy$.

(c) What if the curve is rotated about the y-axis?

$$S = \int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\, dx$$

or

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy$$

3. Describe how we can find the hydrostatic force against a vertical wall submersed in a fluid.

We divide the wall into horizontal strips of equal height Δx and approximate each by a rectangle with horizontal length $f(x_i)$ at depth x_i. If δ is the weight density of the fluid, then the hydrostatic force is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i f(x_i)\, \Delta x = \int_a^b \delta x f(x)\, dx$$

4. (a) What is the physical significance of the center of mass of a thin plate?

The center of mass is the point at which the plate balances horizontally.

(b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leqslant x \leqslant b$, write expressions for the coordinates of the center of mass.

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\, dx \qquad \text{and} \qquad \bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}[f(x)]^2\, dx$$

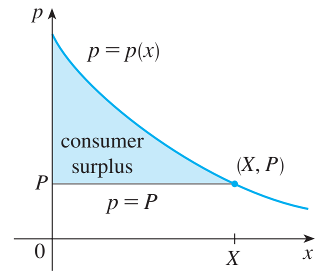
where $A = \int_a^b f(x)\, dx$.

5. What does the Theorem of Pappus say?

If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. Given a demand function $p(x)$, explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P [when they were willing to purchase it at price $p(x)$], corresponding to an amount demanded of X.



7. (a) What is the cardiac output of the heart?

It is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

(b) Explain how the cardiac output can be measured by the dye dilution method.

An amount A of dye is injected into part of the heart and its concentration $c(t)$ leaving the heart is measured over a time interval $[0, T]$ until the dye has cleared. The cardiac output is given by $A/\int_0^T c(t)\, dt$.

8. What is a probability density function? What properties does such a function have?

Given a random variable X, its probability density function f is a function such that $\int_a^b f(x)\, dx$ gives the probability that X lies between a and b. The function f has the properties that $f(x) \geqslant 0$ for all x, and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

9. Suppose $f(x)$ is the probability density function for the weight of a female college student, where x is measured in pounds.

(a) What is the meaning of the integral $\int_0^{130} f(x)\, dx$?

It represents the probability that a randomly chosen female college student weighs less than 130 pounds.

(b) Write an expression for the mean of this density function.

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x f(x)\, dx$$

[since $f(x) = 0$ for $x < 0$]

(c) How can we find the median of this density function?

The median of f is the number m such that

$$\int_m^{\infty} f(x)\, dx = \frac{1}{2}$$

10. What is a normal distribution? What is the significance of the standard deviation?

A normal distribution corresponds to a random variable X that has a probability density function with a bell-shaped graph and equation given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and the positive constant σ is the standard deviation. σ measures how spread out the values of X are.

CHAPTER 9 CONCEPT CHECK ANSWERS

1. (a) What is a differential equation?

It is an equation that contains an unknown function and one or more of its derivatives.

(b) What is the order of a differential equation?

It is the order of the highest derivative that occurs in the equation.

(c) What is an initial condition?

It is a condition of the form $y(t_0) = y_0$.

2. What can you say about the solutions of the equation $y' = x^2 + y^2$ just by looking at the differential equation?

The equation tells us that the slope of a solution curve at any point (x, y) is $x^2 + y^2$. Note that $x^2 + y^2$ is always positive except at the origin, where $y' = x^2 + y^2 = 0$. Thus there is a horizontal tangent at $(0, 0)$ but nowhere else and the solution curves are increasing everywhere.

3. What is a direction field for the differential equation $y' = F(x, y)$?

A direction field (or slope field) for the differential equation $y' = F(x, y)$ is a two-dimensional graph consisting of short line segments with slope $F(x, y)$ at point (x, y).

4. Explain how Euler's method works.

Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field until the approximation is complete.

5. What is a separable differential equation? How do you solve it?

It is a differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)f(y)$. We can solve the equation by rewriting it as $[1/f(y)]\,dy = g(x)\,dx$, integrating both sides, and solving for y.

6. What is a first-order linear differential equation? How do you solve it?

A first-order linear differential equation is a differential equation that can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. To solve such an equation, we multiply both sides by the integrating factor $I(x) = e^{\int P(x)\,dx}$ to put it in the form $\big(I(x)\,y\big)' = I(x)\,Q(x)$. We then integrate both sides and solve for y.

7. (a) Write a differential equation that expresses the law of natural growth. What does it say in terms of relative growth rate?

If $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then $\dfrac{dP}{dt} = kP$.

In this case the relative growth rate, $\dfrac{1}{P}\dfrac{dP}{dt}$, is constant.

(b) Under what circumstances is this an appropriate model for population growth?

It is an appropriate model under ideal conditions: unlimited environment, adequate nutrition, absence of predators and disease.

(c) What are the solutions of this equation?

If $P(0) = P_0$, the initial value, then the solutions are $P(t) = P_0e^{kt}$.

8. (a) Write the logistic differential equation.

The logistic differential equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where M is the carrying capacity.

(b) Under what circumstances is this an appropriate model for population growth?

It is an appropriate model for population growth if the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.

9. (a) Write Lotka-Volterra equations to model populations of food-fish (F) and sharks (S).

$$\frac{dF}{dt} = kF - aFS \qquad \text{and} \qquad \frac{dS}{dt} = -rS + bFS$$

(b) What do these equations say about each population in the absence of the other?

In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is $dS/dt = -rS$, where r is a positive constant.

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CHAPTER 10 CONCEPT CHECK ANSWERS

1. (a) What is a parametric curve?

A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are functions of a variable t, the parameter.

(b) How do you sketch a parametric curve?

Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) How do you find the slope of a tangent to a parametric curve?

You can find dy/dx as a function of t by calculating

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{if } dx/dt \neq 0$$

(b) How do you find the area under a parametric curve?

If the curve is traced out once by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then the area is

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

[or $\int_\beta^\alpha g(t)f'(t)\,dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. Write an expression for each of the following:

(a) The length of a parametric curve

If the curve is traced out once by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then the length is

$$L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

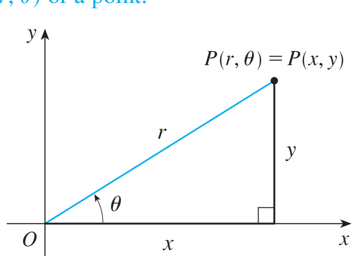
$$= \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

(b) The area of the surface obtained by rotating a parametric curve about the x-axis

$$S = \int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

$$= \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.



(b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

(c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?

To find a polar representation (r, θ) with $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\tan\theta = y/x$. Be sure to choose θ so that (r, θ) lies in the correct quadrant.

5. (a) How do you find the slope of a tangent line to a polar curve?

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{d}{d\theta}(y)}{\dfrac{d}{d\theta}(x)} = \frac{\dfrac{d}{d\theta}(r\sin\theta)}{\dfrac{d}{d\theta}(r\cos\theta)}$$

$$= \frac{\left(\dfrac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\dfrac{dr}{d\theta}\right)\cos\theta - r\sin\theta} \qquad \text{where } r = f(\theta)$$

(b) How do you find the area of a region bounded by a polar curve?

$$A = \int_a^b \tfrac{1}{2}r^2\,d\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

(c) How do you find the length of a polar curve?

$$L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta$$

$$= \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$$

$$= \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$

6. (a) Give a geometric definition of a parabola.

A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line l (the directrix) are equal.

(b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?

In the first case an equation is $x^2 = 4py$ and in the second case, $y^2 = 4px$.

7. (a) Give a definition of an ellipse in terms of foci.

An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

(b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \geqslant b > 0$ and $c^2 = a^2 - b^2$.

(continued)

8. (a) Give a definition of a hyperbola in terms of foci.

A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.

(b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$.

(c) Write equations for the asymptotes of the hyperbola in part (b).

$$y = \pm \frac{b}{a} x$$

9. (a) What is the eccentricity of a conic section?

If a conic section has focus F and corresponding directrix l, then the eccentricity e is the fixed ratio $|PF|/|Pl|$ for points P of the conic section.

(b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?

$e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola

(c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

$$\text{directrix } x = d\text{:} \quad r = \frac{ed}{1 + e\cos\theta}$$

$$x = -d\text{:} \quad r = \frac{ed}{1 - e\cos\theta}$$

$$y = d\text{:} \quad r = \frac{ed}{1 + e\sin\theta}$$

$$y = -d\text{:} \quad r = \frac{ed}{1 - e\sin\theta}$$

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CHAPTER 11 CONCEPT CHECK ANSWERS

1. (a) What is a convergent sequence?

A convergent sequence $\{a_n\}$ is an ordered list of numbers where $\lim_{n\to\infty} a_n$ exists.

(b) What is a convergent series?

A series $\Sigma\, a_n$ is the *sum* of a sequence of numbers. It is convergent if the partial sums $s_n = \sum_{i=1}^{n} a_n$ approach a finite value, that is, $\lim_{n\to\infty} s_n$ exists as a real number.

(c) What does $\lim_{n\to\infty} a_n = 3$ mean?

The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) What does $\Sigma_{n=1}^{\infty} a_n = 3$ mean?

By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) What is a bounded sequence?

A sequence $\{a_n\}$ is bounded if there are numbers m and M such that $m \leqslant a_n \leqslant M$ for all $n \geqslant 1$.

(b) What is a monotonic sequence?

A sequence is monotonic if it is either increasing or decreasing for all $n \geqslant 1$.

(c) What can you say about a bounded monotonic sequence?

Every bounded, monotonic sequence is convergent.

3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?

A geometric series is of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

It is convergent if $|r| < 1$ and its sum is $\dfrac{a}{1-r}$.

(b) What is a p-series? Under what circumstances is it convergent?

A p-series is of the form $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$. It is convergent if $p > 1$.

4. Suppose $\Sigma\, a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?

If $\Sigma\, a_n = 3$, then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$.

5. State the following.

(a) The Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

- If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) The Comparison Test

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

- If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.
- If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

(d) The Limit Comparison Test

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If $\lim_{n\to\infty} a_n/b_n = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge.

(e) The Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

where $b_n > 0$ satisfies (i) $b_{n+1} \leqslant b_n$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

(f) The Ratio Test

- If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

(g) The Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

6. (a) What is an absolutely convergent series?

A series $\Sigma\, a_n$ is called absolutely convergent if the series of absolute values $\Sigma\, |a_n|$ is convergent.

(b) What can you say about such a series?

If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

(c) What is a conditionally convergent series?

A series $\Sigma\, a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

(continued)

7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?

The sum s can be estimated by the inequality

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

where s_n is the nth partial sum.

(b) If a series is convergent by the Comparison Test, how do you estimate its sum?

We first estimate the remainder for the comparison series. This gives an upper bound for the remainder of the original series (as in Example 11.4.5).

(c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?

We can use a partial sum s_n of an alternating series as an approximation to the total sum. The size of the error is guaranteed to be no more than $|a_{n+1}|$, the absolute value of the first neglected term.

8. (a) Write the general form of a power series.

A power series centered at a is

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

(b) What is the radius of convergence of a power series?

Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$,

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) What is the interval of convergence of a power series?

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is (i) the single point $\{a\}$, (ii) $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ that can contain neither, either, or both of the endpoints.

9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.

(a) How do you differentiate f? What is the radius of convergence of the series for f'?

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ with radius of convergence R.

(b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?

$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$ with radius of convergence R.

10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

(b) Write an expression for the Taylor series of f centered at a.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(c) Write an expression for the Maclaurin series of f.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \qquad [a = 0 \text{ in part (b)}]$$

(d) How do you show that $f(x)$ is equal to the sum of its Taylor series?

If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f and $R_n(x)$ is the remainder of the Taylor series, then we must show that

$$\lim_{n\to\infty} R_n(x) = 0$$

(e) State Taylor's Inequality.

If $|f^{(n+1)}(x)| \leqslant M$ for $|x-a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leqslant d$$

11. Write the Maclaurin series and the interval of convergence for each of the following functions.

(a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad R = 1$

(b) $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}, \quad R = \infty$

(c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}, \quad R = \infty$

(d) $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}, \quad R = \infty$

(e) $\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}, \quad R = 1$

(f) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}, \quad R = 1$

12. Write the binomial series expansion of $(1+x)^k$. What is the radius of convergence of this series?

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The radius of convergence for the binomial series is 1.

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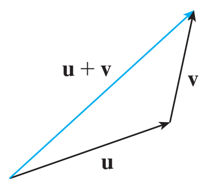
CHAPTER 12 CONCEPT CHECK ANSWERS

1. What is the difference between a vector and a scalar?

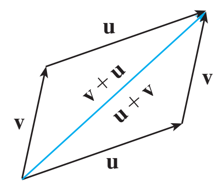
A scalar is a real number, whereas a vector is a quantity that has both a real-valued magnitude and a direction.

2. How do you add two vectors geometrically? How do you add them algebraically?

To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law:



Triangle Law



Parallelogram Law

Algebraically, we add the corresponding components of the vectors.

3. If **a** is a vector and c is a scalar, how is $c\mathbf{a}$ related to **a** geometrically? How do you find $c\mathbf{a}$ algebraically?

For $c > 0$, $c\mathbf{a}$ is a vector with the same direction as **a** and length c times the length of **a**. If $c < 0$, $c\mathbf{a}$ points in the direction opposite to **a** and has length $|c|$ times the length of **a**. Algebraically, to find $c\mathbf{a}$ we multiply each component of **a** by c.

4. How do you find the vector from one point to another?

The vector from point $A(x_1, y_1, z_1)$ to point $B(x_2, y_2, z_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

5. How do you find the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

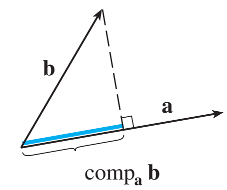
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

6. How are dot products useful?

The dot product can be used to find the angle between two vectors. In particular, it can be used to determine whether two vectors are orthogonal. We can also use the dot product to find the scalar projection of one vector onto another. Additionally, if a constant force moves an object, the work done is the dot product of the force and displacement vectors.

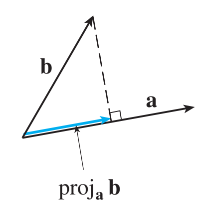
7. Write expressions for the scalar and vector projections of **b** onto **a**. Illustrate with diagrams.

Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}}\, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$



Vector projection of **b** onto **a**:

$$\text{proj}_{\mathbf{a}}\, \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\, \mathbf{a}$$



8. How do you find the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors if you know their lengths and the angle between them? What if you know their components?

If θ is the angle between **a** and **b** $(0 \leqslant \theta \leqslant \pi)$, then $\mathbf{a} \times \mathbf{b}$ is the vector with length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ and direction orthogonal to both **a** and **b**, as given by the right-hand rule. If

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \text{and} \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

9. How are cross products useful?

The cross product can be used to create a vector orthogonal to two given vectors and it can be used to compute the area of a parallelogram determined by two vectors. Two nonzero vectors are parallel if and only if their cross product is **0**. In addition, if a force acts on a rigid body, then the torque vector is the cross product of the position and force vectors.

(continued)

10. (a) How do you find the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$?

The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

(b) How do you find the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$?

The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

11. How do you find a vector perpendicular to a plane?

If an equation of the plane is known, it can be written in the form $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c\rangle$ (or any nonzero scalar multiple of $\langle a, b, c\rangle$). If an equation is not known, we can use points on the plane to find two nonparallel vectors that lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.

12. How do you find the angle between two intersecting planes?

The angle between two intersecting planes is defined as the acute angle θ between their normal vectors. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

13. Write a vector equation, parametric equations, and symmetric equations for a line.

A vector equation for a line that is parallel to a vector $\mathbf{v}$ and that passes through a point with position vector $\mathbf{r}_0$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c\rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

while symmetric equations are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

14. Write a vector equation and a scalar equation for a plane.

A vector equation of a plane that passes through a point with position vector $\mathbf{r}_0$ and that has normal vector $\mathbf{n}$ (meaning $\mathbf{n}$ is orthogonal to the plane) is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ or, equivalently, $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$.

A scalar equation of a plane through a point (x_0, y_0, z_0) with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

15. (a) How do you tell if two vectors are parallel?

Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$.

(b) How do you tell if two vectors are perpendicular?

Two vectors are perpendicular if and only if their dot product is 0.

(c) How do you tell if two planes are parallel?

Two planes are parallel if and only if their normal vectors are parallel.

16. (a) Describe a method for determining whether three points P, Q, and R lie on the same line.

Determine the vectors $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PR} = \mathbf{b}$. If there is a scalar t such that $\mathbf{a} = t\mathbf{b}$, then the vectors are parallel and the points must all lie on the same line.

Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.

An algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

(b) Describe a method for determining whether four points P, Q, R, and S lie in the same plane.

Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. Then $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q, and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$.

Alternatively, we can check if the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is 0 (see Example 12.4.5).

An algebraic method is to find an equation of the plane determined by three of the points, and then check whether or not the fourth point satisfies this equation.

17. (a) How do you find the distance from a point to a line?

Let P be a point not on the line L that passes through the points Q and R and let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QP}$. The distance from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

(b) How do you find the distance from a point to a plane?

Let $P_0(x_0, y_0, z_0)$ be any point in the plane $ax + by + cz + d = 0$ and let $P_1(x_1, y_1, z_1)$ be a point not in the plane. If $\mathbf{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$, then the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the plane's normal vector $\mathbf{n} = \langle a, b, c\rangle$:

$$D = |\text{comp}_{\mathbf{n}}\,\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(c) How do you find the distance between two lines?

Two skew lines L_1 and L_2 can be viewed as lying on two parallel planes, each with normal vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors of L_1 and L_2. After choosing one point on L_1 and determining the equation of the plane containing L_2, we can proceed as in part (b). (See Example 12.5.10.)

(continued)

CHAPTER 12 CONCEPT CHECK ANSWERS *(continued)*

18. What are the traces of a surface? How do you find them?

The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

19. Write equations in standard form of the six types of quadric surfaces.

Equations for the quadric surfaces symmetric with respect to the z-axis are as follows.

Ellipsoid: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Cone: $$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Elliptic paraboloid: $$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperboloid of one sheet: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets: $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Hyperbolic paraboloid: $$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

CHAPTER 13 CONCEPT CHECK ANSWERS

1. What is a vector function? How do you find its derivative and its integral?

A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component function of the vector function.

2. What is the connection between vector functions and space curves?

A continuous vector function $\mathbf{r}$ defines a space curve that is traced out by the tip of the moving position vector $\mathbf{r}(t)$.

3. How do you find the tangent vector to a smooth curve at a point? How do you find the tangent line? The unit tangent vector?

The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

4. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function, write the rules for differentiating the following vector functions.

(a) $\mathbf{u}(t) + \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

(b) $c\mathbf{u}(t)$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

(c) $f(t)\,\mathbf{u}(t)$

$$\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$$

(d) $\mathbf{u}(t) \cdot \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

(e) $\mathbf{u}(t) \times \mathbf{v}(t)$

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

(f) $\mathbf{u}(f(t))$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$$

5. How do you find the length of a space curve given by a vector function $\mathbf{r}(t)$?

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leqslant t \leqslant b$, and the curve is traversed exactly once as t increases from a to b, then the length is

$$L = \int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

6. (a) What is the definition of curvature?

The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

(b) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

(c) Write a formula for curvature in terms of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

(d) Write a formula for the curvature of a plane curve with equation $y = f(x)$.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

7. (a) Write formulas for the unit normal and binormal vectors of a smooth space curve $\mathbf{r}(t)$.

Unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

(b) What is the normal plane of a curve at a point? What is the osculating plane? What is the osculating circle?

The normal plane of a curve at a point P is the plane determined by the normal and binormal vectors $\mathbf{N}$ and $\mathbf{B}$ at P. The tangent vector $\mathbf{T}$ is orthogonal to the normal plane.

The osculating plane at P is the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$. It is the plane that comes closest to containing the part of the curve near P.

The osculating circle at P is the circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature). It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

(continued)

8. (a) How do you find the velocity, speed, and acceleration of a particle that moves along a space curve?

If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(b) Write the acceleration in terms of its tangential and normal components.

$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = v'$ and $a_N = \kappa v^2$ ($v = |\mathbf{v}|$ is speed and κ is the curvature).

9. State Kepler's Laws.

- A planet revolves around the sun in an elliptical orbit with the sun at one focus.
- The line joining the sun to a planet sweeps out equal areas in equal times.
- The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

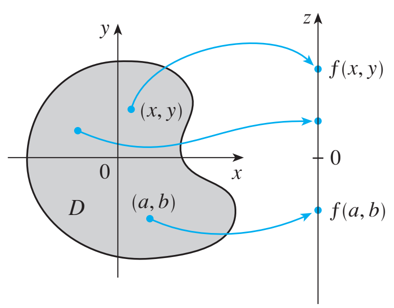
CHAPTER 14 CONCEPT CHECK ANSWERS

1. (a) What is a function of two variables?

A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

(b) Describe three methods for visualizing a function of two variables.

One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method is a contour map, consisting of level curves $f(x, y) = k$ (k a constant), which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as the one below.



2. What is a function of three variables? How can you visualize such a function?

A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. What does

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

mean? How can you show that such a limit does not exist?

$\lim_{(x, y)\to(a, b)} f(x, y) = L$ means that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) What does it mean to say that f is continuous at (a, b)?

A function f of two variables is continuous at (a, b) if

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

(b) If f is continuous on $\mathbb{R}^2$, what can you say about its graph?

If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) Write expressions for the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ as limits.

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

(b) How do you interpret $f_x(a, b)$ and $f_y(a, b)$ geometrically? How do you interpret them as rates of change?

If $f(a, b) = c$, then the point $P(a, b, c)$ lies on the surface S given by $z = f(x, y)$. We can interpret $f_x(a, b)$ as the slope of the tangent line at P to the curve of intersection of the vertical plane $y = b$ and S. In other words, if we restrict ourselves to the path along S through P that is parallel to the xz-plane, then $f_x(a, b)$ is the slope at P looking in the positive x-direction. Similarly, $f_y(a, b)$ is the slope of the tangent line at P to the curve of intersection of the vertical plane $x = a$ and S.

If $z = f(x, y)$, then $f_x(x, y)$ can be interpreted as the rate of change of z with respect to x when y is fixed. Thus $f_x(a, b)$ is the rate of change of z (with respect to x) when y is fixed at b and x is allowed to vary from a. Similarly, $f_y(a, b)$ is the rate of change of z (with respect to y) when x is fixed at a and y is allowed to vary from b.

(c) If $f(x, y)$ is given by a formula, how do you calculate f_x and f_y?

To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. What does Clairaut's Theorem say?

If f is a function of two variables that is defined on a disk D containing the point (a, b) and the functions f_{xy} and f_{yx} are both continuous on D, then Clairaut's Theorem states that $f_{xy}(a, b) = f_{yx}(a, b)$.

7. How do you find a tangent plane to each of the following types of surfaces?

(a) A graph of a function of two variables, $z = f(x, y)$

If f has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(b) A level surface of a function of three variables, $F(x, y, z) = k$

The tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

(continued)

8. Define the linearization of f at (a, b). What is the corresponding linear approximation? What is the geometric interpretation of the linear approximation?

The linearization of f at (a, b) is the linear function whose graph is the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear approximation of f at (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Geometrically, the linear approximation says that function values $f(x, y)$ can be approximated by values $L(x, y)$ from the tangent plane to the graph of f at $(a, b, f(a, b))$ when (x, y) is near (a, b).

9. (a) What does it mean to say that f is differentiable at (a, b)?

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. In other words, a differentiable function is one for which the linear approximation as stated above is a good approximation when (x, y) is near (a, b).

(b) How do you usually verify that f is differentiable?

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

10. If $z = f(x, y)$, what are the differentials dx, dy, and dz?

The differentials dx and dy are independent variables that can be given any values. If f is differentiable, the differential dz is then defined by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

11. State the Chain Rule for the case where $z = f(x, y)$ and x and y are functions of one variable. What if x and y are functions of two variables?

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

12. If z is defined implicitly as a function of x and y by an equation of the form $F(x, y, z) = 0$, how do you find $\partial z/\partial x$ and $\partial z/\partial y$?

If F is differentiable and $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

13. (a) Write an expression as a limit for the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$. How do you interpret it as a rate? How do you interpret it geometrically?

The directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

We can interpret it as the rate of change of f (with respect to distance) at (x_0, y_0) in the direction of $\mathbf{u}$.

Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction of $\mathbf{u}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is the slope of the tangent line to C at P.

(b) If f is differentiable, write an expression for $D_{\mathbf{u}} f(x_0, y_0)$ in terms of f_x and f_y.

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\,a + f_y(x_0, y_0)\,b$$

14. (a) Define the gradient vector ∇f for a function f of two or three variables.

If f is a function of two variables, then

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

For a function f of three variables,

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

(b) Express $D_{\mathbf{u}} f$ in terms of ∇f.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

or $$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

(continued)

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(c) Explain the geometric significance of the gradient.

The gradient vector of f gives the direction of maximum rate of increase of f. On the graph of $z = f(x, y)$, ∇f points in the direction of steepest ascent. Also, the gradient vector is perpendicular to the level curves or level surfaces of a function.

15. What do the following statements mean?

(a) f has a local maximum at (a, b).

f has a local maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b).

(b) f has an absolute maximum at (a, b).

f has an absolute maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b).

f has a local minimum at (a, b) if $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b).

f has an absolute minimum at (a, b) if $f(x, y) \geqslant f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b).

f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) If f has a local maximum at (a, b), what can you say about its partial derivatives at (a, b)?

If f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) What is a critical point of f?

A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. State the Second Derivatives Test.

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. The point (a, b) is a saddle point of f.

18. (a) What is a closed set in $\mathbb{R}^2$? What is a bounded set?

A closed set in $\mathbb{R}^2$ is one that contains all its boundary points. If one or more points on the boundary curve are omitted, the set is not closed.

A bounded set is one that is contained within some disk. In other words, it is finite in extent.

(b) State the Extreme Value Theorem for functions of two variables.

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

(c) How do you find the values that the Extreme Value Theorem guarantees?

- Find the values of f at the critical points of f in D.
- Find the extreme values of f on the boundary of D.
- The largest of the values from the above steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

19. Explain how the method of Lagrange multipliers works in finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. What if there is a second constraint $h(x, y, z) = c$?

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$], we first find all values of x, y, z, and λ where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$. (Thus we are finding the points from the constraint where the gradient vectors ∇f and ∇g are parallel.) Evaluate f at all the resulting points (x, y, z); the largest of these values is the maximum value of f, and the smallest is the minimum value of f.

If there is a second constraint $h(x, y, z) = c$, then we find all values of x, y, z, λ, and μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

Again we find the extreme values of f by evaluating f at the resulting points (x, y, z).

CHAPTER 15 CONCEPT CHECK ANSWERS

1. Suppose f is a continuous function defined on a rectangle $R = [a, b] \times [c, d]$.

(a) Write an expression for a double Riemann sum of f. If $f(x, y) \geqslant 0$, what does the sum represent?

A double Riemann sum of f is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x, y) \geqslant 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) Write the definition of $\iint_R f(x, y)\,dA$ as a limit.

$$\iint_R f(x, y)\,dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

(c) What is the geometric interpretation of $\iint_R f(x, y)\,dA$ if $f(x, y) \geqslant 0$? What if f takes on both positive and negative values?

If $f(x, y) \geqslant 0$, $\iint_R f(x, y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. If f takes on both positive and negative values, then $\iint_R f(x, y)\,dA$ is $V_1 - V_2$, where V_1 is the volume above R and below the surface $z = f(x, y)$, and V_2 is the volume below R and above the surface.

(d) How do you evaluate $\iint_R f(x, y)\,dA$?

We usually evaluate $\iint_R f(x, y)\,dA$ as an iterated integral according to Fubini's Theorem:

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

(e) What does the Midpoint Rule for double integrals say?

The Midpoint Rule for double integrals says that we approximate the double integral $\iint_R f(x, y)\,dA$ by the double Riemann sum $\sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$, where the sample points $(\bar{x}_i, \bar{y}_j)$ are the centers of the subrectangles.

(f) Write an expression for the average value of f.

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\,dA$$

where $A(R)$ is the area of R.

2. (a) How do you define $\iint_D f(x, y)\,dA$ if D is a bounded region that is not a rectangle?

Since D is bounded, it can be enclosed in a rectangular region R. We define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then we define

$$\iint_D f(x, y)\,dA = \iint_R F(x, y)\,dA$$

(b) What is a type I region? How do you evaluate $\iint_D f(x, y)\,dA$ if D is a type I region?

A region D is of type I if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Then

$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

(c) What is a type II region? How do you evaluate $\iint_D f(x, y)\,dA$ if D is a type II region?

A region D is of type II if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous on $[c, d]$. Then

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy$$

(d) What properties do double integrals have?

- $$\iint_D [f(x, y) + g(x, y)]\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA$$
- $$\iint_D cf(x, y)\,dA = c \iint_D f(x, y)\,dA$$
  where c is a constant
- If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then
  $$\iint_D f(x, y)\,dA \geqslant \iint_D g(x, y)\,dA$$
- If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then
  $$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$$
- $\iint_D 1\,dA = A(D)$, the area of D.
- If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then
  $$mA(D) \leqslant \iint_D f(x, y)\,dA \leqslant MA(D)$$

(continued)

3. How do you change from rectangular coordinates to polar coordinates in a double integral? Why would you want to make the change?

We may want to change from rectangular to polar coordinates in a double integral if the region D of integration is more easily described in polar coordinates:

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

To evaluate $\iint_R f(x, y)\, dA$, we replace x by $r\cos\theta$, y by $r\sin\theta$, and dA by $r\, dr\, d\theta$ (and use appropriate limits of integration):

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

4. If a lamina occupies a plane region D and has density function $\rho(x, y)$, write expressions for each of the following in terms of double integrals.

(a) The mass: $m = \iint_D \rho(x, y)\, dA$

(b) The moments about the axes:

$$M_x = \iint_D y\rho(x, y)\, dA \qquad M_y = \iint_D x\rho(x, y)\, dA$$

(c) The center of mass:

$$(\bar{x}, \bar{y}), \qquad \text{where } \bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

(d) The moments of inertia about the axes and the origin:

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

$$I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$$

5. Let f be a joint density function of a pair of continuous random variables X and Y.

(a) Write a double integral for the probability that X lies between a and b and Y lies between c and d.

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

(b) What properties does f possess?

$$f(x, y) \geqslant 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

(c) What are the expected values of X and Y?

The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA$

The expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$

6. Write an expression for the area of a surface with equation $z = f(x, y)$, $(x, y) \in D$.

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

(assuming that f_x and f_y are continuous).

7. (a) Write the definition of the triple integral of f over a rectangular box B.

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where ΔV is the volume of each sub-box and $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in each sub-box.

(b) How do you evaluate $\iiint_B f(x, y, z)\, dV$?

We usually evaluate $\iiint_B f(x, y, z)\, dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals: If f is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Note that there are five other orders of integration that we can use.

(c) How do you define $\iiint_E f(x, y, z)\, dV$ if E is a bounded solid region that is not a box?

Since E is bounded, it can be enclosed in a box B as described in part (b). We define a new function F with domain B by

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \text{ is in } E \\ 0 & \text{if } (x, y, z) \text{ is in } B \text{ but not in } E \end{cases}$$

Then we define

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

(continued)

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(d) What is a type 1 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A region E is of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\right] dA$$

(e) What is a type 2 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A type 2 region is of the form

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

where D is the projection of E onto the yz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx\right] dA$$

(f) What is a type 3 solid region? How do you evaluate $\iiint_E f(x, y, z)\, dV$ if E is such a region?

A type 3 region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy\right] dA$$

8. Suppose a solid object occupies the region E and has density function $\rho(x, y, z)$. Write expressions for each of the following.

(a) The mass:

$$m = \iiint_E \rho(x, y, z)\, dV$$

(b) The moments about the coordinate planes:

$$M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV$$

$$M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

(c) The coordinates of the center of mass:

$$(\bar{x}, \bar{y}, \bar{z}), \text{ where } \bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}$$

(d) The moments of inertia about the axes:

$$I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\, dV$$

$$I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\, dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\, dV$$

(continued)

9. (a) How do you change from rectangular coordinates to cylindrical coordinates in a triple integral?

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

where

$$E = \{(r, \theta, z) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \leqslant z \leqslant u_2(r\cos\theta, r\sin\theta)\}$$

Thus we replace x by $r\cos\theta$, y by $r\sin\theta$, dV by $r\,dz\,dr\,d\theta$, and use appropriate limits of integration.

(b) How do you change from rectangular coordinates to spherical coordinates in a triple integral?

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

where

$$E = \{(\rho, \theta, \phi) \mid \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d,\ g_1(\theta, \phi) \leqslant \rho \leqslant g_2(\theta, \phi)\}$$

Thus we replace x by $\rho\sin\phi\cos\theta$, y by $\rho\sin\phi\sin\theta$, z by $\rho\cos\phi$, dV by $\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$, and use appropriate limits of integration.

(c) In what situations would you change to cylindrical or spherical coordinates?

We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates. Regions that involve symmetry about the z-axis are often simpler to describe using cylindrical coordinates, and regions that are symmetrical about the origin are often simpler in spherical coordinates. Also, sometimes the integrand is easier to integrate using cylindrical or spherical coordinates.

10. (a) If a transformation T is given by $x = g(u, v)$, $y = h(u, v)$, what is the Jacobian of T?

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(b) How do you change variables in a double integral?

We change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Thus, under the appropriate conditions,

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

where R is the image of S under the transformation.

(c) How do you change variables in a triple integral?

Similarly to the case of two variables in part (b),

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

is the Jacobian.

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CHAPTER 16 CONCEPT CHECK ANSWERS

1. What is a vector field? Give three examples that have physical meaning.

A vector field is a function that assigns a vector to each point in its domain.

A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vector of the earth's gravitational field at a location in space.

2. (a) What is a conservative vector field?

A conservative vector field $\mathbf{F}$ is a vector field that is the gradient of some scalar function f, that is, $\mathbf{F} = \nabla f$.

(b) What is a potential function?

The function f in part (a) is called a potential function for $\mathbf{F}$.

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.

If C is given by the parametric equations $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$, we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width. The ith subinterval corresponds to a subarc of C with length Δs_i. Then

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

where (x_i^*, y_i^*) is any sample point in the ith subarc.

(b) How do you evaluate such a line integral?

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Similarly, if C is a smooth space curve, then

$$\int_C f(x, y, z)\, ds$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.

The mass is $m = \int_C \rho(x, y)\, ds$.

The center of mass is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m} \int_C x\,\rho(x, y)\, ds$$

$$\bar{y} = \frac{1}{m} \int_C y\,\rho(x, y)\, ds$$

(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.

$$\int_C f(x, y, z)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta x_i$$

$$\int_C f(x, y, z)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta y_i$$

$$\int_C f(x, y, z)\, dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$

(We have similar results when f is a function of two variables.)

(e) How do you evaluate these line integrals?

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$$

$$\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.

If $\mathbf{F}$ is a continuous vector field and C is given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

(b) If $\mathbf{F}$ is a force field, what does this line integral represent?

It represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) If $\mathbf{F} = \langle P, Q, R \rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

5. State the Fundamental Theorem for Line Integrals.

If C is a smooth curve given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, and f is a differentiable function whose gradient vector ∇f is continuous on C, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial points and the same terminal points.

(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

We know that $\mathbf{F}$ is a conservative vector field, that is, there exists a function f such that $\nabla f = \mathbf{F}$.

(continued)

7. State Green's Theorem.

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.

(a) Define curl $\mathbf{F}$.

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \nabla \times \mathbf{F}$$

(b) Define div $\mathbf{F}$.

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

At a point in the fluid, the vector curl $\mathbf{F}$ aligns with the axis about which the fluid tends to rotate, and its length measures the speed of rotation; div $\mathbf{F}$ at a point measures the tendency of the fluid to flow away (diverge) from that point.

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

If P and Q have continuous first-order derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

11. (a) What is a parametric surface? What are its grid curves?

A parametric surface S is a surface in $\mathbb{R}^3$ described by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

of two parameters u and v. Equivalent parametric equations are

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

The grid curves of S are the curves that correspond to holding either u or v constant.

(b) Write an expression for the area of a parametric surface.

If S is a smooth parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

where $(u, v) \in D$ and S is covered just once as (u, v) ranges throughout D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

(c) What is the area of a surface given by an equation $z = g(x, y)$?

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.

We divide S into "patches" S_{ij}. Then

$$\iint_S f(x, y, z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\,\Delta S_{ij}$$

where ΔS_{ij} is the area of the patch S_{ij} and P_{ij}^* is a sample point from the patch. (S is divided into patches in such a way that ensures that $\Delta S_{ij} \to 0$ as $m, n \to \infty$.)

(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?

$$\iint_S f(x, y, z)\,dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where D is the parameter domain of S.

(c) What if S is given by an equation $z = g(x, y)$?

$$\iint_S f(x, y, z)\,dS$$
$$= \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

The mass is

$$m = \iint_S \rho(x, y, z)\,dS$$

The center of mass is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x, y, z)\,dS$$
$$\bar{y} = \frac{1}{m}\iint_S y\,\rho(x, y, z)\,dS$$
$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x, y, z)\,dS$$

(continued)

CHAPTER 16 CONCEPT CHECK ANSWERS *(continued)*

13. (a) What is an oriented surface? Give an example of a nonorientable surface.

An oriented surface S is one for which we can choose a unit normal vector $\mathbf{n}$ at every point so that $\mathbf{n}$ varies continuously over S. The choice of $\mathbf{n}$ provides S with an orientation.

A Möbius strip is a nonorientable surface. (It has only one side.)

(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

We multiply by -1 if the opposite orientation of S is desired.

(d) What if S is given by an equation $z = g(x, y)$?

If $\mathbf{F} = \langle P, Q, R \rangle$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

for the upward orientation of S; we multiply by -1 for the downward orientation.

14. State Stokes' Theorem.

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

15. State the Divergence Theorem.

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

In each theorem, we integrate a "derivative" over a region, and this integral is equal to an expression involving the values of the original function only on the *boundary* of the region.

CHAPTER 17 CONCEPT CHECK ANSWERS

Cut here and keep for reference

1. (a) Write the general form of a second-order homogeneous linear differential equation with constant coefficients.

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

(b) Write the auxiliary equation.

$$ar^2 + br + c = 0$$

(c) How do you use the roots of the auxiliary equation to solve the differential equation? Write the form of the solution for each of the three cases that can occur.

If the auxiliary equation has two distinct real roots r_1 and r_2, the general solution of the differential equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

If the roots are real and equal, the solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

where r is the common root.

If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

2. (a) What is an initial-value problem for a second-order differential equation?

An initial-value problem consists of finding a solution y of the differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) What is a boundary-value problem for such an equation?

A boundary-value problem consists of finding a solution y of the differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) Write the general form of a second-order nonhomogeneous linear differential equation with constant coefficients.

$ay'' + by' + cy = G(x)$, where a, b, and c are constants and G is a continuous function.

(b) What is the complementary equation? How does it help solve the original differential equation?

The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the nonhomogeneous differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) Explain how the method of undetermined coefficients works.

To determine a particular solution y_p of $ay'' + by' + cy = G(x)$, we make an initial guess that y_p is a general function of the same type as G. If $G(x)$ is a polynomial, choose y_p to be a general polynomial of the same degree. If $G(x)$ is of the form Ce^{kx}, choose $y_p(x) = Ae^{kx}$. If $G(x)$ is $C \cos kx$ or $C \sin kx$, choose $y_p(x) = A \cos kx + B \sin kx$. If $G(x)$ is a product of functions, choose y_p to be a product of functions of the same type. Some examples are:

$G(x)$	$y_p(x)$
x^2	$Ax^2 + Bx + C$
e^{2x}	Ae^{2x}
$\sin 2x$	$A \cos 2x + B \sin 2x$
xe^{-x}	$(Ax + B)e^{-x}$

We then substitute y_p, y_p', and y_p'' into the differential equation and determine the coefficients.

If y_p happens to be a solution of the complementary equation, then multiply the initial trial solution by x (or x^2 if necessary).

If $G(x)$ is a sum of functions, we find a particular solution for each function and then y_p is the sum of these.

The general solution of the differential equation is

$$y(x) = y_p(x) + y_c(x)$$

(d) Explain how the method of variation of parameters works.

We write the solution of the complementary equation $ay'' + by' + cy = 0$ as $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, where y_1 and y_2 are linearly independent solutions. We then take $y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$ as a particular solution, where $u_1(x)$ and $u_2(x)$ are arbitrary functions. After computing y_p', we impose the condition that

$$u_1' y_1 + u_2' y_2 = 0 \qquad \textbf{(1)}$$

and then compute y_p''. Substituting y_p, y_p', and y_p'' into the original differential equation gives

$$a(u_1' y_1' + u_2' y_2') = G \qquad \textbf{(2)}$$

We then solve equations **(1)** and **(2)** for the unknown functions u_1' and u_2'. If we are able to integrate these functions, then a particular solution is $y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$ and the general solution is $y(x) = y_p(x) + y_c(x)$.

4. Discuss two applications of second-order linear differential equations.

The motion of an object with mass m at the end of a spring is an example of simple harmonic motion and is described by the second-order linear differential equation

$$m\frac{d^2x}{dt^2} + kx = 0$$

(continued)

where k is the spring constant and x is the distance the spring is stretched (or compressed) from its natural length. If there are external forces acting on the spring, then the differential equation is modified.

Second-order linear differential equations are also used to analyze electrical circuits involving an electromotive force, a resistor, an inductor, and a capacitor in series.

See the discussion in Section 17.3 for additional details.

5. How do you use power series to solve a differential equation?

We first assume that the differential equation has a power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Differentiating gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n + 1)c_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2} x^n$$

We substitute these expressions into the differential equation and equate the coefficients of x^n to find a recursion relation involving the constants c_n. Solving the recursion relation gives a formula for c_n and then

$$y = \sum_{n=0}^{\infty} c_n x^n$$

is the solution of the differential equation.

LIES MY CALCULATOR AND COMPUTER TOLD ME

■ See Section 1.4 for a discussion of graphing calculators and computers with graphing software.

A wide variety of pocket-size calculating devices are currently marketed. Some can run programs prepared by the user; some have preprogrammed packages for frequently used calculus procedures, including the display of graphs. All have certain limitations in common: a limited range of magnitude (usually less than 10^{100} for calculators) and a bound on accuracy (typically eight to thirteen digits).

A calculator usually comes with an owner's manual. Read it! The manual will tell you about further limitations (for example, for angles when entering trigonometric functions) and perhaps how to overcome them.

Program packages for microcomputers (even the most fundamental ones, which realize arithmetical operations and elementary functions) often suffer from hidden flaws. You will be made aware of some of them in the following examples, and you are encouraged to experiment using the ideas presented here.

PRELIMINARY EXPERIMENTS WITH YOUR CALCULATOR OR COMPUTER

To have a first look at the limitations and quality of your calculator, make it compute $2 \div 3$. Of course, the answer is not a terminating decimal so it can't be represented exactly on your calculator. If the last displayed digit is 6 rather than 7, then your calculator approximates $\frac{2}{3}$ by truncating instead of rounding, so be prepared for slightly greater loss of accuracy in longer calculations.

Now multiply the result by 3; that is, calculate $(2 \div 3) \times 3$. If the answer is 2, then subtract 2 from the result, thereby calculating $(2 \div 3) \times 3 - 2$. Instead of obtaining 0 as the answer, you might obtain a small negative number, which depends on the construction of the circuits. (The calculator keeps, in this case, a few "spare" digits that are remembered but not shown.) This is all right because, as previously mentioned, the finite number of digits makes it impossible to represent $2 \div 3$ exactly.

A similar situation occurs when you calculate $(\sqrt{6})^2 - 6$. If you do not obtain 0, the order of magnitude of the result will tell you how many digits the calculator uses internally.

Next, try to compute $(-1)^5$ using the y^x key. Many calculators will indicate an error because they are built to attempt $e^{5 \ln(-1)}$. One way to overcome this is to use the fact that $(-1)^k = \cos k\pi$ whenever k is an integer.

Calculators are usually constructed to operate in the decimal number system. In contrast, some microcomputer packages of arithmetical programs operate in a number system with base other than 10 (typically 2 or 16). Here the list of unwelcome tricks your device can play on you is even larger, since not all terminating decimal numbers are represented exactly. A recent implementation of the BASIC language shows (in double precision) examples of incorrect conversion from one number system into another, for example,

$$8 \times 0.1 \stackrel{?}{=} 0.79999\,99999\,99999\,9$$

whereas $$19 \times 0.1 \stackrel{?}{=} 1.90000\,00000\,00001$$

Yet another implementation, apparently free of the preceding anomalies, will not calculate standard functions in double precision. For example, the number $\pi = 4 \times \tan^{-1}1$, whose representation with sixteen decimal digits should be $3.14159\,26535\,89793$, appears as $3.14159\,29794\,31152$; this is off by more than 3×10^{-7}. What is worse, the cosine function is programmed so badly that its "cos" $0 = 1 + 2^{-23}$. (Can you invent a situation when this could ruin your calculations?) These or similar defects exist in other programming languages too.

THE PERILS OF SUBTRACTION

You might have observed that subtraction of two numbers that are close to each other is a tricky operation. The difficulty is similar to this thought exercise: Imagine that you walk blindfolded 100 steps forward and then turn around and walk 99 steps. Are you sure that you end up exactly one step from where you started?

The name of this phenomenon is "loss of significant digits." To illustrate, let's calculate

$$8721\sqrt{3} - 10{,}681\sqrt{2}$$

The approximations from my calculator are

$$8721\sqrt{3} \approx 15105.21509 \qquad \text{and} \qquad 10{,}681\sqrt{2} \approx 15105.21506$$

and so we get $8721\sqrt{3} - 10{,}681\sqrt{2} \approx 0.00003$. Even with three spare digits exposed, the difference comes out as 0.00003306. As you can see, the two ten-digit numbers agree in nine digits that, after subtraction, become zeros before the first nonzero digit. To make things worse, the formerly small errors in the square roots become more visible. In this particular example we can use rationalization to write

$$8721\sqrt{3} - 10{,}681\sqrt{2} = \frac{1}{8721\sqrt{3} + 10{,}681\sqrt{2}}$$

(work out the details!) and now the loss of significant digits doesn't occur:

$$\frac{1}{8721\sqrt{3} + 10{,}681\sqrt{2}} \approx 0.00003310115 \qquad \text{to seven digits}$$

(It would take too much space to explain why all seven digits are reliable; the subject *numerical analysis* deals with these and similar situations.) See Exercise 7 for another instance of restoring lost digits.

Now you can see why in Exercises 2.2 (Exercise 44) your guess at the limit of $(\tan x - x)/x^3$ was bound to go wrong: $\tan x$ becomes so close to x that the values will eventually agree in all digits that the calculator is capable of carrying. Similarly, if you start with just about any continuous function f and try to guess the value of

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

long enough using a calculator, you will end up with a zero, despite all the rules in Chapter 2!

WHERE CALCULUS IS MORE POWERFUL THAN CALCULATORS AND COOMPUTERS

One of the secrets of success of calculus in overcoming the difficulties connected with subtraction is symbolic manipulation. For instance, $(a + b) - a$ is always b, although the calculated value may be different. Try it with $a = 10^7$ and $b = \sqrt{2} \times 10^{-5}$. Another powerful tool is the use of inequalities; a good example is the Squeeze Theorem as demonstrated in Section 2.3. Yet another method for avoiding computational difficulties is provided by the Mean Value Theorem and its consequences, such as l'Hospital's Rule (which helps solve the aforementioned exercise and others) and Taylor's Inequality.

The limitations of calculators and computers are further illustrated by infinite series. A common misconception is that a series can be summed by adding terms until there is "practically nothing to add" and "the error is less than the first neglected term." The latter statement is true for certain alternating series (see the Alternating Series Estimation Theorem) but not in general; a modified version is true for another class of series (Exercise 10). As an example to refute these misconceptions, let's consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$$

which is a convergent p-series ($p = 1.001 > 1$). Suppose we were to try to sum this series, correct to eight decimal places, by adding terms until they are less than 5 in the ninth decimal place. In other words, we would stop when

$$\frac{1}{n^{1.001}} < 0.00000\,0005$$

that is, when $n = N = 196{,}217{,}284$. (This would require a high-speed computer and increased precision.) After going to all this trouble, we would end up with the approximating partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n^{1.001}} < 19.5$$

But, from the proof of the Integral Test, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.001}} > \int_1^{\infty} \frac{dx}{x^{1.001}} = 1000$$

Thus, the machine result represents less than 2% of the correct answer!

Suppose that we then wanted to add a huge number of terms of this series, say, 10^{100} terms, in order to approximate the infinite sum more closely. (This number 10^{100}, called a *googol*, is outside the range of pocket calculators and is much larger than the number of elementary particles in our solar system.) If we were to add 10^{100} terms of the above series (only in theory; a million years is less than 10^{26} microseconds), we would still obtain a sum of less than 207 compared with the true sum of more than 1000. (This estimate of 207 is obtained by using a more precise form of the Integral Test, known as the Euler-Maclaurin Formula, and only then using a calculator. The formula provides a way to accelerate the convergence of this and other series.)

If the two preceding approaches didn't give the right information about the accuracy of the partial sums, what does? A suitable inequality satisfied by the remainder of the series, as you can see from Exercise 6.

Computers and calculators are not replacements for mathematical thought. They are just replacements for some kinds of mathematical labor, either numerical or symbolic. There are, and always will be, mathematical problems that can't be solved by a calculator or computer, regardless of its size and speed. A calculator or computer does stretch the human capacity for handling numbers and symbols, but there is still considerable scope and necessity for "thinking before doing."

EXERCISES

A Click here for answers. S Click here for solutions.

1. Guess the value of

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$$

and determine when to stop guessing before the loss of significant digits destroys your result. (The answer will depend on your calculator.) Then find the precise answer using an appropriate calculus method.

2. Guess the value of

$$\lim_{h\to 0}\frac{\ln(1+h)}{h}$$

and determine when to stop guessing before the loss of significant digits destroys your result. This time the detrimental subtraction takes place inside the machine; explain how (assuming that the Taylor series with center $a = 1$ is used to approximate $\ln x$). Then find the precise answer using an appropriate calculus method.

3. Even innocent-looking calculus problems can lead to numbers beyond the calculator range. Show that the maximum value of the function

$$f(x) = \frac{x^2}{(1.0001)^x}$$

is greater than 10^{124}. [*Hint:* Use logarithms.] What is the limit of $f(x)$ as $x \to \infty$?

4. What is a numerically reliable expression to replace $\sqrt{1 - \cos x}$, especially when x is a small number? You will need to use trigonometric identities. (Recall that some computer packages would signal an unnecessary error condition, or even switch to complex arithmetic, when $x = 0$.)

5. Try to evaluate

$$D = \ln\ln(10^9 + 1) - \ln\ln(10^9)$$

on your calculator. These numbers are so close together that you will likely obtain 0 or just a few digits of accuracy. However, we can use the Mean Value Theorem to achieve much greater accuracy.

(a) Let $f(x) = \ln \ln x$, $a = 10^9$, and $b = 10^9 + 1$. Then the Mean Value Theorem gives

$$f(b) - f(a) = f'(c)(b - a) = f'(c)$$

where $a < c < b$. Since f' is decreasing, we have $f'(a) > f'(c) > f'(b)$. Use this to estimate the value of D.

(b) Use the Mean Value Theorem a second time to discover why the quantities $f'(a)$ and $f'(b)$ in part (a) are so close to each other.

6. For the series $\Sigma_{n=1}^{\infty} n^{-1.001}$, studied in the text, exactly how many terms do we need (in theory) to make the error less than 5 in the ninth decimal place? You can use the inequalities from the proof of the Integral Test:

$$\int_{N+1}^{\infty} f(x)\,dx < \sum_{n=N+1}^{\infty} f(n) < \int_{N}^{\infty} f(x)\,dx$$

7. Archimedes found an approximation to 2π by considering the perimeter p of a regular 96-gon inscribed in a circle of radius 1. His formula, in modern notation, is

$$p = 96\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}$$

(a) Carry out the calculations and compare with the value of p from more accurate sources, say $p = 192\sin(\pi/96)$. How many digits did you lose?

(b) Perform rationalization to avoid subtraction of approximate numbers and count the exact digits again.

8. This exercise is related to Exercise 2. Suppose that your computing device has an excellent program for the exponential function $\exp(x) = e^x$ but a poor program for $\ln x$. Use the identity

$$\ln a = b + \ln\left(1 + \frac{a - e^b}{e^b}\right)$$

and Taylor's Inequality to improve the accuracy of $\ln x$.

9. The cubic equation

$$x^3 + px + q = 0$$

where we assume for simplicity that $p > 0$, has a classical solution formula for the real root, called Cardano's formula:

$$x = \frac{1}{3}\left[\left(\frac{27q + \sqrt{729q^2 + 108p^3}}{2}\right)^{1/3} + \left(\frac{27q - \sqrt{729q^2 + 108p^3}}{2}\right)^{1/3}\right]$$

For a user of a pocket-size calculator, as well as for an inexperienced programmer, the solution presents several stumbling blocks. First, the second radicand is negative and the fractional power key or routine may not handle it. Next, even if we fix the negative radical problem, when q is small in magnitude and p is of moderate size, the small number x is the difference of two numbers close to $\sqrt{p/3}$.

(a) Show that all these troubles are avoided by the formula

$$x = \frac{-9q}{a^{2/3} + 3p + 9p^2 a^{-2/3}}$$

where $$a = \frac{27|q| + \sqrt{729q^2 + 108p^3}}{2}$$

Hint: Use the factorization formula

$$A + B = \frac{A^3 + B^3}{A^2 - AB + B^2}$$

(b) Evaluate

$$u = \frac{4}{(2 + \sqrt{5})^{2/3} + 1 + (2 + \sqrt{5})^{-2/3}}$$

If the result is simple, relate it to part (a), that is, restore the cubic equation whose root is u written in this form.

10. (a) Consider the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{100^n + 1}$$

It is easy to show that its radius of convergence is $r = 100$. The series will converge rather slowly at $x = 99$: find out how many terms will make the error less than 5×10^{-7}.

(b) We can speed up the convergence of the series in part (a). Show that

$$f(x) = \frac{x}{100 - x} - f\left(\frac{x}{100}\right)$$

and find the number of the terms of this transformed series that leads to an error less than 5×10^{-7}. [*Hint:* Compare with the series $\Sigma_{n=1}^{\infty}(x^n/100^n)$, whose sum you know.]

11. The positive numbers

$$a_n = \int_0^1 e^{1-x} x^n\,dx$$

can, in theory, be calculated from a reduction formula obtained by integration by parts: $a_0 = e - 1$, $a_n = na_{n-1} - 1$. Prove, using $1 \leqslant e^{1-x} \leqslant e$ and the Squeeze Theorem, that $\lim_{n\to\infty} a_n = 0$. Then try to calculate a_{20} from the reduction formula using your calculator. What went wrong?

The initial term $a_0 = e - 1$ can't be represented exactly in a calculator. Let's call c the approximation of $e - 1$ that we can enter. Verify from the reduction formula (by observing the pattern after a few steps) that

$$a_n = \left[c - \left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right]n!$$

and recall from our study of Taylor and Maclaurin series that

$$\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n}$$

converges to $e - 1$ as $n \to \infty$. The expression in square brackets

converges to $c - (e - 1)$, a nonzero number, which gets multiplied by a fast-growing factor $n!$. We conclude that even if all further calculations (after entering a_0) were performed without errors, the initial inaccuracy would cause the computed sequence $\{a_n\}$ to diverge.

12. (a) A consolation after the catastrophic outcome of Exercise 11: If we rewrite the reduction formula to read

$$a_{n-1} = \frac{1 + a_n}{n}$$

we can use the inequality used in the squeeze argument to obtain improvements of the approximations of a_n. Try a_{20} again using this reverse approach.

(b) We used the reversed reduction formula to calculate quantities for which we have elementary formulas. To see that the idea is even more powerful, develop it for the integrals

$$\int_0^1 x^{n-\theta} e^{1-x}\, dx$$

where θ is a constant, $0 < \theta < 1$, and $n = 0, 1, \ldots$. For such θ the integrals are no longer elementary (not solvable in "finite terms"), but the numbers can be calculated quickly. Find the integrals for the particular choice $\theta = \frac{1}{3}$ and $n = 0, 1, \ldots, 5$ to five digits of accuracy.

13. An advanced calculator has a key for a peculiar function:

$$E(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \end{cases}$$

After so many warnings about the subtraction of close numbers, you may appreciate that the definition

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

gives inaccurate results for small x, where $\sinh x$ is close to x. Show that the use of the accurately evaluated function $E(x)$ helps restore the accuracy of $\sinh x$ for small x.

ANSWERS

S Click here for solutions.

1. $\frac{1}{3}$ **3.** 0 **5.** (a) $4.82549424 \times 10^{-11}$

9. (b) 1, $x^3 + 3x - 4 = 0$

SOLUTIONS

1. The computer results are from Maple, with `Digits:=16`. The last column shows the values of the sixth-degree Taylor polynomial for $f(x) = \csc^2 x - x^{-2}$ near $x = 0$. Note that the second arrangement of Taylor's polynomial is easier to use with a calculator:

$$T_6(x) = \tfrac{1}{3} + \tfrac{1}{15}x^2 + \tfrac{2}{289}x^4 + \tfrac{1}{675}x^6 = \left[\left(\tfrac{1}{675}x^2 + \tfrac{2}{289}\right)x^2 + \tfrac{1}{15}\right]x^2 + \tfrac{1}{3}$$

x	$f(x)_{\text{calculator}}$	$f(x)_{\text{computer}}$	$f(x)_{\text{Taylor}}$
0.1	0.33400107	0.3340010596845	0.33400106
0.01	0.333341	0.33334000010	0.33334000
0.001	0.3334	0.333333400	0.33333340
0.0001	0.34	0.3333334	0.33333333
0.00001	2.0	0.33334	0.33333333
0.000001	100 or 200	0.334	0.33333333
0.0000001	10,000 or 20,000	0.4	0.33333333
0.00000001	1,000,000	0	0.33333333

We see that the calculator results start to deteriorate seriously at $x = 0.0001$, and for smaller x, they are entirely meaningless. The different results "100 or 200" etc. depended on whether we calculated $[(\sin x)^2]^{-1}$ or $[(\sin x)^{-1}]^2$. With Maple, the result is off by more than 10% when $x = 0.0000001$ (compare with the calculator result!) A detailed analysis reveals that the values of the function are always greater than $\frac{1}{3}$, but the computer eventually gives results less than $\frac{1}{3}$.

The polynomial $T_6(x)$ was obtained by patient simplification of the expression for $f(x)$, starting with $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$, where $\cos 2x = 1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \cdots - \dfrac{(2x)^{10}}{10!} + R_{12}(x)$. Consequently, the exact value of the limit is $T_6(0) = \frac{1}{3}$. It can also be obtained by several applications of l'Hospital's Rule to the expression $f(x) = \dfrac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ with intermediate simplifications.

3. From $f(x) = \dfrac{x^{25}}{(1.0001)^x}$ (We may assume $x > 0$; Why?), we have $\ln f(x) = 25 \ln x - x \ln(1.0001)$ and $\dfrac{f'(x)}{f(x)} = \dfrac{25}{x} - \ln(1.0001)$. This derivative, as well as the derivative $f'(x)$ itself, is positive for $0 < x < x_0 = \dfrac{25}{\ln(1.0001)} \approx 249{,}971.015$, and negative for $x > x_0$. Hence the maximum value of $f(x)$ is $f(x_0) = \dfrac{x_0^{25}}{(1.0001)^{x_0}}$, a number too large to be calculated directly. Using decimal logarithms, $\log_{10} f(x_0) \approx 124.08987757$, so that $f(x_0) \approx 1.229922 \times 10^{124}$. The actual value of the limit is $\lim_{x\to\infty} f(x) = 0$; it would be wasteful and inelegant to use l'Hospital's Rule twenty-five times since we can transform $f(x)$ into $f(x) = \left(\dfrac{x}{(1.0001)^{x/25}}\right)^{25}$, and the inside expression needs just one application of l'Hospital's Rule to give 0.

5. For $f(x) = \ln \ln x$ with $x \in [a, b]$, $a = 10^9$, and $b = 10^9 + 1$, we need $f'(x) = \dfrac{1}{x \ln x}$, $f''(x) = -\dfrac{\ln x + 1}{x^2 (\ln x)^2}$.

(a) $f'(b) < D < f'(a)$, where $f'(a) \approx 4.8254942434 \times 10^{-11}$, $f'(b) \approx 4.8254942383 \times 10^{-11}$.

(b) Let's estimate $f'(b) - f'(a) = (b - a) f''(c_1) = f''(c_1)$. Since f'' increases (its absolute value decreases), we have $|f'(b) - f'(a)| \le |f''(a)| \approx 5.0583 \times 10^{-20}$.

7. (a) The 11-digit calculator value of $192 \sin \frac{\pi}{96}$ is 6.2820639018, while the value (on the same device) of p before rationalization is 6.282063885, which is 1.68×10^{-8} less than the trigonometric result.

(b) $$p = \frac{96}{\sqrt{2+\sqrt{3}} \cdot \sqrt{2+\sqrt{2+\sqrt{3}}} \cdot \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} \cdot \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}$$

but of course we can avoid repetitious calculations by storing intermediate results in a memory:

$$p_1 = \sqrt{2+\sqrt{3}}, \quad p_2 = \sqrt{2+p_1}, \quad p_3 = \sqrt{2+p_2}, \quad p_4 = \sqrt{2+p_3}, \quad \text{and so} \quad p = \frac{96}{p_1 p_2 p_3 p_4}$$

According to this formula, a calculator gives $p \approx 6.2820639016$, which is within 2×10^{-10} of the trigonometric result. With `Digits:=16;`, Maple gives $p \approx 6.282063901781030$ before rationalization (off the trig result by about 1.1×10^{-14}) and $p \approx 6.282063901781018$ after rationalization (error of about 1.7×10^{-15}), a gain of about one digit of accuracy for rationalizing. If we sets `Digits:=100;`, the difference between Maple's calculation of $192 \sin \frac{\pi}{96}$ and the radical is only about 4×10^{-99}.

9. (a) Let $A = \left[\frac{1}{2}\left(27q + \sqrt{729q^2 + 108p^3}\right)\right]^{1/3}$ and $B = \left[\frac{1}{2}\left(27q - \sqrt{729q^2 + 108p^3}\right)\right]^{1/3}$.

Then $A^3 + B^3 = 27q$ and $AB = \frac{1}{4}[729q^2 - (729q^2 + 108p^3)]^{1/3} = -3p$. Substitute into the formula $A + B = \dfrac{A^3 + B^3}{A^2 - AB + B^2}$ where we replace B by $\dfrac{-3p}{A}$:

$$x = \tfrac{1}{3}(A+B) = \frac{27q/3}{\left[\frac{1}{2}\left(27q + \sqrt{729q^2+108p^3}\right)\right]^{2/3} + 3p + 9p^2\left[\frac{1}{2}\left(27q + \sqrt{729q^2+108p^3}\right)\right]^{-2/3}}$$

which almost yields the given formula; since replacing q by $-q$ results in replacing x by $-x$, a simple discussion of the cases $q > 0$ and $q < 0$ allows us to replace q by $|q|$ in the denominator, so that it involves only positive numbers. The problems mentioned in the introduction to this exercise have disappeared.

(b) A direct attack works best here. To save space, let $\alpha = 2 + \sqrt{5}$, so we can rationalize, using $\alpha^{-1} = 2 + \sqrt{5}$ and $\alpha - \alpha^{-1} = 4$ (check it!):

$$u = \frac{4}{\alpha^{2/3} + 1 + \alpha^{-2/3}} \cdot \frac{\alpha^{1/3} - \alpha^{-1/3}}{\alpha^{1/3} - \alpha^{-1/3}} = \frac{4(\alpha^{1/3} - \alpha^{-1/3})}{\alpha - \alpha^{-1}} = \alpha^{1/3} - \alpha^{-1/3}$$

and we cube the expression for u: $u^3 = \alpha - 3\alpha^{1/3} + 3\alpha^{-1/3} - \alpha^{-1} = 4 - 3u$, $u^3 + 3u - 4 = (u-1)(u^2 + u + 4) = 0$, so that the only real root is $u = 1$. A check using the formula from part (a): $p = 3$, $q = -4$, so $729q^2 + 108p^3 = 14{,}580 = 54^2 \times 5$, and

$$x = \frac{36}{\left(54 + 27\sqrt{5}\right)^{2/3} + 9 + 81\left(54 + 27\sqrt{5}\right)^{-2/3}},$$ which simplifies to the given form.

11. Proof that $\lim_{n\to\infty} a_n = 0$: From $1 \le e^{1-x} \le e$ it follows that $x^n \le e^{1-x}x^n \le x^n e$, and integration gives

$$\frac{1}{n+1} = \int_0^1 x^n\,dx \le \int_0^1 e^{1-x}x^n\,dx \le \int_0^1 x^n e\,dx = \frac{e}{n+1},$$ that is, $\frac{1}{n+1} \le a_n \le \frac{e}{n+1}$, and since

$\lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{e}{n+1} = 0$, it follows from the Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$. Of course, the expression $1/(n+1)$ on the left side could have been replaced by 0 and the proof would still be correct.

Calculations: Using the formula $a_n = \left[e - 1 - \left(\frac{1}{1!} + \frac{2}{2!} + \cdots + \frac{1}{n!}\right)\right] n!$ with an 11-digit calculator:

n	a_n	n	a_n	n	a_n
0	1.7182818284	7	0.1404151360	14	−5.07636992
1	0.7182818284	8	0.1233210880	15	−77.1455488
2	0.4365636568	9	0.1098897920	16	−1235.3287808
3	0.3096909704	10	0.0988979200	17	−21001.589274
4	0.2387638816	11	0.0878771200	18	−378029.60693
5	0.1938194080	12	0.0545254400	19	−7182563.5317
6	0.1629164480	13	−0.2911692800	20	−143651271.63

It is clear that the values calculated from the direct reduction formula will diverge to $-\infty$. If we instead calculate a_n using the reduction formula in Maple (with `Digits:=16`), we get some odd results: $a_{20} = -1000$, $a_{28} = 10^{14}$, $a_{29} = 0$, and $a_{30} = 10^{17}$, for examples. But for larger n, the results are at least small and positive (for example, $a_{1000} \approx 0.001$.) For $n > 32{,}175$, we get the delightful `object too large` error message. If, instead of using the reduction formula, we integrate directly with Maple, the results are much better.

13. We can start by expressing e^x and e^{-x} in terms of $E(x) = (e^x - 1)/x$ $(x \ne 0)$, where $E(0) = 1$ to make E continuous at 0 (by l'Hospital's Rule). Namely, $e^x = 1 + xE(x)$, $e^{-x} = 1 - xE(-x)$ and

$\sinh x = \frac{1 + xE(x) - [1 - xE(-x)]}{2} = \frac{1}{2}x[E(x) + E(-x)]$, where the addition involves only positive numbers $E(x)$ and $E(-x)$, thus presenting no loss of accuracy due to subtraction.

Another form, which calls the function E only once:

We write $\sinh x = \frac{(e^x)^2 - 1}{2e^x} = \frac{[1 + xE(x)]^2 - 1}{2[1 + xE(x)]} = \frac{x\left[1 + \frac{1}{2}\,|x|\,E(|x|)\right]\,E(|x|)}{1 + |x|\,E(|x|)}$, taking advantage of the fact that $\frac{\sinh x}{x}$ is an even function, so replacing x by $|x|$ does not change its value.